Math. Res. Lett.
Volume 25, Number 6, 1957–1975, 2018

There may be no minimal non-σ-scattered linear orders

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In this paper we demonstrate that it is consistent, relative to the existence of a supercompact cardinal, that there is no linear order which is minimal with respect to being non-σ-scattered. This shows that a theorem of Laver, which asserts that the class of σ-scattered linear orders is well quasi-ordered, is sharp. We also prove that PFA^+ implies that every non-σ-scattered linear order either contains a real type, an Aronszajn type, or a ladder system indexed by a stationary subset of ω_1, equipped with either the lexicographic or reverse lexicographic order. Our work immediately implies that CH is consistent with "no Aronszajn tree has a base of cardinality $\aleph_1$." This gives an affirmative answer to a problem due to Baumgartner.

1. Introduction

In [9], Laver verified a longstanding conjecture of Fraïssé: the countable linear orders are *well quasi-ordered* by embeddability. That is to say if L_i $(i < \infty)$ is an infinite sequence of countable linear orderings, then there is an $i < j$ such that L_i is embeddable into L_j. In fact, Laver proved the following stronger result.

Theorem 1.1. *[9] The class $\mathscr{M}$ of σ-scattered linear orders is well quasi-ordered by embeddability.*

Recall that a linear order is *scattered* if it does not contain an isomorphic copy of the linear order $(\mathbb{Q}, \leq)$ and is *σ-scattered* if it is a union of countably many scattered suborders.

In the final paragraph of [9], Laver writes, "Finally, the question arises as to how the order types outside of $\mathscr{M}$ behave under embeddability." For instance, is there a class of linear orders which is closed under taking suborders, which properly includes the class of σ-scattered linear orders, and which is well quasi-ordered by embeddability? Cast in another way, is there a non-σ-scattered linear order which embeds into all of its non-σ-scattered suborders?

Already in [1], Baumgartner proved that it is consistent that any two $\aleph_1$-*dense* sets of reals are isomorphic; in fact this conclusion is a consequence of the Proper Forcing Axiom (PFA). Here a linear order is κ-*dense* if all of its intervals have cardinality κ. It is not difficult to show that any suborder of $\mathbb{R}$ of cardinality $\aleph_1$ is bi-embeddable with an $\aleph_1$-dense set of reals and thus in Baumgartner's model, any set of reals of cardinality $\aleph_1$ is minimal with respect to being non-σ-scattered. On the other hand, it follows easily from work of Dushnik and Miller [5] that the Continuum Hypothesis (CH) implies that there are no minimal uncountable linear orders which are separable. (In fact Dushnik and Miller show in ZFC that there is no minimal separable linear order of cardinality continuum.)

The main result of this paper is that Theorem 1.1 is consistently sharp.

Theorem 1.2. *If there is a supercompact cardinal, then there is a forcing extension which satisfies CH in which there are no minimal non-σ-scattered linear orders.*

This result builds on work of Moore [13] and Ishiu-Moore [7]. In [13] it was proved that it is consistent with CH that ω_1 and $-\omega_1$ are the only minimal uncountable linear orderings. In fact, this conclusion is derived from the conjunction of CH and a certain combinatorial consequence (A) of PFA. Notice that if ω_1 and $-\omega_1$ are the only minimal uncountable linear orders, then any minimal non-σ-scattered linear order must have the property that it does not contain an uncountable separable suborder or an *Aronszajn suborder*. Here an Aronszajn line is an uncountable linear order which does not contain uncountable separable or scattered suborders.

In [7] it was proved that PFA$^+$, a strengthening of PFA, implies that every minimal non-σ-scattered linear order is either isomorphic to a set of reals of cardinality $\aleph_1$ or else is an *Aronszajn line*. Moreover, Martinez-Ranero [10], building on work of Moore [12] [14] proved that that PFA implies that the class of Aronszajn lines is well quasi-ordered by embeddability. In [7], it was pointed out that if the consequences of PFA$^+$ needed to carry out the analysis in that paper were consistent with the conjunction of (A) and CH, then one could establish the consistency of "there are no minimal non-σ-scattered linear orders." In fact these consequence of PFA$^+$ followed from a weaker axiom CPFA$^+$ which had been expected to be consistent with CH; this was later refuted in [15]. The strategy of the present paper for proving

Theorem 1.2 also utilizes the combination of (A) and CH, but involves a re-examination of the hypotheses sufficient to obtain the results of [7].

In addition to proving Theorem 1.2, we will also establish a result concerning the structure of non-σ-scattered linear orders under the assumption of PFA^+. Baumgartner proved in [3] that there exist non-σ-scattered linear orders which do not contain real or Aronszajn types. His construction can be described as the lexicographic ordering on a family $\{x_\alpha : \alpha \in S\}$ where $S \subseteq \omega_1$ is stationary and x_α is a cofinal strictly increasing ω-sequence in α for each α in S. We will refer to such a linear ordering as a *Baumgartner type* and we will refer to S as its *index set*.

Theorem 1.3. *Assume* PFA^+ *and let $X \subseteq \mathbb{R}$ have cardinality $\aleph_1$ and C be a Countryman line. If L is a non-σ-scattered linear order, then L contains an isomorphic copy of one of the following linear orders: X, C, $-C$, a Baumgartner type or its reverse.*

The proof of Theorem 1.2 immediately yields the following result.

Theorem 1.4. *It is consistent with CH that no Aronszajn tree has a base of cardinality $\aleph_1$.*

Here a collection $\mathcal{B}$ of uncountable downward closed subtrees of an Aronszajn tree T is called a *base* if whenever U is an uncountable downward closed subtree of T, there is $V \in \mathcal{B}$ such that $V \subseteq U$. This answers a problem posed in [2], where it is proved that every Aronszajn tree has a base of cardinality $\aleph_1$ after Levy collapsing an inaccessible cardinal to $\aleph_2$.

The paper will be organized as follows. Section 2 will review some notation, definitions, and results concerning linear orders. In Section 3 we will prove Theorem 1.3. Section 4 contains the analysis needed to derive the conclusion of Theorem 1.2 from a list of axioms. Section 5 gives a proof that the collection of axioms used in Section 4 is consistent. This section also includes a proof of theorem 1.4 as a remark. The paper closes with some open problems in Section 6.

2. Preliminaries

This section is devoted to some background and conventions on trees, linearly ordered sets and forcing axioms. More discussion can be found in [7], [11], [13], [16] and [17]. We will also introduce two set-theoretic axioms $(*)$ and $(\dagger)$ which will play an important role in the proofs of Theorem 1.2 and 1.3.

We first recall the notion of a forcing axiom associated to a class of partial orders.

Notation 2.1. If $\mathfrak{P}$ is a class of partial orders, then by FA($\mathfrak{P}$) we mean the forcing axiom for the class $\mathfrak{P}$: whenever P is in $\mathfrak{P}$ and $\mathcal{D}$ is a collection of $\aleph_1$-many dense subsets of P, there is a filter $G \subseteq P$ which intersects all of the dense sets in $\mathcal{D}$. FA$^+$($\mathfrak{P}$) is the assertion that whenever P is in $\mathfrak{P}$, $\mathcal{D}$ is a collection of $\aleph_1$-many dense subsets of P, and $\dot{S}$ is a name for a stationary subset of ω_1, then there is a filter $G \subseteq P$ which intersects all the dense sets in $\mathcal{D}$ and satisfies that the set

$$\{\xi \in \omega_1 : \exists p \in G(p \Vdash \check{\xi} \in \dot{S})\}$$

is stationary.

The following axiom is a consequence of FA$^+$(σ-closed) and will play an important role in our analysis of non-σ-scattered linear orders of cardinality $\aleph_1$.

Definition 2.2. ($\dagger$) is the assertion that if $S \subseteq \omega_1$ is stationary and for each $\alpha \in S$, $U_\alpha \subseteq \alpha$ is open, then there a club $E \subseteq \omega_1$ such that for stationarily many $\alpha \in S \cap E$ there is an $\bar{\alpha} < \alpha$ such that either $E \cap (\bar{\alpha}, \alpha) \subseteq U_\alpha$ or $E \cap (\bar{\alpha}, \alpha) \cap U_\alpha = \emptyset$.

Let P be the poset consisting of all countable closed subsets of ω_1, ordered by end extension and let $\dot{E}$ be the P-name for the union of the generic filter. By using the arguments of [11], it is possible to show that if $S \subseteq \omega_1$ is stationary and $\langle U_\alpha : \alpha \in S\rangle$ is as in the formulation of ($\dagger$), then every condition forces $\dot{E}$ satisfies the conclusion of ($\dagger$) for $\langle U_\alpha : \alpha \in S\rangle$. In particular FA$^+$($\sigma$-closed) implies ($\dagger$). Moreover, ($\dagger$) holds in the model obtained by adding $\aleph_2$ Cohen subsets of ω_1 to a model of GCH. It should be noted that while this shows that it is easy to obtain models of ($\dagger$) and CH, it remains an open problem whether the strengthening of ($\dagger$) in which a relative club of $\alpha \in S \cap E$ are required to satisfy the conclusion is consistent with CH (see [6]).

It will often be convenient to let, for each set X, θ_X denote the least regular cardinal such that all finite iterates of the power set applied to X are in $H(\theta_X)$, the collection of sets of hereditary cardinality less than θ_X. Let $\mathcal{E}(X)$ denote the collection of all countable elementary submodels of $H(\theta_X)$ which have X as an element.

We will now recall a number of definitions from [7]. For a linearly ordered set L we will use $\hat{L}$ to denote the completion of L. Formally this is the set of all Dedekind cuts of L with z identified with the cut $\{x \in L : x < z\}$. The purpose of the following definitions is to abstractly recover the set of indices from a Baumgartner type, purely from its order-theoretic properties.

Definition 2.3. Whenever L is a linearly ordered set and Z is some arbitrary set we say that Z *captures* $x \in L$ if there is a $z \in Z \cap \hat{L}$ such that there is no element of $Z \cap L$ which is strictly in between z and x.

Fact 2.4. *[7] Suppose L is a linear order and let λ be a regular cardinal such that $\hat{L}$ is in $H(\lambda)$. If M is a countable elementary submodel of $H(\lambda)$ with $L \in M$ and $x \in \hat{L} \setminus M$, then M captures x if and only if there is a unique $z \in \hat{L} \cap M$ such that there is no element of $M \cap L$ which is strictly in between x and z. In this case we say M captures x via z.*

Definition 2.5. [7] If L is a linear order, define $\Gamma(L)$ to be the set of all countable subsets Z of $\hat{L}$ such that for some $x \in L$, Z does not capture x. (This is the relative complement of the set $\Omega(L)$ in [7].)

If $B = \langle x_\alpha : \alpha \in S \rangle$ is a Baumgartner type and M is a countable elementary submodel of $H(\theta)$ for some regular cardinal $\theta \geq \omega_2$ with $B \in M$, then $M \in \Gamma(B)$ if and only if $M \cap \omega_1 \in S$. This is because M captures all elements of B except x_δ, where $\delta = M \cap \omega_1$. So $\Gamma(B)$ is equivalent to S modulo the equivalence induced by the following quasi-order.

Definition 2.6. Let A, B be two collections of countable sets and $X = \bigcup A$, $Y = \bigcup B$. we say $B \leq A$ if there is an injection $\iota : X \to Y$ such that for club many M in $[Y]^\omega$, if $M \in B$ then $\iota^{-1}M$ is in A. We let $B < A$ if $B \leq A$ but not $A \leq B$; A and B are *equivalent* if $A \leq B$ and $B \leq A$.

The following results summarize the properties of the map $L \mapsto \Gamma(L)$ and the quasi-order $\leq$.

Theorem 2.7. *[7] A linear order L is not σ-scattered if and only if $\Gamma(L)$ is stationary.*

Proposition 2.8. *[7] If L_0 and L are linearly ordered sets and L_0 embeds into L, then $\Gamma(L_0) \leq \Gamma(L)$.*

A key feature of Baumgartner types L is that it is always possible to find a non-σ-scattered suborder L_0 such that $\Gamma(L_0) < \Gamma(L)$. This is not always possible in the more general class of non-σ-scattered orders as the next proposition shows.

Proposition 2.9. *[7] If a linear order L contains a real or Aronszajn type, then $\Gamma(L)$ contains a club.*

Definition 2.10. If L is a linear order and M is in $\mathcal{E}(L)$, then we say that an element x of L is *internal* (respectively *external*) to M, if there is a club $E \subseteq [\hat{L}]^\omega$ in M such that every (respectively no) element of $E \cap M$ captures x.

The next definition will play a central role in the proofs of our results. It abstracts the property of Baumgartner types needed to allow us to decrease Γ by thinning out the linear order.

Definition 2.11. A linear order L is said to be *amenable* if whenever M is in $\mathcal{E}(L)$ and $x \in L$, then x is internal to M.

Observe that by Theorem 2.7, σ-scattered linear orders are amenable. It is also true that Baumgartner types are amenable.

Proposition 2.12. *[7] If L is a non-σ-scattered amenable linear order of cardinality $\aleph_1$ and $S \subseteq \Gamma(L)$ is stationary, then there is a non-σ-scattered $L_0 \subseteq L$ such that $\Gamma(L_0) \leq S$.*

In particular, non-σ-scattered amenable linear orders of cardinality $\aleph_1$ are not minimal. The next theorem shows that the existence of external elements of a linear order characterizes the presence of either a real or Aronszajn suborder. In particular amenable linear orders do not contain real or Aronszajn types.

Theorem 2.13. *[7] The following are equivalent for a linear order L:*

- *L contains a real or Aronszajn type.*
- *There are M in $\mathcal{E}(L)$ and $x \in L$ such that x is external to M.*

We are now ready to formulate the other set-theoretic hypothesis which will be needed in our analysis.

Definition 2.14. $(*)$ is the assertion that for every non-σ-scattered linear order L there is a continuous $\in$-chain $\langle M_\xi : \xi \in \omega_1\rangle$ in $\mathcal{E}(L)$ such that:

- the set of all $\xi \in \omega_1$ such that $M_\xi \cap \hat{L} \in \Gamma(L)$ is a stationary set,
- $\hat{L_0} \subseteq \bigcup_{\xi\in\omega_1} M_\xi$, where $L_0 = L \cap (\bigcup_{\xi\in\omega_1} M_\xi)$,
- for every ξ if $M_\xi \cap \hat{L} \in \Gamma(L)$ then there is an $x \in L_0$ such that M_ξ does not capture x.

Observe that if $L_0 \subseteq L$ are as in the statement of $(*)$, then L_0 is also non-σ-scattered. Thus $(*)$ implies every non-σ-scattered linear order contains a non-σ-scattered suborder of cardinality $\aleph_1$. Also, if we apply $(*)$ to a linear order of cardinality at most $\aleph_1$, then $L \subseteq \bigcup_{\xi\in\omega_1} M_\xi$ and consequently $\hat{L} \subseteq \bigcup_{\xi\in\omega_1} M_\xi$. This gives the following fact.

Fact 2.15. *Assume $(*)$. If L is a linear order of cardinality at most $\aleph_1$ which does not contain a real type, then $|\hat{L}| \leq \aleph_1$.*

In particular $(*)$ implies that CH is true. A consequence of the work in [5] and [13] is that by iterating certain forcings over a model of CH, it is possible to obtain a generic extension in which there is no minimal real or Aronszajn type. We briefly review this result and recall some of the relevant definitions and terminology. If T is an Aronszajn tree, then a *subtree* of T is an uncountable downward closed subset of T.

Notation 2.16. If T is a tree, $t \in T$ and α is an ordinal, then $t \restriction \alpha$ is defined to be t if α is greater than the height of t otherwise it is the unique $s \leq t$ with height α.

Definition 2.17. A sequence $\langle f_\alpha : \alpha \in \lim(\omega_1)\rangle$ is called *ladder system coloring* if $\langle \mathrm{dom}(f_\alpha) : \alpha \in \omega_1\rangle$ is a ladder system and the range of each f_α is contained in ω.

Definition 2.18. If T is an ω_1-tree, then a ladder system coloring $\langle f_\alpha : \alpha \in \lim(\omega_1)\rangle$ can be *T-uniformized* if there is a subtree U of T and function from $\phi : U \to \omega$ such that whenever height of $u \in U$ is a limit ordinal α, f_α agrees with $\xi \mapsto \phi(u \restriction \xi)$ at all except for finitely many $\xi \in \mathrm{dom}(f_\alpha)$.

Definition 2.19. (A) is the assertion that every ladder system coloring can be T-uniformized for every Aronszajn tree T.

The significance of (A) lies in the following theorem, along with the fact that it is consistent with CH.

Theorem 2.20. *[13] Assume* (A) *and* $2^{\aleph_0} < 2^{\aleph_1}$*. There is no minimal Aronszajn line.*

In [13], a forcing $Q_{T,\bar{f}}$ was introduced which T-uniformizes a given ladder system coloring $\bar{f}$. We will recall the definition of this poset in Section 5 when we need to analyze it, but for now we will simply summarize its important properties.

While $(<\omega_1)$-properness and complete properness play a role in the proof of the main result of this paper, they can be treated as black boxes via the following results, along with the straightforward fact that σ-closed posets are both $(<\omega_1)$-proper and completely proper.

Lemma 2.21. *[13] For every ladder system coloring* $\bar{f}$ *and Aronszajn tree* T*, the forcing* $Q_{T,\bar{f}}$ *is completely proper and* $(<\omega_1)$*-proper.*

Theorem 2.22. *[16] A countable support iteration of* $(<\omega_1)$*-proper, completely proper forcing is proper and does not introduce new real numbers.*

We will also need the following iteration theorem of Shelah.

Theorem 2.23. *[16, III.8.5] If the iterands of a countable support iteration are proper and don't add new uncountable branches to* ω_1*-trees, then the iteration is proper and does not add uncountable branches to* ω_1*-trees.*

3. A rough classification of non-σ-scattered orders

In [7] it was shown that under PFA^+, every non-σ-scattered linear order contains an amenable non-σ-scattered suborder of cardinality $\aleph_1$. In this section we prove that under a fragment of PFA^+ every non-σ-scattered amenable linear order contains a copy of a Baumgartner type or its reverse. Taken together, these results determines a basis for the class of non-σ-scattered linear orders under PFA^+: if X is any set of reals of cardinality $\aleph_1$ and C is any Countryman type, then any non-σ-scattered linear order must contain an isomorphic copy of either X, C, $-C$, or a Baumgartner type of cardinality $\aleph_1$ or its reverse.

Theorem 3.1. *Assume the conjunction of* $\mathrm{MA}_{\aleph_1}$ *and* $(\dagger)$*. If* L *is an amenable non-*σ*-scattered linear order of size* $\aleph_1$*, then it contains a copy of a Baumgartner type or its reverse.*

First we will prove the following lemma.

Lemma 3.2. *Suppose that L is a an amenable linear order of cardinality $\aleph_1$. If $\langle M_\xi : \xi \in \omega_1 \rangle$ is a continuous $\in$-chain of elements of $\mathcal{E}(L)$ which is in $N \in \mathcal{E}(L)$ and $N \cap \omega_1 = \delta$, then M_δ and N capture the same elements of L.*

Proof. First observe that by continuity of the $\in$-chain and the fact that $\{\nu \in \omega_1 : M_\nu \cap \omega_1 = \nu\}$ is a club in N, $M_\delta \subseteq N$ and $M_\delta \cap \omega_1 = \delta$. Next observe that since L has cardinality $\aleph_1$, $N \cap L = M_\delta \cap L$ and thus any element of L captured by M_δ is captured by N. Now suppose that N captures $x \in L$ and let $z \in \hat{L} \cap N$ be such that there is no element of $N \cap L$ which is strictly in between x and z. Since L is amenable, there is a club $E \subseteq [\hat{L}]^\omega$ in M_δ such that for all $Z \in M_\delta \cap E$, Z captures x. Let $\lambda \in \theta_L \cap M_0$ be a regular cardinal such that the powerset of $[\hat{L}]^\omega$ is in $H(\lambda)$. Let $\overline{M} \in N$ be a countable elementary submodel of $H(\lambda)$ such that $\langle M_\xi \cap H(\lambda) : \xi \in \omega_1 \rangle$, E, and z are in $\overline{M}$. Observe that for sufficiently large $\xi < \delta$, $M_\xi \cap \hat{L}$ is in E and if $\nu = \overline{M} \cap \omega_1$ then $L \cap \overline{M} = M_\nu \cap L$. Notice that $\overline{M}$ captures x via z. Since $M_\nu \cap \hat{L}$ is in $E \cap M_\delta$, M_ν also captures x. By Fact 2.4, it must be that z is in M_ν and hence M_δ. $\square$

Proof of Theorem 3.1. Now let $\langle M_\xi : \xi \in \omega_1 \rangle$ be a continuous $\in$-chain of elements of $\mathcal{E}(L)$. Since L is amenable it does not contain any real types, there is a countable set $X_\xi \subseteq L$ such that if $M_\xi \cap L \subseteq X_\xi$ and if $y \in L \setminus M_\xi$, there is a unique $x \in X_\xi \setminus M_\xi$ such that $x \neq y$. Let $x : \omega \times \omega_1 \to L$ be such that for all $\xi \in \omega_1$, $X_\xi = \{x(n, M_\xi \cap \omega_1) : n \in \omega\}$. Now let $\langle N_\xi : \xi \in \omega_1 \rangle$ be a continuous $\in$-chain of elements of $\mathcal{E}(L)$ such that $\langle M_\xi : \xi \in \omega_1 \rangle$ and x are in N_0. Note that there is a club of ξ in ω_1 such that $M_\xi \cap \omega_1 = \xi = N_\xi \cap \omega_1$ and hence M_ξ and N_ξ capture the same elements of L. Since $\Gamma(L)$ is stationary, then by applying the pressing down lemma there is a stationary set $S_0 \subseteq \omega_1$, an $n \in \omega$, and a club $E \subseteq [\hat{L}]^\omega$ such that if $\xi \in S_0$:

- $M_\xi \cap \omega_1 = \xi = N_\xi \cap \omega_1$;
- $x(n, \xi)$ is not captured by N_ξ;
- E is in N_ξ and if Z is in $N_\xi \cap E$, then Z captures $x(n, \xi)$.

Set $y_\xi = x(n, \xi)$ for all $\xi \in S_0$. Now it is easy to see that for all ξ and η in S_0, N_ξ captures y_η if and only if $\xi \neq \eta$.

Let z_ξ ($\xi \in \omega_1$) be an enumeration of all $z \in \hat{L}$ for which there are $\eta \in \omega_1$ and $\alpha \in S_0$ such that N_η captures y_α via z. We can assume without loss of generality that this enumeration is in N_0. For every $\alpha \in S_0$ define $g_\alpha : \alpha \to \{z_\xi : \xi \in \omega_1\}$ by letting $g_\alpha(\xi)$ be the unique $z \in N_\xi$ such that N_ξ captures y_α via z. Note that if $g_\alpha(\xi) = z_\eta$ then $\eta \in \alpha$.

Claim 3.3. *The following are true for $\alpha, \beta \in S_0$:*

1) $\{\xi \in \alpha : g_\alpha(\xi) \neq g_\alpha(\xi+1)\}$ *has order type ω and supremum α*
2) *If $y_\alpha < y_\beta$, then $g_\alpha(\xi) \leq g_\beta(\xi)$ for all $\xi < \min(\alpha, \beta)$.*
3) *If $\alpha < \beta$, then there is a $\xi < \alpha$ such that $g_\alpha(\xi) \neq g_\beta(\xi)$.*
4) *If $\xi < \eta < \min(\alpha, \beta)$ and $g_\alpha(\xi) \neq g_\beta(\xi)$, then $g_\alpha(\eta) \neq g_\beta(\eta)$.*

Proof. First observe that for each $\alpha \in S_0$ and limit $\eta \in \alpha$ there is an $\bar{\eta} \in \eta$ such that $g_\alpha \restriction (\bar{\eta}, \eta]$ is constant. On the other hand N_α does not capture y_α and therefore the set $\{\xi \in \alpha : g_\alpha(\xi) \neq g_\alpha(\xi+1)\}$ must have ordertype ω and supremum α. This proves (1); the remainder of the items follow easily from (1) and the definitions. $\square$

Define C_α to be the set of all $\xi \in \alpha$ such that z_ξ is in the range of g_α and equip the set $\{C_\alpha : \alpha \in S_0\}$ with the lexicographic order. For each $\alpha \in S_0$ let $U_\alpha = \{\xi \in \alpha : g_\alpha(\xi) < y_\alpha\}$ and observe that U_α is an open subset of α. So by (†) there is a stationary set $S \subseteq S_0$ such that either

- for every $\alpha \in S$ and $\xi \in S \cap \alpha$, $g_\alpha(\xi) > y_\alpha$ or,
- for every $\alpha \in S$ and $\xi \in S \cap \alpha$, $g_\alpha(\xi) < y_\alpha$.

Without loss of generality assume that for every $\alpha \in S$ and $\xi \in \alpha \cap S$, $g_\alpha(\xi) > y_\alpha$. Define S' to be the set of all elements of S which are limit points of elements of S.

Let Q be the set of all finite $p \subseteq S'$ such that whenever $\alpha \neq \beta$ are in p, $C_\alpha <_{\text{lex}} C_\beta$ if and only if $y_\alpha < y_\beta$. We will prove that Q is c.c.c..

Suppose for a contradiction that X is an uncountable antichain in Q. By applying the Δ-System Lemma and removing the root if necessary, we may assume that X is pairwise disjoint and consists of elements of some fixed cardinality n. Let M be an element of $\mathcal{E}(Q)$ such that X, L, x, $\langle N_\xi : \xi \in \omega_1\rangle$, $\langle y_\xi : \xi \in S\rangle$, and $\langle z_\xi : \xi \in \omega_1\rangle$ are all in M. Set $\delta = M \cap \omega_1$ and let $p = \{\alpha_1, ..., \alpha_n\}$ be in X such that $\delta < \alpha_i$ for all $i \leq n$. Let $\zeta \in \delta \cap S$ be such that:

- if $i, j \leq n$, then $g_{\alpha_i} \restriction \delta \neq g_{\alpha_j} \restriction \delta$ implies $g_{\alpha_i}(\zeta) \neq g_{\alpha_j}(\zeta)$;
- the range of $g_{\alpha_i} \restriction \zeta + 1$ coincides with the range of $g_{\alpha_i} \restriction \delta$ for each $i \leq n$ (i.e. $C_{\alpha_i} \cap \delta \subseteq \zeta + 1$ for each $i \leq n$).

Notice that the existence of ζ follows from the observation that if $g_\alpha(\xi) \neq g_\beta(\xi)$, then $g_\alpha(\eta) \neq g_\beta(\eta)$ for all $\eta > \xi$. By elementarity of M there exists a $p' = \{\alpha'_1, ..., \alpha'_n\}$ in $M \cap X$ such that:

- for all $i, j \leq n$, $y_{\alpha_i} < y_{\alpha_j}$ if and only if $y_{\alpha'_i} < y_{\alpha'_j}$;
- if $i \leq n$, then $g_{\alpha_i}(\zeta) = g_{\alpha'_i}(\zeta)$.

We will now show that $p \cup p' \in Q$. Let $i, j \leq n$. There are two cases, depending on whether $g_{\alpha_i}(\zeta)$ and $g_{\alpha_j}(\zeta)$ are the same. If $g_{\alpha_i}(\zeta) \neq g_{\alpha_j}(\zeta)$, then observe that $g_{\alpha_j}(\zeta) = g_{\alpha'_j}(\zeta)$ and

$$C_{\alpha_i} \cap (\zeta + 1) \neq C_{\alpha_j} \cap (\zeta + 1) = C_{\alpha'_j} \cap (\zeta + 1).$$

Since p and p' are both in Q, it follows that $y_{\alpha_i} < y_{\alpha'_j}$ is equivalent to $C_{\alpha_i} <_{\text{lex}} C_{\alpha'_j}$

If $g_{\alpha_i}(\zeta) = g_{\alpha_j}(\zeta)$, then observe that $g_{\alpha_i} \restriction \delta = g_{\alpha_j} \restriction \delta$ and thus that $C_{\alpha_i} \cap \delta = C_{\alpha_j} \cap \delta$. Observe that

$$g_{\alpha'_j} \restriction \zeta = g_{\alpha_j} \restriction \zeta = g_{\alpha_i} \restriction \zeta$$

and that g_{α_i} is constant on the interval $[\zeta, \delta)$. Also, $g_{\alpha'_j}$ is not constant on $[\zeta, \delta)$ by Claim 3.3. Observe that there is a $\xi \in S$ such that $\zeta < \xi < \alpha'_j$ and

$$g_{\alpha'_j}(\xi) < g_{\alpha'_j}(\zeta) = g_{\alpha_j}(\zeta) = g_{\alpha_j}(\xi) = g_{\alpha_i}(\xi)$$

It follows that $y_{\alpha_i} > y_{\alpha'_j}$. On the other hand,

$$C_{\alpha_i} \cap \delta = C_{\alpha_i} \cap (\zeta + 1) = C_{\alpha'_j} \cap (\zeta + 1) \neq C_{\alpha'_j} \cap \delta$$

and consequently $C_{\alpha'_j} <_{\text{lex}} C_{\alpha_i}$. Since $i, j \leq n$ were arbitrary, p and p' are compatible and thus Q is c.c.c..

By applying $\mathrm{MA}_{\aleph_1}$ to the finite support product $Q^{<\omega}$ of countably many copies of Q, it is possible to find a partition of S into countably many pieces such that whenever α and β are in the same piece of the partition, $C_\alpha <_{\text{lex}} C_\beta$ if and only if $y_\alpha < y_\beta$. Since there is a piece of this partition which is stationary, it shows that L contains a Baumgartner type. $\square$

We finish this section by noting if we add a Cohen real r to a model of ZFC, then Theorem 3.1 will not hold in the resulting generic extension. To see this, suppose that $r \in 2^\omega$ and $\langle x_\xi : \xi \in \lim(\omega_1) \rangle$ is such that $x_\xi : \omega \to \xi$ is increasing and has cofinal range for each ξ. Define a linear ordering on $\lim(\omega_1)$ by $\xi <_r \eta$ if and only if

$$x_\xi(n) < x_\eta(n) \text{ is equivalent to } r(n) = 0$$

where n is minimal such that $x_\xi(n) \neq x_\eta(n)$. It is left to the reader to check that if $S \subseteq \lim(\omega_1)$ is stationary, then there is a comeager set of r such that

$(S, <_r)$ contains both a copy of ω_1 and of $-\omega_1$. Furthermore, if S is nonstationary, then $(S, <_r)$ is σ-scattered and thus not a Baumgartner type. On the other hand, it is not hard to show that every uncountable subset of a Baugmartner type contains a copy of ω_1; in particular, Baumgartner types do not contain $-\omega_1$. Since every stationary subset of ω_1 in the generic extension by a Cohen real contains a ground model stationary set, this proves the claim.

4. An axiomatic analysis of non-σ-scattered orders

In this section we will prove the following proposition.

Proposition 4.1. *Assume* (†) *and* $(*)$. *If L is a non-σ-scattered linear order which does not contain a real or Aronszajn type, then there is a non-σ-scattered suborder $L' \subseteq L$ with $\Gamma(L') < \Gamma(L)$.*

Proof. As noted in Section 2, $(*)$ implies that L contains a non-σ-scattered suborder L_0 such that $\hat{L}_0$ has cardinality $\aleph_1$. We may therefore assume without loss of generality that $|L| = |\hat{L}| = \aleph_1$. This implies, in particular that if M and N are in $\mathcal{E}(L)$ and $M \cap \omega_1 = N \cap \omega_1$, then $M \cap \hat{L} = N \cap \hat{L}$. If $Z \subseteq \hat{L}$ is countable, let $\{x(n, Z) : n \in \omega\} \subseteq L$ be such that $Z \cap L \subseteq \{x(n, Z) : n \in \omega\}$ and if $y \in L \setminus Z$, then there is an n such that no element of $L \cap Z$ is between $x(n, Z)$ and y. This is possible since L does not contain a real type. Let $\langle N_\xi : \xi \in \omega_1 \rangle$ be a continuous $\in$-chain in $\mathcal{E}(L)$ with the map $Z \mapsto \{x(n, Z) : n \in \omega\}$ in N_0. Since L is not σ-scattered, there is an $n \in \omega$ such that

$$S_0 = \{\xi \in \omega_1 : N_\xi \cap \omega_1 = \xi \text{ and } N_\xi \text{ does not capture } x(n, N_\xi \cap \hat{L})\}$$

is stationary. Fix such an n and set $x_\xi = x(n, N_\xi \cap \hat{L})$. For each $\alpha \in S_0$ let U_α be the set of all $\xi \in \alpha$ such that N_ξ captures x_α. Clearly U_α is an open subset of α so by (†) there is a stationary subset $S \subseteq S_0$ and a club $E \subseteq \omega_1$ such that for every $\alpha \in S$ there is an $\bar{\alpha} \in \alpha$ such that either $E \cap (\bar{\alpha}, \alpha) \subseteq U_\alpha$ or $E \cap (\bar{\alpha}, \alpha) \cap U_\alpha = \emptyset$. The second alternative can only happen for at most nonstationary many $\alpha \in S$, because L has no external element by Theorem 2.13. By applying the Pressing Down Lemma and thinning S down if necessary, we can assume that for every $\alpha, \beta \in S$, N_α captures x_β if and only if $\alpha \neq \beta$.

Now let $S' \subseteq S$ be stationary such that $S \setminus S'$ is also stationary and define $L' = \{x_\xi : \xi \in S'\}$. We will show that L' is not σ-scattered and that $\Gamma(L') < \Gamma(L)$.

Fix an $M \in \mathcal{E}(L)$ which has $\langle N_\xi : \xi \in \omega_1 \rangle$, S, and S' as elements and has the property that $M \cap \omega_1 = \delta \in S'$. To show that L' is not σ-scattered we prove that M does not capture x_δ in L'. Suppose for contradiction that M captures x_δ in L' via $z \in \hat{L} \cap M$. By replacing L with $-L$ if necessary, we may assume that $z < x_\delta$. Let

$$A = \{x_\xi : \xi \in S' \text{ and } z < x_\xi\}.$$

Observe that A is in M and hence $\inf(A)$ is also in M. Since $\inf(A) \leq x_\delta$ and x_δ is not in M, it follows that $\inf(A) < x_\delta$. Since M does not capture x_δ in L, there is $y \in L \cap M$ such that $z \leq \inf(A) < y < x_\delta$. By elementarity of M, there is a $\xi \in S' \cap M$ such that $z < x_\xi < y < x_\delta$. But this contradicts our assumption that M captures x_δ in L' via z.

To see that $\Gamma(L) \not\leq \Gamma(L')$ it suffices to show that the set of all $M \in \mathcal{E}(L)$ which capture all elements of L' but does not capture some elements of L forms a stationary set. To this end let $M \in \mathcal{E}(L)$ with $L' \in M$ and $M \cap \omega_1 \in S \setminus S'$ and observe that M does not capture x_δ in L but it captures all elements of L'. $\square$

5. The consistency of the axioms

In this section we will prove that if there is a supercompact cardinal, then there is a forcing extension with the same reals which satisfies $(*)$, $(\dagger)$, and (A). By results of the previous section this will finish the proof of Theorem 1.2. Our forcing construction will resemble the consistency proof of PFA^+ and will involve a countable support iteration of forcings which are completely proper, $({<}\omega_1)$-proper, and which do not add new uncountable branches through ω_1-trees. By results of Shelah discussed in the introduction, the resulting iteration will not introduce new reals or uncountable branches through ω_1-trees.

All of the iterands used in building the iteration will either be σ-closed or else be of the following form.

Definition 5.1. [13] For an Aronszajn tree T and ladder system coloring $\bar{f}$ let $Q_{T,\bar{f}}$ be the set of all conditions $q = (\phi_q, \mathcal{U}_q)$ such that:

- ϕ_q is a function from $X_q \subseteq T$ into ω such that X_q is a countable downward closed subset of T which has a last level of height α_q,
- if $t \in X_q$ has limit height δ, f_δ agrees with $\xi \mapsto \phi_q(t \restriction \xi)$ at all $\xi \in C_\alpha$ except for finitely many $\xi \in C_\alpha$.

- $\mathcal{U}_q$ is a non-empty countable collection of pruned subtrees of $T^{[n]}$ for some n.
- for every $U \in \mathcal{U}_q$ there is some $\sigma \in U$ which is a subset of the last level of X_q.

($T^{[n]}$ is the collection of all weakly increasing n-tuples from some level of T, regarded as a tree with the coordinatewise order.) We let $p \leq q$, in Q if $X_p \restriction \alpha_q = X_q$, $\mathcal{U}_q \subseteq \mathcal{U}_p$, and $\phi_p \restriction X_q = \phi_q$.

Remark 5.2. A simplification of this type of forcings can be used to prove Theorem 1.4. For an Aronszajn tree T let Q_T be the set of all conditions as above forgetting the information about the ladder system coloring. More precisely Q_T consists of all conditions $q = (X_q, \mathcal{U}_q)$ such that,

- X_q is a countable downward closed subset of T which has a last level of height α_q,
- $\mathcal{U}_q$ is a non-empty countable collection of pruned subtrees of $T^{[n]}$ for some n.
- for every $U \in \mathcal{U}_q$ there is some $\sigma \in U$ which is a subset of the last level of X_q.

We let $p \leq q$, in Q if $X_p \restriction \alpha_q = X_q, \mathcal{U}_q \subseteq \mathcal{U}_p$. It is easy to see that the forcing $Q_{T,\bar{f}}$ projects onto Q_T for every Aronszajn tree T, so by the work in [13], Q_T is completely proper, $< \omega_1$-proper and satisfies proper isomorphism condition. Now let $\mathbb{P}$ be the countable support iteration of all posets of Q_T of length ω_2 such that whenever T is an Aronszajn tree in some intermediate model, Q_T is repeated in the iteration cofinally often. Let $\mathbb{V}$ be a model satisfying $2^\omega = \omega_1 + 2^{\omega_1} = \omega_2$, and let G be $\mathbb{P}$-generic over $\mathbb{V}$. Then it is easy to see that ω_2 is preserved and in $\mathbb{V}[G]$

- $2^\omega = \omega_1 + 2^{\omega_1} = \omega_2$,
- if T is an Aronszajn tree, there is a sequence $\langle V_i : i \in \omega_2 \rangle$ of uncountable downward closed subtrees of T such that whenever $i \in j$, V_i contains no subtree of V_j.

This proves Theorem 1.4.

The following lemma asserts that these forcings $Q_{T,\bar{f}}$ do not add new uncountable branches to ω_1-trees.

Lemma 5.3. *Suppose T is Aronszajn and S is an ω_1-tree, and $\bar{f}$ is a ladder system coloring. Then $Q_{T,\bar{f}}$ does not add new uncountable branches to S. Consequently, if L is a linear order of size $\aleph_1$, then forcing with $Q_{T,\bar{f}}$ does not introduce new elements to $\hat{L}$.*

Proof. Let Q denote $Q_{T,\bar{f}}$ and let $\dot{b}$ be a Q-name which is forced by some $p \in Q$ to be an uncountable branch in S which is not in the ground model. If q is in Q and σ is in $T^{[n]}$ for some n, then we say that σ is *consistent* with q if the range of $\sigma \restriction \alpha_q$ is contained in X_q.

Let $M \in \mathcal{E}(Q)$ with $p, \dot{b} \in M$ and set $\delta = M \cap \omega_1$.

Claim 5.4. *If $\sigma \in T_\delta^{[n]}$ is consistent with p and $s \in S_\delta$, then there is a condition $q \leq p$ in $M \cap Q$ such that $q \Vdash \check{s} \notin \dot{b}$ and such that σ is consistent with q.*

Proof. By Lemma 5.5 in [13] we can find a decreasing sequence $\langle p_k : k \in \omega \rangle$ in M such that:

- $p_0 = p$,
- p_{k+1} decides $\dot{b} \restriction \alpha_{p_k}$,
- σ is consistent with p_k for all k,
- $\langle p_k : k \in \omega \rangle$ has a lower bound in M.

Thus without loss of generality we can assume that p forces $\check{s} \restriction \check{\alpha}_p \in \dot{b}$.

Suppose for contradiction that for every $q \leq p$ in $M \cap Q$, if $q \Vdash \check{s} \notin \dot{b}$, then σ is not consistent with q. Define W to be the set of all $\tau \in T^{[n]}$ which are compatible with $\sigma \restriction \alpha_p$ and such that there exists an $\bar{s} \in S_{\mathrm{ht}(\tau)}$ compatible with $s \restriction \alpha_p$ and for all $q \leq p$, if $q \Vdash \bar{s} \notin \dot{b}$ and $\alpha_q \leq \mathrm{ht}(\tau)$, then $\mathrm{range}(\tau \restriction \alpha_q) \nsubseteq X_q$. Since W is definable from parameters in M, it is in M. Observe that W is downwards closed and that s witnesses that σ is in W. Hence by elementarily of M, W is uncountable. Let U be the set of all $\tau \in W$ which have uncountably many extensions in W. Notice that $\sigma \restriction \alpha_p$ is in U and thus $p' = (\varphi_p, X_p, \mathcal{U}_p \cup \{U\})$ is a condition in Q.

For each $\tau \in U_\delta$ and $t \in S_\delta$, let $\varphi(\tau, t)$ be the assertion: *"whenever $r \leq p'$ is (M, Q)-generic with* $\mathrm{range}(\tau) \subseteq X_r$, $r \Vdash \check{t} \in \dot{b}$.*"* Notice that if r is (M, Q)-generic, then so is $r \restriction \delta$. It is easy to see that for every $\tau \in W_\delta$ there exists a unique $t \in S_\delta$ which extends $s \restriction \alpha_p$ such that $\varphi(\tau, t)$. Moreover, observe that if τ_1 and τ_2 are in U_δ and s_1, s_2 are such that $\phi(\tau_1, s_1)$ and $\phi(\tau_2, s_2)$, then we can find an (M, Q)-generic condition $r \leq p'$ which is consistent to both

τ_1 and τ_2. This implies $r \Vdash \check{s_1} = \check{s_2}$. Thus $s \in S_\delta$ satisfies that $\phi(\tau, s)$ holds for every $\tau \in U_\delta$.

We now claim that $p' \Vdash \check{s}' \in \dot{b}$ for all $s' < s$. Since such an s' is necessarily in M, by elementarity it suffices to show that if $p'' \leq p'$ is in M, then p'' has an extension r which forces that $\check{s}' \in \dot{b}$. Because Q is proper, p'' has an (M, Q)-generic extension r. Let $\tau \in U_\delta$ be such that range$(\tau) \subseteq X_r$. Since $r \leq p'$ and $\varphi(\tau, s')$ holds, $r \Vdash \check{s}' \in \dot{b}$. Thus we have established that $p' \Vdash \check{s}' \in \dot{b}$ for all $s' < s$. By elementarily, $\{t \in S : p' \Vdash \check{t} \in \dot{b}\}$ is uncountable, which implies that p' decides $\dot{b}$, a contradiction. □

Returning to the main proof, by the claim we can find a condition $\bar{p} \leq p$ such that $\bar{p} \Vdash S_\delta \cap \dot{b} = \emptyset$. □

Theorem 5.5. *Assume there is a supercompact cardinal. Then there is forcing extension in which* (*A*), (†), *and* (∗) *hold.*

Proof. Let V be a ground model with a supercompact cardinal κ. By performing some preparatory forcing if necessary, we may assume that CH is true. Mimicking the consistency proof of PFA (see [4] or [8]), use a Laver function ψ to build a countable support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha \in \kappa\rangle$ such that:

- $\dot{Q}_\alpha$ is a P_α-name in V_κ for a partial order which is either σ-closed or of the form $Q_{\dot{T},\bar{f}}$;
- if $\psi(\alpha)$ is a P_α-name and $p \in P_\alpha$ forces that $\psi(\alpha)$ either σ-closed or of the form $Q_{\dot{T},\bar{f}}$, then p forces $\dot{Q}_\alpha = \psi(\alpha)$.

Let $G \subseteq P_\kappa$ be a V-generic filter. It is immediate that $V[G]$ satisfies (A). By Lemma 2.21 and Theorem 2.22 the iteration does not add new reals and thus $V[G]$ satisfies CH. By Lemmas 2.23 and 5.3, every final segment of the iteration does not add new uncountable branches to ω_1-trees. Arguing as in [4], $V[G]$ satisfies FA$^+$(σ-closed) and in particular (†).

We will now show that (∗) holds in $V[G]$. Fix for a moment a non-σ-scattered linear order L in $V[G]$ and let Q be the set of all countable continuous $\in$-chains in $\mathcal{E}(L)$ ordered by end extension. It is obvious that Q is σ-closed and easily verified that

$$\dot{S} = \{(\check{\xi}, q) : \xi \in \mathrm{dom}(q) \text{ and } q(\xi) \cap \hat{L} \in \Gamma(L)\}$$

is a Q-name for a stationary subset of ω_1. Since Q is countably closed, it does not add new elements to $\hat{L}$. Thus if $H \subseteq Q$ is a $V[G]$-generic filter, then $V[G][H]$ contains the desired witness to (∗) for L. Moreover, this witness is preserved in any further generic extension by a proper forcing in which $\hat{L}$ does not gain new elements.

The proof that $\mathrm{FA}^+(\sigma\text{-closed})$ holds in $V[G]$ can now be applied in this situation to show that $(*)$ holds in V. The only difference is that while in the verification of $\mathrm{FA}^+(\sigma\text{-closed})$ it is sufficient to know that the factor forcings are proper, in our setting it is necessary to know that, additionally, the factor forcings do not add new elements to the completions of linear orders. As noted already, this follows from Lemmas 2.23 and 5.3. $\square$

6. Open questions

We will conclude this paper by mentioning some open questions which are natural in light of the results obtained here and in [7]. The first question is closely related to a problem due to Galvin [3, Problem 4].

Question 6.1. *Must every minimal non-σ-scattered linear order be a real type nor an Aronszajn type?*

Of course it is consistent that this question has positive answer (this was first shown in [7]), but at present it seems possible that this question could have a positive answer in ZFC. (Added in proof: the first author has shown that this question can consistently have a negative answer.)

Question 6.2. *Must every minimal non-σ-scattered linear order have cardinality $\aleph_1$?*

Notice that if $\kappa > \aleph_1$ is a regular cardinal and $L = \{x_\alpha : \alpha \in S\}$ is a ladder system indexed by a nonreflecting stationary set $S \subseteq \kappa$. consisting of ordinals of countable cofinality, then the lexicographical ordering on L is non-σ-scattered but has no σ-scattered suborder of cardinality $\aleph_1$ (of course this example fails to be minimal).

Finally, it is unclear whether Theorem 1.3 can be sharpened so that the Baumgartner types are all realized as suborders of a single Baumgartner type.

Question 6.3. *Assume* PFA^+. *If two Baumgartner types are indexed by a common stationary subset of ω_1, must there be a non-σ-scattered order which embeds into both of them?*

Acknowledgements

The research presented in this paper was supported in part by NSF grants DMS-1262019 and DMS-1600635.

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Received April 20, 2016