

Symmetries and regularity for holomorphic maps between balls

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Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a holomorphic map. We study subgroups $\Gamma_f \subseteq \text{Aut}(\mathbb{B}^n)$ and $T_f \subseteq \text{Aut}(\mathbb{B}^N)$. When f is proper, we show both these groups are Lie subgroups. When Γ_f contains the center of $\mathbf{U}(n)$, we show that f is spherically equivalent to a polynomial. When f is minimal we show that there is a homomorphism $\Phi : \Gamma_f \rightarrow T_f$ such that f is equivariant with respect to Φ . To do so, we characterize minimality via the triviality of a third group H_f . We relate properties of $\text{Ker}(\Phi)$ to older results on invariant proper maps between balls. When f is proper but completely non-rational, we show that either both Γ_f and T_f are finite or both are noncompact.

1. Introduction

Let \mathbb{B}^n denote the unit ball in complex Euclidean space \mathbb{C}^n . We consider bounded holomorphic maps $f : \mathbb{B}^n \rightarrow \mathbb{C}^N$; after division by a constant we assume that the image of the ball under f lies in the unit ball \mathbb{B}^N . Following [DX] we study various groups associated with such an f . Our set-up and many of the results require neither that the map be proper nor that it be rational. Under these assumptions, however, we obtain additional information.

The holomorphic automorphism group $\text{Aut}(\mathbb{B}^n)$ is transitive. It is a Lie group that can be regarded as the quotient of $\mathbf{SU}(n, 1)$ by its center or as a collection of linear fractional transformations. Holomorphic maps $f, g : \mathbb{B}^n \rightarrow \mathbb{B}^N$ are **spherically equivalent** if there are automorphisms γ and ψ such that $\psi \circ g = f \circ \gamma$.

In [DX] the authors associated a subgroup A_f of $\text{Aut}(\mathbb{B}^n) \times \text{Aut}(\mathbb{B}^N)$ to a proper map f . This group A_f is defined to be those pairs (γ, ψ) for which $f \circ \gamma = \psi \circ f$. Note that this definition makes sense whenever the image of f is contained in \mathbb{B}^N . One of our key regularity results holds in this more general setting.

Most of the results in [DX] involve properties of Γ_f , the projection of A_f onto its first factor. In this paper we develop further uses of this group, including results that make no regularity assumptions about f . In Proposition 2.3 we show that Γ_f is a Lie subgroup of $\text{Aut}(\mathbb{B}^n)$, without assuming f is rational. We also use properties of T_f , the projection onto the second factor. Here properness is used to show that T_f is a (closed) Lie subgroup.

A map $f : \mathbb{B}^n \rightarrow \mathbb{C}^N$ is called **minimal** if its image lies in no lower dimensional affine linear subspace. See the survey article [HJY]. Let H_f denote the subgroup of T_f consisting of those target automorphisms ψ for which $\psi \circ f = f$. In Proposition 3.1 we show that H_f is trivial if and only if f is minimal. When $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ is minimal, we prove (Theorem 3.1) that there is a group homomorphism $\Phi : \Gamma_f \rightarrow T_f$, and hence f is equivariant with respect to Φ . When $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ is also proper, the kernel of Φ depends on the boundary regularity of f . The kernel of Φ coincides with the group G_f , as defined in [DX]. Thus G_f is the subgroup of Γ_f consisting of those source automorphisms γ for which $f \circ \gamma = f$.

We next summarize the results in the paper. We start with the following result.

Corollary 1.1. *Suppose $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ is holomorphic. If Γ_f contains the center of $\mathbf{U}(n)$, then f is spherically equivalent to a polynomial.*

This corollary follows from the next theorem, where we give a general condition implying that a holomorphic map f is a polynomial. Since Γ_f is a Lie group, $\Gamma_f \cap \mathbf{U}(n)$ also is. The Lie algebra \mathfrak{g} of $\Gamma_f \cap \mathbf{U}(n)$ consists of skew-Hermitian matrices M . For $M \in \mathfrak{g}$ put $L = -iM$. Then L has real eigenvalues. We prove Theorem 1.1 in Section 4 and obtain the consequences in Corollary 1.2.

Theorem 1.1. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be holomorphic with $f(0) = 0$. If the Lie algebra \mathfrak{g} of $\Gamma_f \cap \mathbf{U}(n)$ contains a matrix M such that $L = -iM$ has k positive eigenvalues, then there is a k -dimensional linear subspace V such that the restriction of f to V is a polynomial.*

Corollary 1.2. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a proper holomorphic mapping.*

- *Assume Γ_f contains the maximal n -torus in $\text{Aut}(\mathbb{B}^n)$. Then f is spherically equivalent to a monomial map.*
- *Assume Γ_f contains $\mathbf{U}(n)$. Then f is spherically equivalent to an orthogonal sum of tensor products.*

- Suppose Γ_f contains the center of $\mathbf{U}(n)$ and is noncompact. Then f is a linear fractional transformation.

These statements appear in [DX] when f is assumed to be rational. Recall from [F1], when $n \geq 2$, that a proper holomorphic map between balls, with sufficient boundary regularity, must be rational. Here we show that Hermitian invariance under a large group forces rationality. In particular, if the group Γ_f contains the maximal n -torus in $\text{Aut}(\mathbb{B}^n)$, then f is rational. Once we know that the map is rational, the results from [DX] yield the additional information about spherical equivalence in this corollary. At the end of this paper we mention a rigidity problem discussed in [S] and [CM] and related to the third conclusion of the corollary.

The projections of A_f onto each of the factors will be significant in this paper. It is natural to ask when there is a homomorphism from Γ_f to T_f . We answer this question (Theorem 3.1) as follows.

Theorem 1.2. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a minimal proper map. Then there is a homomorphism $\Phi : \Gamma_f \rightarrow T_f \subseteq \text{Aut}(\mathbb{B}^N)$. Thus f is equivariant with respect to Φ .*

Proposition 3.1 shows that f is minimal if and only if H_f is trivial. Another name for minimal is *target essential*. The related notion of *source essential* appears in [DX] and is expressed there in terms of Γ_f .

Section 3 also gathers some older known results (see for example [D, F2, Li1, Li2]) about the groups G_f and places them in the context of this paper. As noted above, for minimal proper maps, G_f is the kernel of Φ . The possible invariant groups G_f for proper maps depend upon the regularity assumptions. As has been long known, the invariant group for a rational proper map must be cyclic ([Li2]) and the list of possible representations is short ([D]). On the other hand, every finite group arises when we drop the assumption of rationality.

Section 4 provides the proof of Theorem 4.1, the first theorem stated above.

Section 5 considers **completely non-rational** proper maps f ; that is, for each positive-dimensional affine linear subspace V of \mathbb{C}^n , the restriction of f to V is not rational. We prove the following result.

Theorem 1.3. *Let f be a minimal completely non-rational proper mapping between balls. Then Γ_f and T_f are either both finite or both noncompact.*

2. preliminaries

The inner product of z, w in complex Euclidean space is denoted by $\langle z, w \rangle$ and the Euclidean squared norm by $\|z\|^2$. We use the same notation in each dimension. Let \mathbb{B}^n denote the unit ball in \mathbb{C}^n and let $\mathbf{U}(n)$ denote the unitary group. As noted in the first paragraph of the introduction, f is not assumed to be proper in the following crucial definitions.

Definition 2.1. Suppose $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ is holomorphic. Then A_f is the subgroup of $\text{Aut}(\mathbb{B}^n) \times \text{Aut}(\mathbb{B}^N)$ consisting of those pairs (γ, ψ) for which

$$f \circ \gamma = \psi \circ f.$$

Definition 2.2. In the setting of Definition 2.1, we consider the following groups:

- Γ_f is the projection of A_f on the first factor.
- T_f is the projection of A_f on the second factor.
- $G_f = \{\gamma \in \text{Aut}(\mathbb{B}^n) : f \circ \gamma = f\}$.
- $H_f = \{\psi \in \text{Aut}(\mathbb{B}^N) : \psi \circ f = f\}$.

We first observe the following simple facts about spherical equivalence.

Proposition 2.1. Let $f, g : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be spherically equivalent. Put $g = \alpha \circ f \circ \beta$. Then $H_g = \alpha \circ H_f \circ \alpha^{-1}$.

Proof. Suppose $\psi_f \circ f = f$. Then

$$\alpha \circ \psi_f \circ \alpha^{-1} \circ g = \alpha \circ \psi_f \circ \alpha^{-1} \circ \alpha \circ f \circ \beta = \alpha \circ \psi_f \circ f \circ \beta = \alpha \circ f \circ \beta = g.$$

□

Proposition 2.2. Let $f, g : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be spherically equivalent. Put $g = \alpha \circ f \circ \beta$. Then

$$A_g = \{(\beta^{-1} \circ \gamma \circ \beta, \alpha \circ \psi \circ \alpha^{-1}) : (\gamma, \psi) \in A_f\}.$$

Proof. The proof is a formal calculation similar to the previous proof. □

Proposition 2.3. Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a proper holomorphic map. Then

- Γ_f is a Lie subgroup of $\text{Aut}(\mathbb{B}^n)$.
- T_f is a Lie subgroup of $\text{Aut}(\mathbb{B}^N)$.
- Γ_f is noncompact if and only if T_f is noncompact.

Proof. Both Γ_f and T_f are subgroups; we must therefore show that they are closed. That Γ_f is closed relies only on f being continuous; that T_f is closed relies on f being continuous and proper.

We first note that $\gamma_\nu \rightarrow \gamma$ in $\text{Aut}(\mathbb{B}^k)$ if $\gamma_\nu(z) \rightarrow \gamma(z)$ for each $z \in \mathbb{B}^k$. Here k can be n or N . Recall in each case that an automorphism has the form $L\phi_c$ where $L \in \mathbf{U}(k)$ and ϕ_c is a linear fractional transformation with $\phi_c(0) = c$.

After composing with an automorphism we assume $f(0) = 0$. Consider a sequence $\gamma_\nu \in \Gamma_f$ converging to γ . By definition there is a sequence $\psi_\nu \in \text{Aut}(\mathbb{B}^N)$ such that

$$f \circ \gamma_\nu = \psi_\nu \circ f.$$

Write $\gamma_\nu = U_\nu \phi_{a_\nu}$ and $\psi_\nu = V_\nu \phi_{b_\nu}$ as above. Evaluating at 0 gives

$$(1) \quad f(U_\nu a_\nu) = V_\nu b_\nu.$$

If γ_ν converges to $\gamma \in \text{Aut}(\mathbb{B}^n)$, then the sequence $\{a_\nu\}$ lies in a compact subset of \mathbb{B}^n . Since U_ν is unitary and f is continuous, the left-hand side of (1) lies in a compact subset of \mathbb{B}^N . Therefore, since V_ν is unitary, the sequence $\{b_\nu\}$ lies in a compact subset of \mathbb{B}^N as well.

Next consider a sequence $\psi_\nu \in T_f$ converging in $\text{Aut}(\mathbb{B}^N)$. Using the above notation, now $\{b_\nu\}$, and hence $V_\nu b_\nu$, lies in a compact subset of the target ball \mathbb{B}^N . Since f is continuous and proper, the inverse image of a compact set is compact. Hence (1) implies that $\{U_\nu a_\nu\}$, and hence $\{a_\nu\}$, lies in a compact subset of \mathbb{B}^n .

Thus, if $\gamma_\nu \in \Gamma_f$ and γ_ν converges to γ , or if $\psi_\nu \in T_f$ and ψ_ν converges to ψ , then the sequences $\{a_\nu\}$ or $\{b_\nu\}$ are bounded and hence have convergent subsequences. The unitary groups are compact, and hence the U_ν and the V_ν also have convergent subsequences. Thus there is a subsequence of source automorphisms converging to a source automorphism γ and a corresponding target automorphism ψ with $f \circ \gamma = \psi \circ f$. Thus both Γ_f and T_f are closed subgroups of their respective automorphism groups. Hence each is a Lie subgroup.

The third statement has a similar proof. If Γ_f contains a sequence of automorphisms $U_\nu \phi_{a_\nu}$ where $\{a_\nu\}$ tends to the boundary sphere in the source,

then the corresponding sequence $V_\nu \varphi_{b_\nu}$, by properness, must have $\{b_\nu\}$ tending to the sphere in the target, and conversely. \square

An automorphism of the ball preserves the origin if and only if it is unitary. This fact leads to the following lemma.

Lemma 2.1. *Suppose $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ is holomorphic and $f(0) = 0$. Let $\gamma \in \mathbf{U}(n)$. Then $\gamma \in \Gamma_f$ if and only if $\|f \circ \gamma\|^2 = \|f\|^2$.*

Proof. Suppose first that $\|f \circ \gamma\|^2 = \|f\|^2$. Then, by a well-known result from [D], there is a $U \in \mathbf{U}(N)$ such that $f \circ \gamma = U \circ f$. Thus $\gamma \in \Gamma_f$. Conversely, suppose $\gamma \in \Gamma_f$. Then $f \circ \gamma = \psi_\gamma \circ f$ for some target automorphism ψ_γ . Evaluating at 0 and using $f(0) = 0$ gives $\psi_\gamma(0) = 0$. Hence ψ_γ is unitary and

$$\|f \circ \gamma\|^2 = \|\psi_\gamma \circ f\|^2 = \|f\|^2.$$

\square

3. Equivariance

The primary purpose of this section is to prove the following theorem.

Theorem 3.1. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a minimal holomorphic map. Then there is a homomorphism $\Phi : \Gamma_f \rightarrow T_f \subseteq \text{Aut}(\mathbb{B}^N)$. Thus f is equivariant with respect to Φ .*

We also have the following characterization of minimality.

Proposition 3.1. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be holomorphic. Then H_f is the trivial group if and only if f is minimal.*

Proof. Lemma 3.1 below states that f minimal implies H_f trivial. Lemma 3.2 shows that f not minimal implies H_f not trivial. \square

Lemma 3.1. *Let f be a minimal map from \mathbb{B}^n to \mathbb{B}^N . Then H_f is the trivial group consisting of the identity map \mathbf{I}_N .*

Proof. By Proposition 2.1, the statement is true for f if and only if it is true for $\psi \circ f \circ \gamma$ with $\gamma \in \text{Aut}(\mathbb{B}^n)$ and $\psi \in \text{Aut}(\mathbb{B}^N)$. Thus, without loss of generality, we can compose with a target automorphism to make $f(0) = 0$.

Consider the span:

$$\text{Span}_{\mathbb{C}}(f) = \left\{ \sum_{i=1}^s \lambda_i f(z_i) : z_i \in \mathbb{B}^n \text{ and } \lambda_i \in \mathbb{C} \right\}.$$

Since f is minimal, $\text{Span}_{\mathbb{C}}\{f\} = \mathbb{C}^N$. Let $\psi \in H_f$. Then $\psi \circ f = f$. We put $z = 0$ to conclude $\psi(0) = 0$. Thus $\psi \in \mathbf{U}(N)$. Since ψ is linear and preserves every $f(z)$, it must preserve every element in $\text{Span}_{\mathbb{C}}(f) = \mathbb{C}^N$. Thus ψ is the identity map. \square

Lemma 3.2. *Let f be a minimal map from \mathbb{B}^n to \mathbb{B}^m . Let g be a holomorphic map from \mathbb{B}^n to \mathbb{B}^m that is spherically equivalent to $0 \oplus f$. Then H_g is conjugate to $\mathbf{U}(k) \oplus \mathbf{I}_m$.*

Proof. It suffices to assume $g = 0 \oplus f$. The general case follows from this special case and Proposition 2.1. As usual, we can assume $f(0) = 0$ and $g(0) = 0$. Since f is minimal, we have $\text{Span}_{\mathbb{C}}\{g\} = 0 \oplus \mathbb{C}^m$. Let $\psi \in H_f$. Then $\psi \circ f = f$. Again we let $z = 0$ to get $\psi(0) = 0$. Thus ψ is unitary. As above, since ψ is linear and preserves every $g(z)$, it preserves $\text{Span}_{\mathbb{C}}\{g\} = 0 \oplus \mathbb{C}^m$. Thus $\psi \in \mathbf{U}(k) \oplus \mathbf{I}_m$. Hence $H_g \subseteq \mathbf{U}(k) \oplus \mathbf{I}_m$. The definition of g yields the opposite inclusion. We conclude $H_g = \mathbf{U}(k) \oplus \mathbf{I}_m$. \square

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We use minimality to show, for each $\gamma \in \Gamma_f$, that there is a unique $\psi \in T_f$ such that $(\gamma, \psi) \in A_f$. Assume $(\gamma, \psi_1), (\gamma, \psi_2) \in A_f$. Then

$$(\gamma \circ \gamma^{-1}, \psi_1 \circ \psi_2^{-1}) = (\mathbf{I}_n, \psi_1 \circ \psi_2^{-1}) \in A_f.$$

Thus $\psi_1 \circ \psi_2^{-1} \in H_f$. Proposition 3.1 implies that H_f is trivial and hence $\psi_1 = \psi_2$.

By the uniqueness of ψ , there is a well-defined map $\Phi : \Gamma_f \rightarrow \text{Aut}(\mathbb{B}^N)$ with $\Phi(\gamma) = \psi$ where $(\gamma, \psi) \in A_f$. We next verify that Φ is a homomorphism. Assume $\Phi(\gamma_1) = \psi_1$ and $\Phi(\gamma_2) = \psi_2$. By the definition of Φ , we have $\psi_1 \circ f = f \circ \gamma_1$ and $\psi_2 \circ f = f \circ \gamma_2$. Consequently,

$$\psi_2 \circ \psi_1 \circ f = \psi_2 \circ f \circ \gamma_1 = f \circ \gamma_2 \circ \gamma_1.$$

We conclude that $\Phi(\gamma_2 \circ \gamma_1) = \psi_2 \circ \psi_1 = \Phi(\gamma_2) \circ \Phi(\gamma_1)$. Hence Φ is a homomorphism. The definition of equivariance now yields the equivariance of f with respect to Φ , completing the proof. \square

Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a minimal map. By Theorem 3.1, f induces a homomorphism $\Phi : \Gamma_f \rightarrow \text{Aut}(\mathbb{B}^N)$. We say Φ *represents* $\Gamma = \Gamma_f$ in $\text{Aut}(\mathbb{B}^N)$.

We discuss the kernel of the induced homomorphism Φ when f is proper. The behavior of Φ then depends on the boundary regularity of f . Propositions 3.4 and 3.5 are known but are expressed in different language in the literature. The proof of Proposition 3.2 uses the following simple fact.

Remark 3.1. A proper holomorphic map is *finite*: the inverse image of a point is a finite set.

Proposition 3.2. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a minimal proper map. Let Φ be the induced homomorphism. Then $\text{Ker}(\Phi)$ a finite subgroup of $\text{Aut}(\mathbb{B}^N)$.*

Proof. That $\text{Ker}(\Phi)$ is a subgroup of $\text{Aut}(\mathbb{B}^N)$ is clear. First we show that $\text{Ker}(\Phi)$ is closed in $\text{Aut}(\mathbb{B}^N)$. Assume $\gamma_\nu \in \text{Ker}(\Phi)$ and γ_ν converges to $\gamma \in \text{Aut}(\mathbb{B}^N)$. By continuity, we conclude that $\gamma \in \text{Ker}(\Phi)$.

Next we claim $\text{Ker}(\Phi)$ is compact. If not, then $\text{Ker}(\Phi)$ moves some point a in \mathbb{B}^n arbitrarily close to the boundary. By Remark 3.1, $f^{-1}(f(a))$ is a finite set, yielding a contradiction.

Thus $\text{Ker}(\Phi)$ is a compact Lie subgroup of $\text{Aut}(\mathbb{B}^N)$. By standard Lie group theory (cf. [HT]), it is contained in a conjugate of $\mathbf{U}(n)$.

We finally claim that $\text{Ker}(\Phi)$ is finite. It is contained in a conjugate of $\mathbf{U}(n)$. If it were infinite, then there would exist $a \in \mathbb{B}^n$ such that $\{\gamma(a) : \gamma \in \text{Ker}(\Phi)\}$ is infinite. This conclusion contradicts f being a finite map. \square

Proposition 3.3. *Let G be a finite subgroup of $\text{Aut}(\mathbb{B}^N)$. Then there exists an N and a holomorphic proper map $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ such that $\text{Ker}(\Phi) = G$.*

Proof. We first assume G is a subgroup of $\mathbf{U}(n)$. In this case, the statement was proved in [Li1] (See Theorem 4.3.4 there). Next we assume G is an arbitrary finite subgroup of $\text{Aut}(\mathbb{B}^N)$. By Lie group theory, G is contained in a conjugate of $\mathbf{U}(n)$. For some $\chi \in \text{Aut}(\mathbb{B}^N)$, we thus have $G_0 = \chi \circ G \circ \chi^{-1} \subseteq \mathbf{U}(n)$. By the result above for $\mathbf{U}(n)$, there is a holomorphic proper map g for which $\text{Ker}(\Phi_g) = G_0$. Then $\text{Ker}(\Phi_f) = G$ if $f = g \circ \chi$. \square

We recall the notion of fixed-point-free subgroup of $\mathbf{U}(n)$. A finite subgroup G of $\mathbf{U}(n)$ is *fixed-point-free* if the only element in G with an eigenvalue of 1 is the identity. Thus the origin is the only fixed point in \mathbf{C}^n under the action of G . Equivalently, G is fixed point free if each element of the group other than the identity has no fixed points on the unit sphere. The next

result appears in [Lil]. In the language of this paper, it shows that a boundary regularity assumption on f puts restrictions on $\text{Ker}(\Phi)$. We sketch the proof.

Proposition 3.4. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a holomorphic proper map that is continuously differentiable on the closed ball. Then $\text{Ker}(\Phi)$ is conjugate to a fixed-point-free finite subgroup of $\mathbf{U}(n)$.*

Proof. After composing f with a source automorphism, we can assume that $\text{Ker}(\Phi)$ is contained in $\mathbf{U}(n)$. Suppose $\gamma \in \text{Ker}(\Phi)$ and that γ is not the identity. We show that 1 cannot be an eigenvalue of γ .

Since f is C^1 -smooth to the boundary, by a classical Hopf lemma argument, f is of full rank at each boundary point and thus is a local embedding there. Assume for some $\gamma \in \text{Ker}(\Phi)$ that 1 is an eigenvalue of γ . Let E be the corresponding eigenspace. If γ is not the identity, then E is a proper subspace. Then γ maps any point q close to but not on E to a different point nearby. Since $f \circ \gamma = f$, the map f cannot be injective near E . But E intersects the sphere, contradicting f being a local embedding there. \square

When f is C^∞ -smooth up to the boundary, $\text{Ker}(\Phi)$ is quite restricted. The following result gives a complete list of the possible groups that can arise. See [D] for a proof and for the corresponding maps. See also [G] for related results when the target is a generalized ball.

Proposition 3.5. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a rational holomorphic proper map. Then $\text{Ker}(\Phi)$ is cyclic. Furthermore $\text{Ker}(\Phi)$ is conjugate to one of the following:*

- G is the cyclic group generated by $\eta \mathbf{I}_n$, where η is a primitive m -th root of unity. (Each m is possible.)
- G is the cyclic group generated by $\eta \mathbf{I}_j \oplus \eta^2 \mathbf{I}_k$, where η is a primitive odd root of unity. Here $j + k = n$. (Each odd natural number is possible.)
- G is the cyclic group of order 7 generated by $\eta \mathbf{I}_j \oplus \eta^2 \mathbf{I}_k \oplus \eta^4 \mathbf{I}_l$. Here $j + k + l = n$.

Remark 3.2. The map $z \mapsto z^{\otimes m}$ provides an example of the first type. When $n = 2$, maps of the second type have the largest possible degree given the target dimension. See [DKR]. The simplest example of the third type maps \mathbb{B}_3 to \mathbb{B}_{17} and appears in [D].

When $f : \mathbb{B}^n \rightarrow \mathbb{B}^m$, one can artificially increase the target dimension by considering the map $0 \oplus f$. This map is not minimal. Lemma 3.2 shows how this construction impacts the group H_f . We close this section with a simple result showing that this construction does not change Γ_f .

Proposition 3.6. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^m$ be minimal. For $N > m$, put $g = 0 \oplus f$. Then $\Gamma_f = \Gamma_g$.*

Proof. Put $k = N - m$. We may assume that $f(0) = 0$. If $(\gamma, \psi) \in A_f$, then

$$(2) \quad 0 \oplus (f \circ \gamma) = \psi(0 \oplus f).$$

Let $\phi \in \text{Aut}(\mathbb{B}^n)$ be such that $(\phi \circ f \circ \gamma)(0) = 0$. We may regard (\mathbf{I}_k, ϕ) as an element φ of $\text{Aut}(\mathbb{B}^N)$. Then $\varphi(0 \oplus (f \circ \gamma)) = 0 \oplus (\phi \circ f \circ \gamma)$. Hence φ maps 0 to 0. By (2),

$$(3) \quad 0 \oplus (\phi \circ f \circ \gamma) = (\varphi \circ \psi)(0 \oplus f) = (\varphi \circ \psi)(g).$$

Evaluating at 0 shows that the automorphism $\varphi \circ \psi$ maps 0 to 0 and hence is unitary. Since f is minimal, $\text{Span}_{\mathbb{C}}\{g\} = 0 \oplus \mathbb{C}^m$. As in the proof of Lemma 3.2, we conclude that $\varphi \circ \psi$ lies in $\mathbf{U}(k) \oplus \mathbf{I}_m$. Now (3) implies $\phi \circ f \circ \gamma = f$. Thus $\gamma \in \Gamma_f$ if and only if $\gamma \in \Gamma_g$. \square

4. Groups and regularity

We develop the tools to prove Theorem 4.1. The next lemma illustrates how group invariance allows one to prove that certain power series are in fact polynomials. We establish Theorem 4.1 by generalizing the proof of this lemma.

Lemma 4.1. *Let Ω be an open ball about $0 \in \mathbb{C}^n$ and suppose that $H : \Omega \rightarrow \mathbb{C}^N$ is holomorphic. Assume for each $\gamma \in \mathbf{U}(1) \oplus \cdots \oplus \mathbf{U}(1)$ that $\|H \circ \gamma\|^2 = \|H\|^2$. Then there is a monomial map G and a unitary U such that $H = U \circ G$.*

Proof. Since H is holomorphic and Ω is a ball about 0, we can expand H in a convergent power series about 0. Put

$$H(z) = \sum_{\alpha} C_{\alpha} z^{\alpha}.$$

Then we have

$$\|H(z)\|^2 = \sum_{\alpha, \beta} \langle C_\alpha, C_\beta \rangle z^\alpha \bar{z}^\beta.$$

Let γ be a diagonal unitary matrix with eigenvalues $e^{i\theta_j}$. Then the assumption $\|H \circ \gamma\|^2 = \|H\|^2$ implies

$$(4) \quad \sum_{\alpha, \beta} \langle C_\alpha, C_\beta \rangle z^\alpha \bar{z}^\beta = \sum_{\alpha, \beta} \langle C_\alpha, C_\beta \rangle z^\alpha \bar{z}^\beta e^{i\theta(\alpha-\beta)}.$$

By assumption, equation (4) holds for all choices of the θ_j . Equating Taylor coefficients then shows, whenever $\alpha \neq \beta$, that $\langle C_\alpha, C_\beta \rangle = 0$. Since each coefficient C_α lies in the finite-dimensional space \mathbb{C}^N , they are linearly dependent and orthogonal. Hence there are only finitely many non-zero coefficient vectors. Thus H is a polynomial. Since the coefficient vectors are orthogonal, there is a unitary map U such that $H = U \circ G$, where G is the monomial map whose components are the monomials arising in H . \square

The idea of using circular symmetries to study power series appears already in [BM]. We next establish Theorem 4.1 from the introduction.

Definition 4.1. Let $\mathcal{C} = \mathcal{C}(n)$ denote the collection of continuously differentiable maps $\gamma : (-\pi, \pi) \rightarrow \mathbf{U}(1) \oplus \cdots \oplus \mathbf{U}(1)$ such that $\gamma(0) = \mathbf{I}_n$.

For $\gamma \in \mathcal{C}$, we note that $\gamma'(0) = iL$, where $i^2 = -1$ and $L = L^*$ is real and diagonal. (The Lie algebra of the unitary group is the skew-Hermitian matrices.) The eigenvalues of L are real. When they all have the same sign, we obtain a criterion guaranteeing that a holomorphic map is in fact a polynomial.

Proposition 4.1. *Let Ω be a ball about 0 in \mathbb{C}^n and suppose $f : \Omega \rightarrow \mathbb{C}^N$ is holomorphic. For some $\gamma \in \mathcal{C}$, assume that $\|f \circ \gamma\|^2 = \|f\|^2$. Put $L = -i\gamma'(0)$. If all the eigenvalues of L have the same sign, then f is a polynomial. More generally, if L has k eigenvalues of the same sign, then there is a k -dimensional vector subspace V such that the restriction of f to V is a polynomial.*

Proof. Put $\theta(t) = (\theta_1(t), \dots, \theta_n(t))$, where $\gamma(t)$ is diagonal with eigenvalues $e^{i\theta_j(t)}$. We expand f in a (vector-valued) power series convergent in Ω :

$$f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}.$$

We are given that $\|f(z)\|^2 = \|f(\gamma(t)z)\|^2$ for all $t \in (-\pi, \pi)$ and all $z \in \Omega$. Equating Taylor coefficients yields, for each pair α, β of multi-indices, that

$$\langle c_\alpha, c_\beta \rangle = e^{i\theta(t) \cdot (\alpha - \beta)} \langle c_\alpha, c_\beta \rangle$$

and hence we have

$$(5) \quad 0 = \langle c_\alpha, c_\beta \rangle \left(1 - e^{i\theta(t) \cdot (\alpha - \beta)}\right).$$

Write $\mathbf{m} = (m_1, \dots, m_n)$, where the m_j are the eigenvalues of L . Differentiate (5) and evaluate at $t = 0$ to obtain

$$(6) \quad 0 = \langle c_\alpha, c_\beta \rangle (\mathbf{m} \cdot (\alpha - \beta)).$$

We will show that $c_\alpha = 0$ for all but finitely many α .

Suppose first that the eigenvalues of L have the same sign. After replacing t by $-t$ we may assume they are all positive. Let $K_1 = \min(m_j)$ and $K_2 = \max(m_j)$. Thus $K_2 \geq K_1 > 0$. Let α and β be multi-indices for which $\mathbf{m} \cdot (\alpha - \beta) = 0$; then

$$(7) \quad K_1 |\alpha| = K_1 \sum \alpha_j \leq \sum m_j \alpha_j = \sum m_j \beta_j \leq K_2 \sum \beta_j = K_2 |\beta|.$$

Let W be the (finite-dimensional) span of the coefficients c_α . Choose $\alpha_1, \dots, \alpha_\nu$ such that the c_{α_j} span W . Choose a multi-index η with $|\eta| > \frac{K_2}{K_1} |\alpha_j|$ for $1 \leq j \leq \nu$. By (7), $\mathbf{m} \cdot (\eta - \alpha_j) \neq 0$ for all j . Since $\langle c_\alpha, c_\beta \rangle \mathbf{m} \cdot (\alpha - \beta) = 0$ for all α, β , we conclude that $\langle c_\eta, c_{\alpha_j} \rangle = 0$ for all j . Therefore $c_\eta = 0$ and hence there are only finitely many non-vanishing coefficient vectors. Thus f is a polynomial.

Next suppose that L has k eigenvalues of the same sign. After renumbering the coordinates and replacing t by $-t$ if necessary, we may assume that these are positive and correspond to the first k coordinates. Setting the remainder of the variables equal to 0 puts us in the situation above. The conclusion follows. \square

Remark 4.1. One can draw stronger conclusions. For example, when $n = 1$, equation (6) implies that the vectors c_α are mutually orthogonal and hence that f is an orthogonal sum of monomials.

Remark 4.2. When L has N_+ positive and N_- negative eigenvalues we can conclude (with obvious notation) that $f(z', 0)$ and $f(0, z'')$ are both polynomials. When L has eigenvalues of both signs, however, f need not be a polynomial.

Example 4.1. Let $\gamma(t)$ be the diagonal unitary matrix with eigenvalues e^{it} and e^{-it} . The eigenvalues of L are then ± 1 . Assume f is a function of the product $z_1 z_2$. Then $f(\gamma(t)z) = f(z)$ but f need not even be rational.

The following results are corollaries of Proposition 4.1.

Theorem 4.1. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be holomorphic with $f(0) = 0$. If the Lie algebra \mathfrak{g} of $\Gamma_f \cap \mathbf{U}(n)$ contains a matrix M such that $L = -iM$ has k positive eigenvalues, then there is a k -dimensional linear subspace V such that the restriction of f to V is a polynomial.*

Proof. Write $M = iPDP^*$ where P is unitary and D is diagonal with k positive eigenvalues. Let W be the linear subspace of \mathfrak{g} spanned over \mathbb{R} by M . Since W is a Lie subalgebra, there is a unique Lie subgroup $G \subseteq \Gamma_f \cap \mathbf{U}(n)$ whose Lie algebra is W . We regard G as the group of transformations of the form $Pe^{iDt}P^*$ for $t \in \mathbb{R}$. Put $h = f \circ P$. By Proposition 2.2, the curve $\gamma = e^{iDt}$ lies in Γ_h . We wish to apply Proposition 4.1. The condition on $\gamma'(0)$ holds by construction. Since $h(0) = 0$, Lemma 2.1 implies that $\|h \circ \gamma\|^2 = \|h\|^2$. Thus both conditions in Proposition 4.1 apply, and we conclude that the restriction of f to some k -dimensional subspace is a polynomial. \square

The corollaries from the introduction follow easily. First, if Γ_f contains the center of $\mathbf{U}(n)$, then the Lie algebra \mathfrak{g} of $\Gamma_f \cap \mathbf{U}(n)$ contains $i\mathbf{I}_n$. The subspace W in Theorem 4.1 is \mathbb{C}^n and hence f is equivalent to a polynomial. The statements in Corollary 1.2 follow as well. In each case the hypotheses imply the hypotheses of Theorem 4.1 with $k = n$. Hence f is rational. The results from [DX] then yield the conclusions about spherical equivalence.

Huang (See [Hu1]) established a linearity result for proper maps in low codimension without assuming rationality. Instead, the maps are assumed to have two continuous derivatives at the boundary sphere. By contrast, we make no regularity assumption; instead we assume that the Hermitian invariant group is large.

We note also the following result from [DX]. Each finite subgroup of $\text{Aut}(\mathbb{B}^n)$ is the Hermitian group of some rational proper map with source \mathbb{B}^n .

5. Completely non-rational proper maps

A proper holomorphic map $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ is **completely non-rational** if, for each positive-dimensional affine linear subspace V of \mathbb{C}^n , the restriction of f to V is not rational.

Example 5.1. Completely non-rational proper maps between balls exist. Begin with an arbitrary bounded holomorphic map g that is completely non-rational but continuous on the closed ball. After dividing by a constant, we may assume $1 - \|g\|^2 > 0$ on the unit sphere. By the work of Løw [Lw], there is a holomorphic map h such that $1 - \|g\|^2 = \|h\|^2$ on the sphere. Hence $g \oplus h$ maps the sphere to the sphere, and hence defines a proper map between balls. It is completely non-rational because g is. One subtlety here involves the regularity of h . In general one cannot conclude that h is any smoother than continuous.

We prove the following result.

Theorem 5.1. *Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a minimal completely non-rational proper mapping. Then Γ_f and T_f are either both finite or both noncompact.*

Proof. By Proposition 2.3, it suffices to prove the conclusion for Γ_f . Assume $f(0) = 0$. Let \mathfrak{g} be the Lie algebra of $\Gamma_f \cap \mathbf{U}(n)$. If $\Gamma_f \cap \mathbf{U}(n)$ is not finite, then its dimension as a Lie group is at least 1. Then \mathfrak{g} contains a non-zero skew-Hermitian matrix M . Put $L = -iM$. After possibly replacing M with $-M$ we may assume that L has a positive eigenvalue. By Theorem 4.1, f is rational on some linear subspace of dimension 1, contradicting the assumption that f is completely non-rational. Hence $\Gamma_f \cap \mathbf{U}(n)$ is finite.

Next let γ be an arbitrary element of $\text{Aut}(\mathbb{B}^n)$. Put $G = f \circ \gamma$. Proposition 2.2 implies that $\Gamma_g = \gamma^{-1} \circ f \circ \gamma$. Thus both $\Gamma_g \cap \mathbf{U}(n)$ and $\Gamma_f \cap \gamma \circ \mathbf{U}(n) \circ \gamma^{-1}$ are finite. By Proposition 2.3, Γ_f is a Lie subgroup. Hence it is either noncompact or contained in a maximal compact Lie subgroup of $\text{Aut}(\mathbb{B}^n)$. In the second case, for some automorphism ϕ , we have

$$(8) \quad \Gamma_f \subseteq \phi \circ \mathbf{U}(n) \circ \phi^{-1}.$$

By the previous argument, the intersection of the group on the right-hand side of (8) with Γ_f is Γ_f and must be finite. \square

We close the paper by mentioning a rigidity problem for holomorphic mappings between compact hyperbolic spaces raised by Siu in [S]. See also [Hu2] for a CR-geometric formulation. Cao and Mok ([CM]) established the following result: Let (X, g) be a compact n -dimensional complex hyperbolic space form and (Y, h) an m -dimensional hyperbolic space form with $m \leq 2n - 1$. Then a holomorphic immersion $f : X \rightarrow Y$ is necessarily a totally geodesic isometric immersion. The conclusion is not known for larger target dimensions. The hyperbolic space form X is the quotient of the unit ball by a lattice.

Siu's question has the following formulation in the language of this paper. Let $f : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a proper holomorphic map, and suppose Γ_f contains a (co-compact) lattice. Must f be a totally geodesic embedding in the Poincaré metric? The third part of Corollary 1.2 from the introduction draws this conclusion when the group Γ_f is non-compact and contains the center of $\mathbf{U}(n)$.

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