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Control for Schrödinger equation on hyperbolic surfaces

Long Jin

We show that any nonempty open set on a hyperbolic surface provides observability and control for the time dependent Schrödinger equation. The only other manifolds for which this was previously known are flat tori [11–13]. The proof is based on the main estimate in [10] and standard arguments in control theory.

1. Introduction

Let M be a compact (connected) hyperbolic surface and Δ the Laplace-Beltrami operator on M. In a recent paper with Dyatlov, [10], we prove the following semiclassical control result which roughly says that any open set in S^*M controls the whole S^*M in the L^2-sense.

Theorem 1.1. *[10, Theorem 2] Assume that $a \in C_0^\infty(T^*M)$ and $a|_{S^*M} \not\equiv 0$, then there exist constants C, $h_0 > 0$ only depending on M and a such that for all $0 < h < h_0$ and $u \in H^2(M)$,*

$$\|u\|_{L^2(M)} \leq C\|\operatorname{Op}_h(a)u\|_{L^2(M)} + C\frac{\log(1/h)}{h}\|(-h^2\Delta - 1)u\|_{L^2(M)}. \tag{1.1}$$

In this short notes, we show that Theorem 1.1 implies the following observability result of the Schrödinger equation on M.

Theorem 1.2. *Let $\Omega \subset M$ be any non-empty open set and $T > 0$, then there exists a constant $K > 0$ depending only on M, Ω and T, such that for any $u_0 \in L^2(M)$, we have*

$$\|u_0\|^2_{L^2(M)} \leq K\int_0^T \|e^{it\Delta}u_0\|^2_{L^2(\Omega)}dt. \tag{1.2}$$

The following control result for the Schrödinger equation then follows immediately by the HUM method of Lions [15].

Theorem 1.3. *Let $\Omega \subset M$ be any non-empty open set and $T > 0$, then for any $u_0 \in L^2(M)$, there exists $f \in L^2((0,T) \times \Omega)$ such that the solution of the equation*

$$(i\partial_t + \Delta)u(t,x) = f 1_{(0,T)\times\Omega}(t,x), \quad u(0,x) = u_0(x) \tag{1.3}$$

satisfies

$$u(T,x) \equiv 0. \tag{1.4}$$

Remark 1.4. In fact, by an elementary perturbation argument, it is not hard to see that Theorem 1.1 still holds if we replace the Laplacian operator $-\Delta$ by a general Schrödinger operator $-\Delta + V$ with $V \in L^\infty(M;\mathbb{R})$. Following the proof, we can also replace $-\Delta$ by $-\Delta + V$ in Theorem 1.2 and 1.3. It is interesting to ask if this can be further extended to L^2-potentials as in the case of two-dimensional tori [5]. Another interesting question is to extend the result to rough control sets as in [9].

1.1. Control for Schrödinger equations

In general, the pioneering work of Lebeau [14] showed that control for Schrödinger equation holds under the geometric control condition (see [4]):

$$\begin{aligned}&\text{There exists } L = L(M,\Omega) > 0\\ &\text{s.t. every geodesic of length } L \text{ on } M \text{ intersects } \Omega.\end{aligned} \tag{1.5}$$

This geometric control condition is necessary when the geodesic flow is periodic (e.g. M is a sphere), see Macia [16]. However in general, it is not necessary for observability and control for Schrödinger equation. In fact, Theorem 1.2 and Theorem 1.3 show that no condition is needed for the nonempty open set Ω on a compact hyperbolic surface.

To our best knowledge, the only other manifold on which this is true is the flat torus. This is first proved by Jaffard [12] and Haraux [11] in dimension two and by Komornik [13] in higher dimensions. These results are further extended to Schrödinger operators $-\Delta + V$ with smooth potential V by Burq–Zworski [8], and L^2-potential V by Bourgain–Burq–Zworski [5] in dimension two; some class of potentials V including continuous ones by Anantharaman–Macia [2] in any dimension. We also mention the recent result of Burq–Zworski [9] on control on two-dimensional tori by any L^4 functions or sets with positive measures.

For certain other manifolds, observability and control for Schrödinger equation is known under weaker dynamical conditions. For example, Anantharaman-Rivière [3] proved the case where M is a manifold with negative sectional curvature and Ω satisfies an entropy condition, i.e. the set of uncontrolled trajectories is thin. A similar dynamical condition appears in the work of Schenck [17] on energy decay of wave equation on such manifolds. In the case of manifolds with boundary, Anantharaman–Léautaud–Macia [1] showed the control and observability for Schrödinger equation on the disk by any nonempty open set touching the boundary.

1.2. Control for wave equation

For control of wave equation, by propagation of singularities, the geometric control condition (1.5) is necessary and sufficient, see Bardos–Lebeau–Rauch [4] and Burq–Gérard [6].

We remark that the same argument as in Proposition 2.1 (or the abstract result in Burq–Zworski [7]) gives the following semiclassical observability result from Theorem 1.1.

Proposition 1.5. *Let $\chi \in C_0^\infty((\frac{1}{2}, 2))$, then there exists C, K and $h_0 > 0$ such that for all $0 < h < h_0$, $u_0 \in L^2(M)$, we have*

(1.6)

$$\|\chi(h\sqrt{-\Delta})u_0\|^2_{L^2(M)} \leq \frac{K}{C\log(1/h)} \int_0^{C\log(1/h)} \|e^{it\sqrt{-\Delta}}\chi(h\sqrt{-\Delta})u_0\|^2_{L^2(\Omega)} dt.$$

However it is unclear to us at the moment whether the HUM method gives a control result for some explicit subspace of L^2-functions.

1.3. Notations

We recall some notations from semiclassical analysis and refer to the book [18] for further references. First, the semiclassical Fourier transform on $\mathbb{R}$ is defined by

$$\mathcal{F}_h\varphi(\tau) = \int_{\mathbb{R}} e^{-it\tau/h}\varphi(t)dt$$

and its adjoint is given by

$$\mathcal{F}_h^*\varphi(\tau) = \int_{\mathbb{R}} e^{it\tau/h}\varphi(t)dt. \tag{1.7}$$

The Parseval identity show that

$$\|\mathcal{F}_h\varphi\|_{L^2} = \|\mathcal{F}_h^*\varphi\|_{L^2} = (2\pi h)^{1/2}\|\varphi\|_{L^2}. \tag{1.8}$$

We also use the standard quantization $a(t, D_t)$ on $\mathbb{R}$ and fix a semiclassical quantization $\mathrm{Op}_h(a)$ on M. We refer to [18] for the standard definition and properties. Finally, as usual, C denotes a constant which may change from line to line.

2. Proof of the theorems

All the parts of the proof are well known in the literature. Here we present them in a self-contained way.

2.1. Semiclassical observability

We first prove a semiclassical version of the observability result

Proposition 2.1. *Let* $\chi \in C_0^\infty((\frac{1}{2}, 2))$, $\psi \in C_0^\infty(\mathbb{R}; [0,1])$ *not identically zero, then there exist* C, $h_0 > 0$ *such that for all* $0 < h < h_0$, $u_0 \in L^2(M)$, *we have*

$$\|\chi(-h^2\Delta)u_0\|^2_{L^2(M)} \leq C \int_{\mathbb{R}} \|\psi(t)e^{it\Delta}\chi(-h^2\Delta)u_0\|^2_{L^2(\Omega)}dt. \tag{2.1}$$

In particular, for any $T > 0$, *there exists* C, $h_0 > 0$ *such that for all* $0 < h < h_0$, $u_0 \in L^2(M)$, *we have*

$$\|\chi(-h^2\Delta)u_0\|^2_{L^2(M)} \leq C \int_0^T \|e^{it\Delta}\chi(-h^2\Delta)u_0\|^2_{L^2(\Omega)}dt. \tag{2.2}$$

Proof. This follows directly from the abstract result in Burq–Zworski [7, Theorem 4] with $G(h) = C\log(1/h)$, $g(h) = C$ and $T(h) = 1/h$. We present the argument in this concrete situation.

First, we put $v(t) = e^{ith\Delta}\chi(-h^2\Delta)u_0$ and write $w(t) = \psi(ht)v(t)$. It is clear that $v(t)$ solves the semiclassical Schrödinger equation $(ih\partial_t + h^2\Delta)v = 0$ and thus

$$(ih\partial_t + h^2\Delta)w = ih^2\psi'(ht)v(t).$$

We take the (adjoint) semiclassical Fourier transform (1.7) to get

$$(-h^2\Delta - \tau)\mathcal{F}_h^* w(\tau) = -ih^2\mathcal{F}_h^*(\psi'(ht)v(t))(\tau). \tag{2.3}$$

For $\tau \in (\frac{1}{2}, 2)$, we use (1.1) and choose $a \in C_0^\infty$ to be supported in $\{(x,\xi) : x \in \Omega\}$ with $\|a\|_{L^\infty} \le 1$. We then choose $\chi \in C_0^\infty(\Omega; [0,1])$ and regard it also as a function on T^*M, such that $\chi \equiv 1$ on a neighborhood of $\operatorname{supp} a$, then for any $u \in L^2(M)$,

$$\operatorname{Op}_h(a)u = \operatorname{Op}_h(a)(\chi u) + \operatorname{Op}_h(a)(1-\chi)u,$$

where $\operatorname{Op}_h(a)(1-\chi) = \mathcal{O}_{L^2\to L^2}(h^\infty)$, so

$$\|\operatorname{Op}_h(a)u\|_{L^2(M)} \le C\|\chi u\|_{L^2(M)} + \mathcal{O}(h^\infty)\|u\|_{L^2(M)}.$$

Now (1.1) gives that, for $0 < h < h_0$,

$$\|u\|_{L^2(M)} \le C\|u\|_{L^2(\Omega)} + C\frac{\log(1/h)}{h}\|(-h^2\Delta - 1)u\|_{L^2(M)}.$$

We can further rescale this estimate to show that uniformly for $\tau \in [1/2, 2]$,

$$\|u\|_{L^2(M)} \le C\|u\|_{L^2(\Omega)} + C\frac{\log(1/h)}{h}\|(-h^2\Delta - \tau)u\|_{L^2(M)}. \tag{2.4}$$

For $\tau \in [\frac{1}{2}, 2]$, applying (2.4) to $u = \mathcal{F}_h^* w(\tau)$, we obtain

$$\begin{aligned}\|\mathcal{F}_h^* w(\tau)\|_{L^2(M)} \le{} & C\|\mathcal{F}_h^* w(\tau)\|_{L^2(\Omega)} \\ & + Ch\log(1/h)\|\mathcal{F}_h^*(\psi'(ht)v(t))(\tau)\|_{L^2(M)}.\end{aligned} \tag{2.5}$$

For $\tau \notin [\frac{1}{2}, 2]$, by definition,

$$\mathcal{F}_h^* w(\tau) = \int_{\mathbb{R}} e^{-it(-h^2\Delta-\tau)/h}\psi(ht)\chi(-h^2\Delta)u_0 dt.$$

Writing

$$e^{-it(-h^2\Delta-\tau)/h} = (h^2\Delta+\tau)^{-N}(hD_t)^N e^{-it(-h^2\Delta-\tau)/h},$$

and noting that for any $u_0 \in L^2(M)$, by functional calculus,

$$\|(h^2\Delta+\tau)^{-N}\chi(-h^2\Delta)u_0\|_{L^2(M)} \le C_N\langle\tau\rangle^{-N}\|\chi(-h^2\Delta)u_0\|_{L^2(M)},$$

we can integrate by parts repeatedly to get

$$\|\mathcal{F}_h^* w(\tau)\|_{L^2(M)} = \mathcal{O}((h\langle\tau\rangle^{-1})^\infty)\|\chi(-h^2\Delta)u_0\|_{L^2(M)}. \tag{2.6}$$

Combining (2.5) and (2.6), we have the following estimate

$$\begin{aligned}\|\mathcal{F}_h^* w(\tau)\|^2_{L^2(\mathbb{R}_\tau,L^2(M))} \le{}& C\|\mathcal{F}_h^* w(\tau)\|^2_{L^2(\mathbb{R}_\tau,L^2(\Omega))}\\ &+ C(h\log(1/h))^2\|\mathcal{F}_h^*(\psi'(ht)v(t))(\tau)\|^2_{L^2(\mathbb{R}_\tau,L^2(M))}\\ &+ \mathcal{O}(h^\infty)\|\chi(-h^2\Delta)u_0\|^2_{L^2(M)}.\end{aligned}$$

By the Parseval identity (1.8), we have

$$\begin{aligned}\|w\|^2_{L^2(\mathbb{R}_t,L^2(M))} \le{}& C\|w\|^2_{L^2(\mathbb{R}_t,L^2(\Omega))}\\ &+ C(h\log(1/h))^2\|\psi'(ht)v(t)\|^2_{L^2(\mathbb{R}_t,L^2(M))}\\ &+ \mathcal{O}(h^\infty)\|\chi(-h^2\Delta)u_0\|^2_{L^2(M)}.\end{aligned} \tag{2.7}$$

From the definition of v and w, we see

$$\begin{aligned}\|w\|^2_{L^2(\mathbb{R}_t,L^2(M))} &= \int_{\mathbb{R}} \psi(ht)^2\|e^{ith\Delta}\chi(-h^2\Delta)u_0\|^2_{L^2(M)}dt\\ &= \left(\int_{\mathbb{R}}\psi(ht)^2dt\right)\|\chi(-h^2\Delta)u_0\|^2_{L^2(M)}\\ &= h^{-1}\|\psi\|^2_{L^2(\mathbb{R})}\|\chi(-h^2\Delta)u_0\|^2_{L^2(M)};\end{aligned}$$

$$\begin{aligned}\|w\|^2_{L^2(\mathbb{R}_t,L^2(\Omega))} &= \int_{\mathbb{R}} \psi(ht)^2\|e^{ith\Delta}\chi(-h^2\Delta)u_0\|^2_{L^2(\Omega)}dt\\ &= h^{-1}\int_{\mathbb{R}}\|\psi(t)e^{it\Delta}\chi(-h^2\Delta)u_0\|^2_{L^2(\Omega)}dt;\end{aligned}$$

and

$$\begin{aligned}\|\psi'(ht)v(t)\|^2_{L^2(\mathbb{R}_t,L^2(M))} &= \int_{\mathbb{R}} |\psi'(ht)|^2\|e^{ith\Delta}\chi(-h^2\Delta)u_0\|^2_{L^2(M)}dt\\ &= \left(\int_{\mathbb{R}}|\psi'(ht)|^2dt\right)\|\chi(-h^2\Delta)u_0\|^2_{L^2(M)}\\ &= h^{-1}\|\psi'\|^2_{L^2(\mathbb{R})}\|\chi(-h^2\Delta)u_0\|^2_{L^2(M)}.\end{aligned}$$

As long as h is small and $\psi \not\equiv 0$, we can absorb the last two terms on the right-hand side of (2.7) into the left-hand side and conclude the proof. □

2.2. Observability with error

Now we prove Theorem 1.2 with an error in $H^{-4}(M)$.

Proposition 2.2. *There exists a constant $C > 0$ such that for any $u_0 \in L^2(M)$, we have*

$$\|u_0\|^2_{L^2(M)} \leq C \left(\int_0^T \|e^{it\Delta} u_0\|^2_{L^2(\Omega)} dt + \|u_0\|^2_{H^{-4}(M)} \right). \tag{2.8}$$

Proof. Again, this argument can be found in Burq–Zworski [7, Theorem 7] or [8, Proposition 4.1]. To pass from the semiclassical observability to the classical one, we use a dyadic decomposition

$$1 = \varphi_0(r)^2 + \sum_{k=1}^{\infty} \varphi_k(r)^2$$

where

$$\varphi_0 \in C_0^\infty((-2,2);[0,1]), \quad \varphi_k(r) = \varphi(2^{-k}|r|), \quad \varphi \in C_0^\infty((1/2,2);[0,1]).$$

Then we have

$$\|u_0\|^2_{L^2(M)} = \sum_{k=0}^{\infty} \|\varphi_k(-\Delta) u_0\|^2_{L^2(M)}, \tag{2.9}$$

and

$$\|u_0\|^2_{H^{-4}(M)} = \|(-\Delta+1)^{-2} u_0\|^2_{L^2(M)} \sim \sum_{k=0}^{\infty} 2^{-4k} \|\varphi_k(-\Delta) u_0\|^2_{L^2(M)}. \tag{2.10}$$

Fix an integer K so that $2^{-K} < h_0^2$, then for $k \geq K$, by (2.1), we have

$$\|\varphi_k(-\Delta) u_0\|^2_{L^2(M)} \leq C \int_{\mathbb{R}} \|\psi(t) e^{it\Delta} \varphi_k(-\Delta) u_0\|^2_{L^2(\Omega)} dt \tag{2.11}$$

uniformly in k where we choose $\psi \in C_0^\infty((0,T);[0,1])$.

The idea is to use the Schrödinger equation to change the frequency localization in space $\varphi_k(-\Delta)$ to frequency localization in time $\varphi_k(D_t)$. More precisely, since $(D_t - \Delta)e^{it\Delta} = 0$ and all φ_k are even, we have

$$e^{it\Delta} \varphi_k(-\Delta) u_0 = \varphi_k(-\Delta) e^{it\Delta} u_0 = \varphi_k(-D_t) e^{it\Delta} u_0 = \varphi_k(D_t) e^{it\Delta} u_0.$$

Now we introduce another cutoff function in time $\widetilde{\psi} \in C_0^\infty((0,T);[0,1])$ such that $\widetilde{\psi} = 1$ on a neighborhood of $\operatorname{supp} \psi$. This allows us to express the pseudolocality of $\psi(t)\varphi_k(D_t)$ as follows:

$$\psi(t)\varphi_k(D_t) = \psi(t)\varphi_k(D_t)\widetilde{\psi}(t) + E_k(t, D_t)$$

where $E_k(t,D_t)=\psi(t)[\tilde{\psi}(t),\varphi(2^{-k}D_t)]$ with symbol satisfying

$$\partial^\alpha E_k(t,\tau)=\mathcal{O}(2^{-kN}\langle t\rangle^{-N}\langle\tau\rangle^{-N}),\quad \forall N. \tag{2.12}$$

Now we have

$$\begin{aligned}\|\psi(t)e^{it\Delta}\varphi_k(-\Delta)u_0\|^2_{L^2(\Omega)}&=\|\psi(t)\varphi_k(D_t)e^{it\Delta}u_0\|^2_{L^2(\Omega)}\\ \le\|\psi(t)\varphi_k(D_t)\widetilde{\psi}(t)e^{it\Delta}u_0\|^2_{L^2(\Omega)}&+\|E_k(t,D_t)e^{it\Delta}u_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore by (2.9) and (2.11), we get

$$\begin{aligned}\|u_0\|^2_{L^2(M)}\le\sum_{k=0}^{K-1}\|\varphi_k(-\Delta)u_0\|^2_{L^2(M)}&+\sum_{k=K}^{\infty}C\int_{\mathbb{R}}\|\varphi_k(D_t)\widetilde{\psi}(t)e^{it\Delta}u_0\|^2_{L^2(\Omega)}dt\\ &+\sum_{k=K}^{\infty}C\int_{\mathbb{R}}\|E_k(t,D_t)e^{it\Delta}u_0\|^2_{L^2(\Omega)}dt.\end{aligned}$$

By (2.10), we see that the first sum is bounded by $C\|u_0\|^2_{H^{-4}(M)}$. The second sum is bounded by

$$C\int_{\mathbb{R}}\sum_{k=0}^{\infty}\langle\varphi_k(D_t)^2\widetilde{\psi}(t)e^{it\Delta}u_0,\widetilde{\psi}(t)e^{it\Delta}u_0\rangle_{L^2(\Omega)}dt=C\int_{\mathbb{R}}\|\widetilde{\psi}(t)e^{it\Delta}u_0\|^2_{L^2(\Omega)}dt.$$

The final sum is bounded by

$$C\sum_{k=K}^{\infty}\int_{\mathbb{R}}\|E_k(t,D_t)e^{it\Delta}u_0\|^2_{L^2(M)}dt=C\sum_{k=K}^{\infty}\|E_k(t,D_t)e^{it\Delta}u_0\|^2_{L^2(\mathbb{R}\times M)}. \tag{2.13}$$

To show this is also bounded by $C\|u_0\|^2_{H^{-4}(M)}$, we write

$$\begin{aligned}E_k(t,D_t)e^{it\Delta}u_0&=E_k(t,D_t)(-D_t+1)^2e^{it\Delta}(-\Delta+1)^{-2}u_0\\ &=\widetilde{E}_k(t,D_t)\langle t\rangle^{-2}e^{it\Delta}(-\Delta+1)^{-2}u_0\end{aligned}$$

where the symbol of $\widetilde{E}_k(t,D_t)=E_k(t,D_t)(-D_t+1)^2\langle t\rangle^2$ also satisfies (2.12) and thus $\widetilde{E}_k(t,D_t)=\mathcal{O}(2^{-k}):L^2(\mathbb{R})\to L^2(\mathbb{R})$. Therefore (2.13) is bounded by

$$C\sum_{k=K}^{\infty} 2^{-2k}\|\langle t\rangle^{-2}e^{it\Delta}(-\Delta+1)^{-2}u_0\|^2_{L^2(\mathbb{R}\times M)}$$
$$\leq C\|(-\Delta+1)^{-2}u_0\|^2_{L^2(M)} = C\|u_0\|^2_{H^{-4}(M)}.$$

This finishes the proof of (2.8). □

Remark 2.3. From the proof, it is clear that the H^{-4} error can be replaced by any H^{-m} error as long as $m > 0$. Here we take $m = 4$ to make the uniqueness-compactness argument in the next section simpler.

2.3. Removing the error

To finish the proof, we use the classical uniqueness-compactness argument of Bardos–Lebeau–Rauch [4] to remove the H^{-4} error term. We remark that a quantitative version of the uniqueness-compactness argument is presented in [5, Appendix A] which can be used to remove any H^{-m} error and compute the constant K in (1.2) from the constant C in (2.8) in principle.

For any $T > 0$, consider the following closed subspaces of $L^2(M)$:

$$N_T = \{u_0 \in L^2(M) : e^{it\Delta}u_0 \equiv 0 \text{ on } (0,T)\times\Omega\}.$$

Lemma 2.4. *We have* $N_T = \{0\}$.

Proof. In fact, if $u_0 \in N_T$, then

$$v_{\varepsilon,0} := \frac{1}{\varepsilon}(e^{i\epsilon\Delta} - I)u_0 \in N_{T-\delta}$$

if $\varepsilon \leq \delta$. Moreover, $v_{\varepsilon,0}$ converges to $v_0 = i\Delta u_0$ in $L^2(M)$. To see this, we only need to show that $v_{\varepsilon,0}$ is a Cauchy sequence in $L^2(M)$. We write the orthonormal expansion of u_0 in terms of the Laplacian eigenfunctions

$$u_0 = \sum_{j=0}^{\infty} u_{0,j}e_j,$$

where $\{e_j\}_{j=0}^{\infty}$ is an orthonormal basis of $L^2(M)$ formed by Laplacian eigenfunctions:

$$-\Delta e_j = \lambda_j e_j, \quad \|e_j\|_{L^2(M)} = 1,$$
$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \nearrow \infty.$$

Then for $\alpha, \beta \in (0, T/2)$, we have by (2.8) (with T replaced by $T/2$),

$$\begin{aligned}\|v_{\alpha,0} - v_{\beta,0}\|_{L^2}^2 &\le C\|v_{\alpha,0} - v_{\beta,0}\|_{H^{-4}}^2 \\ &\le C\sum_{j=1}^{\infty}\left|\frac{e^{-i\alpha\lambda_j}-1}{\alpha} - \frac{e^{-i\beta\lambda_j}-1}{\beta}\right|^2 (1+\lambda_j)^{-4}|u_{0,j}|^2 \\ &\le C\sum_{j=1}^{\infty}|\alpha-\beta|^2\lambda_j^4(1+\lambda_j)^{-4}|u_{0,j}|^2 \le C|\alpha-\beta|^2\|u\|_{L^2(M)}^2.\end{aligned}$$

Now $v_0 = i\Delta u_0 \in N_{T-\delta}$ for any $\delta > 0$, thus also in N_T. As a consequence, N_T is an invariant subspace of Δ in $L^2(M)$. Also, by Proposition 2.2, the $H^{-4}(M)$-norm is equivalent to the $L^2(M)$-norm on N_T, so the unit ball in N_T is compact and thus N_T is of finite dimension. If it is not $\{0\}$, then it must contain some Laplacian eigenfunction φ. But this would mean that $\varphi \equiv 0$ on Ω, which violates the unique continuation for Laplacian eigenfunctions. Therefore $N_T = \{0\}$. □

Now we can proceed by contradiction to finish the proof of Theorem 1.2. Suppose (1.2) is not true, then we can find a sequence $\{u_{n,0}\}$ in $L^2(M)$ such that

$$\|u_{n,0}\|_{L^2(M)} = 1, \quad \text{and} \quad \int_0^T \|e^{it\Delta}u_{n,0}\|_{L^2(\Omega)}^2 dt \le n^{-1}. \tag{2.14}$$

Then we can extract a subsequence $u_{n_k,0}$ converging to u_0 weakly in $L^2(M)$, thus strongly in $H^{-4}(M)$. On one hand, by Proposition 2.2 again, we see

$$\begin{aligned}1 = \|u_{n_k,0}\|_{L^2(M)}^2 &\le C\int_0^T \|e^{it\Delta}u_{n_k,0}\|_{L^2(\Omega)}^2 dt + C\|u_{n_k,0}\|_{H^{-4}(M)}^2 \\ &\le Cn_k^{-1} + C\|u_{n_k,0}\|_{H^{-4}(M)}^2\end{aligned}$$

and thus let $k \to \infty$, we get $\|u_0\|_{H^{-4}(M)} \ge C^{-1/2} > 0$. On the other hand, u_0 must lie in N_T and thus $u_0 \equiv 0$. This contradiction finishes the proof of Theorem 1.2.

2.4. From observability to control: Hilbert Uniqueness Method (HUM)

Now we recall how the Hilbert Uniqueness Method of Lions [15] shows that Theorem 1.2 implies Theorem 1.3.

Consider the following operators $R : L^2([0,T]\times\Omega) \to L^2(M)$ and $S : L^2(M) \to L^2([0,T]\times\Omega))$ defined by $Rg = u|_{t=0}$ where u is the solution to

$$(i\partial_t + \Delta)u = g1_{[0,T]\times\Omega}, \quad u|_{t=T} \equiv 0 \tag{2.15}$$

and $Su_0 = e^{it\Delta}u_0|_{[0,T]\times\Omega}$.

Proposition 2.5. *R and S are continuous and $R^* = -iS$, i.e. for any $g \in L^2((0,T)\times\Omega)$ and $u_0 \in L^2(M)$,*

$$\langle Rg, u_0\rangle_{L^2(M)} = i\langle g, Su_0\rangle_{L^2((0,T)\times\Omega)}. \tag{2.16}$$

In particular, the following statements are equivalent:
(a) (Control) R is surjective;
(b) (Observability) There exists $c > 0$ such that for all $u_0 \in L^2(M)$,

$$\|Su_0\|_{L^2((0,T)\times\Omega)} \geq c\|u_0\|_{L^2(M)}. \tag{2.17}$$

Proof. Let u be the solution to (2.15) and $v = e^{it\Delta}u_0$, then integration by parts gives

$$\begin{aligned}\langle g, Su_0\rangle_{L^2((0,T)\times\Omega)} &= \int_{[0,T]\times M} (i\partial_t + \Delta)u \cdot \bar{v}dtdx \\ &= i\int_M u\bar{v}|_{t=0}^{t=T}\, dx + \int_{[0,T]\times M} u\cdot(-i\partial_t + \Delta)\bar{v}dtdx\end{aligned}$$

By definition of R, we see that

$$i\int_M u\bar{v}|_{t=0}^{t=T}\, dx = -i\langle Rg, u_0\rangle_{L^2(M)}$$

while $(-i\partial_t + \Delta)\bar{v} = 0$. This finishes the proof of (2.16). The equivalence of (a) and (b) follows by standard functional analysis argument. □

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Yau Mathematical Science Center
Tsinghua University, Beijing, China
E-mail address: jinlong@mail.tsinghua.edu.cn

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