Math. Res. Lett.
Volume 25, Number 5, 1673–1693, 2018

Minimizers of the sharp Log entropy on manifolds with non-negative Ricci curvature and flatness

Qi S. Zhang

Consider the scaling invariant, sharp log entropy (functional) introduced by Weissler [W] on noncompact manifolds with nonnegative Ricci curvature. It can also be regarded as a sharpened version of Perelman's W entropy [P] in the stationary case. We prove that it has a minimizer if and only if the manifold is isometric to $\mathbf{R}^n$.

Using this result, it is proven that a class of noncompact manifolds with nonnegative Ricci curvature is isometric to $\mathbf{R}^n$. Comparing with the well known flatness results in [An], [Ba] and [BKN] on asymptotically flat manifolds and asymptotically locally Euclidean (ALE) manifolds, their decay or integral condition on the curvature tensor is replaced by the condition that the metric converges to the Euclidean one in C^1 sense at infinity. No second order condition on the metric is needed.

1. Statement of result

Finding extremals of useful functionals, entropies and inequalities is often an useful task in mathematics. Examples include the Sobolev inequality, Perelman's F and W entropies, log Sobolev inequalities, Yamabe functional etc. In this note we consider the scaling invariant log entropy (functional) introduced by Weissler [We] on noncompact manifolds with nonnegtaive Ricci curvature. It is a scaling invariant version of the log Sobolev functional originally introduced by Gross [Gs] and Federbush [F]. It can also be regarded as a sharpened version of Perelman's W entropy for the Ricci flow in the stationary case.

Definition 1.1. (a). (After Weissler) Let $\mathbf{M}$ be a Riemannian n manifold. The scaling invariant log functional (entropy) is

$$
(1.1) \qquad \begin{aligned} L = L(v,g) &= -\int_{\mathbf{M}} v^2 \ln v^2 dg + \frac{n}{2} \ln \left(\int_{\mathbf{M}} 4|\nabla v|^2 dg \right) + s_n \\ &\equiv -N(v) + \frac{n}{2} \ln(F(v)) + s_n. \end{aligned}
$$

Here $s_n = -\frac{n}{2}\ln(2\pi n) - \frac{n}{2}$ and $v \in W^{1,2}(\mathbf{M})$ with $\|v\|_2 = 1$;

(b). The infimum of the log Sobolev functional is denoted by

$$
\lambda = \lambda(g, \mathbf{M}) = \inf\{L(v,g) \,|\, v \in W_0^{1,2}(\mathbf{M}, g), \quad \|v\|_{L^2(\mathbf{M})} = 1\}.
$$

(c). The infimum of the log Sobolev functional at infinity, in case $\mathbf{M}$ is noncompact, is

$$
\lambda_\infty = \lambda_\infty(g, \mathbf{M}) = \lim_{r\to\infty} \lambda(g, \mathbf{M} - B(x_0, r))
$$

where x_0 is a reference point in M.

When $\mathbf{M} = \mathbf{R^n}$, then $L(v,g)$ is introduced by Weissler [W]. Some existence results for minimizers of a functional similar to $L(v,g)$ (modified with a scalar curvature term) were proven in ([Z14]). Since the functional $L(v,g)$ is scaling invariant, the proof involves an approximation and blow up analysis which is not needed for the usual W functional. Some applications were given on breathers and irreversibility of world sheets. Existence of minimizers for the W functional on compact manifolds was proven by [R]. The situation on noncompact manifolds is different. See [Z12] for existence and nonexistence results of minimizers of the W functional on noncompact manifolds.

Remark 1.2. For general noncompact manifolds, the functional L may not be bounded from below. In fact, if L is bounded from below, then the sharp log Sobolev inequality holds:

$$
\int_{\mathbf{M}} v^2 \ln v^2 dg \le \frac{n}{2} \ln \left(\int_{\mathbf{M}} 4|\nabla v|^2 dg \right) + s_n - \inf L, \quad v \in W^{1,2}(\mathbf{M}).
$$

By now, it is well known (see [BCL] e.g.), that this implies the standard Sobolev inequality, when $n \ge 3$,

$$
\left(\int_{\mathbf{M}} v^{2n/(n-2)} dg \right)^{(n-2)/n} \le C_S \int_{\mathbf{M}} |\nabla v|^2 dg, \quad v \in W^{1,2}(\mathbf{M}).
$$

Here C_S is a positive constant. It follows that the manifold is volume non-collapsed i.e.

$$|B(x,r)| \geq cr^n, \quad x \in \mathbf{M},\, r > 0, \tag{1.2}$$

for a uniform constant $c > 0$.

Remark 1.3. In the above definition, one can write $u = v^2$

$$N = N(v) = \int_{\mathbf{M}} v^2 \ln v^2 dg = \int_{\mathbf{M}} u \ln u dg \tag{1.3}$$

is just Boltzmann's entropy; and

$$F = F(v) = \int_{\mathbf{M}} 4|\nabla v|^2 dg = \int_{\mathbf{M}} \frac{|\nabla u|^2}{u} dg \tag{1.4}$$

is just Perelman's F entropy minus a term involving the scalar curvature.

The first main result of the note is:

Theorem 1.4. *Let* $\mathbf{M}$ *be a noncompact, complete manifold of dimension* $n \geq 3$ *such that Ricci* ≥ 0. *The functional* L *has a minimizer if and only if* $\mathbf{M}$ *is isometric to* $\mathbf{R}^n$.

Normally one would believe that a minimizer exists for many "nice" manifolds with nonnegative Ricci curvature. The theorem unexpectedly shows that the only "nice" one is $\mathbf{R}^n$. The proof is given in the next section. An application on flatness of some noncompact manifolds is given in Section 3.

2. Proof

Proof of Theorem 1.4. It is well known that the Gaussian functions are minimizers for L in $\mathbf{R}^n$. So one only needs to prove that if L on $\mathbf{M}$ has a minimizer then $\mathbf{M}$ is isometric to $\mathbf{R}^n$.

Let v be a minimizer of L on $\mathbf{M}$ such that $Ric \geq 0$. Then we solve the backward heat equation with final value at time $t = 1$.

$$\begin{cases} \Delta u + \partial_t u = 0, & \text{on } \mathbf{M} \times [0,1]; \\ u(x,1) = v^2(x). \end{cases} \tag{2.1}$$

The time interval $[0, 1]$ is chosen for convenience. Any finite interval also works.

The minimizer v satisfies the equation (cf. Theorem 1.9 [Z14]):

$$\begin{aligned}(2.2)\qquad &\frac{n}{2}\frac{4\Delta v}{\int(4|\nabla v|^2)dg}+2v\ln v\\ &+\left(\lambda(g,\mathbf{M})+\frac{n}{2}-\frac{n}{2}\ln\int(4|\nabla v|^2)dg-s_n\right)v=0.\end{aligned}$$

Comparing with that paper, the scalar curvature term is dropped here.

By modifying the proof to Lemma 2.3 in [Z12], we can prove that the function v decays (quadratic) exponentially near infinity, i.e. for one reference point, say x_0.

$$(2.3)\qquad v(x)\le C_1e^{-C_2d^2(x,x_0)},\qquad x\in\mathbf{M},$$

where C_1 and C_2 are positive constants with C_1 depending on $|B(x_0,1)|$. Here we should mention that in the paper [Z12] and [Z14], we assumed that the curvature tensor and/or its gradients are bounded, which are needed in constructing a smooth, distance like function. Since these bounds are no longer available here, we will use the heat kernel to construct this function. Alternatively one can also use the distance like function in Theorem 4.2 of [SY]. To avoid interrupting the flow of the proof, we take (2.3) for granted here and postpone its proof until the end of the section.

Alternatively, one can use the exponential decay property of the Green's function of the operator $\Delta-1$, which follows from the bound of the heat kernel of $\Delta-1-\partial_t$ to prove a weaker decay:

$$v(x)\le C_1e^{-C_2d(x,x_0)},\qquad x\in\mathbf{M}.$$

This is still sufficient for the main result.

From (2.3), by the standard upper bound on heat kernel [LY], we know that $u=u(\cdot,t)$ also decays exponentially for each fixed t. Hence we can use integration by parts to deduce

$$\begin{aligned}(2.4)\qquad \partial_tN=\partial_t\int_{\mathbf{M}}u\ln u dg&=\int_{\mathbf{M}}\partial_tu\ln u dg+\int_{\mathbf{M}}\partial_tu dg\\ &=-\int_{\mathbf{M}}(\Delta u)\ln u dg=\int_{\mathbf{M}}\frac{|\nabla u|^2}{u}dg=F.\end{aligned}$$

Below we give the details of the integration by parts. All we need are Li-Yau type bound for heat kernel and Hamilton type gradient bound [Ha]. Both hold when $Ric\ge0$.

The backward heat kernel $p(x,t;y,1)$ with $t<1$ is the same as the (usual forward) heat kernel $G=G(y,1;x,t)$. Since the manifold has *Ricci* ≥ 0, Li-Yau's [LY] heat kernel bound implies

$$\begin{aligned}(2.5)\qquad \frac{c_3^{-1}}{|B(y,\sqrt{1-t})|}e^{-c_4^{-1}d^2(x,y)/(1-t)} &\leq p(x,t;y,1)\\ &\leq \frac{c_3}{|B(y,\sqrt{1-t})|}e^{-c_4 d^2(x,y)/(1-t)}.\end{aligned}$$

Therefore

$$\begin{aligned}u(x,t) &= \int p(x,t;y,1)v^2(y)dy\\ &\leq \int \frac{c_3 C_1^2}{|B(y,\sqrt{1-t})|}e^{-c_4 d^2(x,y)/(1-t)}e^{-2C_2 d^2(y,x_0)}dy\\ &\leq \int \frac{c_3 C_1^2}{|B(y,\sqrt{1-t})|}e^{-c_4 d^2(x,y)/[2(1-t)]}e^{-c_4 d^2(x,y)/[2(1-t)]}e^{-2C_2 d^2(y,x_0)}dy\\ &\leq c_5 e^{-c_6 d^2(x,x_0)}, \qquad \forall t\in[0,1).\end{aligned}$$

Likewise

$$(2.6)\qquad |\nabla u(x,t)| \leq \frac{c_5}{\sqrt{1-t}}e^{-c_6 d^2(x,x_0)}, \qquad \forall t\in[0,1).$$

Next we can use the local gradient bound in [SZ] backwardly for the backward heat equation, which is the same as for the heat equation, to deduce

$$\frac{|\nabla u|^2}{u}(x,t) \leq u(x,t)\frac{C}{1-t}\ln^2\frac{A}{u(x,t)}$$

where $A=\sup_{B(x,2\sqrt{1-t})\times[t,(1+t)/2]}u$. Using (2.5) and volume doubling condition, direct computation shows that

$$(2.7)\qquad \frac{|\nabla u|^2}{u}(x,t) \leq \frac{c_7}{(1-t)^3}e^{-c_6 d^2(x,x_0)}, \quad t\in[0,1).$$

Here c_7 may depend on $|B(x_0,1)|$. Also the dependence on $(1-t)^{-3}$ can be improved but there is no need to do it here. Now for any large $r>0$, for each fixed $t\in[0,1)$, integration by parts shows

$$\int_{B(x_0,r)}(\Delta \ln u)u dg = -\int_{B(x_0,r)}\frac{|\nabla u|^2}{u}dg + \int_{\partial B(x_0,r)}\frac{\partial u}{\partial n}dS.$$

Since $|\partial B(x_0, r)| \le C_n r^{n-1}$ by Bishop-Gromov volume comparison, letting $r \to \infty$, we can use (2.6) and (2.7) to conclude that

$$\int_{\mathbf{M}} (\Delta \ln u) u dg = - \int_{\mathbf{M}} \frac{|\nabla u|^2}{u} dg,$$

justifying the integration by parts.

Similar to Hamilton's calculation [Ha] for the forward heat equation case, we also have

$$(2.8) \qquad (\Delta + \partial_t) \left(\frac{|\nabla u|^2}{u} \right) = 2|Hess \ \ln u|^2 u + 2Ric(\nabla u, \nabla u)/u.$$

Hence, after integration by parts on large balls of radius say $R > 0$ and let $R \to \infty$, we can deduce,

$$(2.9) \qquad \partial_t F = 2 \int_{\mathbf{M}} |Hess \ \ln u|^2 u dg + 2 \int_{\mathbf{M}} Ric(\nabla u, \nabla u)/u dg.$$

Again the integration is justified due the exponential decay of u. Since the integration by parts apparently involves the 3rd derivative of u, we carry it out in detail.

Let R be a large positive number and $\eta = \eta(t)$ be a smooth compactly supported function in $(0, 1)$. Since the Ricci curvature is nonnegative, according to Cheeger and Colding (Theorem 6.33 [CC]), there exists a good cut off function $\phi_R = \phi_R(x)$ such that $\phi_R \in C^2_0(B(x_0, 2R))$, $0 \le \phi_R \le 1$, $\phi_R = 1$ on $B(x_0, R)$ and $|\nabla \phi_R|^2 + |\Delta \phi_R| \le C/R^2$. We comment that although the bound C/R^2 was not specified in [CC], for manifolds with nonnegative Ricci curvature, one can just scale a large ball to a unit ball and construct the good cut off function on the unit ball. After scaling back to the original size, one can obtain the bound. Then (2.8) and integration by parts imply that

$$(2.10) \qquad - \int_0^1 \int_{\mathbf{M}} \frac{|\nabla u|^2}{u} \phi_R dg \eta'(t) dt = \int_0^1 \partial_t \int_{\mathbf{M}} \frac{|\nabla u|^2}{u} \phi_R dg \, \eta(t) dt$$
$$= \int_0^1 \int_{\mathbf{M}} \left(-\Delta \frac{|\nabla u|^2}{u} \right) \phi_R dg \eta dt$$
$$+ 2 \int_0^1 \int_{\mathbf{M}} \left(|Hess \ \ln u|^2 u dg + \frac{Ric(\nabla u, \nabla u)}{u} \right) \phi_R \eta dg dt$$

$$
\begin{aligned}
&= \int_0^1 \int_{\mathbf{M}} \left(-\frac{|\nabla u|^2}{u} \right) \Delta \phi_R dg \eta dt \\
&\quad + 2 \int_0^1 \int_{\mathbf{M}} \left(|Hess\ \ln u|^2 u dg + \frac{Ric(\nabla u, \nabla u)}{u} \right) \phi_R \eta dg dt \\
&\equiv T_1(R) + T_2(R).
\end{aligned}
$$

Notice that the above integrations are justified since the domain of integration is compact and $u > 0$. Since

$$
|T_1(R)| \leq \frac{C}{R^2} \int_{supp\,\eta} \int_{\mathbf{M}} \frac{|\nabla u|^2}{u} dg \eta dt,
$$

from (2.7) and $supp\,\eta \subset\subset (0,1)$, we see that

$$
\lim_{R \to \infty} T_1(R) = 0.
$$

Also, from the properties of the good cut off function, we know that $\phi_{2R}(x) \geq \phi_R(x)$ for $x \in \mathbf{M}$. For positive integers i, we take $R = 2^i$ and let $i \to \infty$. Then $\phi_{2^i}(x) \to 1$ when $i \to \infty$. Using the monotone convergence theorem on the $T_2(R)$ term and dominated convergence theorem on the left hand side of (2.10), we deduce that

$$
\begin{aligned}
- \int_0^1 F(t) \eta'(t) dt &= \int_0^1 \int_{\mathbf{M}} \frac{|\nabla u|^2}{u} dg \eta'(t) dt \\
&= 2 \int_0^1 \int_{\mathbf{M}} \left(|Hess\ \ln u|^2 u dg + \frac{Ric(\nabla u, \nabla u)}{u} \right) \eta dg dt.
\end{aligned}
$$

This proves (2.9) since η is a arbitrary.

Therefore, from (2.4), (2.9) and the definition of L in (1.1), it follows that

$$
\begin{aligned}
\partial_t L &= \left[-F^2 + n \int_{\mathbf{M}} |Hess\ \ln u|^2 u dg + n \int_{\mathbf{M}} Ric(\nabla u, \nabla u)/u dg \right] F^{-1} \\
&= F^{-1} \left[-F^2 + \int_{\mathbf{M}} (\Delta \ln u)^2 u dg \right] \\
&\quad + nF^{-1} \int_{\mathbf{M}} |Hess\ \ln u - \frac{1}{n} (\Delta \ln u) g|^2 u dg \\
&\quad + nF^{-1} \int_{\mathbf{M}} Ric(\nabla u, \nabla u)/u dg.
\end{aligned} \tag{2.11}
$$

Using integration by parts and Cauchy-Schwarz inequality,

$$F^2 = \left(\int_{\mathbf{M}} \frac{|\nabla u|^2}{u} dg \right)^2 = \left(\int_{\mathbf{M}} (\Delta \ln u) u dg \right)^2 \tag{2.12}$$
$$\le \int_{\mathbf{M}} (\Delta \ln u)^2 u dg \int u dg = \int_{\mathbf{M}} (\Delta \ln u)^2 u dg.$$

The equality is reached only if

$$(\Delta \ln u)\sqrt{u} = C\sqrt{u}$$

i.e. $\Delta \ln u = C$, since $u > 0$. As explained above, integration by parts can be justified by modifying the proof of Corollary 4.1 in [Z12].

Hence all three terms on the right hand side of (2.11) are non-negative. From definition, we know $L(v, g) = L(\sqrt{u(\cdot, 1)}, g) = \lambda$ and $L(\sqrt{u(\cdot, 0)}, g) \ge \lambda$. Hence

$$0 \ge L(\sqrt{u(\cdot, 1)}, g) - L(\sqrt{u(\cdot, 0)}, g) = \int_0^1 \partial_t L(\sqrt{u}, g) dt \ge 0. \tag{2.13}$$

This shows

$$\int_0^1 \partial_t L(\sqrt{u}, g) dt = 0 \tag{2.14}$$

As pointed out by the referee, one can also use a solution of the forward heat equation $u = u(x, t)$, with initial value $u(\cdot, 0) = v^2(\cdot)$, to carry out the above proof.

Substituting (2.11) to the right hand side of (2.14), we find that

$$\Delta \ln u(x, t) = C(t),$$

where $C(t)$ is a function of t only;

$$Hess \ln u - \frac{1}{n}(\Delta \ln u) g = 0;$$

$$Ric(\nabla u, \nabla u) = 0. \tag{2.15}$$

Since $u(\cdot, t)$ decays exponentially, we see that $C(t)$ is a non zero constant. The reason is that if

$$\Delta \ln u = 0$$

then $\ln u$ is harmonic. But $u = v^2$ and v, as a solution to equation (2.2), is bounded from above and nonnegative (cf [Z12]). Hence $\ln u < C$. Yau's Liouville theorem implies $\ln u$ is a constant, which is impossible.

From these equalities, it is known that $\mathbf{M}$ is isometric to $\mathbf{R}^n$. See Tashiro [T] Theorem 2 (I, B). Also Naber [N] shows, by a different method, that a Ricci flat shrinking gradient soliton is $\mathbf{R}^n$. However, one can not assume that $\mathbf{M}$ is Ricci flat yet, since (2.15) holds only in the ∇u direction. So $\mathbf{M}$ is not yet a gradient Ricci soliton to begin with. Here we give a very short proof for completeness. We work with $t = 0$. Then, by considering $f = \ln u$ or $f = -\ln u$ we can assume

$$Hess\, f = \lambda g, \tag{2.16}$$

where λ is a positive constant. Fix any point $p \in \mathbf{M}$ and pick a point $x \in \mathbf{M}$. Let $r = r(s)$ be a minimal geodesic connecting p and x, parameterized by arc length. Then the definition of Hessian and (2.16) tell us

$$\lambda = (Hess\, f)(\partial_r, \partial_r) = \nabla_{\partial_r}(\nabla_{\partial_r} f) - (\nabla_{\partial_r}\partial_r) f = \frac{d^2}{ds^2} f(r(s)).$$

Therefore, for $r = d(x, p)$,

$$f(x) = \frac{\lambda}{2} r^2 + r\, \frac{d}{ds} f(r(s))\,|_{s=0}\, + f(p).$$

This shows that f must have a global minimum. Choose p to be a minimal point, then

$$f(x) = \frac{\lambda}{2} r^2 + f(p).$$

Note that the smoothness of f implies that r^2 is smooth. Substituting this to (2.16) and taking trace, we see that

$$\frac{\lambda}{2} \Delta r^2 = \lambda n.$$

Hence, the following holds in the classical sense for all $r > 0$:

$$\Delta r = \frac{n-1}{r}.$$

Let w be the volume element in a spherical coordinate centered at p. Then

$$\partial_r \ln w = \Delta r = \frac{n-1}{r}.$$

Therefore

$$\partial_r(w/r^{n-1}) = 0.$$

This shows, since $\mathbf{M}$ is a smooth manifold, $w = w_n r^{n-1}$ where w_n is the volume of standard unit sphere in $\mathbf{R}^n$. Hence the metric is Euclidean by the equality case of the Bishop-Gromov volume comparison property. This finishes the proof of the theorem modulo (2.3). □

Next we give a proof of (2.3) i.e. a minimizer v satisfies

$$v(x) \le C_1 e^{-C_2 d^2(x,x_0)}, \qquad x \in \mathbf{M},$$

where C_1, C_2 are positive constants depending on x_0.

The proof is divided into a number of steps, which similar to the proof Lemma 2.3 in [Z12]. The main difference is in step 2 where we construct a distance like function by the heat kernel.

Step 1. First, we rewrite (2.2), the equation for v, in a simpler form

$$4\Delta v + \alpha v \ln v + \beta v = 0.$$

Here α is a positive constant and β is another constant. For consistency with Lemma 2.3 in [Z12] and without loss of generality we set $\alpha = 2$ and $\beta = \lambda$. So v solves

$$4\Delta v + 2v \ln v + \lambda v = 0. \tag{2.17}$$

The existence of a minimizer presumes that the functional L is bounded from below, which implies, from Remark 1.2, that the standard Sobolev inequality and volume noncollapsing property hold on $\mathbf{M}$. Hence we can follow Lemma 2.1 in [Z12] verbatim to prove the following bounds. For $m \in \mathbf{M}$, there exists a positive constant C such that

$$\sup_{B(m,1)} v^2 \le C \int_{B(m,2)} v^2 dg, \qquad \sup_{B(m,1/2)} |\nabla v|^2 \le C \int_{B(m,1)} v^2 dg.$$

Note one only needs the Sobolev imbedding and volume noncollapsing property here.

As in Lemma 2.3 of [Z12], since $\ln \int_{B(m,1)} v^2 dg \to -\infty$ as $d(m, x_0) \to \infty$, there exists a large $r_0 > 0$ such that, when $d(x, x_0) \ge r_0$, we have

$$4\Delta v(x) + v(x) \ln v(x) \ge 0, \qquad \text{and} \quad v(x) \le e^{-1}. \tag{2.18}$$

Step 2. We claim that there exist a smooth, distance like function $h = h(x)$ on $\mathbf{M}$, which satisfies,

$$\frac{1}{2}h(x) \le d(x, x_0) \le 2h(x), \quad |\nabla h(x)| \le 1,$$
$$|\Delta h(x)| \le \frac{1}{10}h(x), \quad d(x, x_0) \ge r_0.$$

Comparing with the case of bounded geometry, we can not prove that $|\Delta h(x)|$ is bounded by a constant. However the above bound is sufficient, as indicated in Step 3.

Alternatively, since the Ricci curvature is nonnegative, one can also use the distance like function in Theorem 4.2 of the book by Schoen-Yau [SY]. The gradient and Laplacian of this function is both bounded. We wish to thank the referee for pointing this out. Here we are using the function h since the properties listed above may have independent interest since they rely only on the heat kernel bound together with its gradient and time derivative bound, which hold under weaker assumptions such as certain integral Ricci curvature bound.

Indeed, we can take, for a fixed time t to be chosen later

$$h(x) = \int_{\mathbf{M}} G(x, t, y)d(y, x_0)dg(y), \tag{2.19}$$

where G is the heat kernel of the forward heat equation. Let's verify that the claim is true. First

$$h(x) - d(x, x_0) = \int_{\mathbf{M}} G(x, t, y)[d(y, x_0) - d(x, x_0)]dg(y),$$

which implies, by the triangle inequality, that

$$|h(x) - d(x, x_0)| \le \int_{\mathbf{M}} G(x, t, y)d(x, y)dg(y).$$

By the Li-Yau [LY] bound on heat kernel and the volume noncollapsing property (1.2), the above shows, for some positive constants C, C' that

$$\begin{aligned}|h(x) - d(x, x_0)| &\le \int_{\mathbf{M}} \frac{C}{t^{n/2}} e^{-\frac{d(x,y)^2}{Ct}} \frac{d(x, y)}{\sqrt{t}} \sqrt{t} dg(y) \\ &\le C'\sqrt{t} \int_{\mathbf{M}} \frac{1}{t^{n/2}} e^{-\frac{d(x,y)^2}{2Ct}} dg(y).\end{aligned}$$

Hence

$$|h(x) - d(x, x_0)| \le C_1 \sqrt{t}, \tag{2.20}$$

for some constant $C_1 > 0$.

Next, the Bochner's formula implies

$$(\Delta - \partial_t)|\nabla h|^2 = 2|Hess\, h|^2 + 2Ric(\nabla h, \nabla h) \ge 0.$$

This and the initial condition that $|\nabla h|_{t=0} = |\nabla d(\cdot, 0)| = 1$ infer, via the maximum principle, that

$$|\nabla h| \le 1. \tag{2.21}$$

Thirdly, from (2.19),

$$\Delta h(x) = \int_{\mathbf{M}} \Delta_x G(x, t, y) d(y, x_0) dg(y) = \int_{\mathbf{M}} \partial_t G(x, t, y) d(y, x_0) dg(y).$$

The Li-Yau [LY] bound on the time derivative of the heat kernel and the volume noncollapsing property show that

$$\begin{aligned} |\Delta h(x)| &\le \int_{\mathbf{M}} \frac{C}{t^{(n+2)/2}} e^{-\frac{d(x,y)^2}{Ct}} d(y, x_0) dg(y) \\ &\le \int_{\mathbf{M}} \frac{C}{t^{(n+2)/2}} e^{-\frac{d(x,y)^2}{Ct}} \left[d(x, y) + d(x, x_0)\right] dg(y) \\ &= \int_{\mathbf{M}} \frac{C}{t^{(n+2)/2}} e^{-\frac{d(x,y)^2}{Ct}} \left[\frac{d(x, y)}{\sqrt{t}} \sqrt{t} + d(x, x_0)\right] dg(y) \end{aligned}$$

This infers, for a constant $C > 0$, that

$$|\Delta h(x)| \le C \left(\frac{1}{\sqrt{t}} + \frac{1}{t} d(x, x_0) \right). \tag{2.22}$$

From (2.20) and (2.22), it is clear that we can fix a large $t \ge 100C > 0$ and one large $r_0 >> \sqrt{t}$ so that

$$\frac{1}{2} d(x, x_0) \le h(x) \le 2d(x, x_0), \quad |\Delta h(x)| \le \frac{1}{10} h(x), \qquad \forall\, d(x, x_0) \ge r_0.$$

This and (2.21) prove the claim.

Since $d(x, x_0)$ and $h(x)$ are comparable when they are large, by (2.18), we can choose r_0 sufficiently large so that

$$4\Delta v(x) + v(x)\ln v(x) \geq 0, \qquad \text{and} \quad v(x) \leq e^{-1} \tag{2.23}$$

when $h(x) \geq r_0$.

Step 3.

Next we compare v with a model function

$$J = J(x) = e^{-ah^2(x)+ar_0^2-1}. \tag{2.24}$$

Here the constant $a > 0$ is to be decided later; By direct computation

$$\begin{aligned} \Delta J &= J[4a^2|\nabla h|^2 h^2 - 2ah\Delta h - 2a|\nabla h|^2], \\ J\ln J &= J(-ah^2 + ar_0^2 - 1). \end{aligned}$$

Hence

$$\begin{aligned} 4\Delta J + J\ln J &= J[16a^2|\nabla h|^2 h^2 - 8ah\Delta h - 8a|\nabla h|^2 - ah^2 + ar_0^2 - 1] \\ &\leq J\left[16a^2h^2 + 8a\frac{h^2}{10} - ah^2 + ar_0^2 - 1\right] \\ &= J\left[16a^2h^2 - \frac{a}{5}h^2 + ar_0^2 - 1\right]. \end{aligned}$$

Here we have used the claim in step 2, in particular $|\Delta h| \leq h/10$. This implies, for fixed a sufficiently small that

$$4\Delta J + J\ln J \leq 0 \tag{2.25}$$

when $h(x) \geq r_0$ and $J(x) = e^{-1}$ when $h(x) = r_0$.

The rest of the proof is the same as Lemma 2.3 in [Z12], which is given for completeness. From (2.25) and (2.23), we have

$$\begin{cases} 4\Delta(J - v) + J\ln J - v\ln v \leq 0, & \text{if } \ h(x) \geq r_0, \\ J(x) \leq e^{-1}, \qquad v(x) \leq e^{-1}, & \text{if } \ h(x) \geq r_0 \\ (J - v)(x) \geq 0, & \text{if } \ h(x) = r_0, \\ (J - v)(x) \to 0, & \text{if } \ h(x) \to \infty, \end{cases}$$

Since $J(x), v(x) \leq e^{-1}$, by the mean value theorem, there exists a function $f = f(J(x), v(x))$, $0 < f \leq e^{-1}$ such that

$$J(x)\ln J(x) - v(x)\ln v(x) = (\ln f + 1)(J(x) - v(x)).$$

Observe that

$$\ln f + 1 \le lne^{-1} + 1 \le 0, \qquad \text{when} \quad h(x) \ge r_0.$$

Therefore we can apply the standard maximum principle for the elliptic inequality on

$$4\Delta(J - v)(x) + (\ln f + 1)(J - v)(x) \le 0, \qquad \text{when} \quad h(x) \ge r_0$$

to conclude that

$$v(x) \le J(x) = e^{-ah^2(x) + ar_0^2 - 1}, \qquad \text{when} \quad h(x) \ge r_0.$$

Since $h(x)$ and $d(x, x_0)$ are comparable when they are large, we have proven (2.3). This concludes the proof of Theorem 1.4.

3. flatness of some manifolds with $Ric \ge 0$.

Next we apply the theorem to the study of flatness of manifolds with $Ric \ge 0$. Let us recall the definition of asymptotically flat manifolds (cf p64 [LP]).

Definition 3.1. A complete, noncompact Riemannian manifold M is called Asymptotically Flat of order τ if there is a partition $M = M_0 \cup M_\infty$, which satisfies the following properties.

(i). M_0 is compact.

(ii). M_∞ is the disjoint union of finitely many components each of which is diffeomorphic to $(\mathbf{R}^n - B(0, r_0))$ for some $r_0 > 0$.

(iii). Under the coordinates induced by the diffeomorphism, the metric g_{ij} satisfies, for $x \in M_\infty$,

$$g_{ij}(x) = \delta_{ij}(x) + O(|x|^{-\tau}), \quad \partial_k g_{ij}(x) = O(|x|^{-\tau-1}),$$
$$\partial_k \partial_l g_{ij}(x) = O(|x|^{-\tau-2}).$$

Remark 3.2. For convenience we will equip the compact component M_0 with a reference point 0. We will also assume that M_∞ has only one connected component. This assumption does not reduce any generality

These class of manifolds are quite useful in general relativity and differential geometry. Ricci flat AF manifolds are often the blow up limits in many situations. If one can show these manifolds are $\mathbf{R}^n$, then one usually can prove some useful results by the method of contradiction. In [An], [Ba] and [BKN], these authors showed that Ricci flat (or $Ricc \geq 0$) AF manifolds are isometric to $\mathbf{R}^n$ if the curvature tensor is in $L^{n/2}$ or it decays faster than inverse square of the distance function. Note there are various definitions of AF manifolds. In the definition used in this paper, AF manifolds are special cases of asymptotically locally Euclidean (ALE) manifolds (cf. [BKN]). AF manifolds are ALE manifolds which are simply connected at infinity. Related questions and results on flatness of manifolds with nonnegativity of certain curvatures and faster than quadratic curvature decay can be found in [BGS] (p58-59), [GW], [K] and [KS]. Flatness is also related to properties of a tangent cone at infinity, as indicated in Theorem 0.3 in [Co]. This result involves good estimate of volume of large geodesic balls and hence implicitly Jacobi fields and curvature conditions. See also [MSY], [Ni] and [NT] for results on Kähler manifolds.

Here we replace the AF condition by a much weaker asymptotic condition and remove the curvature condition.

Definition 3.3. A complete, noncompact Riemannian manifold $\mathbf{M}$ is called C^1 Asymptotically Euclidean (C1AE) if there is a partition $\mathbf{M} = M_0 \cup M_\infty$, which satisfies the following properties.

(i). $\mathbf{M}_0$ is compact.

(ii). $\mathbf{M}_\infty$ is the disjoint union of finitely many components each of which is diffeomorphic to $(\mathbf{R}^n - B(0, r_0))$ for some $r_0 > 0$.

(iii). Under the coordinates induced by the diffeomorphism, the metric g_{ij} satisfies, for $x \in M_\infty$,

$$g_{ij}(x) = \delta_{ij}(x) + o(1), \quad \partial_k g_{ij}(x) = o(1).$$

Here $o(1)$ means a quantity that goes to 0 as $|x| \to \infty$. Observe that there is no decay assumption on the second order derivatives of the metric and hence no decay assumption on curvature. The result of this section is:

Theorem 3.4. *A C^1 asymptotically Euclidean manifold with Ric ≥ 0 is isometric to $\mathbf{R}^n$.*

Proof.

If $\lambda = \lambda(g, \mathbf{M}) = 0$, then it is known from Corollary 1.6 [BCL], that $\mathbf{M}$ is isometric to $\mathbf{R}^n$.

So we can assume $\lambda < 0$.

We will show that the functional L has a minimizer. Then Theorem 1.4 will imply that $\mathbf{M} = \mathbf{R}^n$ and hence $\lambda = 0$, reaching a contradiction. So $\mathbf{M} = \mathbf{R}^n$ to begin with.

According to Theorem 1.9 in [Z14], if we can show that

$$-\infty < \lambda < \lambda_\infty = 0 \tag{3.1}$$

then a minimizer exists. So we are left to prove (3.1). We mention that the log functional in [Z14] has an extra scalar curvature term comparing with the current one. However, since the scalar curvature is non-negative, the same conclusion holds and the proof is the same.

First we prove the following

Claim. *Let* $(\mathbf{M}, g)$ *be an C1AE manifold of dimension* $n \ge 3$.

(a). Then there exists a constant $A > 0$*, such that*

$$\left(\int_{\mathbf{M}} v^{2n/(n-2)} dg \right)^{(n-2)/n} \le A \int_{\mathbf{M}} 4|\nabla v|^2 dg, \quad \forall v \in W^{1,2}(\mathbf{M}, g); \tag{3.2}$$

moreover $\lambda(g)$ *is bounded from below i.e.*

$$\int_{\mathbf{M}} v^2 \ln v^2 dg \le \frac{n}{2} \ln \left(A \int_{\mathbf{M}} 4|\nabla v|^2 dg \right), \tag{3.3}$$

$\forall v \in W^{1,2}(\mathbf{M}, g), \|v\|_{L^2(\mathbf{M},g)} = 1.$

(b). $\lambda_\infty(g) \ge 0$.

(a). We just need to prove (3.2) since (3.3) follows from Jensen inequality. A C1AE manifold has maximum volume growth, namely,

$$|B(x, r)| \ge C r^n$$

for some positive constant C and all $r > 0$. Then it is well known that (3.2) holds.

Now we prove part (b).

First we prove the following assertion.

When the radius r *is sufficiently large, we have*

$$\lambda(g, \mathbf{M} - B(0, r)) \ge \lambda(g_E, \mathbf{R}^n - J(B(0, r)) + o(1). \tag{3.4}$$

Here J *is the coordinate map near infinity in the definition of C1AE manifold;* $o(1)$ *is a quantity whose absolute value goes to* 0 *when* $r \to \infty$*;* g_E *is the Euclidean metric.*

Pick a function $v \in C_0^\infty(\mathbf{M} - B(0, r))$ with $\|v\|_{L^2} = 1$. Given any $\epsilon > 0$, by definition of C1AE manifolds, for $x \in \mathbf{M} - B(0, r)$ with r sufficiently large, there are the following relations

$$(3.5)\qquad (1-\epsilon)dx \le dg(x) = \sqrt{detg(x)}dx \le (1+\epsilon)dx,$$

$$(3.6)\qquad (1-\epsilon)|\nabla_{\mathbf{R}^n} f| \le |\nabla v| \le (1+\epsilon)|\nabla_{\mathbf{R}^n} f|$$

where $f = v \circ J^{-1}$ and J is the coordinate map. Also $\nabla_{\mathbf{R}^n}$ is the Euclidean gradient. Hence

$$(3.7)\qquad \int_{\mathbf{M}} 4|\nabla v|^2 dg \ge (1-\epsilon)^2 \int_{\mathbf{R}^n} 4|\nabla_{\mathbf{R}^n} f|^2 \sqrt{detg(x)}dx$$

Write $\sqrt{detg(x)} = w^2$, then

$$(3.8)\qquad \int_{\mathbf{R}^n} 4|\nabla_{\mathbf{R}^n} f|^2 \sqrt{detg(x)}dx = \int_{\mathbf{R}^n} 4|w\nabla_{\mathbf{R}^n} f|^2 dx$$
$$= \int_{\mathbf{R}^n} 4|\nabla_{\mathbf{R}^n}(wf)|^2 dx - 8\int_{\mathbf{R}^n} f\nabla w \nabla(wf)dx + 4\int f^2|\nabla w|^2 dx.$$

By definition of $C1AE$ manifolds, we know that $|\nabla w| \le \eta(r)$ where $\eta = \eta(r)$ is a function going to 0 as $r \to \infty$. Hence, we have

$$(3.9)\qquad \int_{\mathbf{R}^n} 4|\nabla_{\mathbf{R}^n} f|^2 \sqrt{detg(x)}dx$$
$$\ge (1-\eta(r))\int_{\mathbf{R}^n} 4|\nabla_{\mathbf{R}^n}(fw)|^2 dx - 16\eta^{-1}(r)\int f^2|\nabla w|^2 dx,$$
$$\ge (1-\eta(r))\int_{\mathbf{R}^n} 4|\nabla_{\mathbf{R}^n}(fw)|^2 dx - 16\eta(r)\int f^2 dx$$

which implies

$$(3.10)\qquad \int_M 4|\nabla v|^2 dg \ge (1-\epsilon)^2(1-\eta(r))\int_{\mathbf{R}^n} 4|\nabla_{\mathbf{R}^n}(fw)|^2 dx - C\eta(r).$$

Also

$$(3.11)\qquad \int_M v^2 \ln v^2 dg = \int_{\mathbf{R}^n} (fw)^2 \ln f^2 dx$$
$$= \int_{\mathbf{R}^n} (fw)^2 \ln(fw)^2 dx - \int_{\mathbf{R}^n} (fw)^2 \ln w^2 dx$$
$$= \int_{\mathbf{R}^n} (fw)^2 \ln(fw)^2 dx + o(1).$$

This and (3.10) imply that

$$L(v, g, \mathbf{M} - B(0,r)) \geq L(fw, g_E, \mathbf{R}^n - J(B(0,r))) + o(1) - C\eta(r) - n\epsilon. \tag{3.12}$$

Since $\|fw\|_{L^2(\mathbf{R}^n)} = 1$, by taking the infimum of this inequality, it is easy to see that

$$\lambda(g, \mathbf{M} - B(0,r)) \geq \lambda(g_E, \mathbf{R}^n - J(B(0,r)) + o(1) - C\eta(r) - n\epsilon. \tag{3.13}$$

Since ϵ is arbitrary, the assertion is proven.

Using $\lambda(g_E, \mathbf{R}^n - J(B(0,r)) \geq \lambda(g_E, \mathbf{R}^n) = 0$, we see that

$$\lambda_\infty(g) = \lim_{r\to\infty} \lambda(g, \mathbf{M} - B(0,r)) \geq 0. \tag{3.14}$$

This proves part (b) of the claim.

But then

$$\lambda < 0 = \lambda_\infty.$$

This is (3.1) and hence the Theorem follows. □

Remark 3.5. From the proof, it is clear that one only needs the condition $\lambda_\infty = 0$ to get the result. In the Ricci flat case, this condition holds if there is a compact set K such that $\mathbf{M} - K$ is conformal to a domain in $\mathbf{R}^n$. In this situation the $W^{1,2}$ Sobolev constant is the same as the Yamabe constant which stays the same under conformal change. So the best constant in the log Sobolev constant is also the same. This shows $\lambda_\infty = \lambda_{\mathbf{R}^n} = 0$. Thus we obtain a coordinate free result which may also be proven by another method:

Proposition 3.6. *A complete, noncompact Ricci flat $n(\geq 3)$ dimensional manifold which is conformal at infinity to a domain in $\mathbf{R}^n$ is isometric to $\mathbf{R}^n$.*

Acknowledgment. Part of the paper was written when the author was a visiting professor at Fudan University in the summer of 2017. He is grateful to Fudan University and the Simons Foundation for their support. He also wishes to thank Prof. Zhu Meng for helpful conversations. Finally, thanks Prof. Xiaodong Wang and also go to the anonymous referee for checking the paper carefully and for making useful suggestions such as adding key details in the proof, and for the comments below (2.14) and above (2.19).

References

[An] M. T. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. AMS **2** (1989), no. 3, 455–490.

[BCL] D. Bakry, D. Concordet, and M. Ledoux, *Optimal heat kernel bounds under logarithmic Sobolev inequalities*, ESAIM Probab. Statist. **1** (1995/97), 391–407.

[BGS] W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics **61**, Birkhäuser Boston, Inc., Boston, MA, 1985. vi+263pp.

[Ba] R. Bartnik, *The mass of asymptotically flat manifolds*, CPAM **24** (1986), 661–693.

[BKN] S. Bando, A. Kasue, and H. Nakajima, *On construction of coordinates at infinity on manifolds with fast curvature decay and maximum volume growth*, Invent. Math. **97** (1989), 313–349.

[Co] T. H. Colding, *Ricci curvature and volume convergence*, Ann. of Math. (2) **145** (1997), no. 1, 477–501.

[CC] J. Cheeger and T. H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237.

[D] G. Drees, *Asymptotically flat manifolds of nonnegative curvature*, Differential Geom. Appl. **4** (1994), no. 1, 77–90.

[F] P. Federbush, *Partially alternate derivation of a result of Nelson*, J. Math. Physics **10** (1969), no. 1, 50–53.

[Gs] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083.

[GW] R. E. Greene and H. Wu, *Gap theorems for noncompact Riemannian manifolds*, Duke Math. J. **49** (1982), no. 3, 731–756.

[Ha] R. Hamilton, *A matrix Harnack extimate for the heat equation*, Comm. Anal. Geom. **1** (1993), 225–243.

[K] A. Kasue, *Gap theorems for minimal submanifolds of Euclidean space*, J. Math. Soc. Japan **38** (1986), no. 3, 473–492.

[KS] A. Kasue and K. Sugahara, *Gap theorems for certain submanifolds of Euclidean spaces and hyperbolic space forms*, Osaka J. Math. **24** (1987), no. 4, 679–704.

[LP] J. Lee and T. Parker, *The Yamabe problem*, Bull. AMS **17** (1987), no. 1, 37–91.

[LY] P. Li and S.-T. Yau, *On the Parabolic Kernel of the Schödinger operator*, Acta math. **156** (1986), 153–201.

[MSY] N. Mok, Y.-T. Siu, and S.-Y. Yau, *The Poincaré-Lelong equation on complete Kähler manifolds*, Compositio Math. **44** (1981), no. 1-3, 183–218.

[N] A. Naber, *Noncompact shrinking* 4*-solitons with nonnegative curvature*, J. Reine Angew. Math. **645** (2010), 125–153.

[Ni] L. Ni, *An optimal gap theorem*, Invent. Math. **189** (2012), no. 3, 737–761.

[NT] L. Ni and L.-F. Tam, *Poincaré-Lelong equation via the Hodge-Laplace heat equation*, Compos. Math. **149** (2013), no. 11, 1856–1870.

[P] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, `arXiv:math.DG/0211159`.

[R] O. S. Rothaus, *Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators*, J. Funct. Anal. **42** (1981), no. 1, 110–120.

[SY] R. Schoen and S.-T. Yau, Lectures on Differential Geometry, International Press, 1994.

[SZ] P. Souplet and Qi S. Zhang, *Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds*, Bulletin of the London Mathematical Society **38** (6), 1045–1053.

[T] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965), 251–275.

[W] F. B. Weissler, *Logarithmic Sobolev inequalities for the heat-diffusion semigroup*, Trans. Amer. Math. Soc. **237** (1978), 255–269.

[Z12] Qi S. Zhang, *Extremal of Log Sobolev inquality and W entropy on noncompact manifolds*, J. Func. Analysis **263** (2012), 2051–2101.

[Z14] Qi S. Zhang, *A no breathers theorem for some noncompact Ricci flows*, Asian J. Math. **18** (2014), no. 4, 727–755.

School of Mathematical Sciences
Fudan University, Shanghai 200433, P. R. China
and Department of Mathematics
University of California, Riverside
CA 92521, USA
E-mail address: qizhang@math.ucr.edu

Received August 7, 2017