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Tangent cones of Hermitian Yang–Mills connections with isolated singularities

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We give a simple direct proof of uniqueness of tangent cones for singular projectively Hermitian Yang–Mills connections on reflexive sheaves at isolated singularities modelled on a sum of μ–stable holomorphic bundles over $\mathbf{P}^{n-1}$.

1. Introduction

A **projectively Hermitian Yang–Mills (PHYM)** connection A over a Kähler manifold X is a unitary connection A on a Hermitian vector bundle (E, H) over X satisfying

$$\mathrm{F}_A^{0,2} = 0 \quad \text{and} \quad i\Lambda\mathrm{F}_A - \frac{\mathrm{tr}(i\Lambda\mathrm{F}_A)}{\mathrm{rk}\, E} \cdot \mathrm{id}_E = 0. \tag{1.1}$$

Since $\mathrm{F}_A^{0,2} = 0$, $\mathscr{E} := (E, \bar\partial_A)$ is a holomorphic vector bundle, and A is the Chern connection of H. A Hermitian metric H on a holomorphic vector bundle is called **PHYM** if its Chern connection A_H is **P**HYM. The celebrated Donaldson–Uhlenbeck–Yau Theorem [3, 4, 11] asserts that a holomorphic vector bundle $\mathscr{E}$ on a compact Kähler manifold admits a **P**HYM metric if and only if it is μ–polystable; moreover, any two **P**HYM metrics are related by an automorphism of $\mathscr{E}$ and by multiplication with a conformal factor. If H is a **P**HYM metric, then the connection A° on $\mathbf{P}\mathrm{U}(E, H)$, the principal $\mathbf{P}\mathrm{U}(r)$–bundle associated with (E, H), induced by A_H is **Hermitian Yang–Mills (HYM)**, that is, it satisfies $\mathrm{F}_{A^\circ}^{0,2} = 0$ and $i\Lambda\mathrm{F}_{A^\circ} = 0$; it depends only on the conformal class of H. Conversely, any HYM connection A° on $\mathbf{P}\mathrm{U}(E, H)$ can be lifted to a **P**HYM connection A; any two choices of lifts lead to isomorphic holomorphic vector bundles $\mathscr{E}$ and conformal metrics H.

An **admissible** **P**HYM connection is a **P**HYM connection A on a Hermitian vector bundle (E, H) over $X \setminus \mathrm{sing}(A)$ with $\mathrm{sing}(A)$ a closed subset with locally finite $(2n-4)$–dimensional Hausdorff measure and $\mathrm{F}_A \in L^2_{\mathrm{loc}}(X)$.[1] Bando [1] proved that if A is an admissible **P**HYM connection, then $(E, \bar\partial_A)$ extends to X as a reflexive sheaf $\mathscr{E}$ with $\mathrm{sing}(\mathscr{E}) \subset \mathrm{sing}(A)$. Bando and Siu [2] proved that a reflexive sheaf on a compact Kähler manifold admits an admissible **P**HYM metric if and only if it is μ–polystable.

The technique used by Bando and Siu does not yield any information on the behaviour of the admissible **P**HYM connection A_H near the singularities of the reflexive sheaf $\mathscr{E}$ — not even at isolated singularities. The simplest example of a reflexive sheaf on $\mathbf{C}^n$ with an isolated singularity at 0 is $i_*\sigma^*\mathscr{F}$ with $\mathscr{F}$ a holomorphic vector bundle over $\mathbf{P}^{n-1}$; cf. Hartshorne [6, Example 1.9.1]. Here we use the obvious maps summarised in the following diagram:

$$\mathbf{C}^n \xhookleftarrow{\ i\ } \mathbf{C}^n \setminus \{0\} \xrightarrow{\ \pi\ } S^{2n-1} \xrightarrow{\ \rho\ } \mathbf{P}^{n-1}, \qquad \sigma = \rho \circ \pi \colon \mathbf{C}^n \setminus \{0\} \to \mathbf{P}^{n-1}.$$

The main result of this article gives a description of **P**HYM connections near singularities modelled on $i_*\sigma^*\mathscr{F}$ with $\mathscr{F}$ a sum of μ–stable holomorphic vector bundles.

Theorem 1.2. *Let $\omega = \frac{1}{2i}\bar\partial\partial|z|^2 + O(|z|^2)$ be a Kähler form on $\bar B_R(0) \subset \mathbf{C}^n$. Let A be an admissible **P**HYM connection on a Hermitian vector bundle (E, H) over $B_R(0) \setminus \{0\}$ with $\mathrm{sing}(A) = \{0\}$ and $(E, \bar\partial_A) \cong \sigma^*\mathscr{F}$ for some holomorphic vector bundle $\mathscr{F}$ over $\mathbf{P}^{n-1}$. Denote by F the complex vector bundle underlying $\mathscr{F}$.*

If $\mathscr{F}$ is sum of μ–stable holomorphic vector bundles, then there exist a Hermitian metric K on F, a connection A_* on $\sigma^*(F, K)$ which is the pullback of a connection on $\rho^*(F, K)$, and an isometry $(E, H) \cong \sigma^*(F, K)$ such that with respect to this isometry we have

$$|z|^{k+1}|\nabla^k_{A_*}(A^\circ - A^\circ_*)| \leq C_k(-\log|z|)^{-1/2} \quad \textit{for each } k \geq 0;$$

moreover, if $\mathscr{F}$ is μ–stable, then

$$|z|^{k+1}|\nabla^k_{A_*}(A^\circ - A^\circ_*)| \leq D_k|z|^{\alpha} \quad \textit{for each } k \geq 0.$$

[1] It should be pointed out that our notion of admissible **P**HYM connection follows Bando and Siu [2] and not Tian [10]. The notion of admissible Yang–Mills connection introduced by Tian is stronger: it assumes that the Hermitian vector bundle extends to all of X.

The constants $C_k, D_k, \alpha > 0$ *depend on*

$$\omega, \quad \mathscr{F}, \quad A|_{B_R(0)\setminus B_{R/2}(0)}, \quad \textit{and} \quad \|\mathrm{F}_A\|_{L^2(B_R(0))}.$$

Remark 1.3. Using a gauge theoretic Łojasiewicz–Simon gradient inequality, Yang [12, Theorem 1] proved that the tangent cone to a stationary Yang–Mills connection — in particular, a $\mathbf{PU}(r)$ HYM connection — with an isolated singularity at x is unique provided

$$|\mathrm{F}_A| \lesssim d(x,\cdot)^{-2}.$$

In our situation, such a curvature bound can be obtained from Theorem 1.2; our proof of this result, however, proceeds more directly — without making use of Yang's theorem.

The hypothesis that $\mathscr{F}$ be a sum of μ–stable holomorphic vector bundles is optimal. This is a consequence of the following observation, which will be proved in Section 6.

Proposition 1.4. *Let* (F, K) *be a Hermitian vector bundle over* $\mathbf{P}^{n-1}$*. If* B *is a unitary connection on* $\rho^*(F, K)$ *such that* $A_* := \pi^* B$ *is HYM with respect to* $\omega_0 := \frac{1}{2i}\bar\partial\partial|z|^2$*, then there is a* $k \in \mathbf{N}$ *and, for each* $j \in \{1, \dots, k\}$*,* $\mu_j \in \mathbf{R}$*, a Hermitian vector bundle* (F_j, K_j) *on* $\mathbf{P}^{n-1}$*, and an irreducible unitary connection* B_j *on* F_j *satisfying*

$$\mathrm{F}_{B_j}^{0,2} = 0 \quad \textit{and} \quad i\Lambda \mathrm{F}_{B_j} = (2n-2)\pi\mu_j \cdot \mathrm{id}_{F_j}$$

such that

$$F = \bigoplus_{j=1}^{k} F_j \quad \textit{and} \quad B = \bigoplus_{j=1}^{k} \rho^* B_j + i\mu_j \,\mathrm{id}_{\rho^* F_j} \cdot \theta.$$

Here θ *denotes the standard contact structure*[2] *on* S^{2n-1}*. In particular,*

$$\mathscr{E} = (\sigma^* F, \bar\partial_{A_*}) \cong \bigoplus_{j=1}^{k} \sigma^* \mathscr{F}_j$$

with $\mathscr{F}_j = (F_j, \bar\partial_{B_j})$ μ*–stable.*

[2]With respect to standard coordinates on $\mathbf{C}^n$, the standard contact structure θ on S^{2n-1} is such that $\pi^*\theta = \sum_{j=1}^n (\bar z_j \mathrm{d}z_j - z_j \mathrm{d}\bar z_j)/2i|z|^2$.

To conclude the introduction we discuss two concrete examples in which Theorem 1.2 can be applied.

Example 1.5 (Okonek et al. [8, Example 1.1.13]). It follows from the Euler sequence that $H^0(\mathscr{T}_{\mathbf{P}^3}(-1)) \cong \mathbf{C}^4$. Denote by $s_v \in H^0(\mathscr{T}_{\mathbf{P}^3}(-1))$ the section corresponding to $v \in \mathbf{C}^4$. If $v \neq 0$, then the rank two sheaf $\mathscr{E} = \mathscr{E}_v$ defined by

$$0 \to \mathscr{O}_{\mathbf{P}^3} \xrightarrow{s_v} \mathscr{T}_{\mathbf{P}^3}(-1) \to \mathscr{E}_v \to 0$$

is reflexive and $\mathrm{sing}(\mathscr{E}) = \{[v]\}$.

$\mathscr{E}$ is μ–stable. To see this, because $\mu(\mathscr{E}) = 1/2$, it suffices to show that

$$\mathrm{Hom}(\mathscr{O}_{\mathbf{P}^3}(k), \mathscr{E}) = H^0(\mathscr{E}(-k)) = 0 \quad \text{for each } k \geq 1.$$

However, by inspection of the Euler sequence, $H^0(\mathscr{E}(-k)) \cong H^0(\mathscr{T}_{\mathbf{P}^3}(-k-1)) = 0$. It follows that $\mathscr{E}$ admits a **P**HYM metric H with $\mathrm{F}_H \in L^2$ and a unique singular point at $[v] \in \mathbf{P}^3$. To see that Theorem 1.2 applies, pick a standard affine neighborhood $U \cong \mathbf{C}^3$ in which $[v]$ corresponds to 0. In U, the Euler sequence becomes

$$0 \to \mathscr{O}_{\mathbf{C}^3} \xrightarrow{(1,z_1,z_2,z_3)} \mathscr{O}_{\mathbf{C}^3}^{\oplus 4} \to \mathscr{T}_{\mathbf{P}^3}(-1)|_U \to 0,$$

and $s_v = [(1,0,0,0)]$; hence,

$$0 \to \mathscr{O}_{\mathbf{C}^3} \xrightarrow{(z_1,z_2,z_3)} \mathscr{O}_{\mathbf{C}^3}^{\oplus 3} \to \mathscr{E}_v|_U \to 0.$$

On $\mathbf{C}^3 \setminus \{0\}$, this is the pullback of the Euler sequence on $\mathbf{P}^2$; therefore, $\mathscr{E}_v|_U \cong i_*\sigma^*\mathscr{T}_{\mathbf{P}^2}$.

Example 1.6. For $t \in \mathbf{C}$, define $f_t \colon \mathscr{O}_{\mathbf{P}^3}(-2)^{\oplus 2} \to \mathscr{O}_{\mathbf{P}^3}(-1)^{\oplus 5}$ by

$$f_t \coloneqq \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ t \cdot z_3 & z_2 \\ 0 & z_3 \end{pmatrix},$$

and denote by $\mathscr{E}_t$ the cokernel of f_t, i.e.,

$$0 \to \mathscr{O}_{\mathbf{P}^3}(-2)^{\oplus 2} \xrightarrow{f_t} \mathscr{O}_{\mathbf{P}^3}(-1)^{\oplus 5} \to \mathscr{E}_t \to 0. \tag{1.7}$$

If $t \neq 0$, then $\mathscr{E}_t$ is locally free; $\mathscr{E}_0$ is reflexive with $\operatorname{sing}(\mathscr{E}_0) = \{[0:0:0:1]\}$. The proof of this is analogous to that of the reflexivity of $\mathscr{E}_v$ from Example 1.5 given in [8, Example 1.1.13].

For each t, $H^0(\mathscr{E}_t) = H^0(\mathscr{E}_t^*(-1)) = 0$; hence, $\mathscr{E}_t$ is μ–stable according to the criterion of Okonek et al. [8, Remark 1.2.6(b)]. The former vanishing is obvious since $H^0(\mathscr{O}_{\mathbf{P}^3}(-1)) = H^1(\mathscr{O}_{\mathbf{P}^3}(-2)) = 0$. The latter follows by dualising (1.7), twisting by $\mathscr{O}_{\mathbf{P}^3}(-1)$, and observing that the induced map $H^0(f_0^*)\colon H^0(\mathscr{O}_{\mathbf{P}^3})^{\oplus 5} \to H^0(\mathscr{O}_{\mathbf{P}^3}(1))^{\oplus 2}$, which is given by

$$\begin{pmatrix} z_0 & z_1 & z_2 & t\cdot z_3 & 0 \\ 0 & z_0 & z_1 & z_2 & z_3 \end{pmatrix},$$

is injective.

In a standard affine neighborhood $U \cong \mathbf{C}^3$ of $[0:0:0:1]$, we have $\mathscr{E}_0|_U \cong i_*\sigma^*(\mathscr{T}_{\mathbf{P}^2} \oplus \mathscr{O}_{\mathbf{P}^2}(1))$. To see this, note that the cokernel of the map $g\colon \mathscr{O}_{\mathbf{P}^2}^{\oplus 2} \to \mathscr{O}_{\mathbf{P}^2}(1)^{\oplus 4} \oplus \mathscr{O}_{\mathbf{P}^2}$ defined by

$$g := \begin{pmatrix} z_0 & 0 \\ z_1 & z_0 \\ z_2 & z_1 \\ 0 & z_2 \\ 0 & 1 \end{pmatrix}$$

is $\mathscr{T}_{\mathbf{P}^2} \oplus \mathscr{O}_{\mathbf{P}^2}(1)$.

Conventions and notation. Set $B_r := B_r(0)$ and $\dot{B}_r := B_r(0) \setminus \{0\}$. We denote by $c > 0$ a generic constant, which depends only on $\mathscr{F}$, ω, $s|_{B_1 \setminus B_{1/2}}$, $H_\diamond$, and $\|\mathrm{F}_H\|_{L^2(B_R(0))}$ (which will be introduced in the next section). Its value might change from one occurrence to the next. Should c depend on further data we indicate this by a subscript. We write $x \lesssim y$ for $x \leq cy$. The expression $O(x)$ denotes a quantity y with $|y| \lesssim x$. Since reflexive sheaves are locally free away from a closed subset of complex codimension three, without loss of generality, we will assume throughout that $n \geq 3$.

2. Reduction to the metric setting

In the situation of Theorem 1.2, the Hermitian metric H on $\mathscr{E}$ corresponds to a **P**HYM metric on $\sigma^*\mathscr{F}$ via the isomorphism $(E, \bar{\partial}_A) \cong \sigma^*\mathscr{F}$. By slight abuse of notation, we will denote this metric by H as well.

Denote by $\mathscr{F}_1, \dots, \mathscr{F}_k$ the μ–stable summands of $\mathscr{F}$. Denote by K_j the **P**HYM metric on $\mathscr{F}_j$ with

$$i\Lambda_{\omega_{FS}} \mathrm{F}_{K_j} = \lambda_j \cdot \mathrm{id}_{F_j} := \frac{2\pi}{(n-2)!\mathrm{vol}(\mathbf{P}^{n-1})}\mu_j \cdot \mathrm{id}_{F_j} = (2n-2)\pi\mu_j \cdot \mathrm{id}_{F_j}$$

with ω_{FS} denoting the integral Fubini study form and for $\mu_j := \mu(\mathscr{F}_j)$. The Kähler form ω_0 associated with the standard Kähler metric on $\mathbf{C}^n$ can be written as

$$\omega_0 = \frac{1}{2i}\bar{\partial}\partial|z|^2 = \pi r^2 \sigma^*\omega_{FS} + r\mathrm{d}r \wedge \pi^*\theta \tag{2.1}$$

with θ as in Proposition 1.4. Therefore, we have

$$i\Lambda_{\omega_0}\mathrm{F}_{\sigma^* K_j} = (2n-2)\mu_j r^{-2} \cdot \mathrm{id}_{\sigma^* F_j},$$

and $H_{\diamond,j} := r^{2\mu_j} \cdot \sigma^* K_j$ satisfies

$$\begin{aligned} i\Lambda_{\omega_0}\mathrm{F}_{H_{\diamond,j}} &= i\Lambda_{\omega_0}\mathrm{F}_{\sigma^* K_j} + i\Lambda_{\omega_0}\bar{\partial}\partial \log r^{2\mu_j} \cdot \mathrm{id}_{\sigma^* F_j} \\ &= i\Lambda_{\omega_0}\mathrm{F}_{\sigma^* K_j} + \frac{1}{2}\Delta \log r^{2\mu_j} \cdot \mathrm{id}_{\sigma^* F_j} = 0. \end{aligned}$$

Denote by $A_{\diamond,j}$ the Chern connection associated with $H_{\diamond,j}$ and by B_j the Chern connection associated with K_j. The isometry $r^{\mu_j} \colon (\sigma^* F_j, H_{\diamond,j}) \to \sigma^*(F_j, K_j)$ transforms $A_{\diamond,j}$ into

$$A_{*,j} := (r^{\mu_j})_* A_{\diamond,j} = \sigma^* B_j + i\mu_j \,\mathrm{id}_{\sigma^* F_j} \cdot \pi^*\theta.$$

In particular,

$$A_* := \bigoplus_{j=1}^k A_{*,j}$$

is the pullback of a connection B on S^{2n-1}; moreover, A_* is unitary with respect to

$$H_* := \bigoplus_{j=1}^k \sigma^* K_j.$$

Proposition 2.2. *Assume the above situation. Set* $H_\diamond := \bigoplus_{j=1}^k H_{\diamond,j}$ *and fix* $R > 0$*. We have*

$$\left\| |z|^{2+\ell} \nabla^\ell_{H_\diamond} \mathrm{F}_{H_\diamond} \right\|_{L^\infty(B_R)} < \infty \quad \textit{for each } \ell \geq 0. \tag{2.3}$$

Moreover, if $\mathscr{F}$ *is* μ*–stable (that is* $k = 1$*), then*

$$\int_{\partial B_r} |s|^2 \lesssim r^2 \int_{\partial B_r} |\nabla_{H_\diamond} s|^2 \tag{2.4}$$

for all $r \in (0, R]$ *and* $s \in C^\infty(\partial B_r, i\mathfrak{su}(\sigma^* F, H_\diamond))$.

Proof. Using the isometry $g := \bigoplus_{j=1}^k r^{\mu_j}$ both assertions can be translated to corresponding statements for A_*. The first assertion then follows since A_* is the pullback of a connection B on S^{2n-1}. If $k = 1$, then

$$\nabla_B \colon\ C^\infty(S^{2n-1}, i\mathfrak{su}(\rho^* F, K_1)) \to \Omega^1(S^{2n-1}, i\mathfrak{su}(\rho^* F, K_1))$$

agrees with $\nabla_{\rho^* B_1}$ because $i\mu_1 \,\mathrm{id}_{\sigma^* F_1}$ is central. Therefore, any element of $\ker \nabla_B = \ker \nabla_{\rho^* B_1}$ must be invariant under the S^1–action and thus be the pullback of an element of $\ker \nabla_{B_1}$. The latter vanishes because $\mathscr{F}_1$ is μ–stable; hence, simple. This implies the second assertion. □

In the situation of Theorem 1.2, after a conformal change, which does not affect A°, we can assume that $\det H = \det H_\diamond$. Setting

$$s := \log(H_\diamond^{-1} H) \in C^\infty(\dot{B}_r, i\mathfrak{su}(\sigma^* F, H_\diamond))^3$$
$$\text{and} \quad \Upsilon(s) := \frac{e^{\mathrm{ad}_s} - 1}{\mathrm{ad}_s},$$

we have

$$e_*^{s/2} H = H_\diamond \quad \text{and} \quad e_*^{s/2} A = A_\diamond + a$$
$$\text{with} \quad a := \frac{1}{2} \Upsilon(-s/2) \partial_{A_\diamond} s - \frac{1}{2} \Upsilon(s/2) \bar{\partial}_{A_\diamond} s;$$

[3] If H, K are two Hermitian inner products on a complex vector space V, then there is a unique endomorphism $T \in \mathrm{End}(V)$ which is self-adjoint with respect to H and K, has positive spectrum, and satisfies $H(Tv, w) = K(v, w)$. It is customary to denote T by $H^{-1}K$, and thus $\log(H^{-1}K) = \log(T)$.

see, e.g., [7, Appendix A]. Moreover, with $g \coloneqq \bigoplus_{j=1}^{k} r^{\mu_j}$ we have

$$g_* e_*^{s/2} A = A_* + gag^{-1}.$$

Since

$$|\nabla_{A_*}^k gag^{-1}|_{H_*} = |\nabla_{H_\diamond}^k a|_{H_\diamond} \quad \text{for each } k \geq 0,$$

Theorem 1.2 will be a consequence of Proposition 2.2 and the following result.

Theorem 2.5. *Suppose* $\omega = \frac{1}{2i}\bar{\partial}\partial|z|^2 + O(|z|^2)$ *is a Kähler form on* $\bar{B}_R \subset \mathbf{C}^n$, $\mathscr{E}$ *is a holomorphic vector bundle over* $\dot{B}_R$, *and* $H_\diamond$ *is a Hermitian metric on* $\mathscr{E}$ *which is HYM with respect to* ω_0 *and satisfies* (2.3). *If* H *is an admissible HYM metric on* $\mathscr{E}$ *with* $\operatorname{sing}(A_H) = \{0\}$ *and* $\det H = \det H_\diamond$, *then*

$$s \coloneqq \log(H_\diamond^{-1} H) \in C^\infty(\dot{B}_R, i\mathfrak{su}(\pi^* F, H_\diamond))$$

satisfies

$$|s| \leq C_0 \quad \textit{and} \quad |z|^k |\nabla_{H_\diamond}^k s| \leq C_k (-\log|z|)^{-1/2} \quad \textit{for each } k \geq 1.$$

Moreover, if (2.4) *holds, then*

$$|z|^k |\nabla_{H_\diamond}^k s| \leq D_k |z|^\alpha \quad \textit{for each } k \geq 0.$$

The constants $C_k, D_k, \alpha > 0$ *depend on* ω, $H_\diamond$, $s|_{B_R \setminus B_{R/2}}$, *and* $\|\mathrm{F}_H\|_{L^2(B_R)}$.

The next three sections of this paper are devoted to proving Theorem 2.5. Without loss of generality, we will assume that the radius R is one. We set $B \coloneqq B_1$ and $\dot{B} \coloneqq \dot{B}_1$.

3. A priori C^0 estimate

As a first step towards proving Theorem 2.5 we bound $|s|$, using an argument which is essentially contained in Bando and Siu [2, Theorem 2(a) and (b)].

Proposition 3.1. *We have* $|s| \in L^\infty(B)$ *and* $\|s\|_{L^\infty(B)} \leq c$.

Proof. The proof relies on the differential inequality

$$\Delta \log \operatorname{tr} H_0^{-1} H_1 \lesssim |\mathrm{K}_{H_1} - \mathrm{K}_{H_0}| \tag{3.2}$$

for Hermitian metrics H_0 and H_1 with $\det H_0 = \det H_1$, and with

$$\mathrm{K}_H \coloneqq i\Lambda \mathrm{F}_H - \frac{\operatorname{tr}(i\Lambda \mathrm{F}_H)}{\operatorname{rk} E} \cdot \mathrm{id}_E;$$

see [9, p. 13] for a proof.

Step 1. *We have* $\log \operatorname{tr} e^s \in W^{1,2}(B)$ *and* $\|\log \operatorname{tr} e^s\|_{W^{1,2}(B)} \leq c$.

Choose $1 \leq i < j \leq n$ and define the projection $\pi\colon B \to \mathbf{C}^{n-2}$ by

$$\pi(z) \coloneqq (z_1, \ldots, \hat{z}_i, \ldots \hat{z}_j, \ldots, z_n).$$

For $\zeta \in \mathbf{C}^{n-2}$, denote by ∇_ζ and Δ_ζ the derivative and the Laplacian on the slice $\pi^{-1}(\zeta)$ respectively. Set $f_\zeta \coloneqq \log \operatorname{tr} e^s|_{\pi^{-1}(\zeta)}$. Applying (3.2) to $H|_{\pi^{-1}(\zeta)}$ and $H_\diamond|_{\pi^{-1}(\zeta)}$ we obtain

$$\Delta_\zeta f_\zeta \lesssim |\mathrm{F}_H| + |\mathrm{F}_{H_\diamond}|.$$

Fix $\chi \in C^\infty(\mathbf{C}^2; [0,1])$ such that $\chi(\eta) = 1$ for $|\eta| \leq 1/2$ and $\chi(\eta) = 0$ for $|\eta| \geq 1/\sqrt{2}$. For $0 < |\zeta| \leq 1/\sqrt{2}$ and $\varepsilon > 0$, we have

$$\begin{aligned}\int_{\pi^{-1}(\zeta)} |\nabla_\zeta(\chi f_\zeta)|^2 &\lesssim \int_{\pi^{-1}(\zeta)} \chi^2 f_\zeta(|\mathrm{F}_H| + |\mathrm{F}_{H_\diamond}|) + 1 \\ &\leq \varepsilon \int_{\pi^{-1}(\zeta)} |\chi f_\zeta|^2 + \varepsilon^{-1} \int_{\pi^{-1}(\zeta)} |\mathrm{F}_H|^2 + |\mathrm{F}_{H_\diamond}|^2 + 1.\end{aligned}$$

Using the Dirichlet–Poincaré inequality and rearranging, we obtain

$$\int_{\pi^{-1}(\zeta)} |\chi f_\zeta|^2 + |\nabla_\zeta(\chi f_\zeta)|^2 \lesssim \int_{\pi^{-1}(\zeta)} |\mathrm{F}_H|^2 + |\mathrm{F}_{H_\diamond}|^2 + 1.$$

Integrating over $0 < |\zeta| \leq 1/\sqrt{2}$ yields

$$\int_B |\log \operatorname{tr} e^s|^2 + |\nabla' \log \operatorname{tr} e^s|^2 \lesssim \int_B |\mathrm{F}_H|^2 + |\mathrm{F}_{H_\diamond}|^2 + 1$$

with ∇' denoting the derivative along the fibres of π. Using (2.3) and $n \geq 3$, $\mathrm{F}_{H_\diamond} \in L^2(B)$. Since the choice of i, j defining π was arbitrary, the asserted inequality follows.

Step 2. *The differential inequality*

$$\Delta \log \operatorname{tr} e^s \lesssim |\mathrm{K}_{H_\diamond}|$$

holds on B in the sense of distributions.

Fix a smooth function $\chi\colon [0,\infty) \to [0,1]$ which vanishes on $[0,1]$ and is equal to one on $[2,\infty)$. Set $\chi_\varepsilon \coloneqq \chi(|\cdot|/\varepsilon)$. By (3.2), for $\phi \in C_0^\infty(B)$, we have

$$\begin{aligned}
&\int_B \Delta\phi \cdot \log \operatorname{tr} e^s \\
&\quad = \lim_{\varepsilon \to 0} \int_B \chi_\varepsilon \cdot \Delta\phi \cdot \log \operatorname{tr} e^s \\
&\quad \lesssim \int_B \phi \cdot |\mathrm{K}_{H_\diamond}| + \lim_{\varepsilon\to 0} \int_B \phi \cdot \left(\Delta\chi_\varepsilon \cdot \log \operatorname{tr} e^s - 2\langle \nabla\chi_\varepsilon, \nabla \log \operatorname{tr} e^s\rangle\right).
\end{aligned}$$

Since $n \geq 3$, we have $\|\chi_\varepsilon\|_{W^{2,2}(B)} \lesssim \varepsilon^2$. Because $\log \operatorname{tr} e^s \in W^{1,2}(B)$, this shows that the limit vanishes.

Step 3. *We have* $\log \operatorname{tr} e^s \in L^\infty(B)$ *and* $\|\log \operatorname{tr} e^s\|_{L^\infty(B)} \leq c$.

Since $\operatorname{tr} s = 0$, we have $|s| \leq \operatorname{rk}(\mathscr{E}) \cdot \log \operatorname{tr} e^s$; in particular, $\log \operatorname{tr} e^s$ is non-negative. By hypothesis $\mathrm{K}_H = 0$. Since $H_\diamond$ is **P**HYM with respect to ω_0 and $|\mathrm{F}_{H_\diamond}| \lesssim |z|^{-2}$ by hypothesis (2.3), we have $|\mathrm{K}_{H_\diamond}| \leq c$. The asserted inequality thus follows from Step 2 via Moser iteration; see [5, Theorem 8.1]. □

4. A priori Morrey estimates

The following decay estimates are the crucial ingredients of the proof of Theorem 2.5.

Proposition 4.1. *For $r \in [0,1]$, we have*

$$\int_{B_r} |\nabla_{H_\diamond} s|^2 \lesssim r^{2n-2}(-\log r)^{-1}.$$

Proposition 4.2. *If* (2.4) *holds, then there is a constant $\alpha > 0$, depending on $\|s\|_{L^\infty(B)}$ in a monotone decreasing way, such that for $r \in [0,1]$ we have*

$$\int_{B_r} |s|^2 \lesssim r^{2n+2\alpha} \quad \text{and} \quad \int_{B_r} |\nabla_{H_\diamond} s|^2 \lesssim r^{2n-2+2\alpha}.$$

Both of these results rely on the following inequality.

Proposition 4.3. *We have*

$$|\nabla_{H_\diamond} s|^2 \lesssim 1 - \Delta |s|^2.$$

Proof. Since $H = H_\diamond e^s$ is **P**HYM, we have

$$\Delta |s|^2 + 2|\upsilon(-s)\nabla_{H_\diamond} s|^2 \leq -4\langle \mathrm{K}_{H_\diamond}, s\rangle$$

with

$$\upsilon(-s) = \sqrt{\frac{1 - e^{-\operatorname{ad}_s}}{\operatorname{ad}_s}} \in \operatorname{End}(\mathfrak{gl}(E));$$

see, e.g., [7, Proposition A.6]. The assertion follows using

$$\sqrt{\frac{1-e^{-x}}{x}} \gtrsim \frac{1}{\sqrt{1+|x|}},$$

$\|\mathrm{K}_{H_\diamond}\|_{L^\infty} \leq c$, which is a consequence of (2.3) and the fact that $H_\diamond$ is HYM with respect to ω_0, and the bound on $|s|$ established in Proposition 3.1. □

Proof of Proposition 4.2. The proof is very similar to that of [7, Proposition C.2]. Nevertheless, for the reader's convenience we provide the necessary details.

Define $g\colon [0, 1/2] \to [0, \infty]$ by

$$g(r) := \int_{B_r} |z|^{2-2n} |\nabla_{H_\diamond} s|^2.$$

We will show that

$$g(r) \leq c r^{2\alpha},$$

which implies the second asserted inequality and using (2.4) also the first.

Step 1. *We have $g \leq c$.*

Fix a smooth function $\chi\colon [0, \infty) \to [0, 1]$ which is equal to one on $[0, 1]$ and vanishes outside $[0, 2]$. Set $\chi_r(\cdot) := \chi(|\cdot|/r)$. For $r > \varepsilon > 0$, using Proposition 4.3 and Proposition 3.1, and with G denoting Green's function on B

centered at 0, we have

$$\begin{aligned}\int_{B_r\setminus B_\varepsilon}|z|^{2-2n}|\nabla_{H_\diamond}s|^2 &\lesssim \int_{B_{2r}\setminus B_{\varepsilon/2}}\chi_r(1-\chi_{\varepsilon/2})G(1-\Delta|s|^2)\\ &\lesssim \int_{B_{2r}\setminus B_r}|z|^{-2n}|s|^2+r^2+\varepsilon^{-2n}\int_{B_\varepsilon\setminus B_{\varepsilon/2}}|s|^2\\ &\leq c.\end{aligned}$$

Step 2. *There are constants $\gamma\in[0,1)$ and $A>0$ such that*

$$g(r)\leq \gamma g(2r)+Ar^2.$$

Continuing the inequality from Step 1 using (2.4), we have

$$\begin{aligned}\int_{B_r\setminus B_\varepsilon}|z|^{2-2n}|\nabla_{H_\diamond}s|^2 &\lesssim \int_{B_{2r}\setminus B_r}|z|^{2-2n}|\nabla_{H_\diamond}s|^2+r^2+\varepsilon^{2-2n}\int_{B_\varepsilon\setminus B_{\varepsilon/2}}|\nabla_{H_\diamond}s|^2\\ &\lesssim g(2r)-g(r)+r^2+g(\varepsilon).\end{aligned}$$

By Lebesgue's monotone convergence theorem, the last term vanishes as ε tends to zero; hence, the asserted inequality follows with $\gamma=\frac{c}{c+1}$ and $A=c$.

Step 3. *We have $g\leq cr^{2\alpha}$ for some $\alpha\in(0,1)$.*

This follows from Step 1 and Step 2 and as in [7, Step 3 in the proof of Proposition C.2]. □

Proof of Proposition 4.1. We use the same notation as in the proof of Proposition 4.2. It still holds that $g\leq c$. However, the proof of the doubling estimate in Step 2 uses that $\mathscr{F}$ is simple and will not carry over. Instead, using integration by parts and Hölder's inequality we have

$$\begin{aligned}\int_{B_r\setminus B_\varepsilon}|z|^{2-2n}|\nabla_{H_\diamond}s|^2 &\lesssim \int_{B_{2r}\setminus B_{\varepsilon/2}}\chi_r(1-\chi_{\varepsilon/2})G(1-\Delta|s|^2)\\ &\lesssim \int_{B_{2r}\setminus B_r}|z|^{1-2n}\partial_r|s|^2+r^2+\varepsilon^{1-2n}\int_{B_\varepsilon\setminus B_{\varepsilon/2}}\partial_r|s|^2\\ &\lesssim \left(\int_{B_{2r}\setminus B_r}|z|^{2-2n}|\nabla_{H_\diamond}s|^2\right)^{1/2}+r^2\\ &\quad+\left(\int_{B_\varepsilon\setminus B_{\varepsilon/2}}|z|^{2-2n}|\nabla_{H_\diamond}s|^2\right)^{1/2}.\end{aligned}$$

By Lebesgue's monotone convergence theorem, the last term vanishes as ε tends to zero; hence,

$$g(r) \lesssim (g(2r) - g(r))^{1/2} + r^2.$$

The asserted inequality now follows from Proposition 4.4. □

Proposition 4.4. *If* $g\colon [0,1] \to [0,\infty)$ *is monotone increasing and satisfies*

$$g(r) \le A(g(2r) - g(r))^{1/2} + Br^2,$$

then there are constants $c > 0$ *and* $r_0 \in (0,1]$, *depending on* A, B *and* $g(1)$, *such that*

$$g(r) \lesssim c(-\log r)^{-1}$$

for $r \in (0, r_0]$.

Proof. For $r \in (0, r_0]$ the function $h(r) \coloneqq g(r) + B/Ar^2$ satisfies

$$h(r)^2 \le 2A(h(2r) - h(r));$$

hence,

$$h(r) \le \frac{1}{1+\varepsilon h(r)} h(2r)$$

with $\varepsilon = 1/2A$. We can assume that $\varepsilon h(1) \le 1/2$. Using $(1+x)^{-1} \le 1-x$ for $x \ge 0$, and $(1-x)^k \le 1 - \frac{k}{2}x$ for $x \in [0,1/2]$, we derive

$$0 \le h(2^{-k}) \le \left(1 - \frac{k\varepsilon}{2} h(2^{-k})\right) h(1);$$

hence,

$$h(2^{-k}) \le \frac{2}{\varepsilon k}.$$

□

5. Proof of Theorem 2.5

For $r > 0$, define $m_r\colon \mathbf{C}^n \to \mathbf{C}^n$ by $m_r(z) \coloneqq rz$. Set

$$s_r \coloneqq m_r^*(s|_{B_{4r}\setminus B_{r/2}}) \in C^\infty(B_4 \setminus B_{1/2}, i\mathfrak{su}(E, H_*)) \quad \text{and} \quad H_{\diamond,r} \coloneqq m_r^* H_\diamond.$$

The metric $H_{\diamond,r}e^{s_r}$ is **P**HYM with respect to $\omega_r \coloneqq r^{-2} m_r^*\omega$ and $\|\mathrm{F}_{H_\diamond,r}\|_{C^k(B_4\setminus B_{1/2})} \le c_k$.

Proposition 3.1, (2.3) and interior estimates for **P**HYM metrics [7, Theorem C.1] imply that

$$\|s_r\|_{C^k(B_3\setminus B_{3/4})} \leq c_k.$$

By Proposition 4.1, we have

$$\|\nabla_{H_\diamond,r} s_r\|_{L^2(B_3\setminus B_{3/4})} \leq c_k(-\log r)^{-1/2}.$$

Schematically, $\mathrm{K}_{H_\diamond,r e^{s_r}} = 0$ can be written as

$$\nabla^*_{H_\diamond,r}\nabla_{H_\diamond,r} s_r + B(\nabla_{H_\diamond,r} s \otimes \nabla_{H_\diamond,r} s_r) = C(\mathrm{K}_{H_\diamond,r}),$$

where B and C are linear with coefficients depending on s, but not on its derivatives; see, e.g., [7, Proposition A.1]. Since $\|\mathrm{K}_{H_\diamond,r}\|_{C^k(B_3\setminus B_{3/4})} \leq c_k r^2$, as in [7, Step 3 in the proof of Proposition 5.1], standard interior estimates imply that

$$\|\nabla^k_{H_\diamond,r} s_r\|_{L^\infty(B_2\setminus B_1)} \leq c_k(-\log r)^{-1/2}$$

and, hence, the asserted inequalities, for each $k \geq 1$. (The asserted inequality for $k = 0$ has already be proven in Proposition 3.1.)

If (2.4) holds, then by Proposition 4.2 we have

$$\|\nabla_{H_\diamond,r} s_r\|_{L^2(B_4\setminus B_{1/2})} \lesssim r^\alpha \quad \text{and} \quad \|s_r\|_{L^2(B_4\setminus B_{1/2})} \lesssim r^\alpha;$$

hence, using standard interior estimates

$$\|\nabla^k_{H_\diamond,r} s_r\|_{L^2(B_2\setminus B_1)} \lesssim r^\alpha \quad \text{for each } k \geq 0.$$

This concludes the proof of Theorem 2.5. □

6. Proof of Proposition 1.4

We will make use of the following general fact about connections over manifolds with free S^1–actions.

Proposition 6.1. *Let M be a manifold with a free S^1–action. Denote the associated Killing field by $\xi \in \mathrm{Vect}(M)$ and let $q\colon M \to M/S^1$ be the canonical projection. Suppose $\theta \in \Omega^1(M)$ is such that $\theta(\xi) = 1$ and $\mathscr{L}_\xi\theta = 0$. Let A be a unitary connection on a Hermitian vector bundle (E, H) over M. If*

$i(\xi)\mathrm{F}_A = 0$, then there is a $k \in \mathbf{N}$ and, for each $j \in \{1, \dots, k\}$, a Hermitian vector bundles (F_j, K_j) over M/S^1 such that

$$E = \bigoplus_{j=1}^{k} E_j \quad \textit{and} \quad H = \bigoplus_{j=1}^{k} H_j$$

with $E_j \coloneqq q^*F_j$ and $H_j \coloneqq q^*K_j$; moreover, the bundles E_j are parallel and, for each $j \in \{1, \dots, k\}$, there are a unitary connection B_j on F_j and $\mu_j \in \mathbf{R}$ such that

$$A = \bigoplus_{j=1}^{k} q^*B_j + i\mu_j \,\mathrm{id}_{E_j} \cdot \theta.$$

Proof. Denote by $\tilde{\xi} \in \mathrm{Vect}(\mathrm{U}(E))$ the A–horizontal lift of ξ. This vector field integrates to an $\mathbf{R}$–action on $\mathrm{U}(E)$. Thinking of A as an $\mathfrak{u}(r)$–valued 1–form on $\mathrm{U}(E)$ and F_A as an $\mathfrak{u}(r)$–valued 2–form on $\mathrm{U}(E)$, we have

$$\mathscr{L}_{\tilde{\xi}} A = i(\tilde{\xi})\mathrm{F}_A = 0;$$

hence, A is invariant with respect to the $\mathbf{R}$–action on $\mathrm{U}(E)$.

The obstruction to the $\mathbf{R}$–action on $U(E)$ inducing an S^1–action is the action of $1 \in \mathbf{R}$ and corresponds to a gauge transformation $\mathbf{g}_A \in \mathscr{G}(\mathrm{U}(E))$ fixing A. If this obstruction vanishes, i.e., $\mathbf{g}_A = \mathrm{id}_{\mathrm{U}(E)}$, then $E \cong q^*F$ with $F = E/S^1$ and there is a connection A_0 on F such that $A = q^*A_0$.

If the obstruction does not vanish, we can decompose E into pairwise orthogonal parallel subbundles E_j such that $\mathbf{g}_A$ acts on E_j as multiplication with $e^{i\mu_j}$ for some $\mu_j \in \mathbf{R}$. Set $\tilde{A} \coloneqq A - \bigoplus_{j=1}^{k} i\mu_j \,\mathrm{id}_{E_j} \cdot \theta$. This connection also satisfies $i(\tilde{\xi})\mathrm{F}_{\tilde{A}} = 0 \in \Omega^1(M, \mathfrak{g}_E)$ and the subbundles E_j are also parallel with respect to E_j. Since $\mathbf{g}_{\tilde{A}} = \mathrm{id}_E$, the assertion follows. □

In the situation of Proposition 1.4, with $\xi \in S^{2n-1}$ denoting the Killing field for the S^1–action we have $i(\xi)\mathrm{F}_{A_0} = 0$; c.f., Tian [10, discussion after Conjecture 2]. Therefore, we can write

$$A_* = \bigoplus_{j=1}^{k} \sigma^*B_j + i\mu_j \,\mathrm{id}_{E_j} \cdot \pi^*\theta.$$

Since $\mathrm{d}\theta = 2\pi\rho^*\omega_{FS}$, we have

$$\mathrm{F}_{A_*} = \bigoplus_{j=1}^{k} \sigma^*\mathrm{F}_{B_j} + 2\pi i\mu_j \,\mathrm{id}_{E_j} \cdot \sigma^*\omega_{FS}.$$

Using (2.1), A_* being HYM with respect to ω_0 can be seen to be equivalent to

$$\mathrm{F}_{B_j}^{0,2} = 0 \quad \text{and} \quad i\Lambda \mathrm{F}_{B_j} = (2n-2)\pi\mu_j \cdot \mathrm{id}_{E_j}.$$

The isomorphism $\mathscr{E} = (E, \bar{\partial}_{A_*}) \cong \bigoplus_{j=1}^k \rho^* \mathscr{F}_j$ with $\mathscr{F}_j = (F_j, \bar{\partial}_{B_j})$ is given by g^{-1} with $g \coloneqq \bigoplus_{j=1}^k r^{\mu_j}$. □

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