

Remarks on automorphy of residually dihedral representations

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We prove automorphy lifting results for geometric representations $\rho : G_F \rightarrow GL_2(\mathcal{O})$, with F a totally real field, and \mathcal{O} the ring of integers of a finite extension of \mathbb{Q}_p with p an odd prime, such that the residual representation $\bar{\rho}$ is totally odd and induced from a character of the absolute Galois group of the quadratic subfield K of $F(\zeta_p)/F$. Such representations fail the Taylor-Wiles hypothesis and the patching techniques to prove automorphy do not work. We apply this to automorphy of elliptic curves E over F , when E has no F rational 7-isogeny and such that the image of G_F acting on $E[7]$ normalizes a split Cartan subgroup of $GL_2(\mathbb{F}_7)$.

1. Introduction

1.1. The main theorem

Let F be a totally real field, p be an odd prime, and \mathcal{O} the ring of integers of a finite extension of \mathbb{Q}_p . In proving automorphy of geometric representations $\rho : G_F \rightarrow GL_2(\mathcal{O})$ which are residually automorphic, there is an assumption made (the Taylor-Wiles hypothesis) that the residual representation $\bar{\rho}$ is irreducible when restricted to $G_{F(\zeta_p)}$. In [5], Thorne has recently weakened this assumption in many cases to simply asking that $\bar{\rho}$ itself is irreducible but making the auxiliary assumption that the quadratic subfield of $F(\zeta_p)/F$ is totally real.

Under other assumptions on $\bar{\rho}$, but still allowing that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is reducible, we prove automorphy lifting results wherein we do not assume that the quadratic subfield K of $F(\zeta_p)/F$ is totally real. The assumption on $\bar{\rho}$ is that there is a (“level raising”) place v of F that splits in K , such that the ratio of the eigenvalues of $\bar{\rho}(\text{Frob}_v)$ is q_v with q_v not 1 modulo p . This assumption is automatic when K/F is totally real, as exploiting the oddness of $\bar{\rho}$ we can take any v such that $\bar{\rho}(\text{Frob}_v)$ is conjugate to the image of complex conjugation.

The main theorem we prove is the following:

Theorem 1.1. *Let F be a totally real number field, let p be an odd prime, and let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous representation satisfying the following:*

- (1) *The representation ρ is almost everywhere unramified.*
- (2) *For each $v|p$ of F , the local representation $\rho|_{G_{F_v}}$ is de Rham. For each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$, we have $HT_\tau(\rho) = \{0, 1\}$.*
- (3) *For each complex conjugation $c \in G_F$, we have $\det \rho(c) = -1$.*
- (4) *The residual representation $\bar{\rho}$ is absolutely irreducible, but $\bar{\rho}|_{G_{F(\zeta_p)}}$ is a direct sum of two distinct characters. If K denotes the unique quadratic subfield of $F(\zeta_p)/F$ and $\bar{\gamma} : G_K \rightarrow k^\times$ is the ratio of the two characters, then further suppose $F(\zeta_p) \not\subset K(\bar{\gamma}\bar{\epsilon}^{-1}) \cap K(\bar{\gamma}\bar{\epsilon})$.*

Then ρ is automorphic: there exists a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight 2, an isomorphism $\iota : \overline{\mathbb{Q}}_p \rightarrow \mathbb{C}$, and an isomorphism $\rho \cong r_\iota(\pi)$.

The hypothesis that $F(\zeta_p) \not\subset K(\bar{\gamma}\bar{\epsilon}^{-1}) \cap K(\bar{\gamma}\bar{\epsilon})$ is equivalent to the existence of a place v of F such that q_v is not 1 modulo p , v splits in K , and $\bar{\rho}(\mathrm{Frob}_v)$ has eigenvalues with ratio q_v . Furthermore, as remarked above, this hypothesis is automatic when K is totally real as one may, using the oddness of $\bar{\rho}$, take any v such that $\bar{\rho}(\mathrm{Frob}_v)$ is conjugate to $\bar{\rho}(c)$ and q_v is -1 modulo p . Thus the theorem generalizes the main theorem of [5], and in fact its proof is essentially the same as [5]. In loc. cit., the author synthesizes two methods:

- the patching method of Taylor-Wiles, which relies on using places v of F such that q_v is 1 modulo p^N and $\bar{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues to kill certain elements of the mod p dual Selmer group, and
- a method due to Khare, which proves automorphy of ρ by “ p -adic approximation” using Ramakrishna places v , such that q_v is not 1 modulo p and $\rho_N(\mathrm{Frob}_v)$ has eigenvalues with ratio q_v , to perform the task of killing certain elements of the mod p dual Selmer group which cannot be killed with Taylor-Wiles primes. Here ρ_N denotes the mod p^N reduction of ρ .

The Ramakrishna places v that Thorne used were such that $\rho_N(\mathrm{Frob}_v)$ is conjugate to the image of complex conjugation. To ensure that these places could effectively kill the troublesome part of the mod p dual Selmer group,

he had to impose that the quadratic subfield of $F(\zeta_p)/F$ be totally real, which implies $p \equiv 1 \pmod 4$.

The central new remark of this paper is that if we assume there is a place v of F that splits in K , such that the ratio of the eigenvalues of $\bar{\rho}(\text{Frob}_v)$ is q_v with q_v not 1 modulo p , then by choosing “Teichmüller liftings” one may find, for each N , places v such that q_v is not 1 modulo p , $\rho_N(\text{Frob}_v)$ has eigenvalues with ratio q_v , and the v ’s perform the task of killing the troublesome elements of the mod p dual Selmer group which cannot be killed with Taylor-Wiles primes. We present the proof of this remark in Proposition 3.11, and it serves as a replacement for Proposition 5.20 in [5] which needed that K be totally real. Note that this assumption we make is automatic when K is totally real, which explains why our result generalizes the main theorem of loc. cit.

After this remark the proof of our theorem is identical to that of loc. cit. where patching arguments followed by level raising and lowering arguments are used to conclude the automorphy of ρ .

The case when there is no place v of F that splits in K such that the ratio of the eigenvalues of $\bar{\rho}(\text{Frob}_v)$ is q_v with q_v not 1 modulo p cannot be addressed by the methods of [5] nor by the modifications that we carry out in this paper. In a forthcoming work [3] of Khare and Thorne, this case is addressed. Instead of patching and p -adic approximation, the authors use patching together with Wiles’ numerical isomorphism criterion and calculation of η -invariants using monodromy arguments.

1.2. Automorphy of elliptic curves

We get a modest application to automorphy of elliptic curves over totally real fields. Current automorphy lifting theorems prove the modularity of elliptic curves E over F which satisfy the “Taylor-Wiles” hypothesis at some prime $p \in \{3, 5, 7\}$, i.e., $\bar{\rho}_{E,p}(G_{F(\zeta_p)})$ is absolutely irreducible for at least one $p \in \{3, 5, 7\}$. Recently, in [2], the authors prove the modularity of all elliptic curves over real quadratic fields. Moreover, they show that over any totally real field F , there are only finitely many potentially non-modular elliptic curves over F , which necessarily do not satisfy the Taylor-Wiles hypothesis at any prime $p \in \{3, 5, 7\}$.

In [5], Thorne was able to cut down on this list of potentially non-modular elliptic curves by proving a new automorphy lifting theorem and applying it to the above situation of elliptic curves with $p = 5$. However, the results in that paper do not deal with the case $p = 7$ because the requirement was that the quadratic subfield of $F(\zeta_p)/F$ be totally real (which implies

that p is 1 modulo 4). In this note, we try to deal with the cases when this quadratic subfield is not totally real by examining different sorts of auxiliary primes. We use this theorem to prove:

Theorem 1.2. *Let F be a totally real field, and let E be an elliptic curve over F . Suppose:*

- (1) $F \cap \mathbb{Q}(\zeta_7) = \mathbb{Q}$.
- (2) E has no F -rational 7-isogeny.
- (3) *Either $\bar{\rho}_{E,7}(G_{F(\zeta_7)})$ is absolutely irreducible, or it is reducible and is conjugate to a subgroup of a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_7)$.*

Then E is modular.

We further note that one may also deduce by similar arguments the automorphy of elliptic curves E defined over a totally real field F , with a prime p such that E has no F -rational p -isogeny, $\rho_{E,p}(G_F)$ normalizes a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$, and E has bad semistable reduction at a place v such that q_v is not 1 mod p and v splits in the quadratic subfield of $F(\zeta_p)/F$.

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2. Notation

We now introduce the notation we will use throughout. The notation and conventions will be the same as in [5].

Let F be a number field, and let \bar{F} denote a choice of algebraic closure. If v is a place of F , then F_v will denote the completion of F at v and \bar{F}_v will be a choice of algebraic closure (with the algebraic closure of \mathbb{R} being \mathbb{C}). If p is a rational prime, then we use S_p to denote the place of F above p . The p -adic valuation val_p on \mathbb{Q}_p is normalized so that $\mathrm{val}_p(p) = 1$. With all these choices having been made, we define the absolute Galois groups $G_F = \mathrm{Gal}(\bar{F}/F)$ and $G_{F_v} = \mathrm{Gal}(\bar{F}_v/F_v)$, and $I_{F_v} \subset G_{F_v}$ will denote the inertia subgroup.

If v is a finite place of F , then we can fix embeddings $\overline{F} \hookrightarrow \overline{F}_v$ which extend the standard embeddings $F \hookrightarrow F_v$. These determine embeddings $G_{F_v} \rightarrow G_F$. We will use q_v to denote the size of the residue field of F at v .

If S is a finite set of finite places of F , then $F_S \subset \overline{F}$ will denote the maximal extension of F unramified outside S , and $G_{F,S} = \text{Gal}(F_S/F)$ its Galois group, naturally a quotient of G_F . If $v \notin S$ is a finite place of F , then the map $G_{F_v} \rightarrow G_{F,S}$ factors through G_{F_v}/I_{F_v} , and $\text{Frob}_v \in G_{F,S}$ will denote the image of a geometric Frobenius element. We will use $\epsilon : G_F \rightarrow \mathbb{Z}_p^\times$ for the p -adic cyclotomic character, and $\bar{\epsilon} : G_F \rightarrow \mathbb{F}_p^\times$ its reduction mod p . If $v \notin S_p$ is a finite place of F , then $\epsilon(\text{Frob}_v) = q_v^{-1}$.

Suppose K/F is a finite extension (inside \overline{F}) and k is a finite field. If $\chi : G_K \rightarrow k^\times$ is a character, then we will let $K(\chi)$ denote the extension of K given by $\overline{F}^{\ker \chi}$, and $k(\chi)$ will be k endowed with a G_K -module structure given by $\sigma \cdot a = \chi(\sigma)a \in k$. We write $k(m)$ for $k(\bar{\epsilon}^m)$. If $\rho : G_K \rightarrow \text{GL}_n(k)$ is a continuous representation, then $\text{ad } \rho$ will denote the space of $n \times n$ matrices over k with a $k[G_K]$ -module structure whose G_K -action is given by conjugation by ρ , i.e. $\sigma \cdot A = \rho(\sigma)A\rho(\sigma)^{-1}$. We will use $\text{ad}^0 \rho$ to denote the $k[G_K]$ -submodule of trace zero matrices.

Let $\rho : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$ denote a continuous representation. Then ρ is de Rham if for each $v \in S_p$, $\rho|_{G_{F_v}}$ is de Rham. In this case, for each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$, we can associate to ρ an n -element multiset $\text{HT}_\tau(\rho)$ of Hodge-Tate weights. We normalize $\text{HT}_\tau(\rho)$ by declaring that $\text{HT}_\tau(\epsilon) = \{-1\}$ for all embeddings τ .

The automorphic side of our arguments are unchanged from [5], but for completeness, we will introduce the notation necessary for the statements of the theorems in Section 4. Let $\mathbb{A}_F = \prod'_v F_v$ denote the adèle ring of F , and $\mathbb{A}_F^\infty = \prod'_{v \neq \infty} F_v$ the finite part. If v is a finite place of F , then we write rec_{F_v} for the local Langlands correspondence for $\text{GL}_2(F_v)$, normalized as in [5]. If π is an irreducible, admissible representation of $\text{GL}_2(F_v)$ over \mathbb{C} , then we define

$$\text{rec}_{F_v}^T(\pi) = \text{rec}_{F_v}(\pi \otimes |\cdot|^{-1/2}).$$

Let Ω be any field isomorphic to \mathbb{C} , and $\chi : W_{F_v} \rightarrow \Omega^\times$ a character with open kernel. We can consider the Weil-Deligne representation

$$\left(\chi \oplus \chi|\cdot|^{-1}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right).$$

Define $\text{St}_2(\chi \circ \text{Art}_{F_v})$ to be the inverse image under $\text{rec}_{F_v}^T$ of this Weil-Deligne representation (here Art_{F_v} denotes the local Artin map). If (r, N) is a Weil-Deligne representation, then $(r, N)^{\text{F-ss}}$ will denote its Frobenius

semi-simplification. If v is a finite place of F and $\rho : G_{F_v} \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a continuous representation, de Rham if $v|p$, then we write $\mathrm{WD}(\rho)$ for the associated Weil-Deligne representation, uniquely determined up to isomorphism.

Finally, the automorphic representations we consider are those cuspidal automorphic representations $\pi = \bigotimes'_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that for $v|\infty$, the representation π_v is the lowest discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with trivial central character, and such a representation will be called a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight 2. With such a representation π , for every isomorphism $\iota : \overline{\mathbb{Q}}_p \rightarrow \mathbb{C}$, there is an associated Galois representation $r_\iota(\pi) : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that:

- (1) The representation $r_\iota(\pi)$ is de Rham, and for every embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$, we have $\mathrm{HT}_\tau(\rho) = \{0, 1\}$.
- (2) If v is a finite place of F , then

$$\mathrm{WD}(r_\iota(\pi)|_{G_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}_{F_v}^T(\iota^{-1}\pi_v).$$

- (3) If $\omega_\pi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ denotes the central character of π , then

$$\det r_\iota(\pi) = \epsilon^{-1} \iota^{-1}(\omega_\pi \circ \mathrm{Art}_F^{-1}),$$

where Art_F is the global Artin map.

See [5] for the details of the local-global compatibility at the unramified places.

3. Galois theory

3.1. Global deformation problem

Let F be a totally real number field. Let p be an odd prime, and let E be a finite extension of \mathbb{Q}_p with integer ring \mathcal{O} and maximal ideal λ . Write k for the residue field of \mathcal{O} , i.e. $k = \mathcal{O}/\lambda$. If A is a complete Noetherian local \mathcal{O} -algebra with residue field k , then we write CNL_A for the category of complete Noetherian local A -algebras with residue field k .

Fix a continuous, absolutely irreducible representation $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(k)$ and a continuous character $\mu : G_F \rightarrow \mathcal{O}^\times$ lifting $\det \bar{\rho}$. We assume k contains the eigenvalues of all elements in the image of $\bar{\rho}$. Let S be a finite set of finite places of F , containing the places dividing p and the places at which $\bar{\rho}$ and μ

are ramified. For $v \in S$, fix a ring $\Lambda_v \in \text{CNL}_{\mathcal{O}}$. Define $\Lambda = \hat{\otimes}_{v \in S} \Lambda_v$, so that $\Lambda \in \text{CNL}_{\mathcal{O}}$.

If $v \in S$, we write $\mathcal{D}_v^{\square} : \text{CNL}_{\Lambda_v} \rightarrow \text{Sets}$ for the functor which takes $R \in \text{CNL}_{\Lambda_v}$ to the set of continuous homomorphisms $r : G_{F_v} \rightarrow \text{GL}_2(R)$ such that $r \bmod \mathfrak{m}_R = \bar{\rho}|_{G_{F_v}}$ and $\det r$ agrees with the composite

$$G_{F_v} \xrightarrow{\mu|_{G_{F_v}}} \mathcal{O}^{\times} \longrightarrow R^{\times}.$$

This functor \mathcal{D}_v^{\square} is represented by an object $R_v^{\square} \in \text{CNL}_{\Lambda_v}$.

Definition 3.1. Let $v \in S$. A local deformation problem for $\bar{\rho}|_{G_{F_v}}$ is a subfunctor $\mathcal{D}_v \subset \mathcal{D}_v^{\square}$ satisfying:

- (i) The subfunctor \mathcal{D}_v is represented by a quotient R_v of R_v^{\square} ,
- (ii) For all $R \in \text{CNL}_{\Lambda_v}$, $a \in \ker(\text{GL}_2(R) \rightarrow \text{GL}_2(k))$ and $r \in \mathcal{D}_v(R)$, we have $ara^{-1} \in \mathcal{D}_v(R)$.

Definition 3.2. A global deformation problem is a tuple

$$\mathcal{S} = (\bar{\rho}, \mu, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S}),$$

where all the notation is as above, and \mathcal{D}_v is a local deformation problem for $\bar{\rho}|_{G_{F_v}}$ for $v \in S$.

3.2. Galois cohomology

Given a finite place v of F and a local deformation problem \mathcal{D}_v , we denote by \mathcal{L}_v the tangent space of the local deformation problem as a subspace $\mathcal{L}_v \subset H^1(F_v, \text{ad}^0 \bar{\rho})$. We also let $\mathcal{L}_v^{\perp} \subset H^1(F_v, \text{ad}^0 \bar{\rho}(1))$ be the annihilator of \mathcal{L}_v induced by the perfect pairing of Galois modules:

$$\text{ad}^0 \bar{\rho} \times \text{ad}^0 \bar{\rho}(1) \rightarrow k(1), \quad (X, Y) \mapsto \text{tr}(XY).$$

Definition 3.3. Given a global deformation problem

$$\mathcal{S} = (\bar{\rho}, \mu, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S}),$$

we define the dual Selmer group

$$H_{\mathcal{S}, T}^1(\text{ad}^0 \bar{\rho}(1)) = \ker \left(H^1(F_S/F, \text{ad}^0 \bar{\rho}(1)) \rightarrow \prod_{v \in S-T} H^1(F_v, \text{ad}^0 \bar{\rho}(1))/\mathcal{L}_v^{\perp} \right).$$

Let us now take a look at the specific deformation problem we want to consider.

3.3. Special deformations, case $q_v \not\equiv 1 \pmod p$

Let $v \in S$ be a prime not dividing p , and suppose that $q_v \not\equiv 1 \pmod p$. Suppose further that $\bar{\rho}|_{G_{F_v}}$ is unramified, and that $\bar{\rho}(\text{Frob}_v)$ has two distinct eigenvalues $\alpha_v, \beta_v \in k$ such that $\beta_v/\alpha_v = q_v$. Let $\Lambda_v = \mathcal{O}$. We define a subfunctor $\mathcal{D}_v^{\text{St}(\alpha_v)} \subset \mathcal{D}_v^\square$ directly. Let $R \in \text{CNL}_{\mathcal{O}}$ and let $r : G_{F_v} \rightarrow \text{GL}_2(R)$ be an element of $\mathcal{D}_v^\square(R)$. If $\phi_v \in G_{F_v}$ is a choice of geometric Frobenius, then by Hensel’s lemma the characteristic polynomial of $r(\phi_v)$ factors as $(X - A_v)(X - B_v)$, where $A_v, B_v \in R^\times$ with A_v lifting α_v and B_v lifting β_v . We will say $r \in \mathcal{D}_v^{\text{St}(\alpha_v)}(R)$ if $B_v = q_v A_v$ and I_{F_v} acts trivially on $(r(\phi_v) - B_v)R^2$, which is a direct summand R -submodule of R^2 . One checks that this condition is independent of the choice of ϕ_v .

Proposition 3.4. *The functor $\mathcal{D}_v^{\text{St}(\alpha_v)}$ is a local deformation problem. The representing object $R_v^{\text{St}(\alpha_v)}$ is formally smooth over \mathcal{O} of dimension 4.*

Proof. That $\mathcal{D}_v^{\text{St}(\alpha_v)}$ is a local deformation problem is easy. Let $R_v^{\text{St}(\alpha_v)}$ denote the representing object. To see that the dimension of $R_v^{\text{St}(\alpha_v)}$ is 4, consider the unframed deformations of this type and its representing object S_v . Then S_v is smooth of relative dimension 1 over \mathcal{O} . There is a map $S_v \rightarrow R_v^{\text{St}(\alpha_v)}$ which is formally smooth, and $R_v^{\text{St}(\alpha_v)}$ is a power series ring over S_v in

$$\dim_k \text{ad } \bar{\rho} - \dim_k H^0(F_v, \text{ad } \bar{\rho}) = 4 - 2 = 2$$

variables. Thus $R_v^{\text{St}(\alpha_v)}$ has relative dimension 3 over \mathcal{O} , as desired. □

3.4. Existence of auxiliary primes

Continue with the notation from the previous section, and assume further that $\bar{\rho}$ is totally odd, i.e. that $\mu(c) = -1$ for all choices of complex conjugation $c \in G_F$. Write $\zeta_p \in \bar{F}$ for a primitive p -th root of unity, and now fix a choice of complex conjugation $c \in G_F$.

We will assume that $\bar{\rho}|_{G_{F(\zeta_p)}}$ is the direct sum of two distinct characters. By Clifford theory, we know that $\bar{\rho}$ is induced from a continuous character $\bar{\chi} : G_K \rightarrow k^\times$, where K is the unique quadratic subfield of $F(\zeta_p)/F$. That is, $\bar{\rho} \cong \text{Ind}_{G_K}^{G_F} \bar{\chi}$. Write $w \in G_F$ for a fixed choice of element with nontrivial

image in $\text{Gal}(K/F)$. We can assume that, possibly after conjugation, that $\bar{\rho}$ has the form:

$$\bar{\rho}(\sigma) = \begin{pmatrix} \bar{\chi}(\sigma) & 0 \\ 0 & \bar{\chi}^w(\sigma) \end{pmatrix}, \quad \text{for } \sigma \in G_K,$$

$$\bar{\rho}(w) = \begin{pmatrix} 0 & \bar{\chi}(w^2) \\ 1 & 0 \end{pmatrix}.$$

Now let $\bar{\gamma} = \bar{\chi}/\bar{\chi}^w$. By assumption, $\bar{\gamma}$ is nontrivial, even after restriction to $G_{F(\zeta_p)}$. We have the following:

Lemma 3.5. *We have that $\text{ad}^0 \bar{\rho}$ decomposes as $\text{ad}^0 \bar{\rho} \cong k(\delta_{K/F}) \oplus \text{Ind}_{G_K}^{G_F} \bar{\gamma}$ as a G_F -module, where $\delta_{K/F} : \text{Gal}(K/F) \rightarrow k^\times$ is the unique nontrivial character.*

From now on, we will let $M_0 = k(\delta_{K/F})$ and $M_1 = \text{Ind}_{G_K}^{G_F} \bar{\gamma}$. Fix the standard basis for $\text{ad}^0 \bar{\rho}$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

If $M \in \text{ad}^0 \bar{\rho}$, we write $k_M \subset \text{ad}^0 \bar{\rho}$ for the line that it spans.

We have the following easy lemma:

Lemma 3.6. *Let $v \nmid p$ be a finite place of F which splits in K , and suppose that the local deformation problem $\mathcal{D}_v = \mathcal{D}_v^{\text{St}(\alpha_v)}$ is defined.*

- (1) *The subspace $\mathcal{L}_v \subset H^1(F_v, \text{ad}^0 \bar{\rho})$ respects the decomposition $\text{ad}^0 \bar{\rho} = M_0 \oplus M_1$. That is,*

$$\mathcal{L}_v = (\mathcal{L}_v \cap H^1(F_v, M_0)) \oplus (\mathcal{L}_v \cap H^1(F_v, M_1)).$$

- (2) *The subspace $\mathcal{L}_v^\perp \subset H^1(F_v, \text{ad}^0 \bar{\rho}(1))$ respects the decomposition $\text{ad}^0 \bar{\rho}(1) = M_0(1) \oplus M_1(1)$.*

Proof. The second part is dual to the first, so we only prove the first part. The fact that $\mathcal{D}_v^{\text{St}(\alpha_v)}$ is defined means $q_v \not\equiv 1 \pmod p$, that $\bar{\rho}|_{G_{F_v}}$ is unramified, and that $\bar{\rho}(\text{Frob}_v)$ takes two distinct eigenvalues $\alpha_v, \beta_v \in k$ with $\beta_v/\alpha_v = q_v$. The fact that v splits in K means $M_0 = k_H$ and $M_1 = k_E(1) \oplus k_F(-1)$ as $k[G_{F_v}]$ -modules. The case $q_v \equiv -1 \pmod p$ was proved in [5]. If $q_v \not\equiv \pm 1 \pmod p$, then \mathcal{L}_v is 1-dimensional, and is contained in $H^1(F_v, M_1)$, being spanned by $H^1(F_v, k_E(1))$. □

Remark 3.7. The difference between this lemma and the corresponding Lemma 5.18 from [5] is that we do not make the assumption that the inducing field K is totally real. However, we do need to make sure that we choose primes of F which split in K for the rest of the method to work.

Let $\mathcal{S} = (\bar{\rho}, \mu, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S})$ be a global deformation problem, and let $T \subset S$ be a subset containing S_p (the set of places above p). Suppose that for $v \in S - T$, the local deformation problem $\mathcal{D}_v = \mathcal{D}_v^{\text{St}(\alpha_v)}$. The lemma implies we can decompose

$$H_{S,T}^1(\text{ad}^0 \bar{\rho}(1)) = H_{S,T}^1(M_0(1)) \oplus H_{S,T}^1(M_1(1)).$$

We now show that we can kill the $M_1(1)$ portion of dual Selmer using the special deformation problem we defined in the previous section, and then kill the $M_0(1)$ portion using traditional Taylor-Wiles primes.

3.5. Killing the $M_1(1)$ portion

First, we show that lemmas 5.21, 5.22, and 5.23 of [5] continue to hold even if K is not totally real. Indeed, the only one which requires proof is the second, since this is the only place where Thorne used this assumption. However, we will need to impose an additional restriction. We state the other two lemmas here for convenience.

Lemma 3.8. *Let Γ be a group, and $\alpha : \Gamma \rightarrow k^\times$ a character. Let $k' \subset k$ be the subfield generated by the values of α . Then $k'(\alpha)$ is a simple $\mathbb{F}_p[\Gamma]$ -module. If $\beta : \Gamma \rightarrow k^\times$ is another character, then $k'(\alpha)$ is isomorphic to a $\mathbb{F}_p[\Gamma]$ -submodule of $k(\beta)$ if and only if there is an automorphism τ of k such that $\beta = \tau \circ \alpha$.*

Proof. See the proof of Lemma 5.21 in [5]. □

Lemma 3.9. *Let $K(\bar{\gamma})$ be the fixed field of $\ker \bar{\gamma}$, let $L = F(\zeta_p) \cap K(\bar{\gamma})$ and assume that $\#\bar{\epsilon}(G_L) > 1$. Then the $\mathbb{F}_p[G_K]$ -module $k(\bar{\epsilon}\bar{\gamma})$ has no Jordan-Holder factors in common with k , $k(\bar{\gamma})$, or $k(\bar{\gamma}^{-1})$. The characters $\bar{\epsilon}\bar{\gamma}$ and $\bar{\gamma}$ are nontrivial.*

Proof. The second claim follows from the fact that $\bar{\gamma}|_{G_{F(\zeta_p)}}$ is nontrivial. For the first claim, we show there are no $\mathbb{F}_p[G_K]$ -module homomorphisms from $k(\bar{\epsilon}\bar{\gamma})$ to $k(\bar{\gamma})$ or $k(\bar{\gamma}^{-1})$. Let $f : k(\bar{\epsilon}\bar{\gamma}) \rightarrow k(\bar{\gamma})$ be such a homomorphism, choose $a \in k(\bar{\epsilon}\bar{\gamma})$, and assume $f(a) = b$. By the hypothesis of the lemma,

there is an element $\tau \in G_L \subset G_K$ such that $\bar{\epsilon}(\tau) \neq 1$ and $\bar{\gamma}(\tau) = 1$. Since f is a $\mathbb{F}_p[G_K]$ -module homomorphism and $\bar{\epsilon}(\tau) \in \mathbb{F}_p^\times$, we know

$$f(\bar{\epsilon}(\tau)\bar{\gamma}(\tau)a) = \bar{\epsilon}(\tau)f(a).$$

On the other hand,

$$f(\bar{\epsilon}(\tau)\bar{\gamma}(\tau)a) = \bar{\gamma}(\tau)b = b.$$

Thus, $\bar{\epsilon}(\tau)b = b$, which implies $b = 0$. Since $a \in k(\bar{\epsilon}\bar{\gamma})$ was arbitrary, this implies $f = 0$. Thus, there are no nontrivial homomorphisms between $k(\bar{\epsilon}\bar{\gamma})$ and $k(\bar{\gamma})$.

The same proof shows there are no nontrivial homomorphisms between $k(\bar{\epsilon}\bar{\gamma})$ and $k(\bar{\gamma}^{-1})$ or k . □

Lemma 3.10. *Let $N \geq 1$ and let $K_N = F(\zeta_{p^N}, \rho_N)$, i.e. K_N is the splitting field of $\rho_N|_{F(\zeta_{p^N})}$. Then $H^1(K_N/F, M_1(1)) = 0$.*

Proof. When K is totally real, this is proved in Lemma 5.23 of [5]. The same proof proves the lemma in the case K is CM using the preceding lemma. □

The following proposition is the analog of Proposition 5.20 of [5] and is the only place where we argue differently from Thorne because of not having (in the case that K is not totally real) the luxury to choose places v such that $\rho_N(\text{Frob}_v)$ is the image of complex conjugation under ρ_N . The proof relies on the simple observation that given an element g in $\text{GL}_2(\mathcal{O}/p^M)$, then for $N \gg 0$, the element g^{q^N} has (ratio of) eigenvalues that are the Teichmüller lift of the (ratio of) the eigenvalues of the reduction of g .

Proposition 3.11. *Let $\mathcal{S} = (\bar{\rho}, \mu, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S})$ be a global deformation problem, and let $T = S$. Let $N_0 \geq 1$ be an integer. Let $\rho : G_F \rightarrow \text{GL}_2(\mathcal{O})$ be a lifting of type \mathcal{S} . Let $K(\bar{\gamma}\bar{\epsilon})$ (resp. $K(\bar{\gamma}\bar{\epsilon}^{-1})$) be the fixed field of $\ker \bar{\gamma}\bar{\epsilon}$ (resp. $\ker \bar{\gamma}\bar{\epsilon}^{-1}$), and assume that $F(\zeta_p) \not\subset K(\bar{\gamma}\bar{\epsilon}) \cap K(\bar{\gamma}\bar{\epsilon}^{-1})$. Then for any $m \geq h_{\mathcal{S}, T}^1(M_1(1))$, there exists a set Q_0 of primes, disjoint from S , and elements $\alpha_v \in k^\times$, satisfying the following:*

- (1) $\#Q_0 = m$
- (2) For each $v \in Q_0$, the local deformation problem $\mathcal{D}_v^{\text{St}(\alpha_v)}$ is defined. We define the augmented deformation problem

$$\mathcal{S}_{Q_0} = (\bar{\rho}, \mu, S \cup Q_0, \{\Lambda_v\}_{v \in S} \cup \{\mathcal{O}\}_{v \in Q_0}, \{\mathcal{D}_v\}_{v \in S} \cup \{\mathcal{D}_v^{\text{St}(\alpha_v)}\}_{v \in Q_0}).$$

- (3) Let $\rho_{N_0} = \rho \bmod \lambda^{N_0} : G_F \rightarrow \mathrm{GL}_2(\mathcal{O}/\lambda^{N_0}\mathcal{O})$. Then $\rho_{N_0}(\mathrm{Frob}_v)$ has distinct eigenvalues whose ratio is q_v for each $v \in Q_0$.
- (4) $H_{\mathcal{S}_{Q_0}, T}^1(M_1(1)) = 0$.

Proof. We wish to find a set Q_0 of primes such that $h_{\mathcal{S}_{Q_0}, T}^1(M_1(1)) = 0$. Suppose $r = h_{\mathcal{S}, T}^1(M_1(1)) \geq 0$. Using induction, it suffices to find a single prime v satisfying the conditions of the theorem such that $h_{\mathcal{S}_{\{v\}}, T}^1(M_1(1)) = \max(r - 1, 0)$. The case $r = 0$ is easy, so assume $r \geq 1$.

Let $0 \neq [\varphi] \in H_{\mathcal{S}, T}^1(M_1(1))$ be a nonzero class. We wish to find a place $v \notin \mathcal{S}$ such that:

- (i) v splits in K
- (ii) $\rho_{N_0}(\mathrm{Frob}_v)$ has distinct eigenvalues with ratio $q_v \bmod \lambda^{N_0}$.
- (iii) $q_v \not\equiv 1 \bmod \lambda^{N_0}$
- (iv) $\varphi(\mathrm{Frob}_v) \neq 0$ ($\in M_1(1)$).

Indeed, the first three conditions imply that $\mathcal{D}_v^{\mathrm{St}(\alpha_v)}$ is defined for the appropriate choice of α_v . We also have an exact sequence

$$0 \rightarrow H_{\mathcal{S}_{\{v\}}, T}^1(M_1(1)) \rightarrow H_{\mathcal{S}, T}^1(M_1(1)) \rightarrow k.$$

If $q_v \not\equiv \pm 1 \bmod \lambda$, then the last map in the sequence comes from the restriction map $H_{\mathcal{S}, T}^1(M_1(1)) \rightarrow H^1(G_{F_v}, M_1(1)) \cong k$. If $q_v \equiv -1 \bmod \lambda$, then this last map is either $\phi \mapsto \langle E, \phi(\mathrm{Frob}_v) \rangle$ (if $\alpha_v = 1$) or $\phi \mapsto \langle F, \phi(\mathrm{Frob}_v) \rangle$ (if $\alpha_v = -1$). By choosing α_v appropriately, we can ensure the sequence is exact on the right. Condition (iv) implies the final map is surjective, which gives $h_{\mathcal{S}_{\{v\}}, T}^1(M_1(1)) < h_{\mathcal{S}, T}^1(M_1(1))$, as desired.

By the Chebotarev density theorem, it suffices to find an element $\sigma \in G_K$ such that:

- (a) $\rho_{N_0}(\sigma)$ has distinct eigenvalues with ratio $\epsilon(\sigma) \bmod \lambda^{N_0}$.
- (b) $\epsilon(\sigma) \not\equiv 1 \bmod \lambda^{N_0}$
- (c) $\varphi(\sigma) \neq 0$.

If $N_0 = 1$, then the assumption in the Proposition ensures we can find σ_1 in G_K such that $\bar{\gamma}(\sigma_1) = \epsilon(\sigma_1) \bmod \lambda$. Indeed, the assumption $F(\zeta_p) \not\subset K(\bar{\gamma}\bar{\epsilon}) \cap K(\bar{\gamma}\bar{\epsilon}^{-1})$ ensures that either $G_{K(\bar{\gamma}\bar{\epsilon})}$ or $G_{K(\bar{\gamma}\bar{\epsilon}^{-1})}$ is not contained in $G_{F(\zeta_p)}$. This means there exists σ_1 in either $G_{K(\bar{\gamma}\bar{\epsilon})}$ or $G_{K(\bar{\gamma}\bar{\epsilon}^{-1})}$ such that

$\epsilon(\sigma_1) \not\equiv 1 \pmod p$. In the latter case, we find our desired σ_1 . In the former case, by exchanging the roles of the eigenvalues, we get our desired σ_1 .

If $\varphi(\sigma_1) \neq 0$, then take $\sigma = \sigma_1$, so suppose $\varphi(\sigma_1) = 0$. We have the inflation-restriction sequence:

$$0 \rightarrow H^1(K_1/F, M_1(1)^{G_{K_1}}) \rightarrow H^1(F, M_1(1)) \rightarrow H^1(K_1, M_1(1))^{\text{Gal}(K_1/F)}.$$

By the previous lemma, the first group is zero, so the image of φ in $H^1(K_1, M_1(1))$ is nonzero. This restriction is a nonzero homomorphism $\varphi|_{G_{K_1}} : G_{K_1} \rightarrow M_1(1)$. Thus, we can find $\tau \in G_{K_1}$ such that $\varphi(\tau) \neq 0$. Then take $\sigma = \tau\sigma_1$. Then

$$\bar{\rho}(\sigma) = \bar{\rho}(\tau)\bar{\rho}(\sigma_1) = \bar{\rho}(\sigma_1)$$

as $\tau \in \ker(\bar{\rho})$. We also find

$$\epsilon(\sigma) = \epsilon(\tau)\epsilon(\sigma_1) \equiv \epsilon(\sigma_1) \pmod \lambda$$

as $\epsilon(\tau) \equiv 1 \pmod \lambda$. Thus, $\bar{\gamma}(\sigma) \equiv \epsilon(\sigma) \pmod \lambda$. Moreover,

$$\varphi(\sigma) = \varphi(\tau) + \varphi(\sigma_1),$$

meaning $\varphi(\sigma) \neq 0$, as required.

If $N_0 > 1$, then consider the element σ_1 defined above. Then $\rho_{N_0}(\sigma_1)$ has distinct eigenvalues by Hensel’s lemma, and we know this ratio modulo λ is $\bar{\gamma}(\sigma_1) \equiv \epsilon(\sigma_1) \pmod \lambda$. Consider $\sigma_{N_0} = \sigma_1^{q^M}$ for some M to be determined and $q = \#k$. For some sufficiently high power of M , $\epsilon(\sigma_{N_0}) \pmod \lambda^{N_0}$ will be the Teichmuller lifting of $\epsilon(\sigma_1) \pmod \lambda$ to the $\pmod \lambda^{N_0}$ ring (indeed, $M = q^{N_0-1}$ should do). But since $\bar{\gamma}(\sigma_1) \equiv \epsilon(\sigma_1) \pmod \lambda$, we deduce that the ratio of the eigenvalues of $\rho_{N_0}(\sigma_{N_0})$ will be equivalent to $\epsilon(\sigma_{N_0}) \pmod \lambda^{N_0}$.

We still need to make sure $\varphi(\sigma) \neq 0$. If $\varphi(\sigma_{N_0}) \neq 0$, then we can take $\sigma = \sigma_{N_0}$. If $\varphi(\sigma_{N_0}) = 0$, consider $\tau \in G_{K_N}$ with $\varphi(\tau) \neq 0$ as before. Let $\sigma = \tau\sigma_{N_0}$. By the same reasoning as in the $N_0 = 1$ case, the ratio of the eigenvalue of $\rho_N(\sigma)$ will still be equivalent to $\epsilon(\sigma) \pmod \lambda^{N_0}$, and moreover $\varphi(\sigma) = \varphi(\tau) + \varphi(\sigma_{N_0}) \neq 0$ by construction. This concludes the proof. □

Remark 3.12. Note that the assumption $F(\zeta_p) \not\subset K(\bar{\gamma}\bar{\epsilon}) \cap K(\bar{\gamma}\bar{\epsilon}^{-1})$ is implied by the more checkable condition that $(\#\bar{\epsilon}(G_L), \#\bar{\gamma}(G_L)) > 1$, where $L = F(\zeta_p) \cap K(\bar{\gamma})$. Indeed, the condition implies that there exist $\sigma, \tau \in G_L$ such that $\bar{\epsilon}(\sigma) = \bar{\gamma}(\tau)$. Since $\bar{\epsilon}$ and $\bar{\gamma}$ induce maps on $\text{Gal}(F(\zeta_p)/L)$ and $\text{Gal}(K(\bar{\gamma})/L)$, respectively, we can project σ and τ to the quotient groups, yielding (non-identity) elements $\bar{\sigma} \in \text{Gal}(F(\zeta_p)/L)$ and $\bar{\tau} \in \text{Gal}(K(\bar{\gamma})/L)$

such that $\bar{\epsilon}(\bar{\sigma}) = \bar{\gamma}(\bar{\tau}) \in k^\times$. Letting $M = K(\bar{\gamma})F(\zeta_p)$, we can find an element $\bar{\nu} \in \text{Gal}(M/L)$ such that $\bar{\nu}|_{K(\bar{\gamma})} = \bar{\tau}$ and $\bar{\nu}|_{F(\zeta_p)} = \bar{\sigma}$. Lifting $\bar{\nu}$ to G_L produces an element $\nu \in G_L$ such that $\bar{\gamma}(\nu) = \bar{\epsilon}(\nu)$, meaning $\nu \in G_{K(\bar{\gamma}\bar{\epsilon}^{-1})}$. However, $\nu \notin G_{F(\zeta_p)}$ since $\bar{\sigma}$ was not the identity element in $\text{Gal}(F(\zeta_p)/F)$, and the claim follows.

3.6. Killing the $M_0(1)$ portion

Having killed the $M_1(1)$ portion of dual Selmer, we can try and get auxiliary primes that take care of the remaining part of the group. Indeed, we have the following proposition:

Proposition 3.13. *Let $\mathcal{S} = (\bar{\rho}, \mu, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S})$ be a global deformation problem. Let $T \subset S$, and suppose for $v \in S - T$ we have $\mathcal{D}_v = \mathcal{D}_v^{\text{St}(\alpha_v)}$. Suppose further that $h_{\mathcal{S}, T}^1(M_1(1)) = 0$, and let $N_1 \geq 1$ be an integer. Then there exists a finite set Q_1 of finite places of F , disjoint from S , satisfying:*

- (1) *We have $\#Q_1 = h_{\mathcal{S}, T}^1(M_0(1))$, and for each $v \in Q_1$, the norm $q_v \equiv 1 \pmod{p^{N_1}}$ and $\bar{\rho}(\text{Frob}_v)$ has distinct eigenvalues.*
- (2) *Define the augmented deformation problem*

$$\mathcal{S}_{Q_1} = (\bar{\rho}, \mu, S \cup Q_1, \{\Lambda_v\}_{v \in S} \cup \{\mathcal{O}\}_{v \in Q_1}, \{\mathcal{D}_v\}_{v \in S} \cup \{\mathcal{D}_v^\square\}_{v \in Q_1}).$$

Then $h_{\mathcal{S}_{Q_1}, T}^1(\text{ad}^0 \bar{\rho}(1)) = 0$.

Proof. See the proof of Proposition 5.24 in [5]. □

4. The main theorem

From this point, everything from sections 6 of [5] carries over. We can now work towards proving the main theorem. We proceed as in the aforementioned paper, making the necessary modifications. First, some preliminary results.

Lemma 4.1. *Let F be a totally real number field, and let F'/F be a totally real, soluble extension. Let p be a prime and let $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ be a fixed isomorphism.*

- (1) *Let π be a cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ of weight 2, and suppose that $r_\iota(\pi)|_{G_{F'}}$ is irreducible. Then there exists a cuspidal*

automorphic representation $\pi_{F'}$ of $\mathrm{GL}_2(\mathbb{A}_{F'})$ of weight 2, called the base change of π , such that $r_\iota(\pi_{F'}) \cong r_\iota(\pi)|_{G_{F'}}$.

- (2) *Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous representation such that $\rho|_{G_{F'}}$ is irreducible. Let π' be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{F'})$ of weight 2 with $\rho|_{G_{F'}} \cong r_\iota(\pi')$. Then there exists a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight 2 such that $\rho \cong r_\iota(\pi)$.*

Proof. This is stated in [5], Lemma 5.1. The proof follows from results of [4], using arguments of [1], Lemma 1.3. □

Theorem 4.2. *Let F be a totally real field, and let p be an odd prime. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous representation. Suppose that:*

- (1) *[$F : \mathbb{Q}$] is even.*
- (2) *Letting K be the quadratic subfield of $F(\zeta_p)/F$, there exists a continuous character $\bar{\chi} : G_K \rightarrow \overline{\mathbb{F}}_p^\times$ such that $\bar{\rho} \cong \mathrm{Ind}_{G_K}^{G_F} \bar{\chi}$.*
- (3) *Letting $w \in \mathrm{Gal}(K/F)$ be the nontrivial element, the character $\bar{\gamma} = \bar{\chi}/\bar{\chi}^w$ remains nontrivial even after restriction to $G_{F(\zeta_p)}$ (in particular, $\bar{\rho}$ is irreducible).*
- (4) *We have $F(\zeta_p) \not\subset K(\bar{\gamma}\bar{\epsilon}^{-1}) \cap K(\bar{\gamma}\bar{\epsilon})$.*
- (5) *The character $\psi = \epsilon \det \rho$ is everywhere unramified.*
- (6) *The representation ρ is almost everywhere unramified.*
- (7) *For each place $v|p$, $\rho|_{G_{F_v}}$ is semi-stable, and $\bar{\rho}|_{G_{F_v}}$ is trivial. For each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$, we have $HT_\tau(\rho) = \{0, 1\}$.*
- (8) *If $v \nmid p$ is a finite place of F at which ρ is ramified, then $q_v \equiv 1 \pmod p$, $WD(\rho|_{G_{F_v}})^{F-ss} \cong \mathrm{rec}_{F_v}^T(\mathrm{St}_2(\chi_v))$ for some unramified character $\chi_v : F_v^\times \rightarrow \overline{\mathbb{Q}}_p^\times$, and $\bar{\rho}|_{G_{F_v}}$ is trivial. The number of such places is even.*
- (9) *There exists a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight 2 and an isomorphism $\iota : \overline{\mathbb{Q}}_p \rightarrow \mathbb{C}$ satisfying:*
 - (a) *There is an isomorphism $r_\iota(\pi) \cong \bar{\rho}$.*
 - (b) *If $v|p$ and ρ is ordinary, then π_v is ι -ordinary and $\pi_v^{U_0(v)} \neq 0$. If $v|p$ and ρ is non-ordinary, then π_v is not ι -ordinary and π_v is unramified.*

- (c) *If $v \nmid p\infty$ and $\rho|_{G_{F_v}}$ is unramified, then π_v is unramified. If $v \nmid p\infty$ and $\rho|_{G_{F_v}}$ is ramified, then π_v is an unramified twist of the Steinberg representation.*

Then ρ is automorphic: there exists a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight 2 and an isomorphism $\rho \cong r_\iota(\pi')$.

Remark 4.3. Here, $U_0(v)$ is the set of matrices in $\mathrm{GL}_2(\mathcal{O}_{F_v})$ whose reduction modulo a fixed uniformizer of \mathcal{O}_{F_v} is upper triangular.

Proof. This is Theorem 7.2 [5] with the necessary modifications (namely, the addition of condition (4) instead of the condition that K be totally real). One can just repeat the proof the author gives in that paper, replacing Proposition 5.20 of loc. cit. with Proposition 2.11 from the previous section. □

Using this theorem, we arrive at the main theorem.

Theorem 4.4. *Let F be a totally real number field, let p be an odd prime, and let $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a continuous representation satisfying the following:*

- (1) *The representation ρ is almost everywhere unramified.*
- (2) *For each $v|p$ of F , the local representation $\rho|_{G_{F_v}}$ is de Rham. For each embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$, we have $HT_\tau(\rho) = \{0, 1\}$.*
- (3) *For each complex conjugation $c \in G_F$, we have $\det \rho(c) = -1$.*
- (4) *The residual representation $\bar{\rho}$ is absolutely irreducible, but $\bar{\rho}|_{G_{F(\zeta_p)}}$ is a direct sum of two distinct characters. Further suppose that if K is the unique quadratic subfield of $F(\zeta_p)/F$ and $\bar{\gamma} : G_K \rightarrow k^\times$ is the ratio of the two characters, then we have $F(\zeta_p) \not\subset K(\bar{\gamma}\bar{\epsilon}^{-1}) \cap K(\bar{\gamma}\bar{\epsilon})$.*

Then ρ is automorphic: there exists a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight 2, an isomorphism $\iota : \overline{\mathbb{Q}}_p \rightarrow \mathbb{C}$, and an isomorphism $\rho \cong r_\iota(\pi)$.

Proof. The proof is exactly the same as Theorem 7.4 of [5], replacing Theorem 7.2 of loc. cit. with Theorem 4.2 above. The idea is to construct a soluble extension F'/F such that $\bar{\rho}|_{G_{F'}}$ satisfies the conditions of Theorem 4.2 above. We then apply Lemma 4.1 to deduce the automorphy of ρ . We should note that in [5], the author makes use of Corollary 7.4 in loc. cit., but that

goes unchanged for us because that corollary made no assumptions on the quadratic subfield K . □

5. Application to elliptic curves

We can apply this theorem to elliptic curves. In a paper of Freitas, Le Hung, and Siksek (see [2]) the authors prove there are only finitely many non-automorphic elliptic curves over any given totally real field. We can use the theorem above to prove automorphy of elliptic curves E defined over totally real fields F in some cases not covered in the existing literature. This comes about by applying our theorem to the 7-adic representations arising from E . Note that the assumptions of [5] imply that the theorem in that paper can be applied only to p -adic representations of elliptic curves with p congruent to 1 modulo 4.

The following theorem is [2], Theorems 3 and 4.

Theorem 5.1. *Let $p \in \{3, 5, 7\}$. Let E be an elliptic field over a totally real field F and let $\bar{\rho}_{E,p} : G_F \rightarrow \text{GL}_2(\mathbb{F}_p)$ be the representation given by the action on the p -torsion of E . If $\bar{\rho}_{E,p}(G_{F(\zeta_p)})$ is absolutely irreducible, then E is modular.*

We will call an elliptic curve p -bad if $E[p]$ is an absolutely reducible $\mathbb{F}_p[G_{F(\zeta_p)}]$ -module. Otherwise E is p -good. The theorem tells us the only elliptic curves E which are potentially non-modular are those which are p -bad for $p = 3, 5$, and 7. In [5], the author deals with some of these remaining cases:

Theorem 5.2. *Let E be an elliptic curve over a totally real field F . Suppose:*

- (1) 5 is not a square in F .
- (2) E has no F -rational 5-isogeny.

Then E is modular.

We can prove a similar theorem, but before doing so we recall [2], Proposition 9.1.

Proposition 5.3. *Let F be a totally real number field and let E be an elliptic curve over F . Suppose $F \cap \mathbb{Q}(\zeta_7) = \mathbb{Q}$ and write $\bar{\rho} = \bar{\rho}_{E,7}$. Suppose $\bar{\rho}$*

is irreducible but $\bar{\rho}(G_{F(\zeta_7)})$ is absolutely reducible. Then $\bar{\rho}(G_F)$ is conjugate in $\mathrm{GL}_2(\mathbb{F}_7)$ to one of the groups

$$H_1 = \left\langle \left(\begin{array}{cc} 3 & 0 \\ 0 & 5 \end{array} \right), \left(\begin{array}{cc} 0 & 2 \\ 2 & 0 \end{array} \right) \right\rangle, \quad H_2 = \left\langle \left(\begin{array}{cc} 0 & 5 \\ 3 & 0 \end{array} \right), \left(\begin{array}{cc} 5 & 0 \\ 3 & 2 \end{array} \right) \right\rangle.$$

The group H_1 has order 36 and is contained as a subgroup of index 2 in the normalizer of a split Cartan subgroup. The group H_2 has order 48 and is contained as a subgroup of index 2 in the normalizer of a non-split Cartan subgroup. The images of H_1 and H_2 in $\mathrm{PGL}_2(\mathbb{F}_7)$ are isomorphic to $D_3 \cong S_3$ and D_4 , respectively.

Theorem 5.4. *Let F be a totally real field, and let E be an elliptic curve over F . Suppose:*

- (1) $F \cap \mathbb{Q}(\zeta_7) = \mathbb{Q}$.
- (2) E has no F -rational 7-isogeny.
- (3) Either $\bar{\rho}_{E,7}(G_{F(\zeta_7)})$ is absolutely irreducible, or it is reducible and $\bar{\rho}_{E,7}(G_F)$ is conjugate to the group H_1 from the previous proposition.

Then E is modular.

We can extend Theorem 5.4 to primes other than $p = 7$, and we prove this more general version.

Theorem 5.5. *Let F be a totally real field, and let E be an elliptic curve over F . Let $p \geq 7$ be a prime such that $(p - 1)/2 = q^n$ for some odd prime q and $n \geq 1$. Suppose:*

- (1) $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$.
- (2) E has no F -rational p -isogeny.
- (3) $\bar{\rho}_{E,p}(G_F)$ normalizes a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$.

Then E is modular.

Proof. Let $\rho : G_F \rightarrow \mathrm{GL}_2(\mathbb{Q}_p)$ be the representation given by the action of G_F on the étale cohomology $H^1(E_{\bar{F}}, \mathbb{Z}_p)$, after a choice of basis. The goal is to show ρ is automorphic. Hypothesis (2) is equivalent to $\bar{\rho} = \bar{\rho}_{E,p}$ being irreducible, hence absolutely irreducible because of complex conjugation. Hypothesis (3) says $\bar{\rho}(G_F)$ is contained in the normalizer of a split Cartan subgroup. Note that the absolute irreducibility of $\bar{\rho}$ implies the projective

image is non-cyclic, for if it were cyclic, the image of $\bar{\rho}$ would be abelian. Thus, $\bar{\rho}(G_{F(\zeta_p)})$ cannot be scalar, since $\text{Gal}(F(\zeta_p)/F)$ is cyclic, and hence cannot surject onto a non-cyclic group.

Let K be the quadratic subfield of $F(\zeta_p)/F$, so that $[F(\zeta_p) : K] = q^n = (p-1)/2$ by hypothesis (1). Let $\bar{\gamma} : G_K \rightarrow \mathbb{F}_p^\times$ be the character which gives the ratio of eigenvalues of $\bar{\rho}|_{G_K}$. We want to examine $[K(\bar{\gamma}) : K]$, where $K(\bar{\gamma}) = \bar{F}^{\ker(\bar{\gamma})}$ as always. In particular, we will show that $K(\bar{\gamma}) \cap F(\zeta_p)$ is a field L which satisfies $(\#\bar{\epsilon}(G_L), \#\bar{\gamma}(G_L)) > 1$, which implies hypothesis (4) of the main theorem. Note that hypothesis (1) implies that, as a character of G_K , that $\bar{\epsilon}$ takes values in $(\mathbb{F}_p^\times)^2$.

Using the fact that $\det \bar{\rho}$ is the mod p cyclotomic character, we find that $\bar{\chi}\bar{\chi}^w = \bar{\epsilon}$, so that $\bar{\gamma} = \bar{\chi}/\bar{\chi}^w = \bar{\chi}^2\bar{\epsilon}^{-1}$, which is a character $G_K \rightarrow (\mathbb{F}_p^\times)^2$. Thus, the order of $\bar{\gamma}$ divides $(p-1)/2 = q^n$, and moreover cannot equal 1 as $\bar{\gamma}$ is a nontrivial character of G_K . Thus, $1 < [K(\bar{\gamma}) : K] | q^n$. Moreover, $[F(\zeta_p) : K] = q^n$ by hypothesis (1) of the theorem. Lastly, we know $K(\bar{\gamma}) \not\subseteq F(\zeta_p)$ since $\bar{\gamma}$ is nontrivial upon restriction to $G_{F(\zeta_p)}$, and thus $K(\bar{\gamma}) \cap F(\zeta_p)$ is neither $K(\bar{\gamma})$ nor $F(\zeta_p)$. This intersection is therefore a field L which satisfies $(\#\bar{\epsilon}(G_L), \#\bar{\gamma}(G_L)) > 1$ as q divides both quantities. Thus, hypothesis (4) of the main theorem is satisfied, and therefore E is modular. \square

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