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Galois descent for the gonality of curves

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We determine conditions for the invariance of the gonality under base extension, depending on the numeric invariants of the curve. More generally, we study the Galois descent of morphisms of curves to Brauer-Severi varieties, and also of rational normal scrolls.

1. Introduction

Let C be a smooth and projective curve, geometrically connected, defined over a field k (which, by abuse of language, we call just a curve). Recall that the gonality $\gamma(C)$ of C over k is an integer $\gamma \geq 1$ such that there exists a rational non-constant map $f\colon C \to \mathbb{P}^1$ of degree γ defined over k, and there is no rational map defined over k of degree less than γ. Such a rational map f is called a gonal morphism. The gonality is an important invariant of the curve over k, also for its arithmetic properties (see for example [13] and [20]). We define the *conic gonality* $\gamma^{\mathrm{con}}(C)$ as the minimum degree of a rational non-constant map $f\colon C \to D$, where D is a genus 0 curve, all defined over k. Finally, we call the gonality $\gamma(C_{k^{\mathrm{sep}}})$ of C over a separable closure k^{sep} of k *separable gonality*, and denote it by $\gamma^{\mathrm{sep}}(C)$, and the gonality over an algebraic closure $\overline{k}$ the *geometric gonality*, and denote it by $\overline{\gamma}(C)$. Clearly

$$\overline{\gamma}(C) \leq \gamma^{\mathrm{sep}}(C) \leq \gamma^{\mathrm{con}}(C) \leq \gamma(C),$$

since the separably closed fields are pseudo-algebraically closed, so in particular every conic over k^{sep} is isomorphic to $\mathbb{P}^1$.

In this work we study relations between the gonality γ, the conic gonality γ^{con} and the separable gonality γ^{sep} of C, with the aim to obtain sufficient conditions for equalities between them. These emerge as generalizations of the well known results for hyperelliptic curves that a curve of genus $g \geq 2$ and separable gonality $\gamma^{\mathrm{sep}} = 2$ has conic gonality $\gamma^{\mathrm{con}} = 2$, and the result attributed to Mestre [19], that a curve of genus $g \geq 2$ which has even genus and conic gonality $\gamma^{\mathrm{con}} = 2$, has gonality $\gamma = 2$. Observe that $\gamma = \gamma^{\mathrm{sep}}$ implies $\gamma = \gamma^{\mathrm{con}}$, but the converse is not true.

Note also that the geometric gonality $\overline{\gamma}$ can be smaller than the separable gonality γ^{sep}: see Example 17 for an example of a genus 4 curve which has separable gonality 4 and geometric gonality 3. Observe however that for genus ≤ 3 they are equal, as well as for geometric gonality 2.

We say that a gonal map $f\colon C \to \mathbb{P}^1$ is unique if there is a unique subfield F of the function field $k(C)$ isomorphic to $k(\mathbb{P}^1)$ and with $[k(C) : F] = \gamma$, namely the one determined by the function f. The main results in this note are the following.

Theorem 1. *Let C be a curve with genus g and separable gonality γ^{sep}. Suppose that the gonal map $f_{k^{\mathrm{sep}}}$ over k^{sep} is unique. Then*

1) $\gamma^{\mathrm{con}} = \gamma^{\mathrm{sep}}$.
2) *If the curve C has a k-rational divisor of odd degree, then $\gamma = \gamma^{\mathrm{con}} = \gamma^{\mathrm{sep}}$.*
3) *There exists some degree 2 extension L/k such that $\gamma(C_L) = \gamma^{\mathrm{con}} = \gamma^{\mathrm{sep}}$.*
4) *If $\gamma^{\mathrm{sep}} \equiv g \pmod 2$, then $\gamma = \gamma^{\mathrm{sep}}$.*

Brill-Noether theory completely determines the possible gonalities of a curve of genus g over an algebraically closed field k. For every curve C of genus $g > 0$, we have

$$2 \leq \gamma \leq \left\lfloor \frac{g+3}{2} \right\rfloor,$$

and there exists a curve C of genus g and gonality γ for any such number. The uniqueness hypothesis in Theorem 1 is satisfied in most cases with non-maximal geometric gonality, at least when the characteristic of the field is 0.

Theorem (Arbarello-Cornalba, [3, 2.4 and 2.6]). *Let k be algebraically closed and of characteristic 0, and denote $\mathcal{M}^1_{g,d}$ the moduli space of curves of genus g admitting at least a map of degree d to $\mathbb{P}^1$, with $g \geq 3$ and $2 \leq d < \lfloor (g+3)/2 \rfloor$. Then the generic curve in $\mathcal{M}^1_{g,d}$ has a unique map of degree d to $\mathbb{P}^1$. More precisely, the locus in $\mathcal{M}^1_{g,d}$ of curves with two or more such maps has codimension at least $g + 2 - 2d \geq 1$.*

However, it will be much more useful to have effective criteria to decide whether a given curve has a unique gonal map. We derive such criteria from Castelnuovo type inequalities like those of [1]. Recall that a non-constant rational map $f\colon C \to D$ between two algebraic curves C and D is called

simple if there is no smooth curve D' with maps $f_1 : C \to D'$, and $f_2 : D' \to D$ such that $\deg(f_i) \geq 2$, for $i = 1, 2$, and $f = f_2 \circ f_1$. Equivalently, if the corresponding extension of function fields $k(C)/k(D)$ is simple as extension. For example, if the degree of f is a prime number, then f is simple. Under a simplicity hypothesis and a suitable bound on the separable gonality, we can prove that uniqueness holds:

Theorem 2. *Let C be a curve with genus g and separable gonality γ^{sep}. The gonal map over k^{sep} is unique if it is simple (in particular this is always the case if γ^{sep} is prime) and $(\gamma^{\mathrm{sep}} - 1)^2 < g$.*

From now on we say that a curve of genus g has low gonality if $(\gamma^{\mathrm{sep}} - 1)^2 < g$.

Note also that the class of *goneric* curves introduced in [26], have unique gonal maps over $\overline{k}$. Gonericity is a condition expressed in terms of the gonality and the Betti numbers of a minimal resolution of the ideal of the curve in its canonical embedding, and there exist efficient algorithms to decide whether a given curve is goneric. In fact, Proposition 3 in [26] shows that if C is goneric, then $W^1_{\overline{\gamma}}$ is a single reduced point, which shows that in this case (since k^{sep} is pseudo-algebraically closed) $\gamma^{\mathrm{sep}} = \overline{\gamma}$ as well.

We give two proofs for Theorem 1. The first one, given in Section 3, is based on the theory of Brauer-Severi varieties, and it naturally leads to analogous results for maps from C to $\mathbb{P}^r$: in this general case, uniqueness over k^{sep} implies descent to a map to a Brauer-Severi variety (see Theorem 5 for a precise statement).

A curve in projective space over an algebraically closed field is called reflexive if the composition of its Gauss map with that of its dual is an isomorphism, which is always the case in characteristic zero, and in characteristic $p > 2$ is equivalent to the intersection multiplicity with its tangent line at a general point being 2 [15, 3.5]; we say that C is reflexive if $C_{\overline{k}}$ is reflexive. Then, we can give a generalized version of Theorem 1 (4) to maps to $\mathbb{P}^r$ for $r > 1$; in the case $r = 2$ we obtain the following result.

Theorem 3. *Given a curve C defined over k, denote γ_2 (resp. γ_2^{sep}) the smallest degree of a plane model of C (resp. of $C_{k^{\mathrm{sep}}}$).*

1) *Suppose that the corresponding $g^2_{\gamma_2^{\mathrm{sep}}}$ on $C_{k^{\mathrm{sep}}}$ is unique. If $\gamma_2^{\mathrm{sep}} \not\equiv 0 \pmod 3$, then $\gamma_2 = \gamma_2^{\mathrm{sep}}$.*
2) *If the plane model of $C_{k^{\mathrm{sep}}}$ is reflexive (for instance, if $\operatorname{char} k = 0$) and $g > \left\lfloor \frac{(\gamma_2^{\mathrm{sep}})^2 - 3\gamma_2^{\mathrm{sep}} + 3}{3} \right\rfloor$ then the corresponding $g^2_{\gamma_2^{\mathrm{sep}}}$ is unique.*

In Section 4 a second proof of Theorem 1 (4) is obtained, which gives additional information, at least for geometrically trigonal curves (see Theorem 27). This proof is based on the study of Galois descent for rational normal scrolls, which may be of independent interest.

2. Galois descent and uniqueness

Let C be a curve defined over a field k, and fix a separable and an algebraic closure $k \subset k^{\mathrm{sep}} \subset \overline{k}$ for the whole paper.

Given any divisor D defined over k, denote by $\mathcal{L}(D) := H^0(C, \mathcal{O}(D))$ the k-vector space of meromorphic functions f on C such that $\mathrm{div}(f) + D \geq 0$ is effective. Denote also by $|D|$ the set of effective divisors linearly equivalent to D. Then there is a canonical bijection between $\mathbb{P}(\mathcal{L}(D))$ and $|D|$ determined by mapping $f \in \mathcal{L}(D)$ to the divisor $\mathrm{div}(f) + D$.

Recall that an r-dimensional linear series $\mathcal{D}$ over k^{sep} is the family of divisors given by a vector subspace V in $\mathcal{L}(D)$, for some divisor D, and $\mathcal{D}$ is a g^r_d if $\deg D = d$ and $\dim V = r + 1$. The linear series is complete if $V = \mathcal{L}(D)$. Recall also that a base-point-free linear series $\mathcal{D}$ determines a k^{sep}-morphism $\phi_{\mathcal{D}} : C \to \mathbb{P}(V^*) \cong \mathbb{P}^r$, which maps a point $P \in C(k^{\mathrm{sep}})$ to the point corresponding to the hyperplane

$$\phi_{\mathcal{D}}(P) := \{s \in V \mid s(P) = 0\}.$$

A base-point-free linear series g^r_n is called simple if the map ϕ it determines is simple in a suitable sense, namely it can not be factored as $\phi = \phi' \circ f$ with $f : C \to C'$ and $\phi' : C' \to \mathbb{P}^r$ for some curve C', with $\deg(f), \deg(\phi') \geq 2$ (where $\deg(\phi') = n'$ is the degree of the linear series $g^r_{n'}$ induced on C', i.e., $n = \deg(f_1) \cdot n'$). For $r = 1$ this is equivalent to the map of curves $C \to \mathbb{P}^1$ being simple in the sense above, whereas for $r > 1$ it is equivalent to ϕ being birational onto its image.

We say that a complete linear series $\mathcal{D}$ is Galois invariant if for any $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$, the divisor D^σ is linearly equivalent to D. In this case, any $E \in |D|$ is also in $|D^\sigma|$, hence one gets a natural action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ in $\mathbb{P}(\mathcal{L}(D))$. We say that a linear series $\mathcal{D}$ given by $V \subset \mathcal{L}(D)$ is Galois invariant if the corresponding complete linear series and the subspace $\mathbb{P}(V) \subset \mathbb{P}(\mathcal{L}(D))$ are Galois invariant.

It is surely well known to the experts that morphisms $\pi : C \to C'$ of degree d, with C' a genus zero curve, correspond to g^1_d's on $C_{k^{\mathrm{sep}}}$ invariant under the Galois action. We want to extend this result to higher dimensions of the target space.

Recall that a Brauer-Severi variety of dimension r over k is a smooth projective variety $\mathcal{P}$ such that $\mathcal{P} \otimes_k k^{\mathrm{sep}} \cong \mathbb{P}^r_{k^{\mathrm{sep}}}$. Hence the Brauer-Severi varieties $\mathcal{P}$ of dimension 1 are the curves of genus 0.

Lemma 4. *Let X be a variety defined over k. Then the set of morphisms to some r-dimensional Brauer-Severi variety, modulo automorphisms, corresponds bijectively to the base-point free r-dimensional linear series over k^{sep} invariant under the Galois action of the absolute Galois group* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$.

Proof. That morphisms to Brauer-Severi varieties give r-dimensional linear series over k^{sep} invariant under the Galois action is clear.

To show bijectivity, first of all observe that, if a divisor D is invariant by the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, and the associated linear series $\mathcal{D}$ is base-point free, then it determines a k-defined map $\phi_{\mathcal{D}} : X_k \to \mathbb{P}(V^*)_k \cong \mathbb{P}^r_k$.

Now, if the linear series $\mathcal{D}$ is Galois invariant, one gets an action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ in $\mathbb{P}(V)$ by automorphisms; hence also a dual action on $\mathbb{P}(V^*)$. Both actions determine Brauer-Severi varieties $\mathcal{P}$ and $\mathcal{P}^*$ by the classical theory of Galois descent, which become split on the field of definition of the corresponding divisor D.

Finally, we only need to show that the map $\phi_{\mathcal{D}}$ commutes with the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. But

$$\phi_{\mathcal{D}}(P^\sigma) := \{s \in V \mid s(P^\sigma) = 0\} = \{s^{\sigma^{-1}} \in V \mid s^{\sigma^{-1}}(P) = 0\} = \phi_{\mathcal{D}}(P)^\sigma,$$

since the dual action of σ in $\mathbb{P}(V^*)$ sends a point corresponding to a subspace $W \subset V$ to the subspace $W^{\sigma^{-1}}$. □

As a corollary, one immediately obtains:

Theorem 5. *Let C be a (smooth projective) curve defined over k. Suppose that for a fixed r and d there is only one g^r_d, giving a morphism $f : C_{k^{\mathrm{sep}}} \to \mathbb{P}^r_{k^{\mathrm{sep}}}$. Then there exists a Brauer-Severi variety $\mathcal{P}$ defined over k together with a k-morphism $g : C \to \mathcal{P}$ such that $g \otimes_k k^{\mathrm{sep}} : C_{k^{\mathrm{sep}}} \to \mathcal{P}_{k^{\mathrm{sep}}} \cong \mathbb{P}^r_{k^{\mathrm{sep}}}$ is equal to f.*

Determining the uniqueness of g^r_d's for r and d small relative to the genus is a classic problem (see [1], [7], [8], [10], and references therein). In particular it is widely known that the g^2_d of a smooth plane curve of degree $d > 3$ is unique, but the same is true for "mild" singularities, i.e., if d is small enough compared to the genus.

We approach uniqueness of g^r_d's by the classical "Castelnuovo method", see [5], [1], [7], [14, IV.6]. The idea is to estimate the dimension of sums

of linear series, by counting conditions. To begin with, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are linear series, the sum $\mathcal{D}_1 + \mathcal{D}_2$ is the minimal linear series containing all divisors $D_1 + D_2$ with $D_i \in \mathcal{D}_i$. If $\mathcal{D}_i$ is given by the linear subspace $V_i \subset H^0(C, \mathcal{O}(D_i))$, then $\mathcal{D}_1 + \mathcal{D}_2$ is given by the image of the map

$$\phi : V_1 \otimes V_2 \to H^0(C, \mathcal{O}(D_1 + D_2))$$

determined by $\phi(s_1 \otimes s_2) = s_1 s_2$. On the other hand, if $\mathcal{D}$ is a linear series given by the linear subspace $V \subset H^0(C, \mathcal{O}(D))$, and D' is an arbitrary effective divisor, one puts

$$V(-D') = \{s \in V \mid \operatorname{div}(s) \geq D'\}.$$

The linear series determined by $V(-D')$ has D' as a fixed divisor; subtracting D' from it one obtains a series denoted $\mathcal{D} - D'$. The *number of conditions* imposed by D' on $\mathcal{D}$ is $\dim \mathcal{D} - \dim(\mathcal{D} - D')$, or equivalently $\dim V - \dim(V(-D'))$.

Recall that a curve C in projective space $\mathbb{P}^r_k$ is called reflexive if the composition of the Gauss map of $C_{\overline{k}}$ with that of its dual is an isomorphism, and it is called strange if there is a point of $\mathbb{P}^r_{\overline{k}}$ that belongs to every tangent line of $C_{\overline{k}}$. Observe that reflexive curves are not strange. On the other hand, the only nonsingular strange curves over an algebraically closed field are lines, and conics in characteristic 2 [14, IV.3.9], but the plane smooth curve $x^{p+1} + y^{p+1} + z^{p+1} = 0$ is not reflexive in characteristic p (see [15], [21, 2.3] for this and other examples) so there are indeed curves that are neither reflexive nor strange.

Lemma 6. *Let k be an algebraically closed field, and C a curve defined over k. Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are two different base-point-free simple g^r_n's on C, $n > r \geq 2$, and let $f : C \to \mathbb{P}^r_k$ be the map induced by $\mathcal{D}_1$. Assume that at least one of the following is true:*

1) *$f(C)$ is reflexive,*
2) *$f(C)$ is not strange (for instance, $f(C)$ is nonsingular) and $r \geq 4$.*

Then a general divisor $D \in \mathcal{D}_1$ is made up of n distinct points, and every subset of r points in D imposes r conditions to $\mathcal{D}_1$, whereas every subset of $r+1$ points in D imposes $r+1$ conditions to $\mathcal{D}_2$.

Recall that a statement claimed for *a general divisor in $\mathcal{D}$* is meant to hold for all divisors in a nonempty Zariski-open subset of $\mathcal{D} \cong \mathbb{P}^r$.

Proof. That over an algebraically closed field a general divisor is made up of distinct points is well known ([14, IV, Exercise 3.9], [27, Lemma 3.11.2]).

Then, a general divisor $D \in \mathcal{D}_1$ imposes r conditions to $\mathcal{D}_1$ (i.e., D is the unique divisor in $\mathcal{D}_1$ containing D) and $r+1$ conditions to $\mathcal{D}_2$ (otherwise $D \in \mathcal{D}_1 \cap \mathcal{D}_2$, but if this happens for general $D \in \mathcal{D}_1$, the two linear series must coincide). Under the hypotheses, the monodromy group of f contains A_n [21, 2.2, 2.5]. Then the same proof as in [21, 1.8] gives the result. □

Following Accola, we set

$$R(l;r) = l(l+1)r/2 - l(l-1)/2$$

and

$$R(l_1, l_2; r) = R(l_1, r) + R(l_2, r) + l_1 l_2 r.$$

Castelnuovo's method (see [1], [7, Lemma 1.3]) yields the following:

Lemma 7 (Accola, [1, 4.2]). *Suppose $\mathcal{D}_1$, and $\mathcal{D}_2$ are two different simple g^r_n's without fixed points on C, and assume that a general divisor $D_i \in \mathcal{D}_i$ is made up of n distinct points, and*

1) *every subset of r points in D_i imposes r conditions to $\mathcal{D}_i$,*

2) *every subset of $r+1$ points in D_i imposes $r+1$ conditions to $\mathcal{D}_j$, $j \neq i$.*

Then $\dim(l_1\mathcal{D}_1 + l_2\mathcal{D}_2) \geq R(l_1, l_2, r)$ for all non-negative integers l_1, l_2 satisfying $(l_1 + l_2)r + l_1 - 1 \leq d$.

Theorem 8. *Let C be a curve of genus g over an algebraically closed field k, with a simple linear series g^r_d, giving a morphism $f : C \to \mathbb{P}^r_k$. Assume that at least one of the following is true:*

1) *k has characteristic zero,*

2) *$r = 1$,*

3) *$f(C)$ is reflexive, and $r \geq 2$,*

4) *$f(C)$ is not strange (for instance, $f(C)$ is nonsingular) and $r \geq 4$.*

Write $d = m(2r-1) + q$ where q is the residue of d modulo $(2r-1)$ so that $-r+2 \leq q \leq r$. Let $v = 1$ if $q \leq 1$, $v = 0$ otherwise. If

$$g > m^2(2r-1) + m(2q-1-r) - v(q-1) \tag{1}$$

then the given series is the unique simple g^r_d on C.

Remark 9. For $r = 1$, $f(C_{k^{\mathrm{sep}}}) = \mathbb{P}^1_{k^{\mathrm{sep}}}$, $q = v = 1$, $m = d - 1$ and the inequality (1) reads simply $g > (d-1)^2$. In this case, the result of Proposition 8 was already known to Riemann [23] (for $k = \mathbb{C}$). For $r = 2$, the inequality (1) is equivalent to

$$g > \left\lfloor \frac{d^2 - 3d + 3}{3} \right\rfloor .$$

Proof. For $r = 1$, the result follows from the so-called "Castelnuovo-Severi" inequality, see for instance [27, III.11.3]. So assume $r \geq 2$.

When k has characteristic zero, R. Accola has shown that the existence of two distinct simple g^r_d's contradicts the inequality (1), in [1, Theorem 4.3]. The proof relies on a uniform position lemma [1, 4.1], which needs characteristic zero, to show that the number of conditions imposed by divisors satisfies the hypotheses of Lemma 7. In our case they are satisfied thanks to [21, 1.8] and Lemma 6. The rest of Accola's argument consists in matching the dimension estimate of Lemma 7 with Clifford's inequality for special divisors. This does not depend on the characteristic, so the result follows. □

We will use the preceding results, which are stated for algebraically closed fields, in the proof of Theorem 2; therefore, we need to consider the inseparable base change $\overline{k}/k^{\mathrm{sep}}$. It is probably well known to the experts that under the key assumption made throughout the paper that C is a smooth curve, the relevant phenomena are all stable under this base change:

Lemma 10. *Let C be a (smooth, projetive, geometrically connected) curve of genus g over an arbitrary field k, and fix an algebraic closure $\overline{k}$ of k. Let $f : C \to D$ a morphism where D is a (non necessarily smooth) projective curve defined over k, and for every algebraic extension L/k, denote $f_L : C_L \to D_L$, its base change.*

1) *For every algebraic extension L/k, the field of rational functions of C_L is separable over L, of genus g.*
2) *For every algebraic extension L/k, $\deg f_L = \deg f$.*
3) *If $f_{k^{\mathrm{sep}}}$ is simple, then $f_{\overline{k}}$ is simple.*
4) *If a g^r_d is simple on $C_{k^{\mathrm{sep}}}$, then it is simple on $C_{\overline{k}}$.*

Proof. If k is perfect then all claims are well known (see [27, chapter III]) so assume k is an imperfect field of characteristic p.

Since C is smooth, C_L is smooth over L. In particular the field $k(C_L)$ is formally smooth over L, and therefore separable. Moreover, genus does not change for function fields of smooth curves, by [24, §3].

For the second claim, we first prove that the fields L and $k(C)$ are linearly disjoint over k. By the transitivity of linear disjointness [16, VIII,3.1] it is enough to consider the cases that L/k is separable or purely inseparable. In the separable case, as both $k(C)$ and L are separable, the proof of [27, III.6.1] shows that $k(C)$ and L are linearly disjoint over k. In the inseparable case, because $k(C)$ is separable over k, it is linearly disjoint with $k^{p^{-\infty}}$ (MacLane's criterion, [16, VIII,4.1]) and hence with $L \subset k^{p^{-\infty}}$ over k. Now L and $k(C)$ being linearly disjoint over k implies that $k(D_L) = Lk(D)$ and $k(C)$ are linearly disjoint over $k(D)$. This gives the second claim.

For the third claim, assume by way of contradiction that f is simple and there is a nontrivial intermediate field F, $k(C_{\overline{k}}) \supsetneq F \supsetneq k(D_{\overline{k}})$. Then F is purely inseparable over $F \cap k(C_{k^{\mathrm{sep}}})$, and by the simplicity of f, $F \cap k(C_{k^{\mathrm{sep}}}) = k(D_{k^{\mathrm{sep}}})$. So F is purely inseparable over $k(D_{k^{\mathrm{sep}}})$ and therefore over $k(D_{\overline{k}})$ as well. This implies that there is an element $t \in k(D_{\overline{k}})$ which is not a pth power, with $t^{1/p} \in F \subsetneq k(C_{\overline{k}})$. On the other hand, let $x \in k(C_{k^{\mathrm{sep}}}) \setminus k(D_{k^{\mathrm{sep}}})$ and let $P(X)$ be its minimal polynomial over $k(D_{k^{\mathrm{sep}}})$. By the simplicity of f, $k(C_{k^{\mathrm{sep}}}) = k(D_{k^{\mathrm{sep}}})(x)$ and $\deg P = \deg f = n$. Moreover $k(C_{\overline{k}}) = k(D_{\overline{k}})(x)$, and by the second claim, P is also the minimal polynomial of x over $k(D_{\overline{k}})$. In particular, since $k(D_{\overline{k}})(t^{1/p}) \subsetneq k(C_{\overline{k}})$, p is a proper divisor of n. Now let $Q(X)$ be the minimal polynomial of x over $k(D_{\overline{k}})(t^{1/p})$. Considering the degrees of the extensions, one clearly has $\deg Q = n/p$. But then $Q^p \in k(D_{\overline{k}})[X^p]$ is a monic polynomial of degree n in X vanishing at x, i.e., $Q^p = P$. This means that P only involves pth powers of X, i.e., x^p is an element in $k(C_{k^{\mathrm{sep}}})$ whose minimal polynomial has degree $n/p < p$, contradicting the simplicity of f.

Finally, a linear series g^r_d with $r \geq 2$ is simple if and only if the map $f : C \to \mathbb{P}^r$ it defines is birational onto its image. Therefore, by the second claim simplicity does not change under base field extension. The case $r = 1$ has been dealt with in the third claim. □

Remark 11. Counterexamples to Lemma 10 when C is not smooth do exist. For instance one can consult articles on "genus change under inseparable extensions", starting from classical Tate's and Rosenlicht's papers [28], [24].

Proof of Theorem 2. Let f be a gonal map over k^{sep}, which by assumption is simple. By Lemma 10 its base change $f_{\overline{k}} : C_{\overline{k}} \to \mathbb{P}^1_{\overline{k}}$ is simple as well. By Theorem 8 in the case $r = 1$, the unique simple $g^1_{\gamma^{\mathrm{sep}}}$ on $C_{\overline{k}}$ is then the one

determined by $f_{\overline{k}}$. A fortiori, the unique simple $g^1_{\gamma^{\mathrm{sep}}}$ on $C_{k^{\mathrm{sep}}}$ is the one determined by f. □

Example 12. It is well known that there exist curves with genus $g = (\gamma^{\mathrm{sep}} - 1)^2$ and with more than one simple gonal map. For example, a curve C embedded as a smooth curve of type $(\gamma^{\mathrm{sep}}, \gamma^{\mathrm{sep}})$ in the smooth quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. In this case C has gonality γ^{sep} and it has exactly two $g^1_{\gamma^{\mathrm{sep}}}$: the ones induced by the projections of Q onto one of its factors (see for example [18], Theorem 1).

Remark 13. If $r = 2$ and $g > \left\lfloor \frac{d^2-3d+3}{3} \right\rfloor$, so that Theorem 8 holds, every g^1_e on C with $e < d$ is cut out on $f(C_{k^{\mathrm{sep}}}) \subset \mathbb{P}^2_{k^{\mathrm{sep}}}$ by a pencil of lines, and there is no g^2_e with $e < d$. So the separable gonality is $d - m$ where m is the maximal multiplicity of a singular point of Γ.

We include the reference to a last uniqueness result, due to Cilliberto and Lazarsfeld [8], which applies to $r = 3$ in the case of complete intersections.

Theorem (Ciliberto–Lazarsfeld). *Let $C \subset \mathbb{P}^3$ be a smooth curve defined over a field of characteristic 0, which is the complete intersection of two surfaces of degree h and h', both bigger than 4. Then any simple g^s_m, with $m \leq hh'$ and $s \geq 2$ is unique (and, in particular, the canonical $g^3_{hh'}$).*

3. Splitting Brauer-Severi varities

Recall that to any Brauer-Severi variety $\mathcal{P}$ of dimension n over a field k one can assign canonically a central simple algebra of rank $(n+1)^2$ over k. Its class $[x]$ in the Brauer group verifies that in the exact sequence

$$\mathrm{Pic}(\mathcal{P}) \to \mathrm{Pic}(\mathcal{P} \otimes_k k^{\mathrm{sep}}) \cong \mathbb{Z} \to Br(k)$$

the last map sends 1 to $[x]$, and hence the image of some generator of $\mathrm{Pic}(\mathcal{P})$ is equal to m, where m is the order of $[x]$. Hence m divides $n+1$ since $[x]$ has order dividing $n+1$. We say that $\mathcal{P}$ is split over k if $\mathcal{P} \cong \mathbb{P}^n$ already over k. Then $\mathcal{P}$ is split if and only if $[x] = 0$, i.e. its order is equal to 1.

The following result summarizes some properties of $\mathcal{P}$, some of which are well known.

Theorem 14. *Let $\mathcal{P}$ be a Brauer-Severi variety of dimension n over a field k. Then*

1) *If $\mathcal{P}$ contains a hypersurface of degree coprime with $n+1$, then $\mathcal{P}$ is split.*

2) *If there is a Galois invariant element in $\mathbb{Z}[\mathcal{P}(k^{\mathrm{sep}})]$ (e.g. it $\mathcal{P}$ has a k-rational point) whose degree is coprime with $n+1$, then $\mathcal{P}$ is split.*

3) *There exists an immersion $\mathcal{P} \to \mathbb{P}^N$, where $N = \binom{2n+1}{n} - 1$, as a smooth subvariety of degree $(n+1)^n$.*

4) *There exists a finite map $f : \mathcal{P} \to \mathbb{P}^n$ of degree $(n+1)^n$.*

5) *There exists an extension L/k of degree dividing $n+1$ such that $\mathcal{P} \otimes_k L \cong \mathbb{P}^n_L$.*

Proof. The assertion (1) is clear since the hypersurface determines an element in $\mathrm{Pic}(\mathcal{P})$ whose image in $\mathrm{Pic}(\mathcal{P} \otimes_k k^{\mathrm{sep}})$ is equal to the degree of the hypersurface.

The next result (2) is a generalization of a result by Châtelet in [6] (see also [4]), who showed the case of rational points. We will use the dual Brauer-Severi variety $\widehat{\mathcal{P}}$. It is a Brauer-Severi variety together with an inclusion reversing correspondence between twisted linear subvarieties of dimension $d-1$ of $\mathcal{P}$ and those of codimension d in $\widehat{\mathcal{P}}$. Now, a Galois invariant element in $\mathbb{Z}[\mathcal{P}(k^{\mathrm{sep}})]$ determines an element in $\mathrm{Pic}(\widehat{\mathcal{P}})$, whose degree is equal to the degree of the formal sum, since the degree of a linear hypersurface is 1. The result is deduced then from (1).

Result (3) is well known: in fact, the immersion is given by the anticanonical sheaf, which is always defined over k. It is known that the anticanonical sheaf in $\mathbb{P}^n$ is equal to $\mathcal{O}(n+1)$, and it gives the $(n+1)$-tuple Veronese embedding in $\mathbb{P}^N$.

Now, choosing $n-1$ sufficiently general hyperplanes in $\mathbb{P}^N$, we can find some whose intersection, which is a linear subvariety of dimension $N-n-1$, does not intersect the image of $\mathcal{P}$ in $\mathbb{P}^N$. Projecting to a complementary linear subvariety of dimension n we get the desired finite morphism.

The last result is due to F. Châtelet in his thesis, and it is a consequence of the main classical results on central simple algebras. If A denotes a central simple algebra associated to $\mathcal{P}$, then $\mathcal{P}$ splits over an extension L/K if and only if A does. But A always splits over a maximal commutative subfield, which has degree over K equal to the index of A, which is the square root of the dimension of the associated division algebra, which clearly divides $n+1$. □

Corollary 15. *Let C be a curve defined over a field k, with genus g, gonality γ and conic gonality γ^{con}. Then*

1) $\gamma^{\mathrm{con}} \leq \gamma \leq 2\gamma^{\mathrm{con}}$.
2) *If the curve C has a k-rational divisor of odd degree, then $\gamma = \gamma^{\mathrm{con}}$.*
3) *If $\gamma^{\mathrm{con}} \neq \gamma$, then γ is even.*
4) *There exists some degree 2 Galois extension L/k such that $\gamma(C_L) = \gamma^{\mathrm{con}}$.*

Proof. (1) (resp. (2)) follows from (4) (resp. (2)) in Theorem 14, in the case $n = 1$. The point (4) follows from (5), except the fact that we can take the degree 2 extension to be Galois. This follows from the fact that any conic has a point in a separable extension of degree 2. If the characteristic of the field is not 2, this is clear. If it is 2, and it has no k-rational point, then the conic can be described in $\mathbb{P}^2$ by an equation of the form $ax^2 + by^2 + cz^2 + xz + yz = 0$ for some a, b and $c \in k^*$. Then the points which intersect the line $x = 0$ determine the desired extension. Finally, (3) is immediate from (2). □

Observe that it is not true that the gonality is always the conic gonality or its double, as the following example shows.

Example 16. The genus 4 curve over $\mathbb{Q}$ (or even over $\mathbb{R}$) given in canonical form as the intersection in $\mathbb{P}^3$ of the quadric $x^2 + y^2 + z^2 = 0$ with the cubic $x^3 + y^3 + t^3 = 0$, has conic gonality 3 (with unique gonal map given by the projection map to the conic $x^2 + y^2 + z^2 = 0$), and gonality 4 (with gonal map given by the natural projection to the cubic $x^3 + y^3 + z^3 = 0$ followed by the degree two map determined by a rational point of the cubic (e.g. $[1 : -1 : 0]$)).

Example 17. Let $k = \mathbb{F}_2(s)$ be the field of rational functions over the finite field $\mathbb{F}_2$. The genus 4 curve given in canonical form as the intersection in $\mathbb{P}^3$ of the quadric $xy + z^2 + st^2 = 0$ with the cubic $x^3 + y^3 + t^3 = 0$, has geometric gonality 3, and separable gonality 4 (with a gonal map given as in the previous example). In fact the gonal maps over $\overline{k}$ are given by the two rulings of the quadric, which are not defined over k^{sep}.

Proposition 18. *Let C be a curve defined over a field k with genus g and conic gonality γ^{con}. Assume that over a separable closure k^{sep}, the $g^1_{\gamma^{\mathrm{con}}}$ associated to a conic-gonal map is complete. If $\gamma^{\mathrm{con}} \equiv g \pmod 2$, then $\gamma = \gamma^{\mathrm{con}}$.*

Proof. Let $f : C \to D$ be a map of degree γ^{con} to a curve of genus zero as in the claim. If $D \cong \mathbb{P}^1$ there is nothing to prove, so assume D is a conic. By corollary 15 (4), there is a degree 2 Galois extension L of k over which D becomes split. Consider the Galois-invariant $g^1_{\gamma^{\mathrm{con}}}$ on C_L, which by hypothesis is complete; call E a divisor in this $g^1_{\gamma^{\mathrm{con}}}$. Note that, although E is only defined over L, the linear series $|E|$ is Galois-invariant.

Now if K is a canonical divisor the complete linear series $|K - E|$ is Galois-invariant because K and $|E|$ are Galois-invariant. By Riemann-Roch, and because the $g^1_{\gamma^{\mathrm{con}}}$ is complete, $\dim |K - E| = g - \gamma^{\mathrm{con}} = n$, and we get a morphism $C \to \mathcal{P}$ to a Brauer-Severi variety of (even) dimension n.

Since $K - E$ is a divisor defined over L, $\mathcal{P} \otimes_k L \cong \mathbb{P}^n_L$. Now if H is a hyperplane defined over L, and H^σ is its conjugate by $\mathrm{Gal}(L/k)$, $H + H^\sigma$ is a degree 2 hypersurface defined over k. So $\mathcal{P}$ contains a hypersurface of degree 2, which is coprime with $n + 1$, and by Theorem 14, $\mathcal{P} \cong \mathbb{P}^n$. This means that $K - E$ (and hence E) is linearly equivalent to a divisor defined over k, so $D \cong \mathbb{P}^1$ and we are done. □

Remark 19. We have seen in the course of the proof that, if over a separable closure k^{sep}, the $g^1_{\gamma^{\mathrm{con}}}$ associated to a conic-gonal map is complete, then $\gamma^{\mathrm{con}} = g - \dim |K - E| \leq g + 1$.

Proof of Theorem 1. Part (1) is corollary 5 in the case $r = 1$. Parts (2) and (3) are then consequence of the first part and corollary 15.

Finally, to show part (4) we use that by (1), $\gamma^{\mathrm{con}} = \gamma^{\mathrm{sep}}$, and because gonal series over a separably closed field are always complete, Proposition 18 applies, so $\gamma = \gamma^{\mathrm{con}}$. □

Proof of Theorem 3 . By corollary 5, if the $g^2_{\gamma_2^{\mathrm{sep}}}$ is unique there is a morphism $g : C \to \mathcal{P}$, with $\mathcal{P}$ a Brauer-Severi variety of dimension $n = 2$, whose base change to k^{sep} is the generically injective morphism of lowest degree. $g(C)$ is a divisor of degree γ_2^{sep}, coprime with $n + 1 = 3$, so by Theorem 14 (1), $\mathcal{P} \cong \mathbb{P}^2$.

The second claim is a direct application of Theorem 8 in the case $r = 2$, taking into account Lemma 10. □

Remark 20. Let C be a curve such that $C_{k^{\mathrm{sep}}}$ has a plane model Γ of degree d not divisible by 3, such that all singularities of Γ are nodes or ordinary cusps, and assume that either $(d-2)(d-3) < 2g-2$ or $d \geq 8$ and $(d-3)(d-4) < 2g-10$. Then $\gamma_2 = \overline{\gamma}_2 = d$ and Γ is defined over k. In particular, the smooth plane curves over k^{sep} of degree $d > 3$ and not multiple of 3 are already plane curves over k.

Indeed, by Coppens-Kato [10, Theorem 2.4] and [11], the g^2_d on $C_{k^{\mathrm{sep}}}$ is unique, and $\gamma_2^{\mathrm{sep}} = d$. Then by Theorem 3, $\gamma_2 = \overline{\gamma}_2$ and Γ is defined over k. Note that the nature of the proofs in the Coppens-Kato papers is independent of the characteristic, and hold without the reflexivity hypothesis. In contrast, the inequalities on the genus are more restrictive than in Theorem 2.

Example 21. A smooth plane curve of degree 3 has genus 1, and over k^{sep} any such curve has such a plane model and gonality 2. It is well known that there are genus one curves over $\mathbb{Q}$ (or over any number field) with gonality d for any $d \geq 2$ (see [9], and use that the index and the gonality are equal if the index is larger than one by Riemann-Roch). Hence, if the gonality is larger than 3, they do not have a g^2_3 over $\mathbb{Q}$.

4. Rational normal scrolls under base extension

Recall the following well known construction of rational subvarieties of projective spaces (see [12], [29] or [22, chapter 2]). Let $0 \leq a_1 \leq \cdots \leq a_d$ be a list of d integers with $a_d > 0$ for $d \geq 1$. A rational normal scroll of type $(a_1, \ldots, a_d)$ defined over a field k is a d-dimensional subvariety $S_{(a_1,\ldots,a_d)}$ of $\mathbb{P}^n$, for $n = \sum_{i=1}^d a_i + d - 1$, defined as follows: choose d complementary linear subspaces $L_i \subset \mathbb{P}^n$ for $i = 1, \ldots, d$ with $\dim(L_i) = a_i$. If $a_i \neq 0$, choose a rational normal curve $C_i \subset L_i$ and an isomorphism $\phi_i : \mathbb{P}^1 \to C_i$ (if $a_i = 0$, set $C_i = L_i$ and ϕ_i to be the constant map). Then

$$S_{(a_1,\ldots,a_d)} := \bigcup_{t \in \mathbb{P}^1} \overline{\phi_1(t), \ldots, \phi_d(t)}, \tag{2}$$

where $\overline{\phi_1(t), \ldots, \phi_d(t)}$ denotes the linear span of $\{\phi_1(t), \ldots, \phi_d(t)\} \subset \mathbb{P}^n$.

More abstractly, $S_{(a_1,\ldots,a_d)}$ is the image of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_d))$ in projective $n-$space by the map (determined up to projective equivalence) corresponding to the tautological line bundle $\mathcal{O}(1)$. There is a natural morphism

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_d)) \to \mathbb{P}^1$$

which determines a rational map $\pi : S_{(a_1,\ldots,a_d)} \to \mathbb{P}^1$ which we call structural map. With the description of (2), each smooth point $p \in S_{(a_1,\ldots,a_d)}$ is mapped to the unique $t \in \mathbb{P}^1$ such that $p \in \overline{\phi_1(t), \ldots, \phi_d(t)}$. Any two rational normal scrolls of the same type are projectively equivalent. It is also well known that the degree of a rational normal scroll S is equal to $e := \sum_{i=1}^d a_i =$

$n - d + 1 = n - \dim S + 1$, which is the smallest degree of an irreducible non-degenerate d-fold in $\mathbb{P}^n$. In fact, this condition almost determines these subvarieties of $\mathbb{P}^n$ over an algebraically closed field, the other options being the cones over the Veronese surface in $\mathbb{P}^5$ and irreducible quadrics (noting that quadrics of rank 3 or 4 are rational normal scrolls as well).

Recall also that a rational normal scroll is non-singular if and only if $a_1 > 0$ (in which case the structural map is a morphism) or $(a_1, \ldots, a_d) = (0, \ldots, 0, 1)$ (this last case since $S_{(0,\ldots,0,1)} \cong \mathbb{P}^n$). Singular scrolls are cones over nonsingular scrolls. A rational normal scroll of dimension 1 is a rational normal curve; that is, $S(a) \subset \mathbb{P}^a$ is a rational normal curve of degree a.

Quadric scrolls are also well known varieties, easy to describe. Quadrics of rank 3, corresponding to the case $(a_1, \ldots, a_d) = (0, \ldots, 0, 2)$, are cones over a conic if $d > 1$; their structural map is the projection from the vertex of the cone $S(0, \ldots, 0, 2) \to S(2)$. Quadrics of rank 4, corresponding to the case $(a_1, \ldots, a_d) = (0, \ldots, 0, 1, 1)$, are cones over the quadric surface $S(1, 1) \cong \mathbb{P}^1 \times \mathbb{P}^1$, which supports two structures as scrolls, corresponding to two structural maps which are the two projections to $\mathbb{P}^1$. These are all the cases with degree $e = 2$.

Rational normal scrolls can also be characterized as the only linearly normal varieties which contain a pencil of linear spaces of codimension 1 (namely, the fibers of π). Over an algebraically closed field, they can further be characterized as the irreducible varieties determined by the ideal of 2×2 minors of a $2 \times q$ matrix of linear forms.

A proof of the following result can be found in [22], Chapter 2 and Appendix A.

Lemma 22. *Let S be a rational normal scroll. Then* $\mathrm{Pic}(S) = \mathbb{Z}[H] \oplus \mathbb{Z}[F]$, *where $[F]$ is the class of a fiber of a structural map, and $[H]$ is the class of a hyperplane section. Moreover, the canonical class is* $[K] = -d[H] + (e - 2)[F]$.

As a consequence, the structural map is unique whenever $e > 2$.

We say that a subvariety $S \subset \mathbb{P}^n$ is a potential rational normal scroll if there exists a finite algebraic extension L of k such that the base change of S over L is a rational normal scroll over L, and we will say that S splits in L. Or, equivalently, that the base change of S to an algebraic closure becomes a rational normal scroll. As it is the case for rational normal scrolls, every potential rational normal scroll is a cone over a nonsingular potential rational normal scroll. A nonsingular potential rational normal scroll is thus

a k-form S of the abstract variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_d))$, together with a line bundle L on S which is a k-form of the tautological line bundle $\mathcal{O}(1)$.

Example 23. The simplest examples of potential rational normal scrolls arise in degree 2: every quadric of rank 3 or 4 is a potential rational normal scroll. Let us show examples of quadric potential rational normal scrolls defined over some field k which do not split over k. Consider k a field such that there exists some genus 0 curve C without rational points. Then C is isomorphic to a conic in $\mathbb{P}^2$, which we will denote also by C, which is a potential rational normal scroll of dimension 1 (a quadric of rank 3).

From this we construct an example of dimension 2 and rank 4. Put $\mathbb{P}^2$ inside $\mathbb{P}^5$ in two distinct and complementary ways, giving L_1 and L_2 linear subspaces. If the conic is given by an equation $ax_0^2 + bx_1^2 + cx_2^2 = 0$, then the surface is given by

$$S: \left.\begin{array}{rl} ax_0^2 + bx_1^2 + cx_2^2 & = 0 \\ ax_3^2 + bx_4^2 + cx_5^2 & = 0 \\ x_1x_3 - x_0x_4 & = 0 \\ x_1x_5 - x_2x_4 & = 0 \\ x_0x_5 - x_2x_3 & = 0 \\ ax_0x_3 + bx_1x_4 + cx_2x_5 & = 0 \end{array}\right\} \subset \mathbb{P}^5$$

Theorem 24. *Let $S \subset \mathbb{P}^n$ be a potential rational normal scroll of degree e.*

1) *If S is a quadric of rank 4, and* $\operatorname{char} k \neq 2$*, there is a quadratic or a biquadratic extension L/k such that S_L is a rational normal scroll, and the following are equivalent:*
   a) *S is a rational normal scroll.*
   b) *S is a cone over a ruled quadric surface in $\mathbb{P}^3$.*
   c) *S contains a linear subspace of codimension 1.*
2) *If S is not a quadric of rank 4, there is a Galois degree 2 extension L/k such that S_L is a rational normal scroll, and the following are equivalent:*
   a) *S is a rational normal scroll.*
   b) *S contains a linear subspace of codimension 1.*
   c) *S has a k-rational nonsingular point.*

Moreover, if e is odd then S is a rational normal scroll.

Example 25. The hypothesis on the characteristic in the first part of the theorem can not be dropped: let $k = \mathbb{F}_2(s)$ be the field of rational functions

over the finite field $\mathbb{F}_2$. The quadric $xy + z^2 + st^2 = 0$ in $\mathbb{P}^3$ is a potential scroll which is not a scroll over k^{sep}.

Proof. The quadratic cases are classical, but we will give some indications. If $e = 2$, then the rank is 3 or 4. The rank 3 case corresponds to cones over conics, and they are scrolls if and only if the conic has a point. The assertions in (2) are then easy. If the rank is 4, then it is a cone over a quadric in $\mathbb{P}^3$. If the characteristic is not 2, the quadric can be diagonalized, so we can suppose it is given by an equation of the form

$$a_0X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2$$

for some $a_i \in K$ for $0 \leq i \leq 3$ with $a_0a_1a_2a_3 \neq 0$. The quadric is an scroll if and only if a change of variables can be made to get the equation $Y_0Y_1 - Y_2Y_3$, and this can be done for example over the quadratic or biquadratic extension $L = K(\sqrt{a_0a_1}, \sqrt{a_2a_3})$.

Assume that $e > 2$ and S is nonsingular (which is not restrictive, as S is always a cone over a nonsingular potential rational normal scroll).

Consider the divisor class $[R'] = [K] + d[H]$. By Lemma 22, after base change to a field L over which S is a rational normal scroll, $[R']$ equals $(e-2)$ times the class of a fiber of the structural map π. Therefore

$$\dim H^0(S, \mathcal{O}_S(R')) = e - 1,$$

and the linear system $|R'|$ determines a morphism $S \to \mathbb{P}^{e-2}$ whose image is a genus zero curve D (which becomes a rational normal curve over L).

If L/k is any extension where D has points, the fiber over any L-point of D is a linear space of codimension 1 in S_L; thus for every degree 2 extension where D has points, S_L is a rational normal scroll, and there are such extensions L/k which are Galois.

If $S(k) \neq \emptyset$ then obviously D has k-rational points in $f(S(k))$, so S is a rational normal scroll, which proves (2c) $\Rightarrow$ (2a); and obviously, (2a) $\Rightarrow$ (2b) $\Rightarrow$ (2c).

Finally we prove that if e is odd then S has a k-rational point, by induction on the dimension d of S. The case of dimension $d = 1$ is well known, but we give here a short argument for completeness. In this case S is a curve in $\mathbb{P}^e$, which is projectively equivalent to a rational normal curve in the algebraic closure. But then any hyperplane section of S is a k-rational divisor of odd degree e. Since S is a genus 0 curve with a divisor of odd degree, Riemann-Roch tells us that that S is isomorphic to $\mathbb{P}^1$.

Now, in order to do the induction, observe first that, if k is finite, then D has points, so it is not restrictive to assume k infinite. If H is a hyperplane such that H_L does not contain any ruling of the scroll S_L, then $H_L \cap S_L$ is a scroll and $H \cap S$ a potential scroll of the same degree e and dimension $d-1$, which by the induction hypothesis contains a k-point. So we are reduced to showing that such a hyperplane exists. Now, these rulings have dimension $d-1$ so for each of them there is a $(n-d)$-dimensional linear family of hyperplanes containing it, and there is a $(n-d+1)$-dimensional closed subset of the dual space $(\mathbb{P}^n_L)^*$ consisting of the hyperplanes containing *some* ruling. Since $d \geq 2$, $n-d+1 < n$ and by the infiniteness of k, not all hyperplanes in $(\mathbb{P}^n_k)^*$ belong to this closed subset, so we are done. □

5. Gonality and rational normal scrolls

Given a linearly normal projective curve $C \subset \mathbb{P}^{g-1}$ and a map $f : C \to \mathbb{P}^1$ of degree d, there is a classical construction of a rational normal scroll $S \subset \mathbb{P}^{g-1}$ containing C, such that f is induced by the structural pencil of S (see [25] for a detailed exposition). The codimension of S is $\delta = h^0(C, \mathcal{O}_C(1) \otimes f^*(\mathcal{O}_{\mathbb{P}^1}(-1))) - 1$, and it can be described as

$$S := \bigcup_{\lambda \in \mathbb{P}^1} \overline{f^{-1}(\lambda)} \subset \mathbb{P}^{g-1},$$

where $\overline{f^{-1}(\lambda)}$ denotes the linear span of the divisor $f^{-1}(\lambda) \subset C$ seen as a subscheme of $\mathbb{P}^{g-1}$. So, if C is canonically embedded in $\mathbb{P}^{g-1}$ and $d = \gamma^{\mathrm{sep}}$ is the geometric gonality, then S has dimension $\gamma^{\mathrm{sep}} - 1$. We want to show that this generalizes for maps to genus zero curves, as follows:

Proposition 26. *Given a linearly normal projective curve $C \subset \mathbb{P}^{g-1}$ and a map $f : C \to D$ with D a genus zero curve, there is a potential rational normal scroll $S \subset \mathbb{P}^{g-1}$ defined over k, with a map $\bar{f} : S \to D$ extending f, such that, for every extension L/k with $D_L \cong \mathbb{P}^1$,*

1) *S_L is a scroll, whose structural pencil is $f_L : S_L \to D_L$.*
2) *The codimension of S is $\delta = h^0(C_L, \mathcal{O}_{C_L}(1) \otimes f_L^*(\mathcal{O}_{\mathbb{P}^1_L}(-1))) - 1$.*

Proof. Let L/k be a Galois degree 2 extension such that $D_L \cong \mathbb{P}^1_L$, and let $S_L \subset \mathbb{P}^n_L$ be the corresponding scroll containing C_L. We claim that the homogeneous ideal of S_L is invariant by the action of $\mathrm{Gal}(L/k)$; therefore it can be generated over k, defining a potential rational normal scroll which satisfies the conditions.

To prove the claim, let us recall the construction of the ideal of S_L, following [12]. Consider the divisors of $A = f_L^*(p)$ where $p \in \mathbb{P}^1_L$ is a point, and $B = C \cdot H - A$, where $C \cdot H$ is a hyperplane section, on C_L. Denote $V \subset H^0(\mathcal{O}_{C_L}(A))$ the 2-dimensional subspace whose projectivization is the pencil of fibers of f, and $W = H^0(\mathcal{O}_{C_L}(B))$. Note that $\dim W = \delta + 1$. There is a natural morphism $\phi : V \otimes W \to H^0(\mathcal{O}_{\mathbb{P}^n_L}(1))$ which, after a choice of bases, can be described by a $2 \times (\delta + 1)$ matrix M of linear forms on $\mathbb{P}^n$.

The ideal of S_L is generated by the 2×2 minors of M, and it is of course independent on the choice of bases. We need to show that for every such minor, its conjugate also belongs to the ideal.

Conjugation leaves the pencil of fibers $\mathbb{P}(V)$ invariant, and so acts on it. It follows that it also leaves the linear series $|B| = \mathbb{P}(W)$ invariant, and so acts on it. To be precise, conjugation gives an isomorphism $\sigma : H^0(\mathcal{O}_{C_L}(B)) \to H^0(\mathcal{O}_{C_L}(\bar{B}))$, and by invariance there is an isomorphism $i : H^0(\mathcal{O}_{C_L}(\bar{B})) \to H^0(\mathcal{O}_{C_L}(B)))$ (product with a rational function f with $\operatorname{div}(f) = \bar{B} - B$). Then $\sigma_W = i \circ \sigma : W \to W$ is an isomorphism which induces the conjugation action on $|B| = \mathbb{P}(W)$. (σ_W^2 is not in general the identity on W, but multiplication by a scalar, so it does induce the identity on $|B|$). Similarly, there is an isomorphism $\sigma_V : V \to V$ inducing conjugation on the pencil, and if the rational functions giving $\bar{B} \sim B$ and $\bar{A} \sim A$ are chosen adequately, then for every $v \in V, w \in W$, with $\phi(\sigma_V(v) \otimes \sigma_V(w)) = \sigma(\phi(v \otimes w))$. (Other choices just give proportional images).

Given bases on V and W, application of σ_V and σ_W produces new bases and a new matrix M' whose minors equal the minors of M conjugated. Thus, these conjugated minors belong to the ideal of S_L as claimed. □

Now we can give the second proof for Theorem 1 (4).

Proof of Theorem 1(4). We know by Theorem 1 that the gonal map factorizes through a k-defined map $C \to D$ to a genus zero curve. Consider the canonical embedding $C \subset \mathbb{P}^{g-1}$, and the potential rational normal scroll $S \subset \mathbb{P}^{g-1}$ given by Proposition 26. It has dimension $\gamma^{\text{sep}} - 1$ and degree $(g-1) - (\gamma^{\text{sep}} - 1) + 1$, so it is of odd degree, and by Theorem 24, S is actually a scroll. Therefore the gonal map, which is just the restriction to C of the structural map $S \to D$, is actually defined over k. □

The geometry of the scroll S allows to detect other cases in which the gonality of C must agree with its geometric gonality. We illustrate this in the case of geometrically trigonal curves. In this particular situation, the potential rational normal scroll is $S \cong_{\overline{k}} S_{(a_1,a_2)}$, either isomorphic over $\overline{k}$ to the Hirzebruch surface $\mathbb{F}_a$ if $a_1 > 0$, where $a = a_2 - a_1 \geq 0$ is an integer

called the *Maroni invariant* of C, or isomorphic over k to a (singular) cone over a genus zero curve if $a_1 = 0$. Note that in the geometrically trigonal case the potential scroll is defined by the degree 2 part of the ideal of C in $\mathbb{P}^{n-1}$, so it is defined over k even in the case Proposition 26 cannot not be applied.

Theorem 27. *Let C be a geometrically trigonal (i.e., such that $\overline{\gamma} = 3$) curve with genus $g > 4$. Then its gonality γ satisfies $\gamma \leq 6$ and if either*

1) *g is odd, or*
2) *g is even, and the Maroni invariant a of C satisfies $a > 0$ and $g + a \in (4)$,*

then $\gamma = 3$.

Proof. Note that the degree of S equals $e = a_1 + a_2 = g - 2 > 2$, so S is already a scroll over k^{sep} by Theorem 24.

We consider first the singular case $a_1 = 0$. In this case $\gamma^{\text{sep}} = 3$, hence $\gamma \leq 6$ by corollary 15 and if g is odd, $\gamma = 3$ by Proposition 18 . The case (2) cannot occur since $g + a_2 = 2g - 2 \equiv 2(4)$.

Suppose now S is non-singular. Therefore S is isomorphic over k^{sep} to $\mathbb{F}_a$. Let F be a fiber of the ruling on $S_{k^{\text{sep}}}$, defined over k^{sep}. We want to show that under the hypotheses, F is defined over k and so the gonal map (which is the restriction of the map given by $|F|$ to C) is defined over k. In fact it is enough to see that an odd multiple rF is defined over k; indeed, in that case one obtains a map $S \to \mathbb{P}^r$ whose image is the rational normal curve which, if r is odd, is isomorphic to $\mathbb{P}^1$.

Denote by H a hyperplane section, by K a canonical divisor on $S \cong \mathbb{F}_a$, and by E the negative section ($E^2 = -a < 0$, which is the only smooth curve in $\mathbb{F}_a$ with negative selfintersection and is therefore Galois-invariant); all three are divisors defined over k. By Lemma 22 we have over k^{sep} that $\text{Pic}(S_{k^{\text{sep}}}) \cong \mathbb{Z}H \oplus \mathbb{Z}F$. In fact, it is well known (see e.g. [14, V.2] or [22]) that H is linearly equivalent to $E + a_2F$, whereas K is linearly equivalent to $-2E - (a+2)F$. Thus, whenever a_2 or a are odd (which is satisfied under the given hypotheses) there is an odd multiple of F defined over k. □

If $g = 4$, the bound for the gonality γ also holds since $2g - 2 = 6$ in this case; but if S is non-singular, the Maroni invariant is $a = 0$ although the gonal map is not unique and in this case the gonality over k can be 6; in fact the scroll S belongs to the case considered in Theorem 24, (1).

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