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Frobenius–Seshadri constants and characterizations of projective space

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We introduce higher-order variants of the Frobenius–Seshadri constant due to Mustaţă and Schwede, which are defined for ample line bundles in positive characteristic. These constants are used to show that Demailly's criterion for separation of higher-order jets by adjoint bundles also holds in positive characteristic. As an application, we give a characterization of projective space using Seshadri constants in positive characteristic, which was proved in characteristic zero by Bauer and Szemberg. We also discuss connections with other characterizations of projective space.

1. Introduction

Let L be an ample line bundle on a smooth projective variety X defined over an algebraically closed field k. Demailly in [6, §6] introduced the *Seshadri constant* $\varepsilon(L;x)$, which measures the local positivity of L at a closed point $x \in X$. If $\mu\colon X' \to X$ is the blow-up of X at x with exceptional divisor E, then the Seshadri constant is

$$\varepsilon(L;x) \coloneqq \sup\bigl\{t \in \mathbf{R}_{\geq 0} \bigm| \mu^*(L)(-tE) \text{ is nef}\bigr\}.$$

Seshadri constants have received much attention since their inception: see [2] and [17, Ch. 5].

Part of this interest in Seshadri constants stems from the fact that they give effective positivity statements for adjoint bundles. We will be particularly interested in how Seshadri constants can determine when adjoint bundles separate higher-order jets. Recall that we say L *separates ℓ-jets* if the

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restriction map

$$H^0(X,L) \longrightarrow H^0(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1}) \tag{1}$$

is surjective, where $\mathfrak{m}_x \subset \mathcal{O}_X$ is the ideal defining x. Algebraically, this says ℓth-order Taylor polynomials at x are restrictions of global sections, and geometrically, this says L separates ℓth-order tangent directions at x.

We can now state our first main result. Let ω_X denote the canonical bundle on X.

Theorem A. *Let L be an ample line bundle on a smooth projective variety X of dimension n defined over an algebraically closed field of positive characteristic. If $\varepsilon(L;x) > n + \ell$ at a closed point $x \in X$, then $\omega_X \otimes L$ separates ℓ-jets at x.*

Demailly showed this result in characteristic zero using the Kawamata–Viehweg vanishing theorem [6, Prop. 6.8(a)]. Our contribution is that the same result holds in positive characteristic.

As an application of this result, we prove the following:

Theorem B. *Let X be a smooth Fano variety of dimension n defined over an algebraically closed field of positive characteristic. If there exists a closed point $x \in X$ with $\varepsilon(\omega_X^{-1};x) \geq n+1$, then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n$.*

Bauer and Szemberg showed the analogous statement in characteristic zero as an application of the proof of Demailly's result [3, Thm. 2]. One interesting feature of this theorem is that it only requires a positivity condition on the anti-canonical bundle ω_X^{-1} at *one* point $x \in X$. In characteristic zero, Liu and Zhuang in [18, Thm. 2] generalized [3, Thm. 2] to $\mathbf{Q}$-Fano varieties; see Remark 4.10 for a comparison between their result and Theorem B.

There is an interesting connection between Theorem B and the Mori–Mukai conjecture, which states that if X is a smooth Fano variety of dimension n such that the canonical divisor K_X satisfies $(-K_X \cdot C) \geq n+1$ for all rational curves $C \subseteq X$, then X is isomorphic to $\mathbf{P}^n$. In characteristic zero, Cho, Miyaoka, and Shepherd-Barron's proof of the conjecture [5, Cor. 0.4] implies [3, Thm. 2]; see Proposition 4.6(i). In positive characteristic, the conjecture is still open, so instead one must assume the lower bound in Theorem B holds at *all* closed points in order to use a weaker result due to Kachi and Kollár [13, Cor. 3]; see Proposition 4.6(ii).

On the other hand, this connection raises the question of whether the opposite relationship holds. More precisely, we ask the following:

Question. Let X be a smooth Fano variety of dimension n defined over an algebraically closed field. If $(-K_X \cdot C) \geq n+1$ for every rational curve $C \subseteq X$, then is there a closed point $x \in X$ with $(-K_X \cdot C) \geq (\operatorname{mult}_x C) \cdot (n+1)$ for all reduced and irreducible curves $C \subseteq X$ passing through x?

This latter condition is equivalent to having $\varepsilon(\omega_X^{-1}; x) \geq n+1$ by [17, Prop. 5.1.5]. In characteristic zero, [5, Cor. 0.4] answers this question affirmatively. If one could do this independently of their result, then [3, Thm. 2] would give an alternative proof of the Mori–Mukai conjecture in characteristic zero, and Theorem B would resolve the conjecture in positive characteristic. See §4.1 for further discussion.

The proofs of Theorems A and B use a new variant of the Seshadri constant. To motivate our definition, we recall the following alternative characterization of the Seshadri constant in terms of separation of jets [17, Prop. 5.1.17]: if for each integer m, we denote by $s(L^m; x)$ the largest integer ℓ such that L^m separates ℓ-jets, then we have the equalities

$$\varepsilon(L; x) = \sup_{m \geq 1} \frac{s(L^m; x)}{m} = \lim_{m \to \infty} \frac{s(L^m; x)}{m}.$$

In positive characteristic, Mustaţă and Schwede [19] defined the Frobenius–Seshadri constant essentially by replacing ordinary powers of $\mathfrak{m}_x$ in the definition of separation of jets (1) with Frobenius powers, in order to take advantage of the Frobenius morphism. Using their definition, Mustaţă and Schwede were able to recover Theorem A when $\ell = 0$, and deduce global generation and very ampleness results for adjoint bundles [19, Thm. 3.1]. However, the full statement of Theorem A remained out of reach; see Remark 3.1.

Our solution is to introduce higher-order variants of the Frobenius–Seshadri constant, which mix both ordinary and Frobenius powers of $\mathfrak{m}_x$. This allows us to more directly deduce separation of higher-order jets. For each integer $\ell \geq 0$ and $m \geq 1$, let $s_F^\ell(L^m; x)$ be the largest integer $e \geq 0$ such that the restriction map

$$H^0(X, L^m) \longrightarrow H^0\bigl(X, L^m \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\bigr)$$

is surjective, where $(\mathfrak{m}_x^{\ell+1})^{[p^e]}$ denotes the eth Frobenius power of $\mathfrak{m}_x^{\ell+1}$. Then, the *ℓth Frobenius–Seshadri constant* of L at x is

$$\varepsilon_F^\ell(L;x) \coloneqq \sup_{m\geq 1} \frac{p^{s_F^\ell(L^m;x)}-1}{m/(\ell+1)} = \limsup_{m\to\infty} \frac{p^{s_F^\ell(L^m;x)}-1}{m/(\ell+1)}.$$

These constants are related to the ordinary Seshadri constant in the following manner (Proposition 2.9):

$$\frac{\ell+1}{\ell+n}\cdot\varepsilon(L;x) \leq \varepsilon_F^\ell(L;x) \leq \varepsilon(L;x). \tag{2}$$

Note that the ℓth Frobenius–Seshadri constant $\varepsilon_F^\ell(L;x)$ converges to the ordinary Seshadri constant $\varepsilon(L;x)$ as $\ell\to\infty$.

Using the ℓth Frobenius–Seshadri constant, we prove the following statement en route to proving Theorem A:

Theorem C. *Let L be an ample line bundle on a smooth projective variety X defined over an algebraically closed field of positive characteristic. If $\varepsilon_F^\ell(L;x) > \ell+1$ at a closed point $x\in X$, then $\omega_X\otimes L$ separates ℓ-jets at x.*

By the comparison (2), Theorem C immediately implies Theorem A. Our proof of Theorem C also works for singular varieties that are *F-injective*; see Remark 3.3. F-injective varieties are related to varieties with Du Bois singularities in characteristic zero [22].

Given Theorem C, it would be very interesting to have non-trivial lower bounds for any of the aforementioned versions of the Seshadri constant at very general points of X. In characteristic zero, it is conjectured that $\varepsilon(L;x)\geq 1$ at all very general $x\in X$. This is known if $n=\dim X=2$ [9]; if $n\geq 3$, then only the lower bound $1/n$ is known [8, Thm. 1]. However, the proofs of both results rely heavily on the characteristic zero assumption, and in arbitrary characteristic, we are only aware of the lower bound $\varepsilon(L;x)\geq 2/\bigl(1+\sqrt{4\sigma(L)+13}\bigr)$ for arbitrary points on surfaces, where

$$\sigma(L) \coloneqq \inf\bigl\{s\in\mathbf{R} \bigm| sL-K_X \text{ is nef}\bigr\}$$

is the *canonical slope* of L [1, Thm. 3.1].

Outline

Our paper is structured as follows: In §2, we define the ℓth Frobenius–Seshadri constant, and prove its basic properties. Most of what we prove is modeled after [19, §2], which studies what would be the zeroth Frobenius–Seshadri constant in our notation. In §3, we prove Theorem C. The main technical tool is the trace map $T\colon F_*\omega_X \to \omega_X$ associated to the (absolute) Frobenius morphism. Finally, in §4, we prove Theorem B, following [3].

Notation

A *variety* is a reduced and irreducible separated scheme of finite type defined over an algebraically closed field k of characteristic $p > 0$, unless stated otherwise. We denote by X a positive-dimensional projective variety, and denote by $F\colon X \to X$ the *(absolute) Frobenius morphism,* which is given by the identity map on points, and the p-power map

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow F_*\mathcal{O}_X(U) \\ f &\longmapsto f^p \end{aligned}$$

on structure sheaves, where $U \subseteq X$ is an open set. If $\mathfrak{a} \subseteq \mathcal{O}_X$ is a coherent ideal sheaf, we define the *eth Frobenius power* $\mathfrak{a}^{[p^e]}$ to be the inverse image of $\mathfrak{a}$ via the eth iterate of the Frobenius morphism. Locally, if $\mathfrak{a}$ is generated by $(h_i)_{i\in I}$, then $\mathfrak{a}^{[p^e]}$ is generated by $(h_i^{p^e})_{i\in I}$. If X is smooth, we denote by ω_X the canonical bundle on X and K_X the canonical divisor on X.

2. Definitions and preliminaries

We start by recalling the definition of the (ordinary) Seshadri constant of a line bundle L at a point. We adopt the "separation of jets" description of the Seshadri constant as our definition, which is equivalent to the other definitions when the line bundle L is ample and the closed point x is smooth [17, Prop. 5.1.17].

Definition 2.1. Let L be a line bundle on a projective variety X, and let $x \in X$ be a closed point with defining ideal $\mathfrak{m}_x \subset \mathcal{O}_X$. For all integers $\ell \geq 0$ and $m \geq 1$, we say that L^m *separates ℓ-jets at x* if the restriction map

$$\rho_{L^m}^{\ell,0}\colon H^0(X, L^m) \longrightarrow H^0(X, L^m \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1})$$

is surjective. Let $s(L^m; x)$ be the largest integer $\ell \geq 0$ such that L^m separates ℓ-jets at x; if no such ℓ exists, set $s(L^m; x) = -\infty$. The *Seshadri constant* of

L at x is

$$\varepsilon(L;x) \coloneqq \sup_{m\geq 1} \frac{s(L^m;x)}{m}. \tag{3}$$

We now give our main definition, which is modeled after the above interpretation of the Seshadri constant in terms of separation of jets. This definition combines both ordinary and Frobenius powers of the ideal $\mathfrak{m}_x$. Compared to the Frobenius–Seshadri constant defined in [19, Def. 2.4], our definition has the advantage of directly encoding information about higher-order jets; see Remark 3.1.

Definition 2.2. Let L be a line bundle on a projective variety X, and let $x \in X$ be a closed point with defining ideal $\mathfrak{m}_x \subset \mathcal{O}_X$. For all integers $\ell, e \geq 0$ and $m \geq 1$, we say that L^m *separates p^e-Frobenius ℓ-jets at x* if the restriction map

$$\rho^{\ell,e}_{L^m} \colon H^0(X, L^m) \longrightarrow H^0\bigl(X, L^m \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\bigr)$$

is surjective. Let $s^\ell_F(L^m;x)$ be the largest integer $e \geq 0$ such that L^m separates p^e-Frobenius ℓ-jets at x; if no such e exists, set $s^\ell_F(L^m;x) = -\infty$. The *$\ell$th Frobenius–Seshadri constant* of L at x is

$$\varepsilon^\ell_F(L;x) \coloneqq \sup_{m\geq 1} \frac{p^{s^\ell_F(L^m;x)} - 1}{m/(\ell+1)}. \tag{4}$$

We refer to the constants $\varepsilon^\ell_F(L;x)$ as *Frobenius–Seshadri constants*. Note that the zeroth Frobenius–Seshadri constant $\varepsilon^0_F(L;x)$ is the Frobenius–Seshadri constant defined in [19, Def. 2.4].

In the rest of this section, we will prove basic formal properties about Frobenius–Seshadri constants, following [19, §2]. The only statements used explicitly in later sections are Lemmas 2.4 and 2.7, and Propositions 2.5(i) and 2.9.

2.1. Separation of Frobenius jets under tensor powers

For the ordinary Seshadri constant, the supremum in (3) is actually a limit when L is ample [6, p. 97]. This property follows from Fekete's lemma [20, Pt. I, n° 98], since for all positive integers m and n, the sequence $s(L^m;x)$

satisfies the superadditivity property (see, e.g., [12, Lem. 3.7] for a proof)

$$s(L^{m+n};x) \geq s(L^m;x) + s(L^n;x). \tag{5}$$

For Frobenius–Seshadri constants, we cannot have an analogous property, since the supremum in (4) may not be a limit; see Example 2.6. Our first goal is to find a replacement for this superadditivity property. This will allow us to show that the supremum in (4) is actually a limit supremum.

We start with the following observation about ideals in a ring of characteristic $p > 0$, which will also be useful later. Note that the first inclusion in (6) is a slight improvement on [21, Lem. 4.6].

Lemma 2.3. *Let R be a commutative ring of characteristic $p > 0$. Then, for any ideal $\mathfrak{a}$ generated by n elements and for any non-negative integers e and ℓ, we have the sequence of inclusions*

$$\mathfrak{a}^{\ell p^e + n(p^e-1)+1} \subseteq (\mathfrak{a}^{\ell+1})^{[p^e]} \subseteq \mathfrak{a}^{(\ell+1)p^e}. \tag{6}$$

Moreover, if R is a regular local ring of dimension n, and $\mathfrak{a}$ is the maximal ideal of R, then

$$\mathfrak{a}^{\ell p^e + n(p^e-1)} \not\subseteq (\mathfrak{a}^{\ell+1})^{[p^e]}.$$

Proof. The second inclusion in (6) is clear; we want to show the first inclusion. Let $y_1, y_2, \ldots, y_n$ be a set of generators for $\mathfrak{a}$. The ideal $\mathfrak{a}^{\ell p^e + n(p^e-1)+1}$ is generated by all elements of the form

$$\prod_{i=1}^n y_i^{a_i} \quad \text{such that} \quad \sum_{i=1}^n a_i = \ell p^e + n(p^e-1)+1, \tag{7}$$

and the ideal $(\mathfrak{a}^{\ell+1})^{[p^e]}$ is generated by all elements of the form

$$\prod_{i=1}^n y_i^{p^e b_i} \quad \text{such that} \quad \sum_{i=1}^n b_i = \ell + 1. \tag{8}$$

We want to show that the elements (7) are divisible by some elements of the form (8). By the division algorithm, we may write $a_i = a_{i,0} + p^e a_i'$ for some non-negative integers $a_{i,0}$ and a'_i such that $0 \leq a_{i,0} \leq p^e - 1$. Then,

$$\prod_{i=1}^{n} y_i^{a_i} = \prod_{i=1}^{n} y_i^{a_{i,0}} \cdot \prod_{i=1}^{n} y_i^{p^e a'_i},$$

and since $a_{i,0} \leq p^e - 1$, we have that $\sum_{i=1}^{n} a_{i,0} \leq n(p^e - 1)$. Thus, we have the inequality

$$\ell p^e + n(p^e - 1) + 1 = \sum_{i=1}^{n} a_i \leq n(p^e - 1) + \sum_{i=1}^{n} p^e a'_i,$$

which implies $\ell + p^{-e} \leq \sum_{i=1}^{n} a'_i$. Since the right-hand side of this inequality is an integer, we have that $\ell + 1 \leq \sum_{i=1}^{n} a'_i$, i.e., the element $\prod_{i=1}^{n} y_i^{p^e a'_i}$ is divisible by one of the form (8). Thus, each element of the form in (7) is divisible by one of the form in (8).

Now suppose R is a regular local ring of dimension n, and $\mathfrak{a}$ is the maximal ideal of R. Let $y_1, y_2, \ldots, y_n$ be a regular system of parameters. Then, we have

$$y_{i_0}^{\ell p^e} \cdot \prod_{i=1}^{n} y_i^{p^e - 1} \in \mathfrak{a}^{\ell p^e + n(p^e - 1)}$$

for any $i_0 \in \{1, 2, \ldots, n\}$. This monomial does not lie in $(\mathfrak{a}^{\ell+1})^{[p^e]}$ since its image is not in the extension of $(\mathfrak{a}^{\ell+1})^{[p^e]}$ in the completion of R at $\mathfrak{a}$, which is isomorphic to a formal power series ring with variables $y_1, y_2, \ldots, y_n$ by the Cohen structure theorem. $\square$

Lemma 2.4 (cf. [19, Lem. 2.5]). *Let L be a line bundle on a projective variety X, and let $x \in X$ be a closed point. Suppose L^m separates p^e-Frobenius ℓ-jets at x for some integers $m \geq 1$, $e \geq 1$, and $\ell \geq 0$, and denote*

$$d_r = \frac{p^{re} - 1}{p^e - 1}$$

for each positive integer r. Then, L^{md_r} separates p^{re}-Frobenius ℓ-jets at x for all $r \geq 1$.

Proof. We want to show the restriction maps

$$\varphi_r \colon H^0(X, L^{md_r}) \longrightarrow H^0\big(X, L^{md_r} \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^{re}]}\big)$$

are surjective for all positive integers r. We prove this by induction on r. The case $r = 1$ is true by assumption, so we consider the case when $r \geq 2$.

Let $\widetilde{y}_1, \widetilde{y}_2, \dots, \widetilde{y}_n$ generate $\mathfrak{m}_x \cdot \mathcal{O}_{X,x}$. After choosing an isomorphism $L_x^m \simeq \mathcal{O}_{X,x}$, we can make the identification

$$L^{md_r} \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^{re}]} \simeq \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^{re}]}.$$

Then, $L^{md_r} \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^{re}]}$ is generated as a vector space over k by the residue classes

$$y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} \in L^{md_r} \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^{re}]} \tag{9}$$

of the monomials $\widetilde{y}_1^{a_1} \widetilde{y}_2^{a_2} \cdots \widetilde{y}_n^{a_n} \in L_x^{md_r}$, where the a_i can be any non-negative integers. As in the proof of Lemma 2.3, write $a_i = a_{i,0} + p^e a_i'$ for each i, where $0 \le a_{i,0} \le p^e - 1$, so that

$$y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} = \prod_{i=1}^n y_i^{a_{i,0}} \cdot \prod_{i=1}^n y_i^{p^e a_i'}.$$

We will show that the elements in (9) lie in the image of φ_r by descending induction on $S \coloneqq \sum_{i=1}^n a_i'$. By Lemma 2.3, if $S \ge (\ell + n)p^{(r-1)e}$, then $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} \equiv 0 \bmod (\mathfrak{m}_x^{\ell+1})^{[p^{re}]}$, and so there is nothing to show. It therefore suffices to consider the inductive case, when $S \le (\ell + n)p^{(r-1)e} - 1$.

By the assumption that φ_1 is surjective, we know that there exists $t_1 \in H^0(X, L^m)$ such that its germ $t_{1,x} \in L_x^m$ satisfies

$$\prod_{i=1}^n \widetilde{y}_i^{a_{i,0}} - t_{1,x} \in (\mathfrak{m}_x^{\ell+1})^{[p^e]} \otimes L_x^m. \tag{10}$$

By the inductive hypothesis with respect to r, we know that φ_{r-1} is surjective, so there exists $t_2 \in H^0(X, L^{md_{r-1}})$ such that its germ $t_{2,x} \in L_x^{md_{r-1}}$ satisfies

$$\prod_{i=1}^n \widetilde{y}_i^{a_i'} - t_{2,x} \in (\mathfrak{m}_x^{\ell+1})^{[p^{(r-1)e}]} \otimes L_x^{md_{r-1}}.$$

Now consider the composition below:

$$\begin{array}{ccccc} H^0(X, L^m) \otimes H^0(X, L^{md_{r-1}}) & \xrightarrow{1 \otimes (F^e)^*} & H^0(X, L^m) \otimes H^0(X, L^{mp^e d_{r-1}}) & \xrightarrow{\text{mult}} & H^0(X, L^{md_r}) \\ t_1 \otimes t_2 & \longmapsto & t_1 \otimes t_2^{p^e} & \longmapsto & t_1 t_2^{p^e} \end{array}$$

Note that $t_1 t_2^{p^e}$ restricts to $t_{1,x} t_{2,x}^{p^e}$ in the stalk $L_x^{md_r}$, and that

$$\prod_{i=1}^{n} \widetilde{y}_i^{p^e a_i'} - t_{2,x}^{p^e} \in (\mathfrak{m}_x^{\ell+1})^{[p^{re}]} \otimes L_x^{mp^e d_{r-1}}. \tag{11}$$

Then, after passing to the stalk $L_x^{md_r}$, we have

$$\begin{aligned}\prod_{i=1}^{n} \widetilde{y}_i^{a_i} - t_{1,x} t_{2,x}^{p^e} &= \prod_{i=1}^{n} \widetilde{y}_i^{a_i} - t_{1,x} \prod_{i=1}^{n} \widetilde{y}_i^{p^e a_i'} + t_{1,x} \prod_{i=1}^{n} \widetilde{y}_i^{p^e a_i'} - t_{1,x} t_{2,x}^{p^e} \\ &= \left(\prod_{i=1}^{n} \widetilde{y}_i^{a_{i,0}} - t_{1,x} \right) \cdot \prod_{i=1}^{n} \widetilde{y}_i^{p^e a_i'} + t_{1,x} \left(\prod_{i=1}^{n} \widetilde{y}_i^{p^e a_i'} - t_{2,x}^{p^e} \right).\end{aligned}$$

To show that $\prod_{i=1}^{n} y_i^{a_i}$ is in the image of φ_r, it suffices to show that the right-hand side of this equation is in the image of φ_r modulo $(\mathfrak{m}_x^{\ell+1})^{[p^{re}]}$. By (11), we know the second term is congruent to zero modulo $(\mathfrak{m}_x^{\ell+1})^{[p^{re}]}$, and so it remains to show the first term is in the image of φ_r modulo $(\mathfrak{m}_x^{\ell+1})^{[p^{re}]}$.

First, for each monomial μ in the $\widetilde{y}_i$ that appears in the difference $\prod_{i=1}^{n} \widetilde{y}_i^{a_{i,0}} - t_{1,x}$, there exists some n-tuple $(b_i)_{1 \le i \le n}$ where $\sum_{i=1}^{n} b_i = \ell + 1$ such that $\prod_{i=1}^{n} \widetilde{y}_i^{p^e b_i}$ divides μ by (10). Thus, the corresponding monomial that appears in the product

$$\left(\prod_{i=1}^{n} \widetilde{y}_i^{a_{i,0}} - t_{1,x} \right) \cdot \prod_{i=1}^{n} \widetilde{y}_i^{p^e a_i'} \tag{12}$$

is divisible by the product $\prod_{i=1}^{n} \widetilde{y}_i^{p^e (a_i' + b_i)}$. Since

$$\sum_{i=1}^{n} (a_i' + b_i) = S + \ell + 1 > S,$$

each monomial that appears in the product (12) is therefore in the image of φ_r modulo $(\mathfrak{m}_x^{\ell+1})^{[p^{re}]}$ by the inductive hypothesis on S. $\square$

This allows us to show that the supremum in (4) can actually be computed as a limit supremum.

Proposition 2.5 (cf. [19, Prop. 2.6]). *Let $\ell \ge 0$ be fixed, and let L be an ample line bundle on a projective variety X. Let $x \in X$ be a closed point.*

(*i*) *The line bundle L^m separates p^e-Frobenius ℓ-jets at x for some positive integers m and e.*

(*ii*) *We have*

$$\varepsilon_F^\ell(L;x) = \sup_{m,e} \frac{p^e-1}{m/(\ell+1)},$$

where the supremum is taken over all positive integers m *and* e *such that* L^m *separates* p^e*-Frobenius* ℓ*-jets at* x*.*

(*iii*) *Given any* $\delta > 0$*, there is a positive integer* e_0 *such that for every positive integer* e *divisible by* e_0*, there is a positive integer* m *such that* L^m *separates* p^e*-Frobenius* ℓ*-jets at* x *and*

$$\frac{p^e-1}{m/(\ell+1)} > \varepsilon_F^\ell(L;x) - \delta. \tag{13}$$

(*iv*) *We have*

$$\varepsilon_F^\ell(L;x) = \limsup_{m\to\infty} \frac{p^{s_F^\ell(L^m;x)}-1}{m/(\ell+1)}.$$

Proof. For (*i*), let $m \geq 1$ be such that L^m is very ample, and let n be the number of generators of $\mathfrak{m}_x \cdot \mathcal{O}_{X,x}$. Then, L^m separates tangent directions (i.e., 1-jets) and so $L^{m(\ell p^e+n(p^e-1))}$ separates $(\ell p^e + n(p^e-1))$-jets by the superadditivity property (5). Thus, $L^{m(\ell p^e+n(p^e-1))}$ separates p^e-Frobenius ℓ-jets at x by the inclusion $\mathfrak{m}_x^{\ell p^e+n(p^e-1)+1} \subseteq (\mathfrak{m}_x^{\ell+1})^{[p^e]}$ in Lemma 2.3.

Assertion (*ii*) follows by (*i*) and the definition of the ℓth Frobenius–Seshadri constant, since $s_F^\ell(L^m;x)$ is defined as the maximum $e \geq 0$ such that L^m separates p^e-Frobenius ℓ-jets at x.

For (*iii*), there exist positive integers m_0 and e_0 such that the inequality (13) holds by (*i*) and the definition of $\varepsilon_F^\ell(L;x)$. For each multiple $e = re_0$ of e_0, let $m = m_0\frac{p^e-1}{p^{e_0}-1}$. Then, by Lemma 2.4, we have that L^m separates p^e-Frobenius ℓ-jets at x. The inequality (13) then follows, since

$$\frac{p^e-1}{m/(\ell+1)} = \frac{p^{e_0}-1}{m_0/(\ell+1)} > \varepsilon_F^\ell(L;x) - \delta.$$

For (*iv*), let m_0 and e_0 be as in (*iii*) for $\delta = 1$. We inductively choose an increasing sequence of positive integers $(m_r)_{r\geq 0}$ as follows: having chosen m_r, we choose m_{r+1} such that (13) holds with $\delta = 1/(r+1)$, and such that $m_r < m_{r+1}$. Note that this increasing property can be ensured by the fact that the m's in (*iii*) increase as e increases. Then, we have

$$\varepsilon_F^\ell(L;x) = \lim_{r\to\infty} \frac{p^{s_F^\ell(L^{m_r};x)}-1}{m_r/(\ell+1)},$$

which implies (*iv*). □

We now give a calculation of both the ordinary Seshadri constant and the ℓth Frobenius–Seshadri constants on projective space, which shows that the limit supremum in Proposition 2.5(iv) cannot be computed as a limit.

Example 2.6. Let $X = \mathbf{P}^n_k$, and let $L = \mathcal{O}_X(1)$ be the line bundle associated to a hyperplane. For every closed point $x \in \mathbf{P}^n_k$, we claim that the restriction map

$$\rho^{\ell,e}_{\mathcal{O}_X(m)} \colon H^0(X, \mathcal{O}_X(m)) \longrightarrow H^0\big(X, \mathcal{O}_X(m) \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\big)$$

is surjective if and only if $m \geq \ell p^e + n(p^e - 1)$. This claim follows from Lemma 2.3 after choosing local affine coordinates in $\mathcal{O}_{X,x}$, and observing that $H^0(X, \mathcal{O}_X(m))$ maps onto all monomials of degree $\leq m$ in $\mathcal{O}_{X,x}$. Note that the inequality $m \geq \ell p^e + n(p^e - 1)$ is equivalent to

$$\frac{m+n}{\ell+n} \geq p^e.$$

Letting $e = 0$ gives $\varepsilon(\mathcal{O}_X(m); x) = 1$. For Frobenius–Seshadri constants, the equality in Proposition 2.5(iv) implies

$$\varepsilon^\ell_F(\mathcal{O}_X(1); x) \leq \limsup_{m \to \infty} \frac{\frac{m+n}{\ell+n} - 1}{m/(\ell+1)} = \frac{\ell+1}{\ell+n} \cdot \limsup_{m \to \infty} \frac{m - \ell}{m} = \frac{\ell+1}{\ell+n},$$

and the reverse inequality holds by computing a lower bound for the limit supremum using the sequence $m_e = \ell p^e + n(p^e - 1)$. On the other hand, we see that the limit supremum is not a limit, since the sequence $m'_e = \ell p^e + n(p^e - 1) - 1$ gives the limit

$$\limsup_{e \to \infty} \frac{p^{s^\ell_F(L^{m'_e}; x)} - 1}{m'_e/(\ell+1)} = \frac{\ell+1}{\ell+n} \cdot \lim_{e \to \infty} \frac{p^{e-1} - 1}{p^e - \frac{n+1}{n+\ell}} = \frac{\ell+1}{(\ell+n)p}.$$

Note that in this example, we have the equality $\frac{\ell+1}{\ell+n} \cdot \varepsilon(L; x) = \varepsilon^\ell_F(L; x)$. We will see in Proposition 2.9 that in fact, the inequality $\leq$ always holds.

We can use Lemma 2.4 and Proposition 2.5(iv) to prove that Frobenius–Seshadri constants are also well-behaved with respect to taking tensor powers. While we will not need this result in the sequel, the proof will use the following lemma, which will also be useful in the proof of Theorem C.

Lemma 2.7 (cf. [19, Rem. 2.3]). *Let X be a projectivty variety, and let $x \in X$ be a closed point. Let L be a line bundle on X, and let M be another* *line bundle that is globally generated at x. If for some integers $e \geq 1$ and $\ell \geq 0$, we have that L separates p^e-Frobenius ℓ-jets at x, then $L \otimes M$ also separates p^e-Frobenius ℓ-jets at x.*

Proof. Since M is globally generated at x, there is a global section $t \in H^0(X, M)$ that does not vanish at x. The commutative diagram

$$\begin{tikzcd} H^0(X,L) \arrow[r, two heads, "\rho_L^{\ell,e}"] \arrow[d, "t\otimes -"'] & H^0\big(X, L \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\big) \arrow[d, "\wr"', "t_x \otimes -"] \\ H^0(X, L\otimes M) \arrow[r, "\rho_{L\otimes M}^{\ell,e}"] & H^0\big(X, L\otimes M \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\big) \end{tikzcd}$$

shows that the restriction map $\rho_{L\otimes M}^{\ell,e}$ is surjective, i.e., we have that $L \otimes M$ separates p^e-Frobenius ℓ-jets. □

Proposition 2.8 (cf. [19, Prop. 2.8]). *Let L be an ample line bundle on a projective variety X, and let $x \in X$ be a closed point. Then, for all integers $\ell \geq 0$ and $s > 0$, we have $\varepsilon_F^\ell(L^r; x) = r \cdot \varepsilon_F^\ell(L; x)$.*

Proof. First, we have

$$\varepsilon_F^\ell(L^r;x) = r \cdot \sup_{m\geq 1} \frac{p^{s_F^\ell(L^{rm};x)-1}}{rm/(\ell+1)} \leq r \cdot \sup_{m'\geq 1} \frac{p^{s_F^\ell(L^{m'};x)-1}}{m'/(\ell+1)} = r\cdot \varepsilon_F^\ell(L;x)$$

by running through all tensor powers of L instead of just the powers that are divisible by r. It therefore remains to show the opposite inequality. We will fix an integer $j > 0$ such that L^j is globally generated.

Let $\delta > 0$ be given, and let m be a positive integer such that L^m separates p^e-Frobenius ℓ-jets for some integers $e, \ell \geq 0$ such that

$$\frac{p^e - 1}{m/(\ell+1)} > \varepsilon_F^\ell(L;x) - \frac{\delta}{r}.$$

Now let i be a positive integer. Denoting $d_i = \frac{p^{ie}-1}{p^e-1}$, we have

$$s_F(L^{md_i+j};x) \geq s_F(L^{md_i};x) \geq ie \tag{14}$$

by Lemmas 2.4 and 2.7. Now denoting

$$a_i = \left\lceil \frac{md_i + j}{r} \right\rceil,$$

we have that $ra_i \geq md_i + j$, hence

$$\begin{aligned}\frac{p^{s_F^\ell(L^{ra_i};x)} - 1}{a_i/(\ell+1)} &\geq \frac{p^{s_F^\ell(L^{md_i+j};x)} - 1}{a_i/(\ell+1)} \\ &\geq \frac{p^{ie} - 1}{a_i/(\ell+1)} = \frac{p^e - 1}{m/(\ell+1)} \cdot \frac{d_i m}{\lceil (md_i + j)/r \rceil},\end{aligned}$$

where the second inequality is by (14). Taking limit suprema as $i \to \infty$ and using Proposition 2.5(iv) gives

$$\varepsilon_F^\ell(L^r; x) \geq \limsup_{i\to\infty} \frac{p^{s_F^\ell(L^{ra_i};x)} - 1}{a_i/(\ell+1)} \geq r \cdot \varepsilon_F^\ell(L; x) - \delta.$$

Since $\delta > 0$ was arbitrary, we have the inequality $\varepsilon_F^\ell(L^r; x) \geq r \cdot \varepsilon_F^\ell(L; x)$. □

2.2. A comparison with the ordinary Seshadri constant

In order to use the ℓth Frobenius–Seshadri constant to prove Theorem A, we require a comparison with the ordinary Seshadri constant. This will allow us to deduce positivity properties of adjoint bundles in §3.

Proposition 2.9 (cf. [19, Prop. 2.12]). *If L is an ample line bundle on a projective variety X of dimension n, then for every smooth point $x \in X$ and integer $\ell \geq 0$, we have the sequence of inequalities*

$$\frac{\ell+1}{\ell+n} \cdot \varepsilon(L; x) \leq \varepsilon_F^\ell(L; x) \leq \varepsilon(L; x). \tag{15}$$

In particular, $\varepsilon_F^\ell(L; x) \to \varepsilon(L; x)$ as $\ell \to \infty$.

Proof. Since $\mathfrak{m}_x \cdot \mathcal{O}_{X,x}$ is generated by n elements, we have the sequence of inclusions

$$\mathfrak{m}_x^{\ell p^e + n(p^e-1)+1} \subseteq (\mathfrak{m}_x^{\ell+1})^{[p^e]} \subseteq \mathfrak{m}_x^{(\ell+1)p^e} \tag{16}$$

by Lemma 2.3. The right inclusion in (16) implies

$$s(L^m; x) \geq (\ell+1)p^{s_F^\ell(L^m;x)} - 1 \geq (\ell+1)(p^{s_F^\ell(L^m;x)} - 1),$$

and so the right inequality in (15) follows after dividing by m throughout, and taking limit suprema as $m \to \infty$.

For the left inequality in (15), let $\delta > 0$ be given, and let m_0 be a positive integer such that

$$\frac{s(L^{m_0};x)}{m_0} > \varepsilon(L;x) - \frac{\ell+n}{\ell+1}\cdot\delta.$$

Given any non-negative integer e, denote

$$d_e = \left\lceil \frac{\ell p^e + n(p^e-1)}{s(L^{m_0};x)} \right\rceil = \left\lceil \frac{(\ell+n)p^e - n}{s(L^{m_0};x)} \right\rceil.$$

By the superadditivity property (5), we have

$$s(L^{m_0 d_e};x) \geq d_e \cdot s(L^{m_0};x) \geq \ell p^e + n(p^e-1).$$

By the left inclusion in (16), this inequality implies

$$s_F^{\ell}(L^{m_0 d_e};x) \geq e,$$

and therefore

$$\varepsilon_F^{\ell}(L;x) \geq \frac{p^{s_F^{\ell}(L^{m_0 d_e};x)}-1}{m_0 d_e/(\ell+1)} \geq \frac{(\ell+1)(p^e-1)}{m_0\bigl\lceil\bigl((\ell+n)p^e-n\bigr)/s(L^{m_0};x)\bigr\rceil}.$$

As $e \to \infty$, the right-hand side converges to

$$\frac{\ell+1}{\ell+n}\cdot\frac{s(L^{m_0};x)}{m_0} > \frac{\ell+1}{\ell+n}\cdot\varepsilon(L;x) - \delta,$$

hence we have the inequality

$$\varepsilon_F^{\ell}(L;x) > \frac{\ell+1}{\ell+n}\cdot\varepsilon(L;x) - \delta$$

for all $\delta > 0$. Since $\delta > 0$ was arbitrary, we obtain the left inequality in (15). □

In light of Example 2.6 and Theorems A and C, it seems more accurate to think of the ℓth Frobenius–Seshadri constant as being closer to $\frac{\ell+1}{\ell+n}\cdot\varepsilon(L;x)$ than to $\varepsilon(L;x)$, just as for the zeroth Frobenius–Seshadri constant [19, p. 869]. We also observe that Example 2.6 shows that the lower bound in (15) is optimal.

We can also compare different Frobenius–Seshadri constants:

Corollary 2.10. *If L is a line bundle on an n-dimensional projective variety X, then for every smooth point $x \in X$ and integers $\ell > m \geq 0$, we have*

$$\frac{\ell+1}{\ell+n} \cdot \varepsilon_F^m(L;x) \leq \varepsilon_F^\ell(L;x) \leq \frac{\ell+1}{m+1} \cdot \varepsilon_F^m(L;x)$$

Proof. If L^r separates p^e-Frobenius ℓ-jets at x, then it separates p^e-Frobenius m-jets at x, giving the right inequality. The left inequality follows by using Proposition 2.9 for different values of ℓ. □

2.3. Numerical invariance

We now prove that Frobenius–Seshadri constants only depend on the numerical equivalence class of a line bundle. This fact will not be used in the sequel. Regularity in the proof below is in the sense of Castelnuovo and Mumford; see [17, Def. 1.8.4] for the definition.

Proposition 2.11 (cf. [19, Prop. 2.14]). *Let X be a projective variety, and let $x \in X$ be a closed point. If L_1 and L_2 are numerically equivalent ample line bundles on X, then $\varepsilon_F^\ell(L_1;x) = \varepsilon_F^\ell(L_2;x)$ for all integers $\ell \geq 0$.*

Proof. We first claim that if A is a globally generated ample line bundle, then there exists m_0 such that $A^m \otimes N$ is globally generated for all integers $m \geq m_0$ and nef line bundles N. First, by Fujita's vanishing theorem [10, Thm. 5.1], there exists an integer m_1 such that for all integers $m \geq m_1$ and nef line bundles N, we have $H^i(X, A^m \otimes N) = 0$ for all $i > 0$. Thus, if $m \geq m_1 + \dim X$, then the line bundle $A^m \otimes N$ is 0-regular with respect to A, hence is globally generated by [17, Thm. 1.8.5(i)]. It therefore suffices to set $m_0 = m_1 + \dim X$.

We now prove the proposition. By hypothesis, there exists a numerically trivial line bundle P such that $L_2 \simeq L_1 \otimes P$. Applying the result of the previous paragraph where A is a large enough power of L_1, we see that there exists a positive integer j such that $L_1^j \otimes N$ is globally generated for all nef line bundles N, hence in particular, $L_1^j \otimes P^i$ is globally generated for all integers i. Now by Proposition 2.5(iv), there exists an increasing sequence $(m_r)_{r\geq 0}$ of positive integers such that

$$\varepsilon_F^\ell(L_1;x) = \lim_{r\to\infty} \frac{p^{s_F^\ell(L_1^{m_r};x)} - 1}{m_r/(\ell+1)}.$$

For each integer $r \geq 0$, since $L_2^{m_r+j} \simeq L_1^{m_r} \otimes L_1^j \otimes P^{m_r+j}$ and $L_1^j \otimes P^{m_r+j}$ is globally generated, we see that

$$s_F^\ell(L_1^{m_r}; x) \leq s_F^\ell(L_2^{m_r+j}; x)$$

by Lemma 2.7. We therefore have that

$$\frac{p^{s_F^\ell(L_1^{m_r};x)} - 1}{m_r/(\ell+1)} \leq \frac{p^{s_F^\ell(L_2^{m_r+j};x)} - 1}{m_r/(\ell+1)} = \frac{p^{s_F^\ell(L_2^{m_r+j};x)} - 1}{(m_r+j)/(\ell+1)} \cdot \frac{m_r+j}{m_r}.$$

Since the limit of the left-hand side is $\varepsilon_F^\ell(L_1; x)$ by choice of the sequence $(m_r)_{r\geq 0}$, taking limit suprema as $r \to \infty$ throughout this inequality yields the inequality $\varepsilon_F^\ell(L_1; x) \leq \varepsilon_F^\ell(L_2; x)$ by Proposition 2.5(iv). Finally, repeating the argument above after switching the roles of L_1 and L_2, we have the equality $\varepsilon_F^\ell(L_1; x) = \varepsilon_F^\ell(L_2; x)$. □

3. Frobenius–Seshadri constants and adjoint bundles

We now turn to the proofs of Theorems A and C, which we restate below. Recall our standing assumption that our ground field k is algebraically closed and of characteristic $p > 0$.

Theorem A. *Let L be an ample line bundle on a smooth projective variety X of dimension n. Let $x \in X$ be a closed point. If the inequality*

$$\varepsilon(L; x) > n + \ell$$

holds, then $\omega_X \otimes L$ separates ℓ-jets at x.

Theorem C. *Let L be an ample line bundle on a smooth projective variety X. Let $x \in X$ be a closed point. If the inequality*

$$\varepsilon_F^\ell(L; x) > \ell + 1$$

holds, then $\omega_X \otimes L$ separates ℓ-jets at x.

As we mentioned in §1, Theorem A is an immediate consequence of Theorem C:

Proof of Theorem A. The inequality $\varepsilon(L; x) > n + \ell$ implies the inequality $\varepsilon_F^\ell(L; x) > \ell + 1$ by Proposition 2.9, hence the assertion immediately follows from Theorem C. □

Demailly proved the analogue of Theorem A in characteristic zero by using the Kawamata–Viehweg vanishing theorem, but only assuming that L is big and nef [6, Prop. 6.8(a)]. We do not know if this assumption suffices in positive characteristic.

Remark 3.1. It is possible to obtain a version of Theorem A using only the zeroth Frobenius–Seshadri constant by naïvely inducing on the order of jets in the proof of [19, Lem. 3.3]. However, the hypothesis needed for separation of ℓ-jets using this method is the stronger lower bound $\varepsilon(L;x) > n + n\ell$. The proof is also more technical and relies on Castelnuovo–Mumford regularity.

There are two main ingredients in the proof of Theorem C. The first is the following reformulation of our results from §2.

Proposition 3.2. *Let L be an ample line bundle on a projective variety X, and let $x \in X$ be a closed point. If $\varepsilon_F^\ell(L;x) > \alpha$ for some real number $\alpha > 0$, then we can find positive integers m and e satisfying*

$$\frac{p^e - 1}{m} > \frac{\alpha}{\ell + 1} \tag{17}$$

such that L^m separates p^e-Frobenius ℓ-jets at x. Furthermore, we may take m and e so that the quantity

$$p^e - 1 - \frac{\alpha}{\ell + 1} m \tag{18}$$

is arbitrarily large.

Proof. By Proposition 2.5(i) and the definition of $\varepsilon_F^\ell(L;x)$, we know there exist $m, e \geq 1$ such that the inequality (17) is satisfied and L^m separates p^e-Frobenius ℓ-jets at x. Moreover, by applying Lemma 2.4, we may make the replacements

$$e \longmapsto re \qquad \text{and} \qquad m \longmapsto \frac{m(p^{re} - 1)}{p^e - 1}$$

and not change the inequality (17) or the condition on separation of jets. Thus, by applying these replacements for integers $r \geq 1$, the quantity in (18) satisfies

$$p^{re} - 1 - \frac{\alpha}{\ell+1} \cdot \frac{m(p^{re}-1)}{p^e - 1} = (p^{re}-1)\left(1 - \frac{\alpha}{\ell+1} \cdot \frac{m}{p^e-1}\right) \longrightarrow \infty$$

as $r \to \infty$ by the inequality (17). We can therefore assume that the quantity (18) is arbitrarily large. $\square$

As in the proof of [19, Thm. 3.1], the other main ingredient in the proof of Theorem C is the *Cartier operator* or the *trace map*

$$T\colon F_*(\omega_X) \longrightarrow \omega_X,$$

which is a morphism of $\mathcal{O}_X$-modules. Here, $F\colon X \to X$ denotes the (absolute) Frobenius morphism. See [4, §1.3] for the definition and basic properties of the map T. Briefly, it can be defined as the trace map for relative duality for the finite flat morphism F as in [11, Ch. III, §6]. We note that F is finite since k is perfect, and F is flat by Kunz's theorem [4, Lem. 1.1.1] since X is smooth. The trace map satisfies the following key properties needed for our proof:

(a) The trace map T and its iterates $T^e\colon F^e_*(\omega_X) \to \omega_X$ are surjective [4, Thm. 1.3.4];

(b) If $\mathfrak{a} \subseteq \mathcal{O}_X$ is a coherent ideal sheaf, then T^e satisfies the equality

$$T^e\bigl(F^e_*(\mathfrak{a}^{[p^e]} \cdot \omega_X)\bigr) = \mathfrak{a} \cdot T^e\bigl(F^e_*(\omega_X)\bigr) = \mathfrak{a} \cdot \omega_X. \tag{19}$$

This follows from (a) by considering the $\mathcal{O}_X$-module structure on $F^e_*(\omega_X)$.

Remark 3.3. The surjectivity of the trace map in (a) is part of the definition for what are called *F-injective* varieties [23, Def. 2.10(iv)], as long as we interpret ω_X as the cohomology sheaf $\mathbf{h}^{-\dim X}\omega_X^\bullet$ of the dualizing complex $\omega_X^\bullet$. F-injective varieties are related to varieties with Du Bois singularities in characteristic zero [22]. Since the justification for (19) still works in this generality, our proof of Theorem C still works for F-injective varieties.

We are now ready to prove Theorem C. The proof closely follows that of [19, Thm. 3.1(i)]. The idea is the following: We can increase powers on L freely so that L^m separates p^e-Frobenius ℓ-jets, and then tensor by an appropriate product of the form $\omega_X \otimes L^{p^e-m}$ that is globally generated. Then, $\omega_X \otimes L^{p^e}$ separates p^e-Frobenius ℓ-jets. The eth iterate T^e of the trace map T allows us to take out these factors of p^e, and thereby deduce that $\omega_X \otimes L$ separates ℓ-jets.

Proof of Theorem C. Let $\mathfrak{m}_x$ denote the defining ideal of x. By Proposition 3.2, we can find m and e such that $m < p^e - 1$ and the restriction map

$$\rho^{\ell,e}_{L^m} \colon H^0(X, L^m) \longrightarrow H^0\bigl(X, L^m \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\bigr)$$

is surjective; moreover, we may assume that $p^e - 1 - m$ is arbitrarily large. In particular, we may assume that $p^e - m$ is arbitrarily large, so that $\omega_X \otimes L^{p^e-m}$ is globally generated. By Lemma 2.7, we then have that $\omega_X \otimes L^{p^e}$ separates p^e-Frobenius ℓ-jets, i.e., the restriction map

$$\varphi \colon H^0(X, \omega_X \otimes L^{p^e}) \longrightarrow H^0\bigl(X, \omega_X \otimes L^{p^e} \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\bigr) \tag{20}$$

is surjective.

We now use the surjectivity of the eth iterate $T^e \colon F^e_*(\omega_X) \to \omega_X$ of the trace map T. By (19), the map T^e induces a surjective morphism

$$F^e_*\bigl((\mathfrak{m}_x^{\ell+1})^{[p^e]} \cdot \omega_X\bigr) \twoheadrightarrow \mathfrak{m}_x^{\ell+1} \cdot \omega_X.$$

Tensoring this by L and applying the projection formula yields a surjective morphism

$$F^e_*\bigl((\mathfrak{m}_x^{\ell+1})^{[p^e]} \cdot \omega_X \otimes L^{p^e}\bigr) \twoheadrightarrow \mathfrak{m}_x^{\ell+1} \cdot \omega_X \otimes L. \tag{21}$$

Since the Frobenius morphism F is affine, the pushforward functor F^e_* is exact, hence we obtain the exactness of the left column in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
F^e_*\bigl((\mathfrak{m}_x^{\ell+1})^{[p^e]} \cdot \omega_X \otimes L^{p^e}\bigr) \arrow[r] \arrow[d] & \mathfrak{m}_x^{\ell+1} \cdot \omega_X \otimes L \arrow[d] \\
F^e_*(\omega_X \otimes L^{p^e}) \arrow[r] \arrow[d] & \omega_X \otimes L \arrow[d] \\
F^e_*\bigl(\omega_X \otimes L^{p^e} \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\bigr) \arrow[r] \arrow[d] & \omega_X \otimes L \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1} \arrow[d] \\
0 & 0
\end{tikzcd}$$

The top horizontal arrow is the map in (21); the middle horizontal arrow is obtained analogously from T^e by tensoring with L, and is therefore surjective. The surjectivity of the middle horizontal arrow also implies the bottom horizontal arrow is surjective. Finally, by taking global sections in the bottom square, we obtain the following commutative square:

$$\begin{tikzcd} H^0(X, \omega_X \otimes L^{p^e}) \arrow[r] \arrow[d, two heads, "\varphi"'] & H^0(X, \omega_X \otimes L) \arrow[d, "\rho^{\ell,0}_{\omega_X \otimes L}"] \\ H^0\big(X, \omega_X \otimes L^{p^e} \otimes \mathcal{O}_X/(\mathfrak{m}_x^{\ell+1})^{[p^e]}\big) \arrow[r, two heads, "\psi"] & H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1}) \end{tikzcd}$$

Note that ψ is surjective because the kernel of the corresponding morphism of sheaves is a skyscraper sheaf supported at x. We have already shown that the restriction map φ is surjective in (20), hence $\rho^{\ell,0}_{\omega_X \otimes L}$ is necessarily surjective. This shows $\omega_X \otimes L$ indeed separates ℓ-jets at x. $\square$

Remark 3.4. It is possible to define a multi-point version of $\varepsilon^{\ell}_F(L; x)$ following [19], which would capture how $\omega_X \otimes L$ simultaneously separates higher-order jets at different points. This method does not improve the result of [19, Thm. 3.1(*iii*),(*iv*)], which says the following:

(*a*) If $\varepsilon^0_F(L; x) > 2$ at some closed point $x \in X$, then $\omega_X \otimes L$ is very big, i.e., the rational map defined by $\omega_X \otimes L$ is birational onto its image;

(*b*) If $\varepsilon^0_F(L; x) > 2$ at all closed points $x \in X$, then $\omega_X \otimes L$ is very ample.

4. Characterizations of projective space

We now give an application of our result on separation of jets. As far as we know, this is the first application of the methods of [19].

Recall that a Fano variety is a projective variety whose anti-canonical bundle ω_X^{-1} is ample. Using Seshadri constants, Bauer and Szemberg showed the following characterization of projective space amongst smooth Fano varieties:

Theorem 4.1 [3, Thm. 2]. *Let X be a smooth Fano variety of dimension n defined over an algebraically closed field of characteristic zero. If there exists a closed point $x \in X$ with*

$$\varepsilon(\omega_X^{-1}; x) \geq n + 1,$$

then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n$.

Our goal in this section is to prove the following positive characteristic version of this result.

Theorem B. *Let X be a smooth Fano variety of dimension n defined over an algebraically closed field of positive characteristic. If there exists a closed point $x \in X$ with*

$$\varepsilon(\omega_X^{-1}; x) \geq n+1,$$

then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n$.

4.1. Comparison with other results and a weaker statement

Before moving on to the proof of Theorem B, we compare our result to other characterizations of projective space. As a consequence of these other characterizations, we also prove a weaker statement (Proposition 4.6) to illustrate why we might expect lower bounds on Seshadri constants to give characterizations of projective space.

Let K_X denote the canonical divisor on X. Theorem B can be restated as follows:

Theorem 4.2. *Let X be a smooth Fano variety of dimension n defined over an algebraically closed field of positive characteristic. If there exists a closed point $x \in X$ with*

$$(-K_X \cdot C) \geq (\operatorname{mult}_x C) \cdot (n+1)$$

for all reduced and irreducible curves $C \subseteq X$ passing through x, then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n$.

Proof. This follows from Theorem B by using [17, Props. 5.1.5, 5.1.17], which say

$$\varepsilon(\omega_X^{-1}; x) = \inf_{x \in C \subseteq X} \left\{ \frac{(-K_X \cdot C)}{\operatorname{mult}_x C} \right\}, \tag{22}$$

where the infimum is taken over all reduced and irreducible curves $C \subseteq X$ passing through x. □

This formulation is reminiscent of the following conjecture due to Mori and Mukai, which we mentioned in §1:

Conjecture 4.3 [15, Conj. V.1.7]. *Let X be a smooth Fano variety of dimension n defined over an algebraically closed field. If the inequality*

$$(-K_X \cdot C) \geq n+1$$

holds for every rational curve $C \subseteq X$, then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n$.

By using results of Kebekus [14] on families of singular rational curves, Cho, Miyaoka, and Shepherd-Barron proved this conjecture in characteristic zero. More precisely, they showed the following stronger statement:

Theorem 4.4 [5, Cor. 0.4]. *Let X be a smooth projective variety of dimension n defined over an algebraically closed field of characteristic zero. If X is uniruled, and the inequality*

$$(-K_X \cdot C) \geq n+1$$

holds for every rational curve $C \subseteq X$ passing through a general point x_0, then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n$.

In arbitrary characteristic, as far as we know the only result in this direction is the following:

Theorem 4.5 [13, Cor. 3]. *Let X be a smooth projective variety of dimension n defined over an algebraically closed field of arbitrary characteristic. Suppose K_X is not nef. If*

(*a*) $(-K_X \cdot C) \geq n+1$ *for every rational curve $C \subseteq X$; and*

(*b*) $(-K_X)^n \geq (n+1)^n$,

then X is isomorphic to the n-dimensional projective space $\mathbf{P}^n$.

Given the similarity between the Mori–Mukai conjecture 4.3 and Theorem 4.2, we asked the following question in §1:

Question. Let X be a smooth Fano variety of dimension n defined over an algebraically closed field of arbitrary characteristic. If the inequality

$$(-K_X \cdot C) \geq n+1$$

holds for every rational curve $C \subseteq X$, then does there exist a closed point $x \in X$ with

$$(-K_X \cdot C) \geq (\operatorname{mult}_x C) \cdot (n+1)$$

for all reduced and irreducible curves $C \subseteq X$ passing through x?

As mentioned in §1, the answer to this question is "yes" in characteristic zero by using Theorem 4.4, since Theorem 4.4 implies $X \simeq \mathbf{P}^n$, and therefore $\varepsilon(\omega_X^{-1}; x) \geq n+1$ for all closed points $x \in X$ by Example 2.6. If one could answer this question affirmatively independently of Theorem 4.4, then Theorem 4.1 would give an alternative proof of the Mori–Mukai conjecture 4.3 in characteristic zero, and Theorem 4.2 would resolve their conjecture in positive characteristic.

Returning to Seshadri constants, we can show the following statement as a consequence of the characterizations of projective space given above. The statement in characteristic zero gives a different proof of [3, Thm. 2].

Proposition 4.6. *Let X be a smooth Fano variety of dimension n defined over an algebraically closed field k. Consider the inequality*

$$\varepsilon(\omega_X^{-1}; x) \geq n+1 \tag{23}$$

for each closed point $x \in X$. Suppose one of the following is satisfied:

- (i) *We have* $\operatorname{char} k = 0$ *and the inequality* (23) *holds for a single closed point $x \in X$; or*
- (ii) *We have* $\operatorname{char} k = p > 0$ *and the inequality* (23) *holds for* all *closed points $x \in X$.*

Then, X is isomorphic to the n-dimensional projective space $\mathbf{P}^n$ over k.

Proof. For (i), we use Theorem 4.4. Since Fano varieties are uniruled [15, Cor. IV.1.15], it suffices to verify the condition $(-K_X \cdot C) \geq n+1$. First, note that $\varepsilon(\omega_X^{-1}; x) > n$ at the given point $x \in X$, and since the locus $\{x \in X \mid \varepsilon(\omega_X^{-1}; x) > n\}$ is open [19, Rem. 2.15], we have $\varepsilon(\omega_X^{-1}; x_0) > n$ at a general point $x_0 \in X$. By the alternative characterization of Seshadri constants in terms of curves in (22), we have the chain of inequalities

$$n < \varepsilon(\omega_X^{-1}; x_0) \leq \frac{(-K_X \cdot C)}{\operatorname{mult}_{x_0} C} \leq (-K_X \cdot C)$$

for any rational curve C containing x_0. Since $(-K_X \cdot C)$ is an integer, we have $(-K_X \cdot C) \geq n+1$.

For (ii), we use Theorem 4.5. The verification of condition (a) proceeds as in (i) by applying (22) to a closed point $x \in C$ contained in a given rational curve $C \subseteq X$. For condition (b), we use the inequality

$$\varepsilon(\omega_X^{-1}; x) \leq \sqrt[n]{(-K_X)^n},$$

which is [17, eq. 5.2]. The inequality $\varepsilon(\omega_X^{-1}; x) \geq n+1$ then implies condition (b). $\square$

4.2. Proof of Theorem B

We now turn to the proof of Theorem B. The main technical tool is the notion of bundles of principal parts, which are also known as jet bundles in the literature. See [16, §4] for a detailed discussion.

Definition 4.7. Let X be a variety defined over an algebraically closed field k of arbitrary characteristic. Denote by p and q the projections

$$\begin{tikzcd} & X \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & X \end{tikzcd}$$

Let $\mathscr{I} \subset \mathcal{O}_{X\times X}$ be the ideal defining the diagonal, and let L be a line bundle on X. For each integer $\ell \geq 0$, the *ℓth bundle of principal parts* associated to L is the sheaf

$$\mathscr{P}^\ell(L) \coloneqq p_*(q^*L \otimes \mathcal{O}_{X\times X}/\mathscr{I}^{\ell+1}).$$

Note that $\mathscr{P}^0(L) \simeq L$, since the diagonal in $X \times X$ is isomorphic to X.

We will use the following facts about these sheaves from [16, §4], assuming X is smooth:

(a) There exists a short exact sequence [16, n° 4.2]

$$0 \longrightarrow \operatorname{Sym}^\ell(\Omega_X) \otimes L \longrightarrow \mathscr{P}^\ell(L) \longrightarrow \mathscr{P}^{\ell-1}(L) \longrightarrow 0, \tag{24}$$

where Ω_X denotes the cotangent bundle on X. By using induction and this short exact sequence, it follows that the sheaf $\mathscr{P}^\ell(L)$ is a vector bundle for all integers $\ell \geq 0$.

(*b*) There exists an identification $\mathscr{P}^\ell(L) \simeq q_*(q^*L \otimes \mathcal{O}_{X\times X}/\mathscr{I}^{\ell+1})$, and by applying adjunction to the map $q^*L \to q^*L \otimes \mathcal{O}_{X\times X}/\mathscr{I}^{\ell+1}$, there is a morphism

$$d^\ell \colon L \longrightarrow \mathscr{P}^\ell(L)$$

of sheaves [16, n° 4.1], such that the diagram

$$\begin{tikzcd} H^0(X,L) \arrow[r, "H^0(d^\ell)"] \arrow[d, "\rho_L^{\ell,0}"'] & H^0\big(X,\mathscr{P}^\ell(L)\big) \arrow[d, "\rho^{0,0}_{\mathscr{P}^\ell(L)}"] \\ H^0(X, L\otimes \mathcal{O}_X/\mathfrak{m}_x^{\ell+1}) & H^0\big(X,\mathscr{P}^\ell(L)\otimes \mathcal{O}_X/\mathfrak{m}_x\big) \arrow[l, "\sim"'] \end{tikzcd}$$

commutes for all closed points $x \in X$ [16, Lem. 4.5(1)]. Thus, if L separates ℓ-jets at x, then $\mathscr{P}^\ell(L)$ is globally generated at x.

We will also use the following description of the determinant of the ℓth bundle of principal parts. This description is stated in [7, p. 1660].

Lemma 4.8. *Let X be a smooth variety of dimension n, and let L be a line bundle. Then, for each $\ell \geq 0$, we have an isomorphism*

$$\det(\mathscr{P}^\ell(L)) \simeq \big(\omega_X^\ell \otimes L^{n+1}\big)^{\frac{1}{n+1}\binom{n+\ell}{n}}.$$

Proof. We proceed by induction on $\ell \geq 0$. If $\ell = 0$, then $\mathscr{P}^0(L) \simeq L$, so we are done.

Now suppose $\ell > 0$. Since X is smooth, the cotangent bundle Ω_X has rank n, and we have isomorphisms

$$\begin{aligned} \det\big(\mathrm{Sym}^\ell(\Omega_X)\otimes L\big) &\simeq \det\big(\mathrm{Sym}^\ell(\Omega_X)\big) \otimes L^{\binom{n+\ell-1}{n-1}} \\ &\simeq \omega_X^{\binom{n+\ell-1}{n}} \otimes L^{\binom{n+\ell-1}{n-1}}. \end{aligned}$$

By induction and taking top exterior powers in the short exact sequence (24), we obtain

$$\begin{aligned} \det(\mathscr{P}^\ell(L)) &\simeq \omega_X^{\binom{n+\ell-1}{n}} \otimes L^{\binom{n+\ell-1}{n-1}} \otimes \det(\mathscr{P}^{\ell-1}(L)) \\ &\simeq \omega_X^{\binom{n+\ell-1}{n}} \otimes L^{\binom{n+\ell-1}{n-1}} \otimes \big(\omega_X^{\ell-1}\otimes L^{n+1}\big)^{\frac{1}{n+1}\binom{n+\ell-1}{n}} \\ &\simeq \big(\omega_X^\ell \otimes L^{n+1}\big)^{\frac{1}{n+1}\binom{n+\ell}{n}}. \end{aligned}$$

Note that the last isomorphism holds because of the identities

$$\binom{n+\ell-1}{n} + \frac{\ell-1}{n+1}\binom{n+\ell-1}{n} = \frac{n+\ell}{n+1}\binom{n+\ell-1}{n} = \frac{\ell}{n+1}\binom{n+\ell}{n},$$
$$\binom{n+\ell-1}{n-1} + \binom{n+\ell-1}{n} = \binom{n+\ell}{n}$$

involving binomial coefficients. □

We now return to the setting where our ground field k is an algebraically closed field of characteristic $p > 0$. We begin with the following key chain of inequalities. Note that our statement is weaker than [3, Prop. 1.1], but it still suffices for our purposes.

Lemma 4.9. *Let X be a smooth Fano variety of dimension n, and let $x \in X$ be a closed point. Denote $\varepsilon = \varepsilon(\omega_X^{-1}; x)$. For every integer $m \geq 1$, we have the chain of inequalities*

$$(m+1)\varepsilon - (n+1) \leq s(\omega_X^{-m}; x) \leq m\varepsilon. \tag{25}$$

In particular, $\varepsilon(\omega_X^{-1}; x) \leq n+1$.

Proof. We have the inequality

$$\frac{s(\omega_X^{-m}; x)}{m} \leq \varepsilon$$

by the definition of the ordinary Seshadri constant in (3). We can then multiply by m throughout to obtain the right inequality in (25).

For the left inequality in (25), we know that if ω_X^{-m} does not separate ℓ-jets, then

$$\varepsilon(\omega_X^{-(m+1)}; x) = (m+1) \cdot \varepsilon(\omega_X^{-1}; x) \leq n+\ell \tag{26}$$

by the contrapositive of Theorem A applied to $L = \omega_X^{-(m+1)}$. Note that the equality in (26) holds by [17, Ex. 5.1.4]. By the definition of $s(\omega_X^{-m}; x)$, the inequality in (26) holds for $\ell = s(\omega_X^{-m}; x) + 1$, hence the left inequality in (25) follows. The last assertion follows by rearranging (25) for $m = 1$. □

We now prove Theorem B. Our proof follows that of [3, Thm. 1.7], although we must be more careful with tensor operations in positive characteristic.

Proof of Theorem B. We first claim that $\mathscr{P}^{n+1}(\omega_X^{-1})$ is a trivial bundle. By Lemma 4.9 for $m=1$, we know that at the given point $x \in X$, we have the equality $\varepsilon(\omega_X^{-1};x) = n+1$, and moreover

$$n+1 = 2\cdot\varepsilon(\omega_X^{-1};x) - (n+1) \le s(\omega_X^{-1};x) \le \varepsilon(\omega_X^{-1};x) = n+1,$$

hence equality holds throughout. By property (*b*) of bundles of principal parts, we therefore have that $\mathscr{P}^{n+1}(\omega_X^{-1})$ is globally generated at x. On the other hand, by Lemma 4.8 applied to $L = \omega_X^{-1}$, we have an isomorphism $\det(\mathscr{P}^{n+1}(\omega_X^{-1})) \simeq \mathcal{O}_X$. Now to show that $\mathscr{P}^{n+1}(\omega_X^{-1})$ is a trivial bundle, consider the following diagram:

$$\begin{tikzcd}
\det\big(\mathscr{P}^{n+1}(\omega_X^{-1})\big) \arrow[r, "\sim"] \arrow[d, two heads] & \mathcal{O}_X \arrow[d, two heads] \\
\det\big(\mathscr{P}^{n+1}(\omega_X^{-1}) \otimes \mathcal{O}_X/\mathfrak{m}_x\big) \arrow[r, "\sim"] & \mathcal{O}_X/\mathfrak{m}_x
\end{tikzcd}$$

Suppose the isomorphism in the top row is given by a non-vanishing global section

$$s \in H^0\big(X, \det\big(\mathscr{P}^{n+1}(\omega_X^{-1})\big)\big).$$

Let $s_{1,x} \wedge s_{2,x} \wedge \cdots \wedge s_{r,x}$ be the image of s in $\det\big(\mathscr{P}^{n+1}(\omega_X^{-1}) \otimes \mathcal{O}_X/\mathfrak{m}_x\big)$, which gives the isomorphism in the bottom row. Then, since $\mathscr{P}^{n+1}(\omega_X^{-1})$ is globally generated at x, each $s_{i,x}$ can be lifted to a global section $\widetilde{s}_i \in H^0\big(X, \mathscr{P}^{n+1}(\omega_X^{-1})\big)$. Because the exterior product $\widetilde{s}_1 \wedge \widetilde{s}_2 \wedge \cdots \wedge \widetilde{s}_r$ does not vanish at x, this exterior product does not vanish anywhere, since $H^0(X, \mathcal{O}_X) = k$. Thus, the global sections $\widetilde{s}_i$ give a frame for $\mathscr{P}^{n+1}(\omega_X^{-1})$, and therefore $\mathscr{P}^{n+1}(\omega_X^{-1})$ is a trivial bundle.

To show $X \simeq \mathbf{P}^n_k$, we use Mori's characterization of projective space [15, Thm. V.3.2]. It suffices to show that for every nonconstant morphism $f\colon \mathbf{P}^1_k \to X$, the pull back f^*T_X is a sum of line bundles of positive degree. Write

$$f^*(T_X) \simeq \bigoplus_{i=1}^{n} \mathcal{O}(a_i) \qquad \text{and} \qquad f^*(\omega_X^{-1}) \simeq \mathcal{O}(b),$$

where b is positive since ω_X^{-1} is ample. We want to show that each a_i is positive. Note that

$$f^*(\Omega_X) \simeq f^*(T_X)^\vee \simeq \bigoplus_{i=1}^{n} \mathcal{O}(-a_i).$$

Dualizing the short exact sequence (24), we have the short exact sequence

$$0 \longrightarrow \mathscr{P}^n(\omega_X^{-1})^\vee \longrightarrow \mathscr{P}^{n+1}(\omega_X^{-1})^\vee \longrightarrow (\operatorname{Sym}^{n+1}\Omega_X)^\vee \otimes \omega_X \longrightarrow 0.$$

The quotient on the right is globally generated because it is a quotient of the trivial bundle $\mathscr{P}^{n+1}(\omega_X^{-1})^\vee$. We have isomorphisms

$$\begin{aligned} f^*\big((\operatorname{Sym}^{n+1}\Omega_X)^\vee \otimes \omega_X\big) &\simeq \big(\operatorname{Sym}^{n+1} f^*(\Omega_X)\big)^\vee \otimes f^*(\omega_X) \\ &\simeq \left(\operatorname{Sym}^{n+1}\bigoplus_{i=1}^n \mathcal{O}(-a_i)\right)^\vee \otimes \mathcal{O}(-b), \end{aligned}$$

and this bundle is globally generated since it is the pullback of a globally generated bundle. By expanding out the symmetric power on the right-hand side, we have a surjection

$$f^*\big((\operatorname{Sym}^{n+1}\Omega_X)^\vee \otimes \omega_X\big) \twoheadrightarrow \bigoplus_{i=1}^n \mathcal{O}\big((n+1)a_i - b\big),$$

hence the direct sum on the right-hand side is also globally generated. Finally, this implies

$$(n+1)a_i - b \geq 0,$$

and therefore since $b > 0$, we have that $a_i > 0$ as required. □

Remark 4.10. Liu and Zhuang's characteristic zero statement in [18, Thm. 2] is stronger than Theorem B: it only assumes that X is $\mathbf{Q}$-Fano, and in particular that X is not necessarily smooth. While Theorem C holds for a large class of singular varieties (see Remark 3.3), the rest of our approach does not generalize to the non-smooth setting, since Mori's characterization of projective space uses bend and break techniques. On the other hand, Liu and Zhuang's methods do not seem to work in positive characteristic without very strong assumptions on dimension and F-singularities since, in particular, they use the Kawamata–Shokurov basepoint-freeness theorem and the Kawamata–Viehweg vanishing theorem.

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