

# Divisibilities among nodal curves

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We prove that there are no effective or anti-effective classes of square  $-1$  or  $-2$  arising from nodal curves on smooth algebraic surfaces by way of divisibility. This general fact has interesting applications to Enriques and K3 surfaces. The proof relies on specific properties of root lattices and their duals.

## 1. Introduction

Nodal curves are among the most intriguing objects on algebraic surfaces. The terminology refers to smooth rational curves  $C$  with self-intersection  $C^2 = -2$ . By the adjunction formula, nodal curves can be contracted to rational double point singularities without affecting the dualizing sheaf; this offers one explanation for our interest in configurations of nodal curves.

Consider the classes of the nodal curves, or more generally  $(-2)$ -curves, in the Néron–Severi lattice

$$\text{Num}(S) = \text{Div}(S) / \cong$$

of divisors modulo numerical equivalence (this equals the Néron–Severi group modulo the torsion). Then divisibilities can lead to the existence of certain coverings of  $S$  which are often interesting in their own right. Classically this problem has been studied for K3 surfaces (which we shall come back to momentarily — think of Kummer surfaces, for instance). Here we will develop a general result which we hope to be of independent relevance:

**Theorem 1.1.** *Let  $R \subset \text{Num}(S)$  be a root lattice generated by  $(-2)$ -curves on a smooth algebraic surface  $S$ . Denote the primitive closure by*

$$R' = (R \otimes \mathbb{Q}) \cap \text{Num}(S) \quad \text{and let } D \in R' \setminus R.$$

*If  $D^2 = -2$  or  $-1$ , then  $D$  is neither effective nor anti-effective.*

The proof of Theorem 1.1 largely builds on basic properties of root lattices and their duals, especially in relation with reflections, which we will

discuss in the subsequent sections before the proof is completed in Section 5. Here we would like to point out two interesting applications to K3 surfaces and Enriques surfaces — where  $(-2)$ -curves are automatically smooth rational (i.e. nodal) by adjunction.

**Corollary 1.2.** *Let  $S$  be a K3 surface and  $R \subset \text{Num}(S)$  be a root lattice generated by nodal curves on  $S$ . Its primitive closure  $R'$  contains no vectors outside  $R$  with square  $-2$ :*

$$D \in R', D^2 > -4 \implies D \in R.$$

The corollary follows immediately from Theorem 1.1 as a consequence of Riemann–Roch (and the evenness of  $\text{Num}(S)$ ). It has received some attention before in special cases, for instance for  $R$  decomposing into orthogonal copies of the root lattice  $A_2$  (see [1], [2], [5]), or for Kummer surfaces.

Theorem 1.1 also has a big impact on possible configurations of nodal curves on Enriques surfaces: here the configuration often forces divisibilities (since  $\text{Num}(S)$  is unimodular), so Theorem 1.1 provides some severe restrictions — without appeal to any K3 surfaces, so that this holds for all Enriques surfaces, even in characteristic two. For detailed applications, the reader is referred to [6], [7], [8].

**Convention 1.3.** Throughout this paper, we take root lattices to be negative-definite (in agreement with the geometric picture from Theorem 1.1).

## 2. Basics on Root lattices

We start by reviewing basic properties of root lattices. Standard references include [3], [4].

Any root lattice  $R$  admits an orthogonal decomposition into irreducible root lattices  $R_i$  of ADE-type, unique up to order:

$$(1) \quad R = \bigoplus_{j=1}^m R_j.$$

Given an embedding into some integral lattice  $L$ ,

$$R \hookrightarrow L$$

(where we will later take  $L = \text{Num}(S)$ ), the primitive closure

$$R' = (R \otimes \mathbb{Q}) \cap L$$

naturally sits inside the dual of  $R$ :

$$R' \hookrightarrow R^\vee = \bigoplus_i R_i^\vee.$$

For ease of notation, assume for the moment that  $R$  is irreducible of rank  $n$ . We can thus interpret the elements in  $R' \setminus R$  as elements of  $R^\vee \setminus R$ , classified modulo  $R$  by the non-zero elements of the discriminant group

$$A_R = R^\vee/R.$$

This is a finite abelian group whose shape depends on  $R$  as detailed in the Table 1. The values modulo  $2\mathbb{Z}$  which the quadratic form assumes on  $R^\vee$  only depend on the representatives of  $A_R$ . These can easily be worked out, for instance in terms of certain dual vectors for the standard basis of  $R$  corresponding to the associated Dynkin diagram:

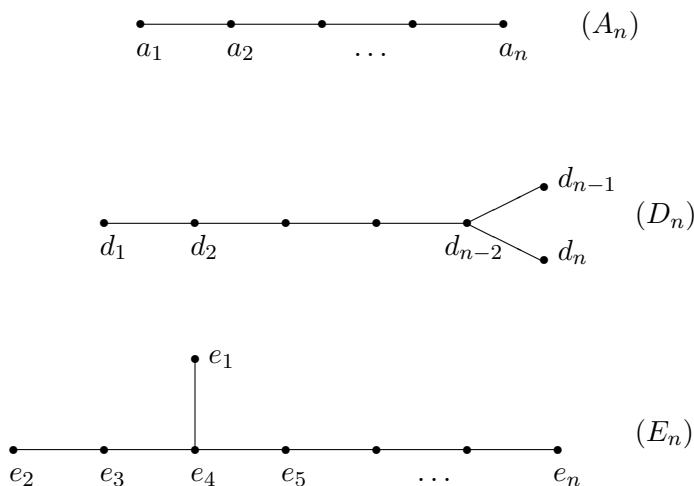


Figure 1: Dynkin diagrams of type  $A_r, D_r, E_r$ .

$R$	$A_n$	$D_n(n \geq 4 \text{ even})$	$D_n(n > 4 \text{ odd})$	$E_6$	$E_7$	$E_8$
$A_R$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
generators	$a_1^\vee$	$d_1^\vee, d_n^\vee$	$d_n^\vee$	$e_6^\vee$	$e_7^\vee$	0
squares	$-\frac{n}{n+1}$	$-1, -\frac{n}{4}$	$-\frac{n}{4}$	$-\frac{4}{3}$	$-\frac{3}{2}$	0

Table 1: Discriminant group data.

Since  $E_8$  is unimodular, in particular  $E_8 = E_8^\vee$ , we will not have to consider  $E_8$  for our purposes and thus omit this lattice for the rest of this paper without further mention.

### 3. Dual vectors

We continue by elaborating on some important properties of the dual vectors of  $R$ . Here we shall mostly be concerned with very specific ones which relate to the appearance of  $R$  in the singular fibers of elliptic fibrations (as classified by Kodaira over  $\mathbb{C}$ ) minus any simple component.

**Definition 3.1.** We call a vertex  $v$  of  $R$  simple if the corresponding fiber component in the resulting singular fiber is simple.

Equivalently  $v$  has multiplicity 1 in the primitive isotropic divisor in the extended Dynkin diagram  $\tilde{R}$ . Concretely, all vertices of root lattices of type  $A_n$  are simple, but only

$$d_1, d_{n-1}, d_n \text{ for } D_n \quad \text{and} \quad e_2, e_6 \text{ for } E_6, \quad e_7 \text{ for } E_7.$$

The meaning of simple vertices is illustrated by the following interrelated classical facts:

**Lemma 3.2.** *The non-zero elements of  $A_R$  are exactly represented by the dual vectors  $v^\vee$  for the simple vertices  $v \in R$ .*

Since  $R^\vee \subset R \otimes \mathbb{Q}$ , we can express each vector  $v \in R^\vee$  in terms of the standard basis  $v_i$  as a sum

$$(2) \quad v = \sum_{i=1}^n \alpha_i v_i, \quad \alpha_i \in \mathbb{Q}.$$

**Lemma 3.3.** *If  $v \in R$  is a simple vertex, then the dual vector  $v^\vee$  has all  $\alpha_i < 0$ .*

Moreover the simple vertices feature an intriguing relation to the roots of  $R$ :

**Lemma 3.4.** *Let  $x \in R$  be a root, expressed in terms of the standard basis  $v_i$  as  $x = \sum_{i=1}^n \alpha_i v_i$ . Then  $\alpha_i \in \{-1, 0, 1\}$  for all simple vertices  $v_i$ .*

We note an important consequence which will be instrumental for our considerations to follow.

**Corollary 3.5.** *If  $v$  is a simple vertex and  $x$  a root in  $R$ , then  $v^\vee \cdot x \in \{-1, 0, 1\}$ .*

**Remark 3.6.** The statement of Corollary 3.5 does no longer hold true if  $v$  is not a simple vertex. Indeed, then there is some root  $x \in R$  such that  $v^\vee \cdot x \geq 2$ .

The statement of Corollary 3.5 does, however, generalize to certain other elements in  $R^\vee$  as we shall see and utilize crucially in Lemma 4.2.

### 4. Small vectors

Recall that in the end we are determined to study vectors of square  $-1$  or  $-2$  in  $R' \subset R^\vee$ . Necessarily these are composed of small (or zero) vectors from the dual lattices  $R_i^\vee$  of the orthogonal summands which we shall thus study in a little more detail here.

**Convention 4.1.** Following the standard terminology inspired from the positive-definite case, we let 'small' refer to the absolute value.

#### 4.1. Smallest vectors

If  $R$  denotes an irreducible root lattice as before, then the value of the smallest non-zero vectors in  $R^\vee$  is exactly given in the Table 1. — except for  $D_n$  for odd  $n > 4$  where  $(d_1^\vee)^2 = -1$  attains the minimum.

It is well known that the smallest vectors form exactly one, two or three orbits under the action of the Weyl group  $W(R)$ , and that each orbit is generated by the dual vector of some simple vertex. In detail, we have:

$R$	$A_1$	$A_n (n > 1)$	$D_4$	$D_n (n > 4)$	$E_6$	$E_7$
# orbits	1	2	3	1	2	1
generators	$a_1^\vee$	$a_1^\vee, a_n^\vee$	$d_1^\vee, d_2^\vee, d_3^\vee$	$d_1^\vee$	$e_2^\vee, e_6^\vee$	$e_7^\vee$

#### 4.2. 2nd smallest vectors

For the second smallest non-zero vectors in the dual  $R^\vee$  of an irreducible root lattice, we did not find a proper reference, but all claims in this paragraph

(and in 4.3, 4.4) can be verified directly (for instance, utilizing the standard model of  $A_n$  as trace zero hypersurface inside euclidean  $\mathbb{Z}^{n+1}$ , up to sign, with Weyl group  $W(A_n) \cong S_{n+1}$  acting by permutation of coordinates).

The second smallest non-zero vectors in  $R^\vee$  have square

- $(a_2^\vee)^2 = -\frac{2(n-1)}{n+1}$  for  $A_n$  ( $n > 2$ ),
- $(d_n^\vee)^2 = -\frac{n}{4}$  for  $D_n$  ( $4 < n \leq 8$ ), and
- $v^2 = -2$ , attained by any root  $v \in R$ , but by no element in  $R^\vee \setminus R$ , for all other irreducible root lattices  $R$ .

Obviously, the third case cannot contribute to our problem, so we simply analyse the first two settings.

$R$	$A_3$	$A_n$ ( $n > 3$ )	$D_n$ ( $4 < n \leq 7$ )	$D_8$
# orbits of 2nd smallest vectors generators	1 $a_2^\vee$	2 $a_2^\vee, a_{n-1}^\vee$	2 $d_{n-1}^\vee, d_n^\vee$	3 $d_{n-1}^\vee, d_n^\vee$ , any root

### 4.3. 3rd smallest vectors

With the third smallest vectors, the problem becomes even easier because there are only a finite number of cases left with value  $\geq -2$  attained by some vector in  $R^\vee \setminus R$ . Indeed, all  $D_n$  and  $E_n$  lattices are excluded anyway, same for  $A_1, \dots, A_4$ , and for  $A_n$  ( $n > 4$ ), the 3rd smallest value in  $A_n^\vee \setminus A_n$  is attained by

$$(a_3^\vee)^2 = -\frac{3(n-2)}{n+1}$$

which exceeds  $-2$  already starting from  $n = 9$ . For the remaining root lattices, we compute:

$R$	$A_5$	$A_n$ ( $n = 6, 7$ )	$A_8$
# orbits of 3rd smallest vectors generators	1 $a_3^\vee$	2 $a_3^\vee, a_{n-2}^\vee$	3 $a_3^\vee, a_6^\vee$ , any root

### 4.4. 4th smallest vectors

Along the same lines, the case of the 4th smallest non-zero vector boils down to the root lattice  $A_7$  right away (as far as our problem is concerned), with value  $-2$  and two orbits generated by  $a_4^\vee$  and any root of  $A_7$ .

For the record, we point out that the fifth smallest non-zero vectors  $w$  in  $R^\vee$  (and beyond) always fulfill  $w^2 < -2$ .

### 4.5. Applications to intersection numbers

We note an innocent, but very useful consequence of our above findings for intersection numbers with roots in  $R$ :

**Lemma 4.2.** *Let  $w \in R^\vee \setminus R$  be a small vector in the range of 4.1–4.4. Then for any root  $x \in R$  we have*

$$w.x \in \{-1, 0, 1\}.$$

*Proof.* By our previous considerations, there is some element  $g \in W(R)$ , with action extended to  $R^\vee \subset R \otimes \mathbb{Q}$ , mapping  $w$  to the dual vector  $v^\vee$  of some simple vertex  $v \in R$ . Hence

$$w.x = g(w).g(x) = v^\vee.g(x) \in \{-1, 0, 1\}$$

by Corollary 3.5 since  $g(x)$  is, of course, again a root in  $R$ . □

Applied to the standard basis  $v_1, \dots, v_n$  of  $R$ , this has an interesting consequence which will become important in the next section:

**Lemma 4.3.** *Let  $w \in R^\vee \setminus R$  be a small non-zero vector in the range of 4.1–4.4. Then*

- either there is some  $i \in \{1, \dots, n\}$  such that  $w.v_i = -1$
- or there is some simple vertex  $v \in R$  such that  $w = v^\vee$ .

*Proof.* By Lemma 4.2, we have  $w.v_i \in \{-1, 0, 1\}$  for all  $i = 1, \dots, n$ . Assume that  $w.v_i \geq 0$  for all  $i$ . Since  $R$  is non-degenerate and  $w \neq 0$ , there is some  $i$  such that  $w.v_i = 1$ . Assume that there is  $j \neq i$  with  $w.v_j = 1$  as well. Then consider the root  $x = v_1 + \dots + v_n$ . By assumption,  $w.x \geq 2$ , contradicting Lemma 4.2. Hence  $w.v_j = 0$  for all  $j \neq i$ , and thus  $w = v_i^\vee$ . Here  $v_i$  is a simple vertex since by Lemma 4.2,  $w.x \in \{-1, 0, 1\}$  for any root in  $R$  which does not hold for any other vertex by Remark 3.6. □

## 5. Proof of Theorem 1.1

We return to the situation from (1) where  $R$  is no longer assumed to be irreducible. Recall that in view of Theorem 1.1, we are interested in vectors

$$(3) \quad w \in R' \setminus R \quad \text{with} \quad w^2 = -1, -2.$$

Writing  $w = (w_1, \dots, w_m)$ , we infer that each  $w_i$  is either zero or a small vector in  $R_i^\vee \setminus R_i$ . For the record, we note the following observation:

**Lemma 5.1.** *Assume that  $w_i \neq 0$ . Then  $w_i$  is in one of the orbits listed in 4.1–4.4.*

**Remark 5.2.** We will not need this in the sequel, but the second smallest vectors in  $A_n$  for  $n \geq 8$  can also be excluded as follows: we have

$$-2 < (a_2^\vee)^2 = -\frac{2(n-1)}{n+1} < -\frac{3}{2}.$$

Since  $v^2 \leq -\frac{1}{2}$  for any non-zero  $v \in R^\vee$ , we infer that  $a_2^\vee$  cannot be complemented by any vector from some dual of a root lattice to have square  $-2$ .

We are now in the position to attack the proof of Theorem 1.1. This puts us in the situation of (3) with the crucial addition that the vertices  $v_j$  of  $R \subset \text{Num}(S)$  are classes of  $(-2)$ -curves  $C_j \subset S$ . Our argument is inspired by discussions with S. Rams on the case of  $R = 4A_2$  on Enriques surfaces (see [5]) which themselves received crucial input from [2, Lem. 1.1].

Assume that  $w \geq 0$  as a divisor on  $S$ . We argue componentwise using the expression  $w = (w_1, \dots, w_m)$ , so let us fix the  $i$ th component and assume that  $R_i^\vee \ni w_i \neq 0$ , so that  $w_i$  is a small vector in the range of 4.1–4.4 by Lemma 5.1. Numbering the vertices of  $R_i$  by  $v_1, \dots, v_n$  as before, there are two cases by Lemma 4.3:

- either  $w_i \cdot v_j \geq 0$  for each  $j = 1, \dots, n$ , so  $w_i = v^\vee$  for some simple vertex  $v \in R_i$ ;
- or there is some vertex  $v_j$  such  $w_i \cdot v_j = -1$ .

We shall now show how the second case successively leads to the first. Since  $w_i \cdot C_j = -1$  and  $w \geq 0$ , we infer that  $C_j$  is contained in the support of  $w_i$ , so  $w_i - C_j$  is still effective. On the other hand, incidentally,  $w_i - C_j$  is the reflection of  $w_i$  in  $C_j$ , so the two vectors have the same square. Thus we can iterate the above process. Necessarily this procedure terminates since at each step, the sum of the  $\mathbb{Q}$ -coefficients of the small vector, expressed in the  $v_j$  as in (2), drops by one while there are only finitely many vectors of given square, of course. In the end, we obtain a vector  $w'_i \in R_i^\vee$  of the same



square as  $w_i$  (equivalent under reflections, in fact), such that

$$w'_i \cdot v_j \geq 0 \quad \text{for all } j = 1, \dots, n.$$

As we have seen, the construction preserves effectivity:

$$(4) \quad w'_i \geq 0.$$

On the other hand,  $w'_i$  equals the dual vector of some simple vertex  $v \in R_i$  by Lemma 4.3. But then all  $\mathbb{Q}$  coefficients of  $w'_i = v^\vee$  are negative by Lemma 3.3.

Since the irreducible root lattices  $R_i$  are orthogonal, we can carry out the above procedure for all  $w_i$  separately. Ultimately, we arrive at an effective divisor  $w' = (w'_1, \dots, w'_m)$  all whose components have zero or negative  $\mathbb{Q}$ -coefficients in terms of the given basis of  $R$  consisting of  $(-2)$ -curves. Hence there is some integral multiple

$$Mw' \leq 0 \quad (M \in \mathbb{N}),$$

giving the required contradiction since  $w'$  was still shown to be effective (as a consequence of (4)), and clearly non-zero.

If  $w \leq 0$ , then reverse the sign and proceed as above. This completes the proof of Theorem 1.1.  $\square$

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