

A note on the cone conjecture for K3 surfaces in positive characteristic

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We prove that, for a K3 surface in characteristic $p > 2$, the automorphism group acts on the nef cone with a rational polyhedral fundamental domain and on the nodal classes with finitely many orbits. As a consequence, for any non-negative integer g , there are only finitely many linear systems of irreducible curves on the surface of arithmetic genus g , up to the action of the automorphism group.

1. Introduction

Given any K3 surface over the complex numbers, it is a theorem of Sterk (based on ideas of Looijenga) [20] that the automorphism group acts on the nef cone of the surface with a rational polyhedral fundamental domain. In this note, we extend these results to characteristic $p > 2$. In particular, any K3 surface in this setting carries only finitely many elliptic pencils, up to the action of the automorphism group, answering a question of A. Kumar to the second author.

We split the problem into three cases, according to the height and Picard number of the surface. For K3 surfaces of finite height, results of Katsura and van der Geer allow us to lift the surface along with its full Picard group to characteristic 0. We can then directly apply Sterk’s theorem by a specialization argument. For K3 surfaces with Picard number 22, we will use Ogus’s supersingular Torelli theorem to reduce the statement to an analogue of Sterk’s original theorem. At this stage, thanks to recent proofs of the Tate conjecture [4–6, 11, 12, 14], we could be done. However, there is a simpler deformation-theoretic argument for dealing with Artin-supersingular K3s of Picard number less than 22 that does not assume the Tate conjecture for supersingular K3 surfaces. We include this argument here in Section 3 in case it proves useful for other purposes.

These results are almost certainly known to experts in the field, but they appear to have escaped appearing in the literature. In characteristic 0, Sterk's theorem is a special case of the cone conjecture of Morrison-Kawamata, predicting such behavior for all smooth projective varieties with numerically trivial canonical bundle. We refer the reader to [22] for a beautiful survey of the history and known results. While the authors are optimistic that several of Totaro's more general results [21] should extend using the methods here, higher-dimensional results remain mysterious even in characteristic 0.

A brief sketch of the contents of this note: In Section 2 we show that for a family of K3s over a dvr, if the specialization map on Picard groups is an isomorphism then it is equivariant with respect to the specialization map on automorphism groups. In Sections 3 and 4 we show how to create various families with constant Picard group and improved generic fiber. In Section 5 we prove the main theorem for K3 surfaces with Picard number 22, and in Section 6 we prove the result in general.

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2. Some results on specializations of automorphisms and Nef cones

Fix a Henselian dvr R with algebraically closed residue field k and fraction field K . Fix an algebraic closure $\bar{K} \supset K$. Given a finite extension $K \subset L$, write R_L for the integral closure of R in L ; this is again a complete dvr.

Theorem 2.1. *Let $\pi : X \rightarrow \text{Spec } R$ be a smooth projective relative surface whose special fiber is not birationally ruled. Write*

$$\text{sp} : \text{Pic}(X_{\bar{K}}) \rightarrow \text{Pic}(X_k)$$

for the natural specialization map. If sp is an isomorphism and there is an ample invertible sheaf \mathcal{L} on $X_{\bar{K}}$ such that $\text{sp}(\mathcal{L})$ is ample then there is a

group homomorphism

$$\sigma : \text{Aut}(X_{\overline{K}}) \rightarrow \text{Aut}(X_k)$$

such that sp is σ -equivariant with respect to the natural pullback actions. In addition, if $H^0(X, T_X) = 0$ then σ is injective.

Proof. Given a pair of field extensions $L'/L/K$, the faithful flatness of L'/L implies that the pullback map

$$\text{Aut}(X_L) \rightarrow \text{Aut}(X_{L'})$$

is injective. On the other hand, given any algebraic field extension L/K , any automorphism of X_L is defined over some finite subextension. We thus see that for a fixed algebraic closure \overline{K}/K the pullback maps define a canonical isomorphism

$$(1) \quad \text{colim}_{K \subset L \subset \overline{K}} \text{Aut}(X_L) \rightarrow \text{Aut}(X_{\overline{K}}),$$

with the colimit taken over all finite subextensions. For each finite extension L/K , let R_L denote the normalization of R in L ; since R is Henselian, R_L is again a Henselian dvr with residue field k .

Since X_k is not birationally ruled, the Matsusaka-Mumford theorem (Corollary 1 of [13]) implies that for each L the restriction map

$$\text{Aut}(X_{R_L}) \rightarrow \text{Aut}(X_L)$$

is an isomorphism. Restricting to X_k thus yields a morphism

$$\text{Aut}(X_L) \rightarrow \text{Aut}(X_k).$$

When $H^0(X_k, T_{X_k}) = 0$ this map is injective, as the (slightly more general) Proposition 2.2 below shows.

For any extension L'/L , the diagram

$$(2) \quad \begin{array}{ccc} \text{Aut}(X_L) & \longleftarrow & \text{Aut}(X_{R_L}) \\ \downarrow & & \searrow \\ & & \text{Aut}(X_k) \\ \downarrow & & \nearrow \\ \text{Aut}(X_{L'}) & \longleftarrow & \text{Aut}(X_{R'_L}) \end{array}$$

induced by base change commutes, yielding the desired group homomorphism

$$\sigma : \text{Aut}(X_{\overline{K}}) \rightarrow \text{Aut}(X_k).$$

Since the colimit in diagram (1) has injective transition maps, we also see that σ is injective when $H^0(X_k, T_{X_k}) = 0$.

The diagram (2) acts in the natural way on the restriction diagram

$$(3) \quad \begin{array}{ccc} \text{Pic}(X_L) & \longleftarrow & \text{Pic}(X_{R_L}) \\ \downarrow & & \searrow \\ & & \text{Pic}(X_k) \\ \downarrow & & \nearrow \\ \text{Pic}(X_{L'}) & \longleftarrow & \text{Pic}(X_{R'_L}) \end{array}$$

of Picard groups. Since the resulting map

$$\text{Pic}(X_{\overline{K}}) \xrightarrow{\sim} \text{colim Pic}(X_L) \rightarrow \text{Pic}(X_k)$$

defines the usual specialization map, we conclude that sp is σ -equivariant, as claimed. □

Proposition 2.2. *Suppose $H^0(X_k, T_{X_k}) = 0$. If $U \subset X$ is the complement of finitely many k -valued points, then any automorphism f of U that acts as the identity on U_k equals the identity.*

Proof. Consider the restriction of f to the formal completion \widehat{U}_{U_k} . Write $U^{(n)}$ for $U \otimes R/\mathfrak{m}_R^{n+1}$. The lifts of $f|_{U^{(n)}}$ to an automorphism of $U^{(n+1)}$ are a torsor under $H^0(U_k, T_{U_k}) = H^0(X_k, T_{X_k}) = 0$, which implies that $f|_{\widehat{U}} = \text{id}$. Writing $\xi \in U_k$ for the generic point, we conclude that the induced automorphism f_ξ of $\widehat{\mathcal{O}}_{U, \xi}$ equals the identity, whereupon the restriction of f to the generic point of X_K fits into a commutative diagram

$$\begin{array}{ccc} K(X_K) & \xrightarrow{f} & K(X_K) \\ \downarrow & & \downarrow \\ K(\widehat{\mathcal{O}}_{U, \xi}) & \xrightarrow{f} & K(\widehat{\mathcal{O}}_{U, \xi}), \end{array}$$

with each vertical arrow the canonical inclusion into the completion. We conclude that $f|_{K(U_K)} = \text{id}$, whence $f = \text{id}$ since U is separated. □

In this paper, we will use Theorem 2.1 for a family of K3 surfaces $X \rightarrow \text{Spec } R$. The special fiber is not birationally ruled, since, for example, it is simply connected and not rational. (Of course, such surfaces can be uniruled.) When the specialization map for the Picard groups of a family of K3 surfaces is an isomorphism, there are a few consequences for the cone of curves that we will also use in the sequel.

Lemma 2.3. *If $X \rightarrow \text{Spec } R$ is a relative K3 surface such that the specialization map sp on Picard groups is an isomorphism, then any integral curve $C_k \subset X_k$ is the closed fiber of a flat family of closed subschemes $C \subset X_{R_L}$ for some finite extension L .*

Proof. By the assumption on specialization, the invertible sheaf $\mathcal{O}_{X_k}(C)$ lifts to an invertible sheaf \mathcal{L} on some such X_{R_L} . Consider the cohomology groups $H^i(X_k, \mathcal{L}_k)$ for $i = 1, 2$. Since \mathcal{L}_k is effective and non-trivial, Serre duality and the fact that X_k is K3 imply that $H^2(X_k, \mathcal{L}_k) = 0$ and $\mathcal{O}_{C_k}(C_k) \cong \omega_{C_k}$. The sequence

$$0 \rightarrow \mathcal{O}_{X_k} \rightarrow \mathcal{O}_{X_k}(C_k) \rightarrow \mathcal{O}_{C_k}(C_k) \rightarrow 0$$

gives rise to a sequence

$$0 \rightarrow H^1(X_k, \mathcal{L}_k) \rightarrow H^1(C_k, \omega_{C_k}) \rightarrow H^2(X_k, \mathcal{O}_X) \rightarrow 0.$$

Grothendieck-Serre duality shows that each of the latter two vector spaces are 1-dimensional, whence we conclude that $H^1(X_k, \mathcal{L}_k) = 0$.

It follows from cohomology and base change that $(\pi_L)_*\mathcal{L}$ is a locally free R_L -module whose sections functorially compute $H^0(\mathcal{L})$. In particular, there is a section $s : \mathcal{O}_{X_L} \rightarrow \mathcal{L}$ lifting the section with vanishing locus C_k . The vanishing locus of s gives the desired closed subscheme $C \subset X_L$. \square

Corollary 2.4. *If $X \rightarrow \text{Spec } R$ is a relative K3 surface such that specialization map sp is an isomorphism then sp induces an isomorphism of nef cones $\text{Nef}(X_{\overline{K}}) \rightarrow \text{Nef}(X_k)$ respecting the ample cones.*

Proof. First observe that the specialization map preserves the effective cones. Indeed, by assumption the cones lie in canonically isomorphic spaces. Specialization inserts one cone into the other. By Lemma 2.3, this inclusion is surjective. But the nef cone is determined by intersecting with elements of the effective cone, giving the result. Since this is an isomorphism that respects the inner product and the ample cone is the interior of the nef cone, this isomorphism respects the ample cones. \square

3. Deforming violations of Artin's conjecture to finite height

This section is not needed for the argument if one accepts the full Tate conjecture for K3 surfaces (which was proven after this was originally written [4–6, 11, 12, 14]). We include it here because it is significantly simpler than the proof of the Tate conjecture and the argument may be of independent interest.

Suppose X is an Artin-supersingular surface. If the Picard rank of X is greater than 4 then X is elliptic and therefore Shioda-supersingular by Theorem 1.7 of [1]. (To see that any X with Picard number at least 5 is elliptic, note that the Picard lattice represents 0 by the Hasse-Minkowski theorem and the Hodge index theorem, and once the lattice represents 0 it is a standard exercise that the surface contains an elliptic curve – e.g., exercise IX.6 of [3].) Thus, we will assume that X has Picard rank $\rho \leq 4$.

Proposition 3.1. *There is a smooth projective relative K3 surface \mathcal{X} over $k[[t]]$ such that*

- 1) \mathcal{X}_0 is isomorphic to X
- 2) the restriction map $\mathrm{Pic}(\mathcal{X}) \rightarrow \mathrm{Pic}(X)$ is an isomorphism
- 3) the generic fiber \mathcal{X}_η is of finite height

Proof. Fix generators L_1, \dots, L_ρ for $\mathrm{Pic}(X)$. By Proposition 1.5 of [7], we know that the locus in $\mathrm{Spf} k[[t_1, \dots, t_{20}]]$ over which each L_i deforms is determined by a single equation. We conclude that there is a closed subscheme $\mathrm{Spec} A \subset \mathrm{Spec} k[[t_1, \dots, t_{20}]]$ of dimension at least 16 and a universal formal deformation of $(X; L_1, \dots, L_\rho)$ over $\mathrm{Spf} A$. Since some linear combination of the L_i is ample, this formal deformation is algebraizable. It suffices to show that the generic fiber is not supersingular.

By Artin approximation, we can realize A as the completion of a scheme S of finite type over k at a closed point that carries a family of K3 surfaces giving rise to our family by completion. Moreover, we may assume that S is unramified over the functor of polarized K3 surfaces. By Proposition 14 in [17], the locus in S parametrizing supersingular K3 surfaces has dimension 9. Thus, a general point of S will parametrize a surface of finite height, and the same will be true of the family restricted to $\mathrm{Spec} A$, as desired. \square

4. Lifting the Picard group to characteristic 0

In this section we prove that any K3 surface X of finite height over a perfect field k of characteristic p admits a lift to characteristic 0 that lifts the entire Picard group. While we could use the Nygaard-Ogus theory of quasi-canonical liftings [15], this bare lifting fact is much simpler to prove by analyzing the first-order deformations.

According to Proposition 10.3 of [9], if X is a K3 surface of finite height over k then the crystalline first Chern class yields an injective linear map

$$\mathrm{NS}(X) \otimes_{\mathbf{Z}} k \rightarrow \mathrm{H}^1(X, \Omega^1).$$

(Note that the related map

$$\mathrm{NS}(X) \otimes \mathbf{Z}/p\mathbf{Z} \rightarrow \mathrm{H}_{\mathrm{dR}}^2(X)$$

is always injective, regardless of the height of X , by results of Deligne and Illusie (Remark 3.5 of [7].)

Serre duality yields an isomorphism $\mathrm{H}^1(X, \Omega^1)^\vee \xrightarrow{\sim} \mathrm{H}^1(X, T_X)$. Given invertible sheaves L_1, \dots, L_n on X , there is a deformation functor $\mathrm{Def}(X; \{L_1, \dots, L_n\})$ parametrizing deformations of X together with deformations of each L_i . Given a subset $J \subset \{1, \dots, n\}$, there is an associated forgetful natural transformation

$$\mathrm{Def}(X; \{L_i\}_{i=1}^n) \rightarrow \mathrm{Def}(X; \{L_j\}_{j \in J}).$$

Proposition 4.1. *Suppose the isomorphism classes of the L_j generate a subgroup of $\mathrm{Pic}(X)$ of rank n . For each subset $J \subset \{1, \dots, n\}$, the functor $\mathrm{Def}(X; \{L_j\}_{j \in J})$ is prorepresentable by a family over $W[[x_1, \dots, x_{20-|J|}]]$. Moreover, there is a choice of coordinates for each subset J such that the forgetful morphism*

$$\mathrm{Def}(X; \{L_j\}_{j \in J}) \rightarrow \mathrm{Def}(X)$$

is identified with the closed immersion associated to the quotient

$$W[[x_1, \dots, x_{20}]] \rightarrow W[[\{x_i\}_{i \notin J}]].$$

In particular, if we fix a \mathbf{Z} -basis of $\mathrm{Pic}(X)$ we see that the deformation functor $\mathrm{Def}(X; L_1, \dots, L_n)$ is smooth over W .

Corollary 4.2. *Any K3 surface of finite height over a perfect field k is the closed fiber of a smooth projective relative K3 surface $\mathcal{X} \rightarrow \text{Spec } W$ in such a way that the restriction map $\text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(X)$ is an isomorphism.*

Proof. Given a basis L_1, \dots, L_n for $\text{Pic}(X)$, we have by Proposition 4.1 that the deformation functor $\text{Def}(X; L_1, \dots, L_n)$ is formally smooth over W of relative dimension $20 - n$. Since W is Henselian, the k -valued point $(X; L_1, \dots, L_n)$ extends to a W -valued point, giving a formal lifting. Since some linear combination of the L_i is ample, we see that the formal family is formally projective. By the Grothendieck Existence Theorem this lift is therefore algebraizable as a projective scheme, as desired. \square

The proof of Proposition 4.1 is an immediate consequence of the following lemma.

Lemma 4.3. *Suppose L_1, \dots, L_j generate a subgroup $\Lambda \subset \text{NS}(X)$ such that the quotient $\text{NS}(X)/\Lambda$ has trivial p -torsion. The cokernel of the induced morphism*

$$d \log |_{\Lambda} : \Lambda \otimes k \rightarrow H^1(X, \Omega_X^1)$$

is naturally identified with the dual of $\text{Def}(X; L_1, \dots, L_j)(k[\varepsilon])$.

Proof. Illusie proved (Proposition IV.3.1.8 of [8]) that cupping the Atiyah class of an invertible sheaf L with the class of a first-order deformation $[X'] \in H^1(X, T_X) = \text{Ext}_X^1(\Omega_X^1, \mathcal{O}_X)$ gives the obstruction class in $H^2(X, \mathcal{O})$ to deforming L to X' . By Serre duality, the cup product pairing $H^1(X, \Omega_X^1) \times H^1(X, T_X) \rightarrow H^2(X, \mathcal{O})$ is perfect. Since $d \log$ computes the Atiyah class of L , we see that the tangent space $\text{Def}(X; L_1, \dots, L_j)(k[\varepsilon])$ is given by the subspace of $\text{Hom}(H^1(X, \Omega^1), H^2(X, \mathcal{O}))$ that annihilates $d \log(\Lambda \otimes k)$, as desired. \square

Proof of Proposition 4.1. By Deligne [7], each L_i imposes one equation f_i on $R := W[[t_1, \dots, t_{20}]]$. We wish to show that $R/(f_1, \dots, f_n)$ is W -flat. Consider the Jacobian ideal $J := (\partial f_i / \partial x_j)$. Reducing modulo p we see from Lemma 4.3 that $\bar{J} \subset R/pR$ is the unit ideal, whence J is itself the unit ideal. It thus follows that $R/(f_1, \dots, f_n)$ is of finite type and formally smooth over W , whence it is smooth, as desired. \square

5. Shioda-supersingular K3s

In this section we assume that $\text{char } k > 2$, so that we can use Ogus’s crystalline theory of supersingular K3 surfaces [16]. Fix a K3 surface over k with

Picard number 22. The intersection pairing makes the Picard group a lattice. Write N for this rank 22 lattice. Inside $N \otimes \mathbf{R}$ is the nef cone $\text{Nef}(X)$. It is the closure of the cone generated by the ample classes in N , and thus there is an arithmetic group O^+ parametrizing isometries of N whose extension to \mathbf{R} preserves the positive cone.

Since $\text{rk } N = 22$ and $\dim_k H^1(X, \Omega^1) = 20$, the map $N \otimes k \rightarrow H_{\text{dR}}^2(X/k)$ has a non-trivial kernel. Ogus has found a canonical semilinear identification of this kernel with a subspace K of $N \otimes k$. He proved the following, among other things.

Theorem 5.1 (Ogus [16]). *An automorphism of X is uniquely determined by its image in O^+ . Moreover, the image of $\text{Aut}(X) \rightarrow O^+$ is precisely the subgroup of elements that preserve $K \subset N \otimes k$.*

Let W be the subgroup of O^+ generated by reflections in (-2) -curves. In [16], Ogus proved for supersingular K3s the following analogue of a well-known classical result.

Proposition 5.2. *The group W is a normal subgroup of O^+ that acts on $\text{Aut}(X)$, and the semidirect product $\text{Aut}(X) \rtimes W$ has finite index in O^+ . Moreover, the nef cone $\text{Nef}(X)$ is a fundamental domain for the action of W on the positive cone \mathcal{C} .*

Proof. That W is normal in O^+ and acts on \mathcal{C} with fundamental domain $\text{Nef}(X)$ is in the series of results proving Proposition 1.10 of [16]. To see the rest of the Proposition, note that reduction modulo p defines a map to a finite group $O^+ \rightarrow \text{GL}(N \otimes \mathbf{F}_p)$. The condition that an element of the latter group preserve K defines an a priori finite subgroup, showing that $\text{Aut}(X)$ has finite index in O^+/W . \square

The following is an immediate consequence of Proposition 5.2

Corollary 5.3. *There is a rational polyhedral fundamental domain for the action of $\text{Aut}(X)$ on $\text{Nef}(X)$ if and only if there is a rational polyhedral fundamental domain for the action of O^+ on \mathcal{C} that lies entirely in $\text{Nef}(X)$.*

These reductions place us in classical territory: the positive cone \mathcal{C} is a standard round cone of the form $x_1 > \sqrt{x_2^2 + \cdots + x_{22}^2}$. This is one of the homogeneous self-adjoint convex cones in the sense of [2].

In fact, Sterk shows slightly more than the bare classical statement about the existence of a rational polyhedral fundamental domain. His proofs carry

over to the present situation verbatim. Write $\text{Nef}(X)_+$ for the convex hull of $\text{Nef}(X) \cap (N \otimes \mathbf{Q})$ in $N \otimes \mathbf{R}$.

Proposition 5.4 (Sterk, Lemma 2.3, Lemma 2.4, Proposition 2.5 of [20]). *Fix an ample divisor H . The locus of $x \in \text{Nef}(X)_+$ such that $H \cdot (\phi(x) - x) \geq 0$ for all $\phi \in O^+$ is a rational polyhedral fundamental domain for the action of O^+ on \mathcal{C} . Moreover, there are finitely many orbits for the action of O^+ on the nodal classes of X .*

The reader is referred to [20] for the proofs.

6. The main theorem

Let X be a K3 surface over an algebraically closed field k with characteristic $p \geq 3$. Recall that an element of $\text{Pic}(X)$ is called a *nodal class* if it comes from a smooth rational curve $C \subset X$.

Theorem 6.1. *Let $O(\text{Pic}(X))$ be the orthogonal group for $\text{Pic}(X)$ with respect to the intersection form and let Γ be the subgroup of $O(\text{Pic}(X))$ consisting of elements preserving the nef cone.*

- 1) *The natural map $\text{Aut}(X) \rightarrow \Gamma$ has finite kernel and cokernel.*
- 2) *The group $\text{Aut}(X)$ is finitely generated.*
- 3) *The action of $\text{Aut}(X)$ on $\text{Nef}(X)$ has a rational polyhedral fundamental domain.*
- 4) *The set of orbits of $\text{Aut}(X)$ in the nodal classes of X is finite.*

Proof. When k has characteristic 0, this is a standard result of Torelli's Theorem and Sterk's results [20]. Now suppose k has positive characteristic p . If we fix a very ample L on X , the kernel of the map $\text{Aut}(X) \rightarrow \Gamma$ is contained in the automorphism group $\text{Aut}(X, L)$ of the pair (because any such element leaves every invertible sheaf invariant). This latter group is finite type, since it is contained in $\text{PGL}(N)$ for some N , and discrete, since X has no nontrivial vector fields, thus finite.

For the remaining claims, we argue as follows. If X has finite height then we have by Theorem 2.1 and Corollary 2.4 that X has a lift X_1 to an algebraically closed field of characteristic 0 such that the specialization isomorphism $\text{Pic}(X_1) \rightarrow \text{Pic}(X)$ is equivariant with respect to the specialization map $\text{Aut}(X_1) \rightarrow \text{Aut}(X)$, and the nef cone specializes to the nef cone.

Since we know all statements for X_1 , it follows immediately that the image of $\text{Aut}(X)$ in Γ is of finite index, and that Γ acts on $\text{Nef}(X) = \text{Nef}(X_1)$ with a rational polyhedral fundamental domain. Once we have finiteness of kernel and cokernel, we have that $\text{Aut}(X)$ is finitely generated because Γ is finitely generated (Schreier's lemma [19]).

If X is Shioda-supersingular, then the proof is given in Section 5. \square

Remark 6.2. If we do not assume the Tate conjecture for supersingular K3 surfaces, we can use Section 3 to handle case of Artin-supersingular X with Picard number less than 5. By Proposition 3.1 X is the specialization of a K3 surface X_1 over an algebraically closed field of characteristic p that has finite height such that the specialization map $\text{Pic}(X_1) \rightarrow \text{Pic}(X)$ is an isomorphism that is equivariant with respect to the specialization map $\text{Aut}(X_1) \rightarrow \text{Aut}(X)$. The same argument as in the previous paragraph concludes the proof.

Remark 6.3. Even in characteristic 0, the kernel of the map $\text{Aut}(X) \rightarrow \Gamma$ of Theorem 6.1(i) can be nontrivial. For instance, for a double cover of a plane sextic with Picard rank 1, the standard involution acts trivially on $\text{Pic}(X)$.

Note that a consequence of the argument is that, after choosing a lift to characteristic 0 which preserves $\text{Pic}(X)$, the specialization map $\text{Aut}(X_1) \rightarrow \text{Aut}(X)$ has finite index.

Since the finite height argument works for all characteristics, the only situation not handled by this argument is the case of supersingular K3 surfaces in characteristic 2. Due to work of Shimada [18], these are all given as inseparable double covers of \mathbf{P}^2 ; it seems plausible that a direct analysis along these lines may handle this case as well.

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