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Section rings of $\mathbb{Q}$-divisors on minimal rational surfaces

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We bound the degrees of generators and relations of section rings associated to arbitrary $\mathbb{Q}$-divisors on projective spaces of all dimensions and Hirzebruch surfaces. For section rings of effective $\mathbb{Q}$-divisors on projective spaces, we find the best possible bound on the degrees of generators and relations.

1. Introduction

For any Weil $\mathbb{Q}$-divisor D on a rational surface X, the graded **section ring** is $R(X, D) := \bigoplus_{d \geq 0} H^0(X, \lfloor dD \rfloor)$. In the case that $D = K_X$, where K_X is the canonical divisor, the graded section ring is referred to as "the canonical ring" and is a classical object of study. For example, if C is a curve of genus $g \geq 4$, Petri's theorem relates the geometry of the curve C to the canonical ring: $R(C, K_C)$ is generated in degree 1 with relations in degree 2 unless C is hyperelliptic, trigonal, or a plane quintic (see [9, p. 157] and [1, Section 3.3]).

In this way, explicit descriptions of generators and relations of section rings yield geometric information about the underlying variety.

One natural way to generalize the classical result of Petri mentioned above is to examine the section rings of stacky curves (i.e., smooth proper geometrically connected 1-dimensional Deligne-Mumford stacks over a field with a dense open subscheme). These were studied by Voight–Zureick-Brown [12] and Landesman–Ruhm–Zhang [7], which provide tight bounds on the degree of generators and relations of log canonical rings and log spin canonical rings on arbitrary stacky curves. All rings of modular forms associated to Fuchsian groups can be realized as canonical rings of such curves, so the above work also yields insight into such rings of modular forms. Further, O'Dorney [8] gives similar descriptions of section rings for arbitrary $\mathbb{Q}$-divisors on $\mathbb{P}^1$ (as opposed to just log canonical $\mathbb{Q}$-divisors and log spin canonical $\mathbb{Q}$-divisors).

Beyond section rings of curves, section rings of certain higher dimensional stacks have also been studied. For example, the Hassett-Keel program [5] studies log canonical rings on $\mathscr{M}_g$ of the form

$$\overline{\mathscr{M}}_g(\alpha) := \bigoplus_{d\geq 0} H^0\left(\overline{\mathscr{M}}_g, \lfloor dK_{\overline{\mathscr{M}}_g} + \alpha\delta\rfloor\right)$$

in terms of certain moduli spaces, where $\overline{\mathscr{M}_g}$ is the moduli space of stable genus g curves.

Moreover, $\mathbb{Q}$-divisors on surfaces not only appear in the context of stacks, but also naturally appear when considering the canonical ring of surfaces of Kodaira dimension 1. In characteristic 0, every surface of Kodaira dimension 1 is an elliptic surface [2, p. 244], and the canonical rings of elliptic surfaces are most naturally described in the setting of $\mathbb{Q}$-divisors [2, Chapter V, Theorem 12.1].

In this paper, we continue the study of section rings of surfaces and higher dimensional varieties, examining section rings of projective spaces $\mathbb{P}^m$ and Hirzebruch surfaces F_m. When D is a general $\mathbb{Q}$-divisor on $\mathbb{P}^m$ or F_m, we give bounds on the generators and relations of $R(X, D)$. In particular, we give a presentation of the section ring when D is any effective $\mathbb{Q}$-divisor on $\mathbb{P}^m$.

Throughout, for ease of notation, we work over a fixed algebraically closed field $\Bbbk$. This is nonessential, see Remark 2.1.1

1.1. Main results and outline

After briefly stating our notation in Section 2, in Section 3, we prove the following two results bounding the degree of generators and relations of section rings on $\mathbb{P}^m$. The first applies to effective $\mathbb{Q}$-divisors and the second applies to arbitrary $\mathbb{Q}$-divisors. Note that here we allow coefficients of divisors to be 0, see Section 3.2.

Theorem 1.1.1. *Let $D = \sum_{i=0}^{n} \alpha_i D_i \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$, with $\alpha_i = \frac{c_i}{k_i} \in \mathbb{Q}_{>0}$ in reduced form and each $D_i \in \operatorname{Div} \mathbb{P}^m$ an integral divisor. Then the section ring $R(\mathbb{P}^m, D)$ is generated in degrees at most $\max_{0 \le i \le n} k_i$ with relations generated in degrees at most $2 \max_{0 \le i \le n} k_i$.*

Theorem 1.1.2. *Let $D = \sum_{i=0}^{n} \alpha_i D_i \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\alpha_i = \frac{c_i}{k_i} \in \mathbb{Q}$ in reduced form. Write $\ell_i := \operatorname{lcm}_{0 \le j \le n, j \ne i}(k_j)$ and $a_i := \deg D_i$. Let $\mathbb{P}^m \cong \operatorname{Proj} \Bbbk[x_0, \ldots, x_m]$ and let $f_i \in \Bbbk[x_0, \ldots, x_m]$ such that $D_i = V(f_i)$. Suppose that $\{f_0, \ldots, f_n\}$ contains a basis for $H^0(\mathbb{P}^m, \mathscr{O}_{\mathbb{P}^m}(1))$, (i.e. $m+1$ independent polynomials with corresponding $a_i = 1$).*

Then $R(\mathbb{P}^m, D)$ is generated in degrees at most $\omega := \sum_{i=0}^{n} \ell_i a_i$ with relations generated in degrees at most

$$\max\left(2\omega, \frac{\max_{0 \le i \le n}(a_i)}{\deg(D)} + \omega\right).$$

Remark 1.1.3. The bounds given in Theorem 1.1.1 are tight and are typically attained, as explained in Remark 3.1.6. Similarly, the bounds given in Theorem 1.1.2 are asymptotically tight to within a factor of two for a class of divisors described in Remark 3.2.10.

Note that the assumption given in Theorem 1.1.2 that $\{f_0, \ldots, f_n\}$ contains a basis for $H^0(\mathbb{P}^m, \mathscr{O}_{\mathbb{P}^m}(1))$ is not a serious assumption: If D is an arbitrary divisor, then $D + 0 \cdot V(x_0) + \cdots + 0 \cdot V(x_m)$ is a divisor satisfying this assumption. Plugging this divisor into the statement of Theorem 1.1.2, we obtain the following bounds for an arbitrary divisor:

Corollary 1.1.4. *Let $D = \sum_{i=0}^{n} \alpha_i D_i \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\alpha_i = \frac{c_i}{k_i} \in \mathbb{Q}$ in reduced form. Write $\ell := \operatorname{lcm}_{0 \le j \le n}(k_j)$, $\ell_i := \operatorname{lcm}_{0 \le j \le n, j \ne i}(k_j)$ and $a_i := \deg D_i$. Let $\mathbb{P}^m \cong \operatorname{Proj} \Bbbk[x_0, \ldots, x_m]$ and let $f_i \in \Bbbk[x_0, \ldots, x_m]$ such that $D_i = V(f_i)$.*

Then $R(\mathbb{P}^m, D)$ is generated in degrees at most $\omega' := (m+1)\ell + \sum_{i=0}^n \ell_i a_i$ *with relations generated in degrees at most*

$$\max\left(2\omega', \frac{\max_{0\le i\le n}(a_i)}{\deg(D)} + \omega'\right).$$

In Section 4, we shift our attention to Hirzebruch surfaces. Recall that for each $m \geq 0$ we define the Hirzebruch surface $F_m := \operatorname{Proj}\operatorname{Sym}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(m))$, viewed as a projective bundle over $\mathbb{P}^1$. Let u, v the projective coordinates on the base $\mathbb{P}^1$ and let z, w be the projective coordinates on the fiber, as defined more precisely at the beginning of 4.

Theorem 1.1.5. *Let* $D = \sum_{i=1}^n \alpha_i D_i \in \operatorname{Div} F_m \otimes_{\mathbb{Z}} \mathbb{Q}$ *where* $\alpha_i = \frac{c_i}{k_i} \in \mathbb{Q}$ *is written in reduced form. Let each* $D_i = V(f_i)$*, where* $f_i \in \mathscr{O}(a_i, b_i)$*. Let* u, v, z, w *be the coordinates for the Hirzebruch surface* F_m*, and suppose that* $\{f_1, \ldots, f_n\}$ *contains bases for* $\mathscr{O}_{F_m}(1,0)$ *and* $\mathscr{O}_{F_m}(0,1)$ *(i.e., two independent linear polynomials in* u, v *and two independent linear polynomials in* w, z *).*

Then $R(F_m, D)$ *is generated in degrees at most*

$$\rho' := \operatorname{lcm}_{1\le i\le n}(k_i) \cdot \left(\sum_{1\le i\le j\le n} a_i b_i\right)$$

with relations generated in degrees at most $2\rho'$*.*

Remark 1.1.6. Theorem 1.1.5 is restated with a more precise bound in Theorem 4.1.4.

Remark 1.1.7. As in Theorem 1.1.2, the condition of Theorem 1.1.5 that D contain independent polynomials in u, v and w, z is easily removed by replacing D with $D' := D + 0 \cdot u + 0 \cdot v + 0 \cdot w + 0 \cdot z$, and computing the resulting bound for D' using either the simpler (but slightly weaker) bound from Theorem 1.1.5 or the more precise bound given in Theorem 4.1.4.

By the classification of minimal rational surfaces as either $\mathbb{P}^2$ or a Hirzebruch surface [4], Theorems 1.1.2 and 1.1.5 provide bounds on generators and relations for the section ring of any $\mathbb{Q}$-divisor on any minimal rational surface. Section 5 discusses further questions.

1.2. Modular forms

The work in this paper was motivated by potential applications to calculating a presentation of certain rings of Hilbert modular forms and Siegel modular forms. Recall that Hilbert and Siegel modular forms are two generalizations of modular forms to higher dimensions, as described in [10] and [3]. If the bounds given in Theorem 1.1.2 and Theorem 1.1.5 could be extended to all rational surfaces, instead of just minimal ones, they would give a bound on generators and relations for rings of Hilbert and Siegel modular forms parametrized by rational Hilbert and Siegel modular surfaces. See Section 5 for a potential approach for generalizing our results to arbitrary rational surfaces. This would be interesting because such modular surfaces tend to be immensely complicated.

Since our results only apply to minimal rational surfaces, and because the rings of Hilbert and Siegel modular forms are so complex, we were unable to use our work to compute the section ring of an explicit modular surface. Although we did not obtain a bound on the degree of generators and relations for general rational surfaces, the restricted class of rational varieties we consider still required significant work.

2. Notation

In this section, we now collect various notation used throughout the paper. Throughout, we work over a fixed algebraically closed field $\Bbbk$ for ease of notation, but our results hold equally well over arbitrary fields.

Remark 2.1.1. Because cohomology commutes with flat base change (in particular with field extensions) the dimensions of the graded pieces of the section ring will be preserved under base change from $\Bbbk$ to $\overline{\Bbbk}$. Therefore generators and relations are preserved under arbitrary base field extension, and so their minimal degrees are preserved. Consequently, there is no harm in assuming $\Bbbk = \overline{\Bbbk}$ for our proofs. The bounds we give hold equally well over arbitrary fields.

Note that if $L/\Bbbk$ is an inseparable extension and X is a scheme over $\Bbbk$, then the base change of the canonical divisor $(K_X)_L$ may be different than the canonical divisor of the base change K_{X_L}. Therefore, the canonical ring may not be preserved under base change along inseparable extensions. Nonetheless, the results we give are not affected because given a divisor on a scheme over some base field, the structure of that particular section ring is unchanged upon base change to the algebraic closure.

Let D be a (Weil) $\mathbb{Q}$-divisor on a rational surface X of the form

$$D = \sum_{i=1}^{n} \alpha_i D_i \in \operatorname{Div} X \otimes_{\mathbb{Z}} \mathbb{Q}.$$

where $n \in \mathbb{Z}_{\geq 0}$ indexes the number of irreducible divisors in the above expansion of D, $\alpha_i \in \mathbb{Q}$, and $D_i \in \operatorname{Div} X$ is an integral codimension 1 closed subscheme of X. When it is convenient to do so, we shall sometimes start the indexing at 0, so that i runs from 0 to n. In the case $X = \mathbb{P}^m$, define the degree of D by $\deg D := \sum_{i=1}^{n} \alpha_i \cdot \deg D_i$. The floor of a $\mathbb{Q}$-divisor D is the divisor $\lfloor D \rfloor := \sum_{i=1}^{n} \lfloor \alpha_i \rfloor D_i$.

Let $R(X, D) := \bigoplus_{d \geq 0} H^0(X, \lfloor dD \rfloor)$ denote the section ring associated to the $\mathbb{Q}$-divisor D. We often alternatively write

$$R(X, D) := \bigoplus_{d \geq 0} u^d H^0(X, \lfloor dD \rfloor),$$

where u is a dummy variable to keep track of the degree. When X is understood from context, we use R_D as notation for $R(X, D)$.

We use $m \in \mathbb{Z}_{\geq 0}$ to index the dimension of a given projective space $\mathbb{P}^m$ and the type of the Hirzebruch surface F_m. If S is a graded ring, we denote the dth graded component of S by S_d. If r is a rational number, we let $\operatorname{frac}(r) := r - \lfloor r \rfloor$ denote the fractional part of r. If $D \in \operatorname{Div} X \otimes_{\mathbb{Z}} \mathbb{Q}$ is an arbitrary divisor, we denote $h^0(X, D) := \dim_{\Bbbk} H^0(X, D)$.

3. Section rings of projective space

Let $\Bbbk$ be a field and let $\mathbb{P}^m$ denote m-dimensional projective space over $\Bbbk$. In this section, we prove Theorem 1.1.2, which bounds the degrees of generators and relations of the section ring of any $\mathbb{Q}$-divisor on $X = \mathbb{P}^m$ for all $m \geq 1$. We also prove Theorem 1.1.1 to give an explicit description of the generators of the section ring R_D when D is an effective divisor.

If $\deg D < 0$, the section ring is concentrated in degree 0, and if $\deg D = 0$, then the section ring has a single generator. Therefore, for the remainder of this section, we shall assume $\deg D > 0$. Note that the $\mathbb{P}^1$ case, in particular, restricts to the results of [8].

For the remainder of this section, we shall fix $m \geq 1$ and choose an isomorphism $\mathbb{P}^m \cong \operatorname{Proj} \Bbbk[x_0, \ldots, x_m]$.

3.1. Effective divisors on projective space

In this subsection, we restrict attention to the case of effective fractional divisors $D \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$. We give an explicit presentation of the section ring when D is an effective divisor.

Convention 3.1.1. Let $\mathbb{P}^m \cong \operatorname{Proj} \Bbbk[x_0, \ldots, x_m]$. Let $\vec{v} = (v_0, \ldots, v_m) \in \mathbb{Z}^{m+1}$. Then write

$$x^{\vec{v}} := \prod_{i=0}^{m} x_i^{v_i}.$$

Definition 3.1.2. For $\vec{v} \in \mathbb{Z}^n$, denote $\deg \vec{v} := \sum_{i=0}^{n} v_i$. For a given sequence of numbers $c_0, \ldots, c_r$, let

$$\mathscr{S}_i := \left\{ \vec{v} \in \mathbb{Z}_{\geq 0}^{m+1} \ : \ \deg \vec{v} = c_i \right\}.$$

Next, we define an ordering on these vectors, which will be used to give a presentation for R_D.

Definition 3.1.3. Let $\vec{v}, \vec{w} \in \mathbb{Z}^{m+1}$. Let $i \in \{0, \ldots, m\}$ be the biggest index such that v_i is nonzero and $j \in \{0, \ldots, m\}$ be the smallest index such that w_j is nonzero. Define a partial ordering on $\mathbb{Z}^{m+1}$ by $\vec{v} \prec \vec{w}$ if $i \leq j$.

We are now ready to give an inductive method for computing the generators and relations of R_D in terms of $R_{D'}$ in the case that $D' = R_D + \alpha H$ for H a hyperplane and α positive. The statement and proofs are natural generalizations of [8, Theorem 6].

Theorem 3.1.4. *Let $\mathbb{P}^m \cong \operatorname{Proj} \Bbbk[x_0, \ldots, x_m]$. Let $D' \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$ and $D = D' + \alpha H$, with $\alpha = \frac{p}{q} \in \mathbb{Q}_{>0}$, $H := V(x_k)$ a hyperplane of $\mathbb{P}^m$, and $H \notin \operatorname{Supp}(D')$. Let*

$$0 = \frac{c_0}{d_0} < \frac{c_1}{d_1} < \cdots < \frac{c_r}{d_r} = \frac{p}{q}$$

be the convergents of the Hirzebruch-Jung continued fraction of α (q.v. [11, Section 2] and [6, Section 3]). Then, the section ring

$$R_D := \bigoplus_{d \geq 0} u^d H^0(\mathbb{P}^m, \lfloor dD \rfloor)$$

has a presentation over $R_{D'}$ consisting of the $\sum_{i=0}^{r} \binom{m+c_i}{c_i}$ generators $F_i^{\vec{v}} := \frac{u^{d_i} x^{\vec{v}}}{x_k^{c_i}}$ where $0 \leq i \leq r$ and $\vec{v} \in \mathbb{Z}_{\geq 0}^{m+1}$ with $\deg \vec{v} = c_i$. Furthermore, the ideal of relations I is generated by the following two classes of elements.

1) For each (i,j) with $j \geq i+2$ and each $\vec{v} \in \mathscr{S}_i, \vec{w} \in \mathscr{S}_j$, there is either a relation of the form

$$G_{i,j}^{\vec{v},\vec{w}} := F_i^{\vec{v}} F_j^{\vec{w}} - \prod_{\vec{y} \in \mathscr{S}_{h_{i,j}}} (F_{h_{i,j}}^{\vec{y}})^{g_{\vec{y}}} \in I,$$

with $i < h_{i,j} < j$, or there is a relation of the form

$$G_{i,j}^{\vec{v},\vec{w}} := F_i^{\vec{v}} F_j^{\vec{w}} - \left(\prod_{\vec{y} \in \mathscr{S}_{h_{i,j}}} (F_{h_{i,j}}^{\vec{y}})^{g_{\vec{y}}} \right) \left(\prod_{\vec{z} \in \mathscr{S}_{h_{i,j}+1}} (F_{h_{i,j}+1}^{\vec{z}})^{g'_{\vec{z}}} \right) \in I,$$

with $i < h_{i,j} < h_{i,j}+1 < j$.

2) For each (i,j) with $j = i$ or $j = i+1$ and each $\vec{v} \in \mathscr{S}_i, \vec{w} \in \mathscr{S}_j$ with $\vec{v} \not\prec \vec{w}$ (see Definition 3.1.3) there is a relation of the form

$$L_{i,j}^{\vec{v},\vec{w}} := F_i^{\vec{v}} F_j^{\vec{w}} - F_i^{\vec{y}} F_j^{\vec{z}} \in I$$

where $\vec{y}$ and $\vec{z}$ are the unique vectors in $\mathscr{S}_i$ and $\mathscr{S}_j$, respectively, such that $\vec{y} + \vec{z} = \vec{v} + \vec{w}$ and $\vec{y} \prec \vec{z}$.

Idea of Proof. The proof follows in three steps. First, since the $F_i^{\vec{v}}$ generate all of R_D over $R_{D'}$, and the leading terms of $G_{i,j}^{\vec{v},\vec{w}}$ lies in $R_{D'}$, we obtain the relations $G_{i,j}^{\vec{v},\vec{w}}$. Then, we derive the relations $L_{i,j}^{\vec{v},\vec{w}}$ by considering when products of generators in neighboring degrees are equal. Finally, we demonstrate that $G_{i,j}^{\vec{v},\vec{w}}$'s and $L_{i,j}^{\vec{v},\vec{w}}$'s generate all of the relations by using them to reduce arbitrary elements of R_D to a canonical form.

Proof. As a first step, we reduce to the case $D' = 0$. Using that D is effective, we can write $R_D = R_{D'} + R_{\alpha H}$ (where the sum is nearly a direct sum, except the $(R_{D'})_d \cap (R_{\alpha H})_d$ is the one dimensional subspace generated by u^d). Then, to give a presentation of R_D over R'_D, it suffices to give a presentation of $R_{\alpha H}$: generators of $R_{\alpha H}$ map to generators of R_D over $R_{D'}$ under the inclusion $\iota : R_{\alpha H} \to R_D$ and relations map to a full set of relations for R_D over $R_{D'}$. Hence, for the remainder of the proof, we can assume $D' = 0$.

Next we show that $F_i^{\vec{v}}$ generate all of R_D. In the case $m = 1$, O'Dorney [8, Theorem 6] demonstrates that each lattice point $(\beta, \gamma) \in \mathbb{Z}^2_{\geq 0}$ with $\gamma \leq \beta\alpha$ lies in the $\mathbb{Z}_{\geq 0}$ span of (d_h, c_h) and (d_{h+1}, c_{h+1}) for some $h \in \{0, \ldots, r\}$. A similar strategy works in the case $m > 1$. Let $(\beta, \gamma) = \lambda(d_h, c_h) + \kappa(d_{h+1}, c_{h+1})$

for $\lambda, \kappa \in \mathbb{Z}_{\geq 0}$. Any element $\frac{u^\beta x^{\vec{v}}}{x_k^\gamma} \in R_D$ is expressible as

$$\frac{u^\beta x^{\vec{v}}}{x_k^\gamma} = \left(\frac{u^{d_h}}{x_k^{c_h}}\right)^\lambda \left(\frac{u^{d_{h+1}}}{x_k^{c_{h+1}}}\right)^\kappa x^{\vec{v}}.$$

We can then write $\vec{v} = \sum_{\tau=1}^{\lambda} \vec{w}_{(\tau)} + \sum_{\eta=1}^{\kappa} \vec{z}_{(\eta)}$ with $\vec{w}_{(\tau)} \in \mathscr{S}_h$ and $\vec{z}_{(\eta)} \in \mathscr{S}_{h+1}$ to give a decomposition

$$\frac{u^\beta x^{\vec{v}}}{x_k^\gamma} = \prod_{\tau=1}^{\lambda} \frac{u^{d_h} x^{\vec{w}_{(\tau)}}}{x_k^{c_h}} \prod_{\eta=1}^{\kappa} \frac{u^{d_{h+1}} x^{\vec{z}_{(\eta)}}}{x_k^{c_{h+1}}} \tag{3.1}$$

consisting of products of generators $F_h^{\vec{w}_{(\lambda)}}$ and $F_{h+1}^{\vec{z}_{(\eta)}}$ which are in the form prescribed in the theorem statement. Since we wrote an arbitrary monomial $\frac{u^\beta x^{\vec{v}}}{x_k^\gamma} \in R_D$ as a product of generators, this shows that the $F_i^{\vec{v}}$ generate R_D.

Next, we show that the relations given in the statement of the theorem generate all relations. In particular, if $j \geq i+2$ then $F_i^{\vec{v}} F_j^{\vec{w}}$ has a decomposition of the form (3.1) of products of generators in adjacent degrees where h depends on i and j, so we denote $h_{i,j} := h \in \{1, \ldots, r\}$. We also have that $i \leq h_{i,j} < j$ since $(d_i + d_j, c_i + c_j)$ is in the $\mathbb{Z}_{\geq 0}$-span of $(d_{h_{i,j}}, c_{h_{i,j}})$ and $(d_{h_{i,j}+1}, c_{h_{i,j}+1})$. Furthermore, $h_{i,j} \neq i$ and $h_{i,j} \neq j-1$ as follows from an analogous proof to that given by O'Dorney [8, Theorem 6] for the case of $\mathbb{P}^1$. This gives the relations $G_{i,j}^{\vec{v},\vec{w}}$.

One can use the relations $G_{i,j}^{\vec{v},\vec{w}}$ to transform any monomial in the $F_i^{\vec{v}}$'s involving indices that differ by more than 1 to a monomial in the $F_i^{\vec{v}}$'s involving indices that differ by at most 1.

We also have relations involving generators in consecutive indices. Suppose $F_i^{\vec{v}}$ and $F_j^{\vec{w}}$ are generators with $j = i$ or $j = i+1$ and $\vec{v} \in \mathscr{S}_i, \vec{w} \in \mathscr{S}_j$ with $\vec{v} \not\prec \vec{w}$. Let $\vec{y}$ and $\vec{z}$ be the unique vectors in $\mathscr{S}_i$ and $\mathscr{S}_j$, respectively, such that $\vec{y} + \vec{z} = \vec{v} + \vec{w}$ and $\vec{y} \prec \vec{z}$ (i.e. the nonzero indices of $\vec{y}$ followed by those of $\vec{z}$ give an increasing sequence). Then we see that

$$F_i^{\vec{v}} F_j^{\vec{w}} = x^{\vec{v}+\vec{w}} \left(\frac{u^{d_i}}{x_k^{c_i}}\right) \left(\frac{u^{d_j}}{x_k^{c_j}}\right) = x^{\vec{y}+\vec{z}} \left(\frac{u^{d_i}}{x_k^{c_i}}\right) \left(\frac{u^{d_j}}{x_k^{c_j}}\right) = F_i^{\vec{y}} F_j^{\vec{z}},$$

which give the relations $L_{i,j}^{\vec{v},\vec{w}}$.

Now, we may apply the relations $L_{i,j}^{\vec{v},\vec{w}}$ to any monomial in the $F_i^{\vec{v}}$'s involving indices that differ by at most 1 to produce the canonical form

$$\prod_{\tau=1}^{\lambda}(F_i^{\vec{v}_{(\tau)}})^{g_{\vec{v}_{(\tau)}}}\prod_{\eta=1}^{\kappa}(F_{i+1}^{\vec{v}_{(\eta)}})^{g_{\vec{v}_{(\eta)}}}$$

where $\vec{v}_{(1)} \prec \vec{v}_{(2)} \prec \ldots \prec \vec{v}_{(\lambda)} \prec \vec{w}_{(1)} \prec \ldots \prec \vec{w}_{(\kappa)}$. Consequently, the relations of form $G_{i,j}^{\vec{v},\vec{w}}$ and the relations of form $L_{i,j}^{\vec{v},\vec{w}}$ generate all the relations among the $F_i^{\vec{v},\vec{w}}$. □

Remark 3.1.5. In the case that $D = \alpha_0 D_0$ is supported on a single hypersurface and $D' = 0$, the relations in Theorem 3.1.4 form a reduced Gröbner basis with respect to the ordering given in Definition 3.1.3.

Theorem 3.1.4 gives an inductive procedure to compute presentations of effective divisors which are supported on hyperplanes. However, for $m \geq 2$, there are hypersurfaces which are not unions of hyperplanes. We now address this general case, giving an inductive presentation of section rings of effective divisors, and a tight bound on the degrees of their generators and relations.

Proof of Theorem 1.1.1. We proceed by induction on n. If $n = 0$, i.e. $D = 0$, then we are done. Now, we inductively add hypersurfaces. Let $D' = \sum_{i=0}^{n} \alpha_i D_i \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$, and assume the theorem holds for D'. It suffices to show the theorem holds for $D \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$, where $D = D' + \alpha C$ for some degree δ hypersurface C. If C were a hyperplane, we would then be done, by Theorem 3.1.4.

To complete the theorem, we reduce the case that C is a general hypersurface to the case that C is a hyperplane, by using the Veronese embedding.

If C is of degree δ, consider the Veronese embedding $\nu_\delta^m \colon \mathbb{P}^m \to \mathbb{P}^{\binom{m+\delta}{\delta}-1}$ so that the image of C is the intersection of a hyperplane in $\mathbb{P}^{\binom{m+\delta}{\delta}-1}$ with $\nu_\delta^m(\mathbb{P}^m)$. Now, the ring $R_{\alpha C}$ is isomorphic to the δ Veronese subring of

$$R_{\alpha V(x_0)} = \bigoplus_{d \geq 0} u^d H^0(\mathbb{P}^m, d\alpha V(x_0)).$$

Therefore, we can bound the degree of generators and relations of R_D over $R_{D'}$ by the degree of generators and relations for $R_C \cong R_H$. This reduces the case of a hypersurface C to a hyperplane H, completing the proof by Theorem 3.1.4. □

Remark 3.1.6. The proof of Theorem 1.1.1 not only gives bounds on the degrees of the generators and relations, but actually gives an explicit method for computing the presentation. Also, a minimal generating set of R_D over $R_{D'}$ in Theorem 3.1.4 can be given as a subset of the generating set given in Theorem 3.1.4. Further, one can verify a minimal generating set necessarily contains generators in each of the degrees $d_0, \ldots, d_r$ using the definition of Hirzebruch-Jung continued fractions. In particular, the bounds of Theorem 1.1.1 are tight. The generator bound is always achieved and the bound on the relations is achieved if $m \geq 2$.

Before giving bounds for arbitrary divisors on projective space in Subsection 3.2, we give a detailed example of the generators and relations for a section ring associated to an effective divisor on projective space.

Example 3.1.7. In this example, we work out generators and relations for the section ring $R(\mathbb{P}^2, D)$ with

$$D := \frac{1}{6}V(x^2 + y^2 + z^2) + \frac{2}{5}V(x).$$

We loosely follow the algorithm described in the course of the proof of Theorem 1.1.1 and use notation from the statement of Theorem 1.1.1.

For ease of notation, let $h := x^2 + y^2 + z^2$ and let $D' := \frac{1}{6}V(h)$. We first compute generators and relations for $R_{D'}$, then compute generators and relations for R_D over $R_{D'}$, and finally put them together to obtain generators and relations for R_D.

To start, we compute generators and relations for $R_{D'}$. Indeed, applying 3.1.4 and scaling the degrees by 2, we see that $\frac{1}{6}$ has convergents given by

$$0 = \frac{0}{1} < \frac{1}{6},$$

and so the generators are given by

$$\begin{aligned} &F_0^{\prime(0,0,0)} = u^1 \quad F_1^{\prime(2,0,0)} = \frac{u^6x^2}{h}, \quad F_1^{\prime(1,1,0)} = \frac{u^6xy}{h}, \\ &F_1^{\prime(1,0,1)} = \frac{u^6xz}{h}, \quad F_1^{\prime(0,2,0)} = \frac{u^6y^2}{h}, \\ &F_1^{\prime(0,1,1)} = \frac{u^6yz}{h}, \quad F_1^{\prime(0,0,2)} = \frac{u^6z^2}{h}. \end{aligned} \tag{3.2}$$

Further, the relations in $R_{D'}$ are given by

$$
\begin{aligned}
L'^{(1,1,0),(1,0,1)}_{1,1} &= \frac{u^6xy}{h}\frac{u^6xz}{h} - \frac{u^6x^2}{h}\frac{u^6yz}{h},\\
L'^{(1,0,1),(0,1,1)}_{1,1} &= \frac{u^6xz}{h}\frac{u^6yz}{h} - \frac{u^6xy}{h}\frac{u^6z^2}{h},\\
L'^{(1,0,1),(0,2,0)}_{1,1} &= \frac{u^6xz}{h}\frac{u^6y^2}{h} - \frac{u^6xy}{h}\frac{u^6yz}{h},\\
L'^{(1,1,0),(1,1,0)}_{1,1} &= \frac{u^6xy}{h}\frac{u^6xy}{h} - \frac{u^6x^2}{h}\frac{u^6y^2}{h},\\
L'^{(1,0,1),(1,0,1)}_{1,1} &= \frac{u^6xz}{h}\frac{u^6xz}{h} - \frac{u^6x^2}{h}\frac{u^6z^2}{h},\\
L'^{(0,1,1),(0,1,1)}_{1,1} &= \frac{u^6yz}{h}\frac{u^6yz}{h} - \frac{u^6y^2}{h}\frac{u^6z^2}{h}.
\end{aligned}
\tag{3.3}
$$

Next, we describe generators and relations for R_D over $R_{D'}$. Indeed, in this case, $\frac{2}{5}$ has convergents given by

$$0 = \frac{0}{1} < \frac{1}{3} < \frac{2}{5}.$$

Therefore, the generators of R_D over $R_{D'}$ are given by

$$
\begin{aligned}
&F^{(0,0,0)}_0 = u^1,\\
&F^{(0,1,0)}_1 = \frac{u^3y}{x}, \quad F^{(0,0,1)}_1 = \frac{u^3z}{x},\\
&F^{(0,2,0)}_2 = \frac{u^5y^2}{x^2}, \quad F^{(0,1,1)}_2 = \frac{u^5yz}{x^2}, \quad F^{(0,0,2)}_2 = \frac{u^5z^2}{x^2}.
\end{aligned}
\tag{3.4}
$$

Note that the generator $F^{(1,0,0)}_3 = \frac{u^3x}{x}$ could be included, but it is redundant as it is equal to $(u^1)^3$. Similarly, the generators $\frac{u^5x^2}{x^2}, \frac{u^5xy}{x^2}, \frac{u^5xz}{x^2}$ are redundant. Furthermore, we have relations given by

$$
(3.5) \qquad
\begin{aligned}
G_{0,2}^{(0,0,0),(0,2,0)} &= u \cdot \frac{u^5y^2}{x^2} - \left(\frac{u^3y}{x}\right)^2 \\
G_{0,2}^{(0,0,0),(0,1,1)} &= u \cdot \frac{u^5yz}{x^2} - \frac{u^3y}{x}\frac{u^3z}{x} \\
G_{0,2}^{(0,0,0),(0,0,2)} &= u \cdot \frac{u^5z^2}{x^2} - \left(\frac{u^3z}{x}\right)^2 \\
L_{1,2}^{(0,0,1),(0,2,0)} &= \frac{u^3z}{x} \cdot \frac{u^5y^2}{x^2} - \frac{u^3y}{x}\frac{u^5yz}{x^2} \\
L_{1,2}^{(0,0,1),(0,1,1)} &= \frac{u^3z}{x} \cdot \frac{u^5yz}{x^2} - \frac{u^3y}{x}\frac{u^5z^2}{x^2} \\
L_{2,2}^{(0,1,1),(0,1,1)} &= \frac{u^5yz}{x^2} \cdot \frac{u^5yz}{x^2} - \frac{u^5y^2}{x^2}\frac{u^5z^2}{x^2}.
\end{aligned}
$$

Then, combining the above, generators for the ring R_D are given by (3.2) together with (3.4) and relations are given by (3.3) together with (3.5).

3.2. Bounds for arbitrary divisors on projective space

We now offer bounds on generators and relations of R_D for a general $\mathbb{Q}$-divisor $D \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$. Write

$$D = \sum_{i=0}^{n} \alpha_i D_i \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$$

For the remainder of this section, we shall make the additional assumption that

$$(3.6) \qquad f_0, \ldots, f_m \text{ are independent linear forms.}$$

This may necessitate the inclusion of “ghost divisors” D_i with coefficients $\alpha_i = 0$.

The main aim of this section is to prove Theorem 1.1.2. Having justified the necessity of adding ghost divisors, we proceed to bound the number of generators and relations of arbitrary $\mathbb{Q}$-divisors in projective space. In Proposition 3.2.4, we record a general proposition describing a basis for $H^0(\mathbb{P}^m, dD)$. We bound the generators in Lemma 3.2.6, and we use Lemmas 3.2.7, 3.2.8, and 3.2.9 to bound the degree of relations in the proof of Theorem 1.1.2. Proposition 3.2.5, Lemma 3.2.7, and Lemma 3.2.8 are quite general and will also be used in Section 4 to bound the degree of generators and relations on Hirzebruch surfaces. However, before moving on to the proof of 1.1.2, we justify the importance of assumption (3.6) with several illustrative examples.

The importance of ghost divisors.

Example 3.2.1. In this example, we show that the naive generalization of [8, Theorem 8] of generation in degree at most $\sum_{i=0}^{n} \ell_i$ cannot possibly hold. The reason for this is that the divisors may be expressible as functions in m of the $m+1$ variables on $\mathbb{P}^m$.

Concretely, take $D = \frac{1}{2}H_0 - \frac{1}{3}H_1$ where $H_0 = V(x_0), H_1 = V(x_1)$ are two coordinate hyperplanes in $\mathbb{P}^2$. Then, R_D has generators in degree 2 and 3 which can be written as $u^2\frac{x_1}{x_0}, u^3\frac{x_1}{x_0}$. In fact, for all degrees less than 5, the elements of R_D can all be expressed as rational functions in x_0, x_1. However, in degree 6, there is $u^6 \cdot \frac{x_1^2 x_2}{x_0^3}$. Since this involves x_2, it must be a generator.

This example generalizes slightly to any divisor of the form $D = \frac{1}{k}H_0 - \frac{1}{k+1}H_1 \in \operatorname{Div}\mathbb{P}^2 \otimes_{\mathbb{Z}} \mathbb{Q}$, with $k \in \mathbb{N}$, showing that there will always exist a generator in degree $k(k+1)$.

This example further generalizes to the following situation: Suppose

$$D = \sum_{i=0}^{n} \frac{\mathrm{p}_i}{\mathrm{q}_i} D_i \in \operatorname{Div}\mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\deg D_i = a_i$ and $\deg D = \frac{1}{\operatorname{lcm}_{0\le i\le n}(\mathrm{q}_i \cdot a_i)}$. Then, if $D_i = V(f_i)$ where all f_i can be written as a polynomial function in $x_0, \ldots, x_{m-1}$, it follows that R_D always has a generator in degree $\operatorname{lcm}(\mathrm{q}_i \cdot a_i)$.

As illustrated in Example 3.2.1, when all components in the support of divisor can be written in terms of m of the $m+1$ variables on $\mathbb{P}^m$, we cannot hope to bound the degree of generation by anything less than the sum of the least common multiples of the denominators. This issue can easily be circumvented by adding in "ghost divisors." That is, we may add divisors of the form $0 \cdot H_i$ to D, and reorder so that if $D = \sum_{i=0}^{n} \alpha_i V(f_i)$, then $f_0, \ldots, f_m$ are independent linear functions in $x_0, \ldots, x_m$.

Remark 3.2.2. We cannot extend Theorem 3.1.4 to the case when D is supported at two hypersurfaces with arbitrary rational (non-effective) coefficients in the same manner that O'Dorney does for the $\mathbb{P}^1$ case [8, Section 4]. As shown in Example 3.2.1, the degrees of generation of the section ring of a general $\mathbb{Q}$-divisor supported on two hyperplanes cannot be bounded so tightly. The two-point $\mathbb{P}^1$ result leverages the fact that $\mathbb{P}^1$ has precisely two independent coordinates, so that two distinct integral subschemes cannot represent equations of only m of the $m+1$ coordinates.

In Example 3.2.3, we show that it is still, in general, necessary to add ghost divisors, even when the irreducible components of a divisor are not all expressible as functions in m of the $m+1$ variables on $\mathbb{P}^m$.

Example 3.2.3. Consider $D := \frac{-1}{5}V(x_0^2+x_1^2+x_2^2) + \frac{1}{7}V(x_0^2+x_1^2+x_3^2) + \frac{1}{17}V(x_0^2+x_2^2+x_3^2) - \frac{1}{596}V(x_1^2+x_2^2+x_3^2)$. In degree $355216 = 5\cdot 7\cdot 17\cdot 596$, R_D has dimension 6. However, for all $d \leq 355216$, $h^0(\mathbb{P}^3, dD) \leq 1$ and $h^0(\mathbb{P}^3, dD) = 1$ precisely when $\lfloor dD \rfloor = 0$ (so in this case, $H^0(\mathbb{P}^3, dD)$ corresponds to the constant functions). Therefore R_D has a generator in degree 355216. Hence, we cannot hope to bound the degree of generation of R_D as a linear combination of ℓ_i, in analogy to [8, Theorem 8] unless we require that D includes ghost divisors. That is, unless $D_0, \ldots, D_m$ are taken to be linearly independent hyperplanes.

A basis for sections on projective space. In order to prepare ourselves to prove Theorem 1.1.2, we will need the following simple description of a basis for $H^0(\mathbb{P}^m, dD)$.

Proposition 3.2.4. *Assuming Equation 3.6, the functions* $u^d \cdot \prod_{i=0}^n f_i^{c_i} \in (R_D)_d$ (recall u is a dummy variable keeping track of the degree) satisfying both of the following conditions

1) $\sum_{i=0}^n c_i \cdot a_i = 0$
2) $c_i \geq -\lfloor d\alpha_i \rfloor$

for $c_0, \ldots, c_n \in \mathbb{Z}$ span $H^0(\mathbb{P}^m, dD)$ over $\Bbbk$. Furthermore, such functions that also satisfy

3) $c_i = -\lfloor d\alpha_i \rfloor$ for all $i > m$

form a basis for $H^0(\mathbb{P}^m, dD)$ over $\Bbbk$.

Proof. By definition of $H^0(\mathbb{P}^m, dD)$, functions satisfying conditions (1) and (2) lie in $H^0(\mathbb{P}^m, dD)$. Conditions (1) and (2) are also necessary for some monomial in the f_i to lie in $H^0(\mathbb{P}^m, dD)$. Since the monomials in the f_i span R_D, it follows the monomials satisfying (1) and (2) span $(R_D)_d =$

$H^0(\mathbb{P}^m, dD)$. To complete the proof, it suffices to check functions satisfying conditions (1)–(3) form a basis of $H^0(\mathbb{P}^m, dD)$. There are $\binom{m+\deg\lfloor dD\rfloor}{m}$ functions satisfying conditions (1)–(3). However we know $h^0(\mathbb{P}^m, dD) = \binom{m+\deg\lfloor dD\rfloor}{m}$, so it suffices to show that those monomials satisfying conditions (1)–(3) are independent. To see why these are independent, observe that monomials satisfying (1)–(3) are all of the form $f_0^{c_0}\cdots f_m^{c_m}\cdot g$ for the fixed monomial $g = f_{m+1}^{-\lfloor d\alpha_{m+1}\rfloor}\cdots f_n^{-\lfloor d\alpha_{m+1}\rfloor}$. For any fixed $N \in \mathbb{Z}$, the set

$$\left\{ f_0^{c_0}\cdots f_m^{c_m} : (c_0,\dots,c_m)\in\mathbb{Z}^{m+1}, \sum_{i=0}^m c_i = N \right\}$$

forms an independent set over $\Bbbk$. Therefore, multiplying the monomials in the above set with the g also forms an independent set, and these are precisely the monomials satisfying (1)–(3) with $N = -\deg g$. □

Bounding the generators. We now develop the tools to bound the degrees of generators for R_D for D an arbitrary divisor on $\mathbb{P}^m$.

Recall that for a semigroup $\Sigma\subset\mathbb{Z}^m$ we say $e_0,\dots,e_n\in\Sigma$ are a set of **extremal rays** for Σ if Σ is contained in the $\mathbb{Q}_{\geq 0}$ span of $e_0,\dots,e_n$.

Proposition 3.2.5. *Let $n\in\mathbb{Z}$, let $\alpha_0,\dots,\alpha_n\in\mathbb{Q}$, and let $a_i, b_i\in\mathbb{Z}$ with $0\leq i\leq n$. Define*

$$\Sigma := \left\{ (d, c_0,\dots,c_n)\in\mathbb{Z}^{n+2} \colon c_i\geq -d\alpha_i, 0\leq i\leq n \text{ and } \sum_{i=0}^n a_i = \sum_{i=0}^n b_i = 0 \right\}.$$

Suppose $e_0,\dots,e_t\in\Sigma$ with $e_i = (\delta_i, c_0^i,\dots,c_n^i)$ are a set of extremal rays of Σ.

Then, as a semigroup, Σ is generated by elements whose first coordinate is less than $\sum_{i=0}^t \delta_i$. Furthermore, every element $\sigma\in\Sigma$ can be written in a canonical form

$$\sigma = \lambda + \sum_{i=0}^t \zeta_i e_i \tag{3.7}$$

with $\zeta_1,\dots,\zeta_t\in\mathbb{Z}_{\geq 0}$, $0\leq s_i<1$, and $\lambda = \sum_{i=0}^r s_i e_i$ so that the first coordinate of λ is less than $\sum_{i=0}^t \delta_i$.

Proof. By assumption, $\sigma\in\Sigma$ can be written as $\sigma = \sum_{i=0}^t r_i e_i$ with $r_i\in\mathbb{Q}$. Let $\operatorname{frac}(r) := r - \lfloor r\rfloor$ denote the fractional part of r. Let $\lambda = \sum_{i=0}^t \operatorname{frac}(r_i)$.

Whence, we can write $\sigma = \lambda + \sum_{i=0}^{t} \lfloor r_i \rfloor e_i$. Consequently, σ lies in the $\mathbb{Z}_{\geq 0}$ span of $\lambda, e_0, \ldots, e_t$, which all have first coordinate less than $\sum_{i=0}^{t} \delta_i$. Ergo, Σ is generated by elements whose first coordinate is less than $\sum_{i=0}^{t} \delta_i$. □

By Proposition 3.2.5, in order to bound the degree of generation of R_D, we only need bound the degrees of extremal rays of an associated cone. We now carry out this strategy.

Lemma 3.2.6. *Let* $D = \sum_{i=0}^{n} \alpha_i D_i \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$, *where* $\deg D_i = a_i$, $\alpha_i = \frac{c_i}{k_i} \in \mathbb{Q}$, *and* $\ell_i = \operatorname{lcm}_{j \neq i}(k_j)$. *Then,* R_D *is generated in degrees at most* $\sum_{i=0}^{n} \ell_i a_i$.

Proof. Let

(3.8)
$$\Sigma = \left\{ (d, c_0, \ldots, c_n) \in \mathbb{Z}^{n+2} \colon c_i \geq -d\alpha_i,\ 0 \leq i \leq n,\ \text{and } \sum_{i=0}^{n} \ell_i a_i = 0 \right\}.$$

Observe that Σ has extremal rays given by the lattice points
(3.9)
$$e_i = \left(\ell_i a_i, -\alpha_0 \ell_i a_i, \ldots - \alpha_{i-1} \ell_i a_i, \ell_i \sum_{j \neq i} \alpha_j a_j, -\alpha_{i+1} \ell_i a_i, \ldots, -\alpha_n, \ell_i a_i \right)$$

for each $i \in \{0, \ldots, n\}$. Therefore, applying Proposition 3.2.5, we see R_D is generated in degrees less than

$$\sum_{i=0}^{n} \ell_i a_i.$$

□

Let $w_1, \ldots, w_r$ be the generators in degrees at most $\sum_{i=0}^{n} \ell_i a_i$ (given by Lemma 3.2.6), and let $\phi \colon \mathbb{k}[w_1, \ldots, w_r] \to R_D$ be the natural surjection. For the remainder of the section, we aim to bound the degree of relations of R_D, or equivalently, the degree of generation of $\ker \phi$. We can factor ϕ through the semigroup ring

$$\mathbb{k}[\Sigma] = \left\langle u^d z_0^{c_0} \cdots z_n^{c_n} \colon c_i \in \mathbb{Z},\ c_i \geq -d\alpha_i,\ \text{and } \sum_{i=0}^{n} a_i c_i = \sum_{i=0}^{n} b_i c_i \right\rangle.$$

by

$$\begin{array}{ccccc} \Bbbk[w_1, \ldots, w_r] & \xrightarrow{\quad\chi\quad} & \Bbbk[\Sigma] & \xrightarrow{\quad\psi\quad} & R_D \\ w_i & \longmapsto & u^{d_i} z_0^{c_{i0}} \cdots z_n^{c_{in}} & \longmapsto & u^{d_i} f_0^{c_{i0}} \cdots f_n^{c_{in}}. \end{array} \tag{3.10}$$

Bounding the relations. We now move on to bounding the degree of relations of a section ring R_D for D a divisor on $\mathbb{P}^m$. In Lemma 3.2.7 we show that the degree of the generators for $\ker \phi$, which is the same as the degree of relations of R_D, is bounded by the maximum of the degree of generators for $\ker \chi$ and for $\ker \psi$. In Lemma 3.2.8, we bound the degree of generation of $\ker \chi$, and we bound the degree of generation of $\ker \psi$ in Lemma 3.2.9

Lemma 3.2.7. *Let X be a $\Bbbk$ variety and let $D = \sum_{i=0}^n \alpha_i D_i \in \operatorname{Div} X \otimes_{\mathbb{Z}} \mathbb{Q}$ where $D_i = V(f_i)$. Suppose we have a surjection $\phi\colon \Bbbk[w_1, \ldots, w_r] \to R_D$ given by $w_i \mapsto p_i(f_0, \ldots, f_n)$, where p_i is a monomial in $f_0, \ldots, f_n$. Let $a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathbb{Z}_{\geq 0}$. Then, define*

$$\Sigma = \left\langle u^d z_0^{c_0} \cdots z_n^{c_n} : c_i \geq -d\alpha_i, \sum_{i=0}^n a_i c_i = \sum_{i=0}^n b_i c_i = 0 \right\rangle.$$

In this case, we can factor ϕ as a composition of χ and ψ defined by

$$\begin{array}{ccccc} \Bbbk[w_1, \ldots, w_r] & \xrightarrow{\quad\chi\quad} & \Bbbk[\Sigma] & \xrightarrow{\quad\psi\quad} & R_D \\ w_i & \longmapsto & u^{d_i} z_0^{c_{i0}} \cdots z_n^{c_{in}} & \longmapsto & u^{d_i} f_0^{c_{i0}} \cdots f_n^{c_{in}}. \end{array}$$

Assuming χ is surjective, the minimal degree of generation of $\ker \phi$ is at most the maximum of the minimal degree of generation of $\ker \chi$ and the minimal degree of generation of $\ker \psi$.

Proof. First, surjectivity of χ implies we have an exact sequence

$$0 \longrightarrow \ker \chi \longrightarrow \ker \phi \longrightarrow \ker \psi \longrightarrow 0.$$

This shows that lifts of generators of $\ker \psi$ together with images of generators of $\ker \chi$ generate all of $\ker \phi$, as desired. $\square$

Lemma 3.2.8. *Retaining the notation of Lemma 3.2.7, if Σ has extremal rays $e_0, \ldots, e_t$ in degrees $d_0, \ldots, d_t$ then $\ker \chi$ is generated in degrees at most $2(\sum_{i=0}^t d_i - 1)$.*

Proof. Since $e_0, \dots, e_t$ are extremal rays, Proposition 3.2.5 implies every element $\sigma \in \Sigma$ can be written in a canonical form $\lambda + \sum_{i=0}^t \zeta_i e_i$ where all $\zeta_i \in \mathbb{Z}_{\geq 0}$. Let $\lambda_0 := 0, \lambda_1, \dots, \lambda_r$ be all elements of Σ which can be written in the form $\lambda_j = \sum_{i=0}^t s_i e_i$ with $0 \leq s_i < 1$. Then, for any $1 \leq j \leq k \leq r$, we can write $\lambda_j + \lambda_k$ in the above canonical form, yielding a (possibly trivial) relation in degree at most $\deg \lambda_j + \deg \lambda_k \leq 2 \cdot \left(\sum_{i=0}^t d_i - 1\right)$. Furthermore, these relations generate all relations, as one can apply a sequence of these relations to put any $\sigma \in \Sigma$ into canonical form $\sigma = \lambda + \sum_{i=0}^t \zeta_i e_i$ from Proposition 3.2.5. □

Lemma 3.2.9. *Let $D = \sum_{i=0}^n \alpha_i D_i \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\deg D_i = a_i$, $\alpha_i = \frac{c_i}{k_i} \in \mathbb{Q}$, and $\ell_i = \operatorname{lcm}_{j \neq i}(k_j)$. Define Σ as in Equation (3.8) and ψ as in Equation (3.10). Then, $\ker \psi$ is generated in degrees at most*

$$\frac{\max_{0 \leq i \leq n}(a_i)}{\deg(D)} + \sum_{i=0}^n \ell_i a_i. \tag{3.11}$$

Proof. We claim there exist $\beta_0, \dots, \beta_n \in \Bbbk[\Sigma]$ such that $\ker \psi$ is generated by

$$u^d (z_i - \beta_i) \prod_{j=0}^n {z_j}^{c_j} \tag{3.12}$$

for all $d \in \mathbb{N}$ and $c_i \geq -\alpha_i d$ satisfying $a_i + \sum_{j=0}^n a_j c_j = 0$.

Indeed, define the β_i as a polynomial in $z_0, \dots, z_m$ such that $\psi(\beta_i) = \psi(z_i) = f_i \in R_D$. This is possible by Proposition 3.2.4. Furthermore, the relations given in Equation (3.12) generate all relations, since they allow us to reduce any $u^d \prod_{j=0}^n z_j^{c_i}$ to a canonical form, with $c_i = -\lfloor d\alpha_i \rfloor$ whenever $i > m$.

For the remainder of the proof, fix $i \in \{0, \dots, n\}$. To complete the proof, it suffices to bound the degree of generation of the relations of the form $u^d (z_i - \beta_i) \prod_{j=0}^n z_j^{c_j}$, by Equation (3.11). For a given monomial

$$u^d (z_i - \beta_i) \prod_{j=0}^n z_j^{c_j} \in \Bbbk[\Sigma],$$

we associate it with the corresponding element $(d, c_0, \dots, c_n) \in \Sigma$. Let $\Sigma_i \subseteq \mathbb{Z}^{n+2}$ be the set of points of the form $(d, c_0, \dots, c_n)$ satisfying $c_j \geq -d\alpha_j$ for all j and $\sum_{j=0}^{n} c_j a_j = -a_i$. Let

$$\delta_i := \left(\frac{a_i}{\deg(D)}, -\frac{\alpha_0 a_0}{\deg(D)}, \ldots, -\frac{\alpha_n a_n}{\deg(D)} \right).$$

Then we see $\Sigma_i = \{\sigma \in \Sigma | \sigma - \delta_i \in \operatorname{span}_{\mathbb{Q}_{\geq 0}}(e_0, \ldots, e_n)\}$ with e_i as defined in Equation 3.9. Therefore, we can write any element of Σ_i uniquely as

$$\delta_i + \sum_{j=0}^{n} c_j e_j$$

where $c_j \in \mathbb{R}$ for each j.

Whenever some there is some j for which $c_j \geq 1$, we can write the relation $u^d(z_i - \beta_i) \prod_{j=0}^{n} z_j^{c_j} = e_j h$, for some $h \in \Sigma_i$. Therefore, for a fixed i, relations of the form $u^d(z_i - \beta_i) \prod_{j=0}^{n} z_j^{c_j} \in \Bbbk[\Sigma]$ are generated by those in degrees less than

$$\frac{a_i}{\deg(D)} + \sum_{i=0}^{n} \ell_i a_i,$$

as $\deg \delta_i = \frac{a_i}{\deg(D)}$. Hence, $\ker \psi$ is generated in degrees less than

$$\frac{\max_{0 \leq i \leq n} a_i}{\deg(D)} + \sum_{i=0}^{n} \ell_i a_i.$$

□

Proving Theorem 1.1.2. By combining the above results, we get our main theorem bounding the generator and relation degrees of the section ring of any $\mathbb{Q}$-divisor on projective space.

Proof of Theorem 1.1.2. The bound on degree of generation is precisely the content of Lemma 3.2.6. It only remains to bound the degree of relations.

By 3.2.8, $\ker \chi$ is generated in degrees at most $2 \sum_{i=0}^{n} \ell_i a_i$ and by Lemma 3.2.9, $\ker \psi$, is generated in degrees up to $\frac{\max_{0 \leq i \leq n} a_i}{\deg(D)} + \sum_{i=0}^{n} \ell_i a_i$. Consequently, Lemma 3.2.7 implies that $\ker \phi$ is generated in degrees less than

$$\max \left(2 \sum_{i=0}^{n} \ell_i a_i, \frac{\max_{0 \leq i \leq n}(a_i)}{\deg(D)} + \sum_{i=0}^{n} \ell_i a_i \right).$$

□

Remark 3.2.10. The bounds given in Theorem 1.1.2 are asymptotically tight to within a factor of two for the following class of divisors. Consider a divisor $D = \sum_{i=0}^{n} \frac{p_i}{2q_i} H_i \in \operatorname{Div} \mathbb{P}^m \otimes_{\mathbb{Z}} \mathbb{Q}$ such that H_i are hyperplanes, q_i

are pairwise coprime integers, and p_i are chosen so that $\deg D = \frac{1}{2\prod_{i=0}^{n} q_i}$. Further, choose a linear subspace $\pi : \mathbb{P}^1 \to \mathbb{P}^m$ generically so that $\pi^* D = \sum_{i=0}^{n} \frac{p_i}{2q_i} P_i$, where P_i are distinct points in $\mathbb{P}^1$. To choose such a map π, we may need to assume that the base field is infinite. By [8, Remark, p. 9], the given bounds on the generators and relations of $R_{\pi^* D}$ are within a factor of two of the degree of generation and relations of $R_{\pi^* D}$. Finally, since restriction map $R_D \to R_{\pi^* D}$ induced by the restriction maps on cohomology $H^0(\mathbb{P}^m, dD) \to H^0(\mathbb{P}^1, \pi^*(dD)) \cong H^0(\mathbb{P}^1, d\pi^* D)$ are surjective, we obtain that the bounds for the generators and relations of R_D given in Theorem 1.1.2 also agree with the degree of generation and relations to within a factor of two.

4. Section rings of Hirzebruch surfaces

Let F_m denote the m-th Hirzebruch surface. The aim of this section is to prove Theorem 1.1.5, which bounds the degree of generators and relations of the section ring of any $\mathbb{Q}$-divisor on $X = F_m$, for all $m \geq 0$.

One way to describe the Hirzebruch surface F_m is as a quotient,

$$F_m \cong (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^2 \setminus \{0\}) / \mathbb{G}_m \times \mathbb{G}_m$$

where $\mathbb{G}_m$ is the multiplicative subgroup of $\mathbb{A}^1$, and the action of $\mathbb{G}_m \times \mathbb{G}_m$ is given by $(\lambda, \mu) \cdot (u\colon v; z\colon w) \mapsto (\lambda u\colon \lambda v; \mu z\colon \lambda^{-m} \mu w)$, as described in [13, p. 6]. Hence, one can think of F_m as a $\mathbb{P}^1$ bundle where u, v are the coordinates on $\mathbb{P}^1$ and z, w are the coordinates on the fiber. Sections of a line bundle $\mathscr{L}$ on F_m can be written as rational functions in z, w, u, v. Furthermore we define the bi-degree of a monomial $u^a v^b z^c w^d$ on F_n to be $(a + b + mc, c + d)$. Rational sections of F_m can be written as rational functions with numerators and denominators of the same bi-degrees. To see this, observe $\mathrm{Pic}(F_m) \cong \mathbb{Z} \times \mathbb{Z}$, where the class of a line bundle in $\mathrm{Pic}(F_m)$ is determined by its bi-degree, as follows from the excision exact sequence for class groups.

Furthermore, we will restrict to the case that D is a divisor for which both of its bi-degrees are positive. We now justify this restriction. If either of the bi-degrees of D are negative, then the section ring is concentrated in degree 0. If one of the bi-degrees is 0, say the first one is 0, then R_D is isomorphic to $R_{D'}$, where $D' \in \operatorname{Div} \mathbb{P}^1 \otimes_{\mathbb{Z}} \mathbb{Q}$, where D' can be written as a sum of divisors whose degrees are multiples of the second bi-degree of D. Since the case of $\mathbb{P}^1$ has already been analyzed in [8], we are justified in assuming that both bi-degrees of D are positive.

For the remainder of this section we will assume D_1, D_2, D_3, and D_4 are distinct divisors with bi-degrees $(1,0)$, $(1,0)$, $(0,1)$, and $(0,1)$ respectively with $D_i = V(f_i)$ for $1 \le i \le 4$ with $f_i \in \mathscr{O}(a_i, b_i)$. In order to achieve the above condition on the bi-degrees of $D_1, \ldots, D_4$, it may be necessary to add in "ghost divisors" (i.e. divisors with of the desired form with a coefficient 0). Also, f_1 and f_2 are independent linear polynomials in u and v and f_3 and f_4 are independent linear polynomials in z and w. Analogously to Proposition 3.2.4 for the case of $\mathbb{P}^m$, all rational functions on F_m can be written uniquely in a form where their numerator is a function of only f_1, f_2, f_3, and f_4.

Definition 4.1.1. Define

$$\mathrm{T}_{=}(D) = \left\{ i \in \{1, \ldots, n\} : a_i \sum_{k=1}^{n} b_k \alpha_k = b_i \sum_{k=1}^{n} a_k \alpha_k \right\},$$

$$\mathrm{T}_{+}(D) = \left\{ i \in \{1, \ldots, n\} \colon a_i \sum_{k=1}^{n} \alpha_k b_k > b_i \sum_{k=1}^{n} \alpha_k a_k \right\},$$

and

$$\mathrm{T}_{\text{-}}(D) = \left\{ i \in \{1, \ldots, n\} \colon a_i \sum_{k=1}^{n} \alpha_k b_k < b_i \sum_{k=1}^{n} \alpha_k a_k \right\}.$$

Lemma 4.1.2. *For $D = \sum_{i=1}^{n} \frac{c_i}{k_i} D_i \in \operatorname{Div} F_m \otimes_{\mathbb{Z}} \mathbb{Q}$, with $\deg D_i = (a_i, b_i)$, $\ell_i = \operatorname{lcm}_{j \neq i}(k_j)$, $\ell_{i,j} = \operatorname{lcm}_{h \neq i,j}(k_h)$. Then, the section ring R_D is generated in degrees at most*

$$\rho := \sum_{i \in \mathrm{T}_{=}(D)} \gcd(a_i, b_i) \ell_i + \sum_{\substack{i \in \mathrm{T}_{+}(D) \\ j \in \mathrm{T}_{\text{-}}(D)}} (a_i b_j - a_j b_i) \ell_{i,j}. \tag{4.1}$$

Proof. Suppose $g \in (R_D)_d$ is a monomial. Then

$$g = u^d \prod_{i=1}^{n} {f_i}^{c_i}$$

for some $c_i \ge -\alpha_i d$ such that $\sum_{i=1}^{n} c_i a_i = 0$ and $\sum_{i=1}^{n} c_i b_i = 0$. We can view g as an element $(d, c_1, \ldots, c_n)$ of the lattice

$$\Sigma = \left\{ (d', c_1', \ldots, c_n') \in \mathbb{Z}_{\ge 0}^{n+1} \colon c_i' \ge -d\alpha_i \text{ for all } i \text{ and } \sum_{i=1}^{n} c_i' a_i = \sum_{i=1}^{n} c_i' b_i = 0 \right\}.$$

In order to determine a generating set for $(R_D)_d$, it suffices to find the extremal rays of Σ. To do this, we extend the method of O'Dorney [8, Theorem 8]. We first consider the sub-cone $\Sigma_1 \subset \Sigma$ given by

$$\Sigma_1 = \left\{ (d, c_1, \ldots, c_n) \in \mathbb{Z}_{\geq 0}^{n+1} \colon c_i \geq -d\alpha_i \text{ for all } i \text{ and } \sum_{i=1}^{n} c_i(a_i + b_i) = 0 \right\},$$

which has extremal rays given by

$$\epsilon_i := \left(1, -\alpha_1, \ldots, -\alpha_{i-1}, \frac{\sum_{j \neq i} \alpha_j(a_j + b_j)}{a_i + b_i}, -\alpha_{i+1}, \ldots, -\alpha_n \right).$$

for $1 \leq i \leq n$.

Let $\Sigma_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ be the $\mathbb{Q}_{\geq 0}$ span of $\epsilon_1, \ldots, \epsilon_n$. We can intersect $\Sigma_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ with the hyperplane $H := V(\sum_{i=1}^{n} a_i x_i)$ to get the subspace $\Sigma \otimes_{\mathbb{Z}} \mathbb{Q} = H \cap (\Sigma_1 \otimes_{\mathbb{Z}} \mathbb{Q})$. Then, the extremal rays of Σ are precisely the extremal rays of $\Sigma \otimes_{\mathbb{Z}} \mathbb{Q}$.

The extremal rays of $\Sigma \otimes_{\mathbb{Z}} \mathbb{Q}$ can be represented by points lying only on the edges $\overline{e_i e_j}$. The extremal rays are given by multiples of those ϵ_i's which are contained in H together with intersection points $e_{i,j}$ which can be expressed as $H \cap \overline{e_i e_j}$, where $i \neq j$ and $e_i, e_j \notin H$. In this case, $e_{i,j}$ is only defined when $\#\{H \cap \overline{e_i, e_j}\} = 1$.

From this geometric description of the extremal rays, we can write the extremal rays algebraically as follows. For $i \in \mathrm{T}_{=}(D)$, define $e_i \in \Bbbk[\Sigma]$ in degree

$$d_i = \ell_i \gcd(a_i, b_i)$$

by

$$e_i := d_i \epsilon_i.$$

For $i \in \mathrm{T}_{+}(D)$ and $j \in \mathrm{T}_{-}(D)$, with $i < j$, define $e_{i,j} \in \Bbbk[\Sigma]$ in degree

$$d_{i,j} = \ell_{i,j}(a_i b_j - a_j b_i)$$

by

$$\begin{aligned} e_{i,j} := {} & \frac{a_j \sum_{k \neq i,j} d_{i,j} \alpha_k b_k - b_j \sum_{k \neq i,j} d_{i,j} \alpha_k a_k}{a_i b_j - b_i a_j} \epsilon_i \\ & + \frac{a_i \sum_{k \neq i,j} d_{i,j} \alpha_k b_k - b_i \sum_{k \neq i,j} d_{i,j} \alpha_k a_k}{a_j b_i - b_j a_i} \epsilon_j. \end{aligned}$$

Since these e_i are multiples of ϵ_i and these $e_{i,j}$ are points of intersection of H with $\overline{e_i e_j}$ such that neither e_i nor e_j are contained in H, these form a set of extremal rays of Σ.

Thus, Proposition 3.2.5 implies that R_D is generated in degrees less than the sum of the degrees of the e_i and $e_{i,j}$, which is

$$\rho = \sum_{i \in \mathrm{T}_=(D)} \gcd(a_i, b_i)\ell_i + \sum_{\substack{i \in \mathrm{T}_+(D) \\ j \in \mathrm{T}_-(D)}} (a_i b_j - a_j b_i)\ell_{i,j}.$$

□

Let $w_1, \ldots, w_r$ be the generators of R_D in degrees less than ρ (as given by Lemma 4.1.2), and let ϕ be the surjection $\Bbbk[w_1, \ldots w_r] \to R_D$. As in Section 3, we can factor ϕ through the semigroup ring

$$\Bbbk[\Sigma] = \left\langle u^d z_1^{c_1} \cdots z_n^{c_n} : c_i \geq -d\alpha_i, \sum_{i=1}^n a_i c_i = \sum_{i=1}^n b_i c_i \right\rangle$$

by

$$\begin{array}{ccccc} \Bbbk[w_1, \ldots, w_r] & \xrightarrow{\quad\chi\quad} & \Bbbk[\Sigma] & \xrightarrow{\quad\psi\quad} & R_D \\ w_i & \longmapsto & u^{d_i} z_1^{c_{i1}} \cdots z_n^{c_{in}} & \longmapsto & u^{d_i} f_1^{c_{i1}} \cdots f_n^{c_{in}}. \end{array}$$

By Lemma 3.2.8, we can bound the degree of generation of $\ker \chi$ below 2ρ. Finally, we calculate the degree of generation of ψ:

Lemma 4.1.3. *Let ρ be as in Equation* (4.1)*. Then,* $\ker \psi$ *is generated in degrees less than*

$$\tau := \rho + \max\left(\max_{i \in \mathrm{T}_=}(\ell_i \gcd(a_i, b_i)), \max_{\substack{i \in \mathrm{T}_+ \\ j \in \mathrm{T}_-}}(\ell_{i,j}(a_i b_j - a_j b_i)) \right).$$

Proof. We first claim that there exist

$$u^{\deg z_1}\beta_1, \ldots, u^{\deg z_n}\beta_n \in \Bbbk[uz_1, uz_2, uz_3, uz_4]$$

such that $\ker \psi$ has relations of the form

$$u^d(z_i - \beta_i) \prod_{j=1}^n z_j{}^{c_j}$$

lying in some degree $d \in \mathbb{N}$ with $c_j \geq -\alpha_j d$ (for all j) satisfying $a_i + \sum_{j=1}^n a_j c_j = 0$ and $b_i + \sum_{j=1}^n b_j c_j = 0$. Specifically, β_i, is the polynomial $\beta_i(z_1, z_2, z_3, z_4)$

so that $\beta_i(z_1, z_2, z_3, z_4) - z_i \in \ker\psi$. Such an element β exists and is unique because rational functions whose numerators are polynomials in f_1, f_2, f_3, f_4 form a basis of all rational functions in R_D. Furthermore, these generate all relations, since they allow us to reduce any monomial $u^d \prod_{j=1}^n z_j^{r_j}$ to the canonical form where $r_j = -\lfloor d\alpha_i \rfloor$ whenever $j > 4$. To bound the degree of generation of these relations, we bound the degree of generation of the ideal $(\beta_i - z_i) \cap \ker(\psi)$ for each i.

For the remainder of this proof: we fix $i \in \{1, \ldots, n\}$ and fix a relation of the form $u^d(z_i - \beta_i) \prod_{j=1}^n z_j^{c_j}$ as we seek to bound the degree of generation of the ideal $(\beta_i - z_i) \cap \ker(\psi)$. There are no relations for $i \in \{1, 2, 3, 4\}$ as then $\beta_i - z_i = 0 \in \Bbbk[\Sigma]$. Thus we restrict attention to $i \geq 5$. Our goal is to show that if $u^d(z_i - \beta_i) \prod_{j=1}^n z_j^{c_j}$ has sufficiently high degree, then there is another relation dividing it. We do so by considering the lattice points corresponding to the monomials appearing in the relation $u^d(z_i - \beta_i) \prod_{j=1}^n z_j^{c_j}$, and finding a fixed $\lambda \in \Sigma$ that we can simultaneously factor out of all monomials.

Consider a relation $u^d(z_i - \beta_i) \prod_{j=1}^n z_j^{c_j}$ and let it correspond to the lattice point $\sigma := (d, c_1, \ldots, c_{i-1}, c_i + 1, c_{i+1}, \ldots, c_n) \in \Sigma$. Then we can write σ as a sum of $s_j e_j$'s for $j \in \mathrm{T}_=$ and $s_{j,k} e_{j,k}$ for $j \in \mathrm{T}_+, k \in \mathrm{T}_-$. For convenience, define $d_j := \deg e_j$ (when it exists) and let the j^{th} component of e_j be $-\alpha_j d_j + \kappa_j$ for some $\kappa_j \in \mathbb{Q}$. Also, let $d_{j,k} := \deg e_{j,k}$ (when it exists) and let the j^{th} component of $e_{j,k}$ be $-\alpha_j d_{j,k} + \kappa'_{j,k}$ and the k^{th} component be $-\alpha_k d_{j,k} + \kappa''_{j,k}$ for $\kappa'_{j,k}, \kappa''_{j,k} \in \mathbb{Q}$.

Since $u^d(z_i - \beta_i) \prod_{j=1}^n z_j^{c_j}$ is a relation, each monomial of it must be an element of Σ. This implies that $s_i\kappa_i \geq 1$, $\sum_{j \in \mathrm{T}_-} s_{i,j}\kappa'_{i,j} \geq 1$, and $\sum_{j \in \mathrm{T}_+} s_{i,j}\kappa''_{j,i} \geq 1$ if $i \in \mathrm{T}_=$, $i \in \mathrm{T}_+$, and $i \in \mathrm{T}_-$ respectively.

If $i \in \mathrm{T}_=$ define $r_i := \frac{1}{\kappa_i}$. If $i \in \mathrm{T}_+$ choose $r_{i,j} \in \mathbb{Q}_{\geq 0}$ for all $j \in \mathrm{T}_-$ such that $\sum_{j \in \mathrm{T}_-} r_{i,j}\kappa'_{i,j} = 1$; similarly, if $i \in \mathrm{T}_-$ choose $r_{j,i} \in \mathbb{Q}_{\geq 0}$ for all $j \in \mathrm{T}_+$ such that $\sum_{j \in \mathrm{T}_-} r_{j,i}\kappa''_{j,i} = 1$. For $j \neq i$, define $r_j := 0$. For all pairs (j, k) so that $j \neq i$ and $k \neq i$ define $r_{j,k} := 0$. Define E by

$$E := \sum_{j \in \mathrm{T}_=} (s_j - \lfloor s_j - r_j \rfloor) e_j + \sum_{\substack{j \in \mathrm{T}_+ \\ k \in \mathrm{T}_-}} (s_{j,k} - \lfloor s_{j,k} - r_{j,k} \rfloor) e_{j,k}. \tag{4.2}$$

Then,

$$\deg(E) \leq \rho + \begin{cases} \ell_i \gcd(a_i, b_i) & \text{if } i \in \mathrm{T}_= \\ \max_{j \in \mathrm{T}_-} \big(\ell_{i,j}(a_i b_j - a_j b_i)\big) & \text{if } i \in \mathrm{T}_+ \\ \max_{j \in \mathrm{T}_+} \big(\ell_{j,i}(a_j b_i - a_i b_j)\big) & \text{if } i \in \mathrm{T}_- \,. \end{cases} \tag{4.3}$$

where ρ is as in Equation (4.1). To obtain the bound given in Equation (4.3), the ρ term corresponds to the sums of fractional parts of $s_j - r_j$'s and $s_{j,k} - r_{j,k}$'s whereas the second term corresponds to the sums of r_j's and $r_{j,k}$'s (noting that in Equation 4.2, $s_j - \lfloor s_j - r_j \rfloor = r_j + \mathrm{frac}(s_j - r_j)$).

Define

$$\lambda := \sigma - E = \sum_{j \in \mathrm{T}_=} (\lfloor s_j - r_j \rfloor) e_j + \sum_{\substack{j \in \mathrm{T}_+ \\ k \in \mathrm{T}_-}} (\lfloor s_{j,k} - r_{j,k} \rfloor) e_{j,k} \in \Sigma.$$

Let M_i be the set of monomials terms of $\beta_i = \beta_i(z_1, \ldots, z_4)$. Let $\mu = \prod_{j=1}^{4} {z_j}^{h_j} \in M_i$ and consider the lattice point

$$\sigma_\mu = (d, c_1 + h_1, \ldots, c_4 + h_4, c_5, \ldots, c_n).$$

where $d = \deg \sigma$. Define

$$E_\mu = \sigma_\mu - \lambda.$$

From the definitions, E_μ lies in Σ and has the same degree as E.

By construction, $E - \sum_{\mu \in M_i} E_\mu \in \ker \psi$ and divides $u^d(z_i - \beta_i) \prod_{j=1}^{n} z_j^{c_j}$. Furthermore, we have already bounded $\deg(E)$ in Equation (4.3). Finally, recall $\ker \psi$ is generated by relations of the form $u^d(z_i - \beta_i) \prod_{j=1}^{n} z_j^{c_j}$ as i ranges between 1 and n. Thus, taking the maximum over all i of our bound in Equation (4.3), we see $\ker \psi$ is generated in degrees at most

$$\tau = \rho + \max \left(\max_{i \in \mathrm{T}_=} \big(\ell_i \gcd(a_i, b_i) \big), \max_{\substack{i \in \mathrm{T}_+ \\ j \in \mathrm{T}_-}} \big(\ell_{i,j} (a_i b_j - a_j b_i) \big) \right) \le 2\rho.$$

□

By combining the above results, we get our main theorem bounding the generator and relation degrees of the section ring of $\mathbb{Q}$-divisors on Hirzebruch surfaces.

Theorem 4.1.4. *Let $D = \sum_{i=1}^{n} \alpha_i D_i \in \mathrm{Div}\, F_m \otimes_{\mathbb{Z}} \mathbb{Q}$ where $\alpha_i = \frac{c_i}{k_i} \in \mathbb{Q}$ is written in reduced form. Write $\ell_{i,j} := \mathrm{lcm}_{h \neq i,j}(k_h)$. Let u, v, z, w be the coordinates for the Hirzebruch surface F_m, as described at the beginning of Section 4 and suppose that $\{f_1, \ldots, f_n\}$ contains two independent linear polynomials in u, v and two independent linear polynomials in w, x (with corresponding α_j possibly zero). Recall $\mathrm{T}_=, \mathrm{T}_+$, and T_- as given in Definition 4.1.1 and let each $D_i = V(f_i)$ where $f_i \in \mathscr{O}(a_i, b_i)$.*

Then R_D is generated in degrees at most

$$\rho = \sum_{i \in \mathrm{T}_=(D)} \gcd(a_i, b_i)\ell_i + \sum_{\substack{i \in \mathrm{T}_+(D) \\ j \in \mathrm{T}_-(D)}} (a_i b_j - a_j b_i)\ell_{i,j}$$

with relations generated in degrees at most 2ρ.

Proof. The generation degree bound is as stated in Lemma 4.1.2. By Proposition 3.2.7, the degree of generation of $\ker\phi$ is at most the maximum of the generation degrees of $\ker\chi$ and $\ker\psi$, giving us the desired relations bound. The bound on $\ker\chi$ follows from Lemma 3.2.8 and the bound on $\ker\psi$ follows from Lemma 4.1.3. □

5. Further questions

Recall that every minimal rational surface is either isomorphic to $\mathbb{P}^2$ or F_m for some $m \geq 0, m \neq 1$ [4]. By Theorems 1.1.2 and 1.1.5, we have given bounds for the generators and relations of arbitrary section rings on any minimal rational surface. A natural extension of our results is the following.

Question 5.1.1. *Can we describe generators and relations of R_D for a divisor D on an arbitrary rational surface X?*

Every rational surface can be obtained from a minimal rational surface by a sequence of blow-ups [4]. Therefore, to answer Question 5.1.1 affirmatively, it suffices to bound the degree of generators and relations of the section ring of a divisor on a blow-up of a given surface in terms of the section rings of some associated divisors on that given surface.

Another direction to generalize the work in this paper would be to try to express section rings of $\mathbb{Q}$-divisors on $X \times Y$ in terms of those on X and Y. In this paper, we bounded the degrees of presentations on section rings on $\mathbb{P}^1 \times \mathbb{P}^1 \cong F_0$. Perhaps similar techniques can be used to bound degrees of presentations on section rings on $(\mathbb{P}^1)^k$ or more generally on $(\mathbb{P}^1)^{i_1} \times \cdots (\mathbb{P}^k)^{i_k}$. One might further try to generalize this to bounding degrees of presentations on bundles over $\mathbb{P}^m$ or on more general products of schemes.

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