Math. Res. Lett.
Volume 25, Number 6, 1977–1992, 2018

Logarithmic vector fields for curve configurations in $\mathbb{P}^2$ with quasihomogeneous singularities

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Let $\mathcal{A} = \bigcup_{i=1}^{r} C_i \subseteq \mathbb{P}^2_{\mathbb{C}}$ be a collection of plane curves, such that each singular point of $\mathcal{A}$ is quasihomogeneous. We prove that if C is an irreducible curve having only quasihomogeneous singularties, such that $C \cap \mathcal{A} \subseteq C_{sm}$ and every singular point of $\mathcal{A} \cup C$ is quasihomogeneous, then there is a short exact sequence relating the $\mathcal{O}_{\mathbb{P}^2}$–module $\mathrm{Der}(-\log \mathcal{A})$ of vector fields on $\mathbb{P}^2$ tangent to $\mathcal{A}$ to the module $\mathrm{Der}(-\log \mathcal{A} \cup C)$. This yields an inductive tool for studying the splitting of the bundles $\mathrm{Der}(-\log \mathcal{A})$ and $\mathrm{Der}(-\log \mathcal{A} \cup C)$, depending on the geometry of the divisor $\mathcal{A}|_C$ on C.

1. Introduction

For a divisor Y in a complex manifold X, Saito [16] introduced the sheaves of logarithmic vector fields and logarithmic one forms with pole along Y:

Definition 1.1. The module of logarithmic vector fields is the sheaf of $\mathcal{O}_X$–modules

$$\mathrm{Der}(-\log Y)_p = \{\theta \in \mathrm{Der}_{\mathbb{C}}(X) | \theta(f) \in \langle f \rangle\},$$

where $f \in \mathcal{O}_{X,p}$ is a local defining equation for Y at p. If $\{x_1, \ldots, x_d\}$ are local coordinates at $p \in X$, then the Jacobian scheme $J_Y(Y)$ is defined locally at p by $\{\partial f/\partial_{x_1}, \ldots, \partial f/\partial_{x_d}, f\}$.

Saito's work generalized earlier work of Deligne [3], who studied the case for Y a normal crossing divisor. In the setting of Definition 1.1, $\mathrm{Der}(-\log Y)_p$ is the kernel of the evaluation map $\theta \mapsto \theta(f) \in \mathcal{O}_{Y,p}$, so there is a short exact sequence [12]

$$0 \longrightarrow \mathrm{Der}(-\mathrm{log}\ Y) \longrightarrow \mathcal{T}_X \longrightarrow J_Y(Y) \longrightarrow 0.$$

Saito shows that $\mathrm{Der}(-\log Y)_p$ is a free $\mathcal{O}_{X,p}$ module iff there exist d elements

$$\theta_i = \sum_{j=1}^{d} f_{ij} \frac{\partial}{\partial x_j} \in \mathrm{Der}(-\log Y)_p$$

such that the determinant of the matrix $[f_{ij}]$ is a unit multiple of the local defining equation for Y; this is basically a consequence of the Hilbert-Burch theorem. A much studied version of this construction occurs when $Y \subseteq \mathbb{P}^d$ is a reduced hypersurface with $I_Y = F$; $V(F)$ may also be studied as a hypersurface in $\mathbb{C}^{d+1}$.

Definition 1.2. For homogeneous $F \in S = \mathbb{C}[x_0, \ldots, x_d]$ with $I_{V(F)} = F$, define $D(V(F)) = \{\theta \in Der_{\mathbb{C}}(S) | \theta(F) \in \langle F \rangle\}$.

Definition 1.2 is a global version of Definition 1.1, and plays a key role [15] in the study of hyperplane arrangements. We first relate the two definitions. Since F is homogeneous, $D(V(F))$ is a graded S–module, with associated sheaf $\mathcal{D}(V(F))$ on $\mathbb{P}^d$. The evaluation map $D(V(F)) \stackrel{\mathrm{ev}}{\to} S$ is defined by $\mathrm{ev}(\theta) = \theta(F)$; the kernel of ev is the syzygy module $D_0(V(F))$ of the Jacobian ideal $J_F = \langle \partial F/\partial_{x_0}, \ldots, \partial F/\partial_{x_d} \rangle$. If F has degree n, then the Euler vector field $E = \sum x_i \partial/\partial x_i$ satisfies $\mathrm{ev}(E) = E(F) = nF$, yielding a surjection

$$D(V(F)) \to \langle F \rangle \to 0,$$

and E generates a free summand $S(-1) \subseteq D(V(F))$, so

$$D(V(F)) \simeq D_0(V(F)) \oplus S(-1) \text{ and } \mathcal{D}(V(F)) \simeq \mathcal{D}_0(V(F)) \oplus \mathcal{O}(-1).$$

The exact sequence

$$0 \longrightarrow D_0(V(F)) \longrightarrow S^{d+1} \stackrel{\mathrm{ev}}{\longrightarrow} S(n-1) \longrightarrow S(n-1)/J_F \longrightarrow 0$$

shows that $\mathcal{D}(V(F))$ is a second syzygy sheaf, so when $d = 2$, $\mathcal{D}(V(F))$ is a vector bundle on $\mathbb{P}^2$. Since the depth of $D(V(F))$ is at least two,

$D(V(F)) = \Gamma_*(\mathcal{D}(V(F)))$. Tensoring $\mathcal{D}(V(F))$ with $\mathcal{O}_{\mathbb{P}^d}(1)$ yields the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^d} & \longrightarrow & \mathcal{O}_{\mathbb{P}^d} & \longrightarrow & 0 \\
 & & \downarrow {\scriptsize\begin{bmatrix} x_0 \\ \vdots \\ x_d \end{bmatrix}} & & \downarrow {\scriptsize\begin{bmatrix} x_0 \\ \vdots \\ x_d \end{bmatrix}} & & \downarrow \\
0 & \longrightarrow & \mathcal{D}(V(F))(1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^d}^{d+1}(1) & \xrightarrow{\gamma} & \mathcal{O}_{V(F)} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{D}(V(F))(1)/\mathcal{O}_{\mathbb{P}^d} & \longrightarrow & \mathcal{T}_{\mathbb{P}^d} & \longrightarrow & \mathcal{O}_{V(F)} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
\end{array}$$

The map γ sends $\theta = \sum f_i e_i$ to $\sum f_i \partial/\partial x_i(F)$ so γ takes the image of $\mathcal{D}(V(F))(1)$ onto $J_F(n)$. Hence,

$$\text{(1.1)} \qquad \mathrm{Der}(-\log\ V(F)) \simeq \mathcal{D}(V(F))(1)/\mathcal{O}_{\mathbb{P}^d} \simeq \mathcal{D}_0(V(F))(1)$$

as sheaves on $\mathbb{P}^d$. A major impetus in studying $D(V(F))$ comes from the setting of hyperplane arrangements, and the isomorphism (1) was noted for generic hyperplane arrangements in [13]. If F is a product of distinct (up to scaling) linear forms, we write $V(F) = \mathcal{A}$. Terao's theorem [20] shows that in this setting, if $D(\mathcal{A})$ is a free S–module, with $D(\mathcal{A}) \simeq \oplus S(-a_i)$, then the cohomology of the *affine* complement of $\mathcal{A}$ satisfies the formula

$$\sum h^i(\mathbb{C}^{d+1} \setminus \mathcal{A}, \mathbb{Q})t^i = \prod (1 + a_i t).$$

1.1. Addition-Deletion theorems

A central tool in the study of hyperplane arrangements is an inductive method due to Terao. For a hyperplane arrangement $\mathcal{A} = \bigcup_{i=1}^n H_i$ and choice of hyperplane $H = V(\ell_H) \in \mathcal{A}$, set

$$\mathcal{A}' = \mathcal{A} \setminus H \text{ and } \mathcal{A}'' = \mathcal{A}|_H.$$

The collection $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ is called a *triple*, and yields a left exact sequence

$$0 \longrightarrow D(\mathcal{A}')(-1) \xrightarrow{\cdot \ell_H} D(\mathcal{A}) \longrightarrow D(\mathcal{A}'').$$

Freeness of a triple is related via:

Theorem 1.3. [*Terao, [21]*] *Let* $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ *be a triple. Then any two of the following imply the third*

- $D(\mathcal{A}) \simeq \oplus_{i=1}^{n} S(-b_i)$
- $D(\mathcal{A}') \simeq S(-b_n + 1) \oplus_{i=1}^{n-1} S(-b_i)$
- $D(\mathcal{A}'') \simeq \oplus_{i=1}^{n-1} S(-b_i)$

For a triple with $\mathcal{A} \subseteq \mathbb{P}^2$, [18] shows that after pruning the Euler derivations and sheafifying, there is an exact sequence

$$0 \longrightarrow \mathcal{D}_0{}'(-1) \longrightarrow \mathcal{D}_0 \longrightarrow i_* \mathcal{D}_0{}'' \longrightarrow 0,$$

where $i : H \hookrightarrow \mathbb{P}^2$; $i_*\mathcal{D}_0{}'' \simeq \mathcal{O}_H(1 - |\mathcal{A}''|)$. In [19], such a sequence is obtained for line-conic arrangements, and [7] studies the situation for nodal curves.

1.2. Statement of results

This paper is motivated by the following example:

Example 1.4. The braid arrangement A_3 is free [22] and $D(\mathrm{A}_3) \simeq \oplus_{i=1}^{3} S(-i)$.

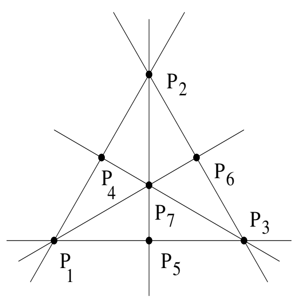

Figure 1: The A_3-arrangement.

The family of homogeneous cubic polynomials vanishing at the seven singular points of A_3 is parameterized by a $\mathbb{P}^2$. A generic member C of this family is smooth. A computer calculation shows $A_3 \cup C$ has quasihomogeneous singularities, and that $D(A_3 \cup C)$ is free, isomorphic to $S(-1) \oplus S(-4)^2$. Our main result is Theorem 1.6, which combined with Proposition 1.9 explains why $D(A_3 \cup C)$ is free, and gives the degrees of the generators.

Definition 1.5. A quasihomogeneous triple $(\mathcal{A}, C, \mathcal{A} \cup C)$ consists of $\mathcal{A} = \bigcup C_i$ with C_i and $C = V(f)$ reduced, irreducible plane curves such that all singulartities of $\mathcal{A}$, C, and $\mathcal{A} \cup C$ are quasihomogeneous, and $\mathcal{A} \cap C \subseteq C_{sm}$.

Theorem 1.6. *A quasihomogeneous triple with $k = |C \cap \mathcal{A}|$ and C degree n gives rise to a short exact sequence*

$$0 \longrightarrow \mathrm{Der}(-\log \mathcal{A})(-n) \xrightarrow{\cdot f} \mathrm{Der}(-\log \mathcal{A} \cup C) \longrightarrow \mathcal{O}_C(D) \longrightarrow 0,$$

with $\mathcal{O}_C(D)$ torsion free and $\deg(D) = \chi(C) - k$.

Remark 1.7. A check shows if $\theta \in \mathrm{Der}(-logA)$, then $f \cdot \theta$ satisfies Definition 1.2.

Remark 1.8. When C is smooth, $\mathcal{O}_C(D)$ is the line bundle $\mathcal{O}_C(-K_C - R)$, with $R = (C \cap \mathcal{A})_{red}$ the reduced scheme of $C \cap \mathcal{A}$.

We prove the theorem in §3, and work out Example 1.4 in detail in §4. For addition-deletion arguments, we will need

Proposition 1.9. *Suppose $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence of graded $S = k[x_0, \ldots, x_2]$–modules, with A and B rank two reflexive modules of projective dimension at most one. Then any two of the following imply the third*

1) *A is free with generators in degrees $\{a, b\}$.*
2) *B is free with generators in degrees $\{c, d\}$.*
3) *C has Hilbert series $\frac{t^c+t^d-t^a-t^b}{(1-t)^3}$.*

Proof. That (1) and (2) imply (3) is trivial. If (1) and (3) hold and B is not free, then $\mathrm{pdim}(B) = 1$, so B has a minimal free resolution of the form

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0.$$

Note that

$$F_1 = \oplus S(-e_i) \text{ and } F_0 = \oplus S(-e_i) \oplus S(-c) \oplus S(-d),$$

since by additivity the Hilbert series of B is $\frac{t^c+t^d}{(1-t)^{n+1}}$. Without loss of generality suppose $d \geq c$. If the largest $e_i > d$, then since the resolution is minimal, no term of F_1 can map to the generator in degree e_i, which forces B to have a free summand of rank one. Since B is reflexive of rank two this forces B to be free.

Next suppose the largest $e_i = d$. Since the Hilbert series of B is $\frac{t^c+t^d}{(1-t)^{n+1}}$, B has a minimal generator of degree d. But then no element of F_1 can be a relation involving that generator, and again B has a free summand. This obviously also works when the largest e_i is less than d, and shows that (1) and (3) imply (2). The argument that (2) and (3) imply (1) is similar. □

Remark 1.10. In arrangement theory, a vector bundle which splits as a direct sum of line bundles is called free, and we follow this convention.

If $\mathrm{Der}(-\log \mathcal{A})$ is free, then applying Γ_* to the sequence of Theorem 1.6 yields a short exact sequence of modules, since $H^1(\oplus \mathcal{O}_{\mathbb{P}^2}(a_i)) = 0$. Then by Proposition 1.9, on $\mathbb{P}^2$ the freeness of $\mathrm{Der}(-\log \mathcal{A} \cup C)$ follows if appropriate numerical conditions hold. In contrast to arrangements of rational curves where the Hilbert series of $\Gamma_*(\mathcal{O}_C(-K_C - R))$ depends only on the degree of R (since $C \simeq \mathbb{P}^1$), for curves of positive genus the Hilbert series of $\Gamma_*(\mathcal{O}_C(-K_C - R))$ depends on subtle geometry.

2. Quasihomogeneous plane curves

Let $C = V(Q)$ be a reduced (but not necessarily irreducible) curve in $\mathbb{C}^2$, let $(0,0) \in C$, and let $\mathbb{C}\{x,y\}$ denote the ring of convergent power series.

Definition 2.1. The Milnor number of C at $(0,0)$ is

$$\mu_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\} / \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

To define μ_p for an arbitrary point p, we translate so that p is the origin.

Definition 2.2. The Tjurina number of C at $(0,0)$ is

$$\tau_{(0,0)}(C) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\} / \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right\rangle.$$

Definition 2.3. A singularity is quasihomogeneous iff there exists a holomorphic change of variables so the defining equation becomes weighted homogeneous; $f(x,y) = \sum c_{ij}x^iy^j$ is weighted homogeneous if there exist rational numbers α, β such that $\sum c_{ij}x^{i\cdot\alpha}y^{j\cdot\beta}$ is homogeneous.

In [17], Saito shows that if f is a convergent power series with isolated singularity at the origin, then f is in the ideal generated by the partial derivatives if and only if f is quasihomogeneous. We call an arrangement of smooth curves quasihomogeneous if all singularities are quasihomogeneous; as noted in §1.3 of [19], $V(Q) \subseteq \mathbb{P}^2$ is quasihomogeneous iff

$$\deg(J_Q) = \sum_{p \in \mathrm{Sing}(V(Q))} \mu_p(Q).$$

Lemma 2.4. *[[23], Theorem 6.5.1] Let X and Y be two reduced plane curves with no common component, meeting at a point p. Then*

$$\mu_p(X \cup Y) = \mu_p(X) + \mu_p(Y) + 2(X \cdot Y)_p - 1,$$

where $(X \cdot Y)_p$ is the intersection number of X and Y at p.

Proposition 2.5. *For a quasihomogeneous triple with $\mathcal{A}$ of degree m and C degree n, the Hilbert polynomial* $\mathrm{HP}(coker(f), t)$ *of the cokernel of the multiplication map*

$$0 \longrightarrow D(\mathcal{A})(-n)/E \xrightarrow{\cdot f} D(\mathcal{A} \cup C)/E$$

is $nt + \frac{7n-3n^2}{2} - k + \beta$, where $k = |C \cap \mathcal{A}|$ and $\beta = \sum_{p \in C} \mu_p(C)$.

Proof. By Equation 1.1 and the exact sequences

$$0 \longrightarrow D_0(\mathcal{A} \cup C) \longrightarrow S^3 \longrightarrow S(m+n-1) \longrightarrow S(m+n-1)/J_{\mathcal{A}\cup C} \longrightarrow 0,$$
$$0 \longrightarrow D_0(\mathcal{A})(-n) \longrightarrow S^3(-n) \longrightarrow S(m-n-1) \longrightarrow S(m-n-1)/J_{\mathcal{A}} \longrightarrow 0,$$

it follows that

$$\mathrm{HP}(D_0(\mathcal{A} \cup C), t) = 3\binom{t+2}{2} - \binom{t+1+m+n}{2} + \deg(J_{\mathcal{A}\cup C})$$
$$\mathrm{HP}(D_0(\mathcal{A})(-n), t) = 3\binom{t+2-n}{2} - \binom{t+1+m-n}{2} + \deg(J_{\mathcal{A}}),$$

so that $\mathrm{HP}(D_0(\mathcal{A}\cup C),t) - \mathrm{HP}(D_0(\mathcal{A})(-n),t)$ is equal to

$$\deg(J_{\mathcal{A}\cup C}) - \deg(J_{\mathcal{A}}) + nt - n(2m+1) + \frac{9n-3n^2}{2}.$$

Since all singularities of C, $\mathcal{A}$ and $\mathcal{A}\cup C$ are quasihomogeneous,

$$\deg(J_{\mathcal{A}\cup C}) = \sum_{p\in \mathrm{Sing}(\mathcal{A}\cup C)} \mu_p(\mathcal{A}\cup C) \text{ and } \deg(J_{\mathcal{A}}) = \sum_{p\in \mathrm{Sing}(\mathcal{A})} \mu_p(\mathcal{A}).$$

If α is the sum of Milnor numbers of singular points of $\mathcal{A}$ not on C, and β is the sum of the Milnor numbers on C, then

$$\deg(J_{A\cup C}) = \alpha + \beta + \sum_{p\in C\cap\mathcal{A}} \mu_p(\mathcal{A}\cup C).$$

By Lemma 2.4, the previous quantity equals

$$\alpha + \beta + \sum_{p\in C\cap\mathcal{A}} (\mu_p(\mathcal{A}) + 2(C\cdot\mathcal{A})_p - 1).$$

As $\deg(J_{\mathcal{A}}) = \alpha + \sum_{p\in C\cap\mathcal{A}} \mu_p(\mathcal{A})$ and $\mid C\cap\mathcal{A}\mid = k$, we obtain:

$$\deg(J_{\mathcal{A}\cup C}) - \deg(J_{\mathcal{A}}) = 2\sum_{p\in C\cap\mathcal{A}} (C\cdot\mathcal{A})_p - k + \beta.$$

By Bezout's theorem,

$$\sum_{p\in C\cap\mathcal{A}} (C\cdot\mathcal{A}) = mn, \text{ so } \deg(J_{\mathcal{A}\cup C}) - \deg(J_{\mathcal{A}}) = 2mn - k + \beta,$$

hence the Hilbert polynomial of the cokernel is

$$nt - n(2m+1) + \frac{9n-3n^2}{2} + 2mn - k + \beta = nt - k + \beta + \frac{7n-3n^2}{2}.$$

□

Corollary 2.6. *For the map*

$$0 \longrightarrow \mathrm{Der}(-\log\mathcal{A})(-n) \xrightarrow{\cdot f} \mathrm{Der}(-\log\mathcal{A}\cup C) \tag{2.1}$$

the Hilbert polynomial of $\Gamma_*(coker(f))$ *is* $nt - k + \beta + \frac{9n-3n^2}{2}$.

Proof. Proposition 2.5 and Equation (1). □

3. Main Theorem

We prove Theorem 1.6. The first technical tool we need is a result stated by Noether in [14], and proved by Hartshorne:

Theorem 3.1. [*2.1, [10]*]. *Let $C \subseteq \mathbb{P}^2$ be an integral (possibly singular) curve of degree n, and D a closed subscheme, with $\mathcal{O}_C(D)$ the associated torsion free sheaf on C. Then C has a dualizing sheaf $K_C \simeq \mathcal{O}_C((n-3)H)$, and $h^0(\mathcal{O}_C(D)) = \deg(D) + 1 - p_a(C)$ as soon as $\deg(D) \geq n(n-3)$.*

We will also need

Theorem 3.2. [*7.1.1, [23]*] *Let $C \subseteq \mathbb{P}^2$ be an integral (possibly singular) curve of degree n. Then*

$$\chi(C) = 3n - n^2 + \sum_{p \in C} \mu_p(C).$$

We first show that the sheaf associated to the cokernel in Equation 2.1 is isomorphic to $\mathcal{O}_C(D)$, where D is a divisor of degree $3n - n^2 - k - \beta$. Consider the commuting diagram below

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Der}(-\log \mathcal{A})(-n) & \longrightarrow & \mathcal{T}_{\mathbb{P}^2}(-n) & \longrightarrow & J_{\mathcal{A}}(-n)(\mathcal{A}) & \longrightarrow & 0 \\
 & & \downarrow{\cdot f} & & \downarrow{\cdot f} & & \downarrow{\cdot f} & & \\
0 & \longrightarrow & \mathrm{Der}(-\log \mathcal{A} \cup C) & \longrightarrow & \mathcal{T}_{\mathbb{P}^2} & \longrightarrow & J_{\mathcal{A} \cup C}(\mathcal{A} \cup C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{coker}(f) & \longrightarrow & \mathcal{T}_{\mathbb{P}^2}/f \cdot \mathcal{T}_{\mathbb{P}^2} & \longrightarrow & J_{\mathcal{A} \cup C}(\mathcal{A} \cup C)/\mathrm{Im}(f) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Exactness of the top two rows follows from the definition, and since $\mathcal{T}_{\mathbb{P}^2}$ is torsion free, the first two vertical maps $\cdot f$ are inclusions. The rightmost vertical map $\cdot f$ is also an inclusion, because f is a nonzerodivisor on $\mathcal{O}_{\mathcal{A}}$. Exactness of the bottom row then follows from the snake lemma. The exact sequence

$$0 \longrightarrow S \longrightarrow S^3(1) \longrightarrow \Gamma_*(\mathcal{T}_{\mathbb{P}^2}) \longrightarrow 0,$$

shows that the Hilbert polynomial of $\Gamma_*(\mathcal{T}_{\mathbb{P}^2}/f \cdot \mathcal{T}_{\mathbb{P}^2})$ is $2nt + 6n - n^2$. Since

$$\mathcal{T}_{\mathbb{P}^2}/f \cdot \mathcal{T}_{\mathbb{P}^2} \simeq \mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{O}_C,$$

and $\mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{O}_C$ is a locally free rank two $\mathcal{O}_C$–module, coker(f) is a torsion free submodule of $\mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{O}_C$. By Corollary 2.6, the Hilbert polynomial of $\Gamma_*(\text{coker}(f))$ is $nt - k + \frac{9n-3n^2}{2} + \beta$, so $\text{coker}(f) \simeq \mathcal{O}_C(D)$. To determine the degree of D, we use Theorem 3.1 to compute

$$\begin{aligned} \text{HP}(\Gamma_*(\text{coker}(f)), t) &= h^0(\mathcal{O}_C(D + tH)), \ t \gg 0 \\ &= \deg(D + tH) + 1 - p_a(C) \\ &= \deg(D) + nt + 1 - p_a(C). \end{aligned}$$

Equating this with the previous expression and using that $1 - p_a(C) = \frac{3n-n^2}{2}$

$$\begin{aligned} \deg(D) &= 3n - n^2 + \beta - k \\ &= \chi(C) - k. \end{aligned}$$

where the second equality follows from Theorem 3.2. □

Corollary 3.3. *If C is smooth, then $\mathcal{O}_C(D) \simeq \mathcal{O}_C(-K_C - R)$.*

Proof. Since $\mathcal{O}_C(D)$ comes from the restriction of $\text{Der}(-\log \mathcal{A} \cup C)$ to C, it must actually be a subbundle of $\mathcal{T}_C$, which by adjunction is isomorphic to $\mathcal{O}_C((3-n)H)$. For the same reason, sections must vanish at points of $\mathcal{A} \cap C$, so that $\mathcal{O}_C(D) \subseteq \mathcal{O}_C((3-n)H - R)$. The degree of this bundle is $3n - n^2 - k$, so equality holds. □

Remark 3.4. An elementary modification ([9], Definition 15) of vector bundles is a short exact sequence $0 \to A \to B \to C \to 0$ with A and B rank two vector bundles on a surface, and C a line bundle supported on a curve. The argument above shows that when C is smooth, the left hand column of the diagram above is an elementary modification.

3.1. Castelnuovo-Mumford regularity

Theorem 1.6 yields bounds on the Castelnuovo-Mumford regularity of logarithmic vector bundles. For the remainder of the paper, we restrict to the case where C is smooth, which is our primary interest.

Definition 3.5. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^d$ is $j-$regular iff $H^i\mathcal{F}(j-i) = 0$ for every $i \geq 1$. The smallest number j such that $\mathcal{F}$ is j-regular is $\operatorname{reg}(\mathcal{F})$.

Lemma 3.6. *With the hypotheses of Corollary 3.3,*

$$\operatorname{reg}(\operatorname{Der}(-\log \mathcal{A} \cup C)) \leq \max\left\{\operatorname{reg}(\operatorname{Der}(-\log \mathcal{A})) + n, 2n - 5 + \frac{k}{n}\right\}.$$

Proof. The short exact sequence

$$0 \longrightarrow \operatorname{Der}(-\log \mathcal{A})(-n) \longrightarrow \operatorname{Der}(-\log \mathcal{A} \cup C) \longrightarrow \mathcal{O}_C(-K_C - R) \longrightarrow 0$$

gives a long exact sequence in cohomology, so if $\operatorname{Der}(-\log \mathcal{A})$ is a-regular, then

$$h^1(\operatorname{Der}(-\log \mathcal{A})(a-1)) = 0 = h^2(\operatorname{Der}(-\log \mathcal{A})(a-2)).$$

So if $t - n - 1 \geq a - 1$ and $t - n - 2 \geq a - 2$ we have that

$$h^1(\operatorname{Der}(-\log \mathcal{A})(t-n-1)) = 0 = h^2(\operatorname{Der}(-\log \mathcal{A})(t-n-2)).$$

This gives vanishings if $t - n \geq a$, that is, if $t \geq \operatorname{reg}\operatorname{Der}(-\log \mathcal{A}) + n$. The result will follow if

$$h^1\mathcal{O}_C((t-1)H - K_C - R) = h^0\mathcal{O}_C((1-t)H + 2K_C + R) = 0.$$

This holds if $\deg((1-t)H + 2K_C + R) < 0$, so using that $K_C = (n-3)H$, it holds when

$$t > 2n - 5 + \frac{k}{n}$$

The result follows. ☐

Proposition 3.7. *With the hypotheses of Corollary 3.3, the Hilbert function of* $\Gamma_*\mathcal{O}_C(-K_C - R)$ *is* $nt + \frac{9n-3n^2}{2} - k$ *for* $t > 2n - 5 + \frac{k}{n}$*, and zero if* $t < n - 3 + \frac{k}{n}$.

Proof. If $t < n - 3 + \frac{k}{n}$, then the degree of $tH - K_C - R$ is negative, so there can be no sections, and the first part follows from Lemma 3.6. ☐

4. Examples

Example 4.1. We analyze Example 1.4 in more detail. Since C is a cubic curve, $g_C = 1$ and $K_C \simeq \mathcal{O}_C$. Since C meets every line of $\mathcal{A}$ in three points, $k = |C \cap \mathcal{A}| = 7$. By Proposition 3.7, the Hilbert function of $\Gamma_*\mathcal{O}_C(-K_C - R)$ is zero for $t < \frac{7}{3}$, and agrees with the Hilbert polynomial for $t > \frac{10}{3}$. Applying Corollary 2.6 yields

t	0	1	2	3	4	5	6	7
$h^0(tH - K_C - R)$	0	0	0	?	5	8	11	14

Now, $H^0((3H - R)$ consists of cubics through the seven singular points of $\mathcal{A}$, and this space has dimension 2, since C is itself one of the three cubics, so is not counted. Thus, in this example the Hilbert polynomial $3t - 7$ agrees with the Hilbert function for $t \geq 3$, and

$$\mathrm{HS}(\Gamma_*\mathcal{O}_C(-K_C - R), t) = \frac{2t^3 + t^4}{(1-t)^2} = \frac{2t^3 - t^4 - t^5}{(1-t)^3}.$$

Terao's result [22] on reflection arrangements shows

$$D(\mathrm{A}_3) \simeq S(-1) \oplus S(-2) \oplus S(-3),$$

so $D_0(\mathrm{A}_3(1)) \simeq S(-1) \oplus S(-2)$, which by Equation (1) is $\Gamma_*(\mathrm{Der}(-\log \mathrm{A}_3))$, hence

$$\Gamma_*(\mathrm{Der}(-\log \mathrm{A}_3(-3))) \simeq S(-4) \oplus S(-5).$$

Taking global sections in Theorem 1.6 and applying Proposition 1.9, we find that $\Gamma_*(\mathrm{Der}(-\log\ \mathrm{A}_3 \cup C))$ is free, with

$$\begin{aligned}\mathrm{HS}(\Gamma_*(\mathrm{Der}(-\log \mathrm{A}_3 \cup C)), t) &= \frac{t^4 + t^5}{(1-t)^3} + \frac{2t^3 - t^4 - t^5}{(1-t)^3} \\ &= \frac{2t^3}{(1-t)^3}.\end{aligned}$$

Example 4.2. The reflection arrangement B_3 consists of the nine planes of symmetry of a cube in $\mathbb{R}^3$. The intersection of B_3 with the chart U_z is below (this does not show the line at infinity $z = 0$). By [22] $D(\mathrm{B}_3) \simeq S(-1) \oplus S(-3) \oplus S(-5)$.

This configuration has 13 singular points, so if the singularities were in general position the quartics passing through the points would be parameterized by $\mathbb{P}^1$. But there are three quadruple points (the intersection of the

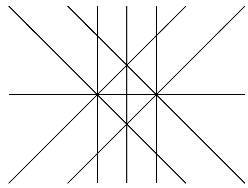

Figure 2: The B_3-arrangement.

three vertical lines with the line at infinity is not pictured), and each set of lines through one of these points is a quartic vanishing on the singularities. A computation shows that a generic quartic C in this net is smooth, and that $\mathrm{B}_3 \cup C$ is quasihomogeneous.

By Proposition 3.7, the Hilbert function of the module $\Gamma_*\mathcal{O}_C(-K_C - R)$ is zero for $t < \frac{17}{4}$, and agrees with the Hilbert polynomial for $t > \frac{25}{4}$. Applying Corollary 2.6, we have

t	4	5	6	7	8	9	10	11
$h^0(tH - K_C - R)$	0	?	?	9	13	17	21	25

It remains to determine $H^0(tH - K - R) = H^0((t-1)H - R)$ for $t \in \{5, 6\}$. The space $H^0(4H - R)$ consists of quartics through the thirteen singular points of B_3. As observed above, this space has dimension 3, but C itself is one of the quartics, so $h^0(4H - R) = 2$. A computer calculation shows that $h^0(5H - R) = 5$, so

$$\mathrm{HS}(\Gamma_*\mathcal{O}_C(-K_C - R), t) = \frac{2t^5 + t^6 + t^7}{(1-t)^2} = \frac{2t^5 - t^6 - t^8}{(1-t)^3}.$$

Since $\Gamma_*(\mathrm{Der}(-\log \mathrm{B}_3)(-4)) \simeq S(-6) \oplus S(-8)$, taking global sections in Theorem 1.6 and applying Proposition 1.9 shows $\Gamma_*(\mathrm{Der}(-\log\ \mathrm{B}_3 \cup C))$ is free, with

$$\begin{aligned}
\mathrm{HS}(\Gamma_*(\mathrm{Der}(-\log \mathrm{B}_3 \cup C)), t) &= \mathrm{HS}(\Gamma_*(\mathrm{Der}(-\log \mathrm{B}_3)(-4)), t) + \frac{2t^5 - t^6 - t^8}{(1-t)^3} \\
&= \frac{t^6 + t^8}{(1-t)^3} + \frac{2t^5 - t^6 - t^8}{(1-t)^3} \\
&= \frac{2t^5}{(1-t)^3}.
\end{aligned}$$

Remark 4.3. In both Example 4.1 and 4.2, $\text{sing}(\mathcal{A}) = \text{sing}(\mathcal{A} \cup C)$. Example 2.2 of [19] gives gives an example of rational curve arrangements where Corollary 3.3 holds, but $\text{sing}(\mathcal{A}) \neq \text{sing}(\mathcal{A} \cup C)$.

Concluding remarks Our work raises a number of questions:

1) As noted, [22] shows $D(\mathcal{A})$ is free for a reflection arrangement $\mathcal{A}$. In this case, is there always a smooth C with $D(\mathcal{A} \cup C)$ free?

2) Is there a connection to the residue map and multiarrangements, as in [24]?

3) Does this generalize to other surfaces? For the Hilbert polynomial arguments to work, the surface should possess an ample line bundle. More generally, does this generalize to higher dimensions? Note that [19] shows the quasihomogeneous property will be necessary.

4) The Hilbert series of $\Gamma_* \mathcal{O}_C(-K_C - R)$ depends solely on a set of reduced points on a plane curve. If $\mathcal{A} = \bigcup_{i=1}^r Y_i$ with Y_i reduced and irreducible, can an iterated construction using linkage yield the Hilbert series?

5) In [12], Liao gives a formula relating Chern classes of logarithmic vector fields to the Chern-Schwartz-MacPherson class of the complement, showing that on a surface the two are equal exactly when the singularities are quasihomogeneous, and in [1], Aluffi gives an explicit relation between the characteristic polynomial of an arrangement and the Segre class of the Jacobian scheme. Can one prove Theorem 1.6 using these methods?

Acknowledgments. Macaulay2 computations were essential to our work. Our collaboration began at the Mathematical Society of Japan summer school on arrangements; Terao and Yoshinaga were organizers and Schenck a participant, and we thank the Mathematical Society of Japan for their generous support. Schenck was supported by NSF 1068754, NSA H98230-11-1-0170. Terao and Yoshinaga were supported by JSPS KAKENHI Grant Numbers JP24244001, JP25400060, JP15KK0144. We thank Alex Dimca and two anonymous referees for useful comments.

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Received November 18, 2013