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Lyapunov exponents of the Hodge bundle over strata of quadratic differentials with large number of poles

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We show an upper bound for the sum of positive Lyapunov exponents of any Teichmüller curve in strata of quadratic differentials with at least one zero of large multiplicity. As a corollary, it holds for any $\mathrm{SL}(2,\mathbb{R})$-invariant submanifold defined over $\mathbb{Q}$ in these strata. This proves Grivaux-Hubert's conjecture about the asymptotics of Lyapunov exponents for strata with a large number of poles in the situation when at least one zero has high multiplicity.

1. Introduction

Lyapunov exponents of translation surfaces were introduced two decades ago by Zorich in [Zor99] and [Zor97] as dynamical invariants which describe how associated leaves *wind around the surface*. On any translation surface we can introduce a translation flow which generalizes the linear flow on a flat torus (see [Zor06] for an introduction to the subject). This flow has a very simple local dynamic — it is a parabolic system. Nonetheless the homology of the translation flow presents a rich asymptotic behaviour and its deviation from the asymptotic cycle is described by Lyapunov exponents of the Hodge bundle over a $\mathrm{SL}_2(\mathbb{R})$ invariant subspace in the moduli space of curves.

Even if a numerical approximation of these exponents is accessible [D^+16], there is *a priori* no hope for an explicit formula to compute them. Yet a breakthrough of Eskin, Kontsevich and Zorich showed an astonishing formula binding the sum of the Lyapunov exponents to the Siegel-Veech constant of the invariant locus [EKZ14]. This theorem followed an insightful observation of [Kon97] that this sum is related to the degree of a Hodge subbundle, which was proven later in [For02]. This was a starting point to evaluate Lyapunov exponents in certain particularly symmetric cases, for example for square-tiled cyclic covers [FMZ14], [EKZ11] and triangle groups [BM10]; and to compute explicitly diffusion rate of wind-tree models [DHL14], [DZ15].

Other advances have since been made to estimate Lyapunov exponents for higher genus. In [Yu14], Yu gave a partial proof of the conjecture of [KZ97] that the second Lyapunov exponent for hyperelliptic components of strata should go to 1 as the genus goes to infinity. His proof was conditional on a conjecture he introduced in the same article which brought new ideas to find bounds from below to these exponents. This conjecture has recently been proven in [EKMZ16] (see [DD15] for a foliation-theoretic point of view). Yu also obtained an upper bound for the sum of Lyapunov exponents with respect to Weierstrass gaps. Yu's idea exposed in [YZ12] and [YZ13] that independently appeared in [CM12] and [CM14] is to use algebraic characterization of sum of Lyapunov exponents and estimate it with homological algebra arguments.

In parallel Grivaux and Hubert remarked that some Teichmüller curves whose Lyapunov exponents are all zero can appear in quadratic strata. Moreover in [GH14] they prove that for this to happen the curve should be in a stratum with at least $\max(2g-2,2)$ poles. The heuristic explanation they provide for this result is that *poles slow down the linear flow*, since passing by a pole makes the flow to retrace its steps. This leads them to the following conjecture transmitted to the author by oral communication.

Conjecture (Grivaux-Hubert). *Positive Lyapunov exponents associated to a translation surface have a uniform bound depending only on the number of poles of the surface. This bound goes to zero when the number of poles goes to infinity.*

Pascal Hubert pointed out later on that Remark 2 of [DZ15] gives a counter-example to the conjecture in this generality. Taking a cover of a base surface, Delecroix and Zorich exhibited a family of surfaces of genus 1 with an arbitrarily large number of poles, in which the first Lyapunov exponent is always equal to $\frac{2}{3}$.

Nevertheless, in [CM14] Theorem 8.1, D. Chen and M. Möller obtained a result in this direction for $\mathcal{Q}(n,-1^n)$ and $\mathcal{Q}\left(n,1,-1^{n+1}\right)$, showing that these strata are non-varying and computing explicitly the sum of Lyapunov exponents for every Teichmüller curve which is equal to $2/(n+2)$.

Inspired by Yu's homological methods we obtain a general upper bound in terms of the highest multiplicity of zeros and genus. This proves the conjecture in this case.

Theorem. *For any Teichmüller curve $\mathcal{C}$ in a quadratic stratum $\mathcal{Q}(m_1, \ldots, m_k, -1^p)$ of genus g where $m_1 \geq m_2 \geq \cdots \geq m_k$, if $m_1 \geq 2g$, then*

$$L^+(\mathcal{C}) \leq \frac{(3g-1)g}{m_1+2},$$

where $L^+(\mathcal{C})$ stands for the sum of its positive Lyapunov exponents.

Using recent advances in the theory of $\mathrm{SL}(2,\mathbb{R})$-invariant submanifold, we obtain as a corollary a very general statement.

Corollary. *For any half-translation surface in the stratum $\mathcal{Q}(m_1, \ldots, m_k, -1^p)$ of genus g where $m_1 \geq m_2 \geq \cdots \geq m_k$, if $m_1 \geq 2g$ and its $\mathrm{SL}(2,\mathbb{R})$-orbit closure $\mathcal{N}$ contains some square tiled surface, then the following inequality holds:*

$$L^+(\mathcal{N}) \leq \frac{(3g-1)g}{m_1+2},$$

where $L^+(\mathcal{N})$ stands for the sum of the positive Lyapunov exponents associated to orbit closure.

Another direction in which the conjecture can be studied is in the case of whole strata. Developing a tool to compute Lyapunov exponents for quadratic differentials (available in [D$^+$16]), we have performed broad computer experiments on strata with number of zeros and poles both going to infinity. These experiments tend to show that the sum of their Lyapunov exponents goes to zero at speed $\frac{1}{\sqrt{p}}$. Consequently, we make the following conjecture,

Conjecture. *Let Q_n be a sequence of connected components of strata for a fixed genus and p_n their number of poles. If p_n goes to infinity, then,*

$$\lambda_1^+(Q_n) = \mathcal{O}(1/\sqrt{p_n}).$$

2. Background material

Strata

A half-translation surface is a pair (X, q) where X is a Riemann surface and q is a quadratic differential on X with at worst simple poles. If $S(q)$ is the set of zeros and poles of q on X, we can endow $\tilde{X} := X \backslash S(q)$ with charts $\phi_i : U_i \to X$ such that $\phi_i^* q = dz^{\otimes 2}$. In such an atlas, the transition functions are translations composed with $\pm\,\mathrm{Id}$. The quadratic differential induces a flat metric $|q|$ on the open surface $\tilde{X}$. This metric can be extended to the whole surface and has conical points at $S(q)$ with angles multiples of π. If we fix integers $m_1, \ldots, m_k, p$ such that $\sum_{i=1}^{k} m_i - p = 4g - 4$ for some positive integer g, the stratum of half-translation surfaces $\mathcal{Q}\,(m_1, \ldots, m_k, -1^p)$ is the set of half-translation surfaces (X, q) where q has k distinct zeros with multiplicity $m_1, \ldots, m_k$ and p simple poles. The projectivized space $P\mathcal{Q}\,(m_1, \ldots, m_k, -1^p)$ is obtained by taking the quotient under the scalar action of $\mathbb{C}^*$ on the quadratic differential.

Teichmüller curves

There is a natural action of $\mathrm{GL}(2, \mathbb{R})$ on each stratum. In the flat atlas above, the charts were constructed in such a way that the transition functions are translations by vectors v_{ij} composed with $\pm\,\mathrm{Id}$. For a matrix $M \in \mathrm{GL}(2, \mathbb{R})$ we define a new surface by multiplying the previous change of charts by M. These new transition maps will be translations by vectors Mv_{ij} composed with $\pm\,\mathrm{Id}$.

Let $\mathbb{H} = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ denote the Poincaré upper half-plane. For any (X, q) the action of $SL(2, \mathbb{R})$ factors to a map

$$\mathbb{H} \to P\mathcal{Q}\,(m_1, \ldots, m_k, -1^p)$$

which is an immersion. The image of this map is called a Teichmüller disk. This map also factors through the Veech group which is its stabilizer. If the Veech group Γ of a given half-translation surface is a lattice in $\mathrm{SL}(2, \mathbb{R})$ we say that the surface is a Veech surface and the image of $\mathbb{H}/\Gamma$ in the projective stratum is called a Teichmüller curve. Another point of view for the Teichmüller curve is to consider the induced algebraic map $\mathcal{C} \to \mathcal{M}_g$. This is the convention used in [Möl13].

By definition, a Teichmüller curve is a surface with hyperbolic structure — which may have cusps — parametrizing a family of curves. A main tool for studying these curves is their compactification at the cusps. It is a general fact that we can extend a family of curves over a punctured disc $f : \mathcal{X} \to B \setminus \{0\}$ by taking a branched cover $B' \to B$ of the base ramified over 0 and a family $f : \mathcal{X} \to B'$ of stable curves extending the fiber product $\mathcal{X} \times_{B\setminus\{0\}} B'$. This is called stable reduction (see [HM98] Theorem 3.47). For a Teichmüller curve $f : \mathcal{X} \to C$, this redution can be made globally on the base curve, which means that there exists a finite-index cover $\overline{B} \to \overline{C}$ ramified over the boundary of C such that we can define a family $\overline{f} : \overline{\mathcal{X}} \to \overline{B}$ which extends f, and that the fibers at the boundary of B are stable curves.

The structure sheaf of $\overline{\mathcal{X}}$ will be denoted by $\mathcal{O}$, the dualizing sheaf of $\overline{\mathcal{X}} \to \overline{B}$ by $\omega_{\overline{\mathcal{X}}/\overline{B}}$, and the opposite of the Euler characteristic of $\overline{B}$ by χ.

Divisors classes

For a Teichmüller curve generated by a quadratic differential q, there is an associated line bundle $\mathcal{L} \subset f_* \omega^{\otimes 2}_{\overline{\mathcal{X}}/\overline{B}}$ on $\overline{B}$ whose fiber over the point corresponding to (X, q) is $\mathbb{C} \cdot q$. This bundle has a maximality property (see [Möl13] Section 2, or Theorem of [Gou12]) which implies that $\deg(\mathcal{L}) = \chi$.

We denote by $Z_1, \dots, Z_k$ and $P_1, \dots, P_p$ sections of $\overline{\mathcal{X}} \to \overline{B}$, which intersect the fibers at each zero with multiplicity $m_1, \dots, m_k$ and at poles with multiplicity 1. For convenience we also introduce the total divisor $\mathcal{A} := \sum_{i=1}^n m_i Z_i - \sum_{j=1}^p P_i$ and the divisor of zeros $\mathcal{Z} := \sum_{i=1}^n m_i Z_i$.

Proposition 4.2 of [CM14] gives a formula for the self-intersection of Z_i. We reproduce the argument of this computation. First notice that $\mathcal{L} \subset f_* \omega^{\otimes 2}_{\overline{\mathcal{X}}/\overline{B}}(P_1 + \cdots + P_p)$. The pullback of this inclusion yields

$$0 \to f^* \mathcal{L} \to \omega^{\otimes 2}_{\overline{\mathcal{X}}/\overline{B}}(P_1 + \cdots + P_p) \to \mathcal{O}_{\mathcal{Z}}(\mathcal{Z}) \to 0.$$

This is true for non-compactified Teichmüller spaces, and it remains true after compactification since the multiplicities of the zeros of the limit differentials stay the same on stable curves (see Section 5.4 of [CM14] for details on cusps of Teichmüller curves). This yields an isomorphism $f^* \mathcal{L}(\mathcal{A}) \simeq \omega^{\otimes 2}_{\overline{\mathcal{X}}/\overline{B}}$.

For completeness, we reproduce below the computation of Proposition 4.2 in [CM14]. According to the adjunction formula, for any singularity Z_i,

$$\omega_{\overline{\mathcal{X}}/\overline{B}}(Z_i)_{|Z_i} = \omega_{Z_i} = 0,$$

which implies $Z_i^2 = -\omega_{\overline{\mathcal{X}}/\overline{B}} \cdot Z_i$. Moreover $Z_i \cdot Z_j = 0$ for any $i \neq j$, since two distinct fibers of f do not intersection. Finally,

$$2 Z_i^2 = -m_i Z_i^2 - \deg \mathcal{L},$$

where m_i is the corresponding multiplicity.

Which leads us to the formula:

$$Z_i^2 = \frac{-\chi}{m_i+2}. \tag{1}$$

3. Lyapunov exponents

In the remainder of the paper, $\mathcal{C}$ will stand for a Teichmüller curve in

$$\mathcal{Q}\left(m_1,\ldots,m_k,-1^p\right).$$

Lyapunov exponents are dynamical invariants for translation surfaces which describe the behaviour of the translation flow on them. For an introduction to their diverse aspects in translation surfaces see [Zor06]. We will adopt an algebraic point of view to compute them. Our strategy is based on the observation of [Kon97] that their sum is related to the degree of a Hodge subbundle on a Teichmüller curve (see [EKZ14] (3.11) or [For02] for a proof). We will use in the following a more precise formula computed in [CM14] as (20),

$$L^+(\mathcal{C}) = \frac{2}{\chi}\cdot \deg f_*\,\mathcal{O}(\mathcal{A}) + (6g-6) - \frac{1}{2}\left(\sum_{i=1}^{n}\frac{m_i(m_i+4)}{m_i+2} - 3p\right),$$

where $L^+(\mathcal{C})$ is the sum of the positive Lyapunov exponents associated to the Teichmüller curve $\mathcal{C}$. Recall that Lyapunov exponents can be defined on both bases C and B and that they will agree. Moreover, here the degrees are always computed on vector bundles over the compact curve $\overline{B}$.

By Gauss-Bonnet,

$$\sum_{i=1}^{n} m_i - p = 4g-4,$$

$$6g-6+\frac{3}{2}p = \frac{3}{2}\sum_{i=1}^{n} m_i.$$

Notice that $m_i(m_i+4) - 3m_i(m_i+2) = -2m_i(m_i+1)$, leading to the following formula which will be convenient for us:

$$L^+(\mathcal{C}) = \frac{2}{\chi}\cdot \deg f_*\,\mathcal{O}(\mathcal{A}) + \sum_{i=1}^{n}\frac{m_i(m_i+1)}{m_i+2}. \tag{2}$$

This equation relates the sum of Lyapunov exponents for a Teichmüller curve to an invariant of the stratum and the Chern class of a sheaf, describing the behaviour of the curve at its cusps.

To estimate the sum of the Lyapunov exponents, we are reduced to bounding the degree of the sheaf $f_* \mathcal{O}(\mathcal{A})$. This will be done based on the two following lemmas,

Lemma 1.

$$\deg f_* \mathcal{O}(\mathcal{A}) = \deg f_* \mathcal{O}(\mathcal{Z}) \, .$$

Proof. The main tool in the introduction of this filtration is the structural short exact sequence associated to any divisor D in $\overline{\mathcal{X}}$

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$$

Assume here that D is one of the divisors P_j. We tensor this exact sequence by $f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A} + D)$, where $\mathcal{A}$ is the total divisor as introduced in Section 2. The functor f_* gives a long exact sequence,

$$\begin{array}{lllll} 0 & \to & f_* \, (\omega^{\otimes 2}) \to \mathcal{L} \otimes f_* \, \mathcal{O}(\mathcal{A}+D) \to \mathcal{L} & & \\ & \stackrel{\delta}{\to} & R^1 f_* \, (\omega^{\otimes 2}) \;\longrightarrow\; R^1 f_* \, (f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A}+D)) \;\longrightarrow\; 0 & & \end{array}$$

Where we use the fact that f induces an isomorphism between D and $\overline{\mathcal{X}}$, which implies the equalities $f_* \, \mathcal{O}_D = \mathcal{O}_{\overline{\mathcal{X}}}$ and $\omega^{\otimes 2} = f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A})$.

Note that $R^1 f_* \, (\omega^{\otimes 2})$ over each stable curve X of the compactified Teichmüller curve has dimension

$$h^1(X, \omega_X^{\otimes 2}) = h^0(X, \omega_X^{\otimes -1}) = 0$$

by definition of the dualizing sheaf (see [Har77] III.7). Hence this sheaf is zero, and so is $R^1 f_* \, (f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A}+D))$. We end up with a short exact sequence which implies a relation on the degrees,

$$\deg (\mathcal{L} \otimes f_* \, \mathcal{O}(\mathcal{A}+D)) = \deg (\mathcal{L} \otimes f_* \, \mathcal{O}(\mathcal{A})) + \deg (\mathcal{L}) \, .$$

We reiterate the previous reasoning with $f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A}+D)$, as we noticed its image under the derived functor $R^1 f_*$ is zero and so will be the image of $f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A}+D+D')$ for any other divisor D' picked amoung the P_j. We do so until all the pole divisors are eliminated, and get a degree formula about the divisor without poles $\mathcal{Z}$,

$$\deg\left(\mathcal{L}\otimes f_*\,\mathcal{O}(\mathcal{Z})\right)=\deg\left(\mathcal{L}\otimes f_*\,\mathcal{O}(\mathcal{A})\right)+p\cdot\deg\left(\mathcal{L}\right).$$

To finish, observe that the fibers of $f_*\,\mathcal{O}(\mathcal{Z})$ have constant dimension $3g-3+p$ by Riemann-Roch theorem, since the degree of the divisor on any fiber will be $4g-4+p>2g-2$ for $g,p\geq 1$. By the Grauert Semicontinuity Theorem (see [Har77] III.12.9) this sheaf will be a vector bundle, and thus, so will be $f_*\,\mathcal{O}(\mathcal{A})$, as it is a subsheaf of the latter. Moreover $\operatorname{rk} f_*\,\mathcal{O}(\mathcal{Z})=3g-3+p$ and $\operatorname{rk} f_*\,\mathcal{O}(\mathcal{A})=3g-3$.

The previous formula becomes

$$\begin{aligned}&(3g-3+p)\deg\left(\mathcal{L}\right)+\deg\left(f_*\,\mathcal{O}(\mathcal{Z})\right)\\ &=(3g-3)\deg\left(\mathcal{L}\right)+\deg\left(f_*\,\mathcal{O}(\mathcal{A})\right)+p\cdot\deg\left(\mathcal{L}\right)\end{aligned}$$

And finally, $\deg\left(f_*\,\mathcal{O}(\mathcal{Z})\right)=\deg\left(f_*\,\mathcal{O}(\mathcal{A})\right)$. □

Thus the pole divisors will not interfere in the degree we need to estimate. By a similar method, we will be able to bound the degree while taking out zeros one after the other. We will do so until no zero is left and end up with the structure sheaf $\mathcal{O}$.

Considering divisors D and D' which are linear combinations of Z_i's with non-negative coefficients such that $D-D'=Z_i$, the dimension of global sections for their corresponding sheaf generally jumps by one when passing from D to D'. Sometimes it remains constant, and this phenomenon is usually referred to as Noether or Weierstrass gaps. Using the Riemann-Roch theorem it can be shown that there are at most g gaps in an infinite sequence of such divisors (see [FK80] III.5.4).

Lemma 2. *Let D and D' be linear combinations of Z_i's with non-negative coefficients, such that $D-D'=Z_i$ for some i; then,*

$$\deg f_*\,\mathcal{O}(D)\leq\deg f_*\,\mathcal{O}\big(D'\big)-\mu_i(D)\frac{\chi}{m_i+2}+R(D,D'),$$

where $R(D,D')$ is zero if there is no gap and $\mu_i(D)\frac{\chi}{m_i+2}$ otherwise.

Proof. Similarly to the previous proof, we tensor the structural exact sequence for divisor Z_i by $\mathcal{O}(D)$, and we get the following long exact sequence:

$$0\to f_*\,\mathcal{O}(D)\to f_*\,\mathcal{O}\big(D'\big)\to\mathcal{O}_{Z_i}(D)\xrightarrow{\delta}\cdots$$

This gives the formula,

$$\deg f_* \mathcal{O}(D) - \deg f_* \mathcal{O}\big(D'\big) = \deg \ker \delta.$$

As $\ker \delta$ is a subsheaf of $\mathcal{O}_{Z_i}(m_i Z_i)$ which is locally free on a complex curve, both are locally free and when $\ker \delta$ is not zero, *i.e.*, when $i \in G(\mathcal{F})^c$,

$$\deg \ker \delta \leq \deg \mathcal{O}_{Z_i}(m_i Z_i) = m_i Z_i \cdot Z_i,$$

Hence by formula (1),

$$\deg f_* \mathcal{O}(D) \leq \deg f_* \mathcal{O}\big(D'\big) - m_i \frac{\chi}{\mu^i + 2}.$$

If $i \in G(\mathcal{F})$, $\ker \delta = 0$, and thus

$$f_* \mathcal{O}(D) \simeq f_* \mathcal{O}\big(D'\big)\,.$$

□

If we pick a sequence of such divisors $\mathcal{F} = (D_j)_{1 \leq j \leq 4g-4+p}$ that removes the zeros one by one, using Lemma 1 and formula (2),

$$L^+(\mathcal{C}) \leq \sum_{l \in G(\mathcal{F})} R(D_l, D_{l+1}),$$

where $G(\mathcal{F})$ is the set of indices of gaps in the sequence $\mathcal{F}$.

This will lead to a proof of the Theorem written in the introduction. Assume we are in the setting of the theorem, we denote the larger zero multiplicity by $m_1 \geq 2g - 1$. We pick a family $\mathcal{F}$ such that we first remove all zeros different from the first one in an arbitrary way, and finish by taking off all the multiplicity of the principal one. Thus,

$$L^+(\mathcal{C}) \leq \sum_{l \in G(\mathcal{F})} R(D_l, D_{l+1}) \leq \sum_{i=1}^{g} \frac{2(2g-i)}{m_1 + 2} \leq \frac{(3g-1)g}{m_1 + 2},$$

where we use the fact that Weierstrass gaps happen for multiplicity smaller than $2g$.

Remark. If we can show that one of the zeros with multiplicity larger than g is not a Weierstrass point generically, the bound becomes $\frac{(g+1)g}{m_1+2}$.

We finish by proving the corollary claimed in the introduction. An *affine invariant manifold* is an immersed manifold in a stratum of abelian differentials whose image is locally defined by real linear equations in *period coordinates* (see [Zor06], Chapter 3). Each affine invariant manifold $\mathcal{N}$ is the support of an ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure which is locally in period coordinates the restriction of Lebesgue measure on the affine subspace (see [EM11]).

Theorem ([BEW17], Theorem 2.8 and [EMM15], Theorem 2.3). *Let $\mathcal{N}_n$ be a sequence of affine invariant manifolds and suppose the sequence of affine measures $\nu_{\mathcal{N}_n}$ converges to ν in the weak-* topology. Then ν is a probability measure, and it is an affine measure $\nu_{\mathcal{N}}$ where $\mathcal{N}$ is the smallest submanifold such that there exists n_0 with $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$. Moreover the Lyapunov exponents of $\nu_{\mathcal{N}_n}$ converge to the Lyapunov exponents of ν.*

According to [Wri14][Theorem 1.1], if $\mathcal{N}$ contains one square tiled surface, its field of definition is $\mathbb{Q}$; thus it contains an infinite number of square tiled surfaces, which implies corollary 1.

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