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Configurations of FK Ising interfaces and hypergeometric SLE

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In this paper, we show that the interfaces in the FK Ising model at criticality in a domain with 4 marked boundary points and wired–free–wired–free boundary conditions conditioned on a specific internal arc configuration of interfaces converge in the scaling limit to the hypergeometric SLE (hSLE). The arc configuration consists of a pair of interfaces and the scaling limit of their joint law can be described by an algorithm to sample the pair from an hSLE curve and a chordal SLE (in a random domain defined by the hSLE).

1. Introduction

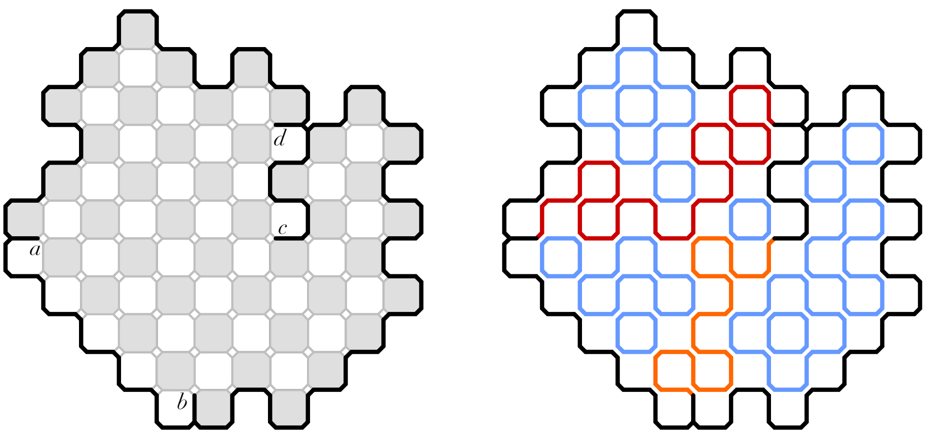
In the seminal paper [17], Oded Schramm introduced SLE as a one-parameter family of conformally invariant random fractal curves, and showed that those are the only possible conformally invariant scaling limits of the interfaces in the lattice models at criticality. The SLEs are dynamically grown, by running the Loewner evolution with a random driving term. The original definition was formulated in two setups: chordal (curves between two boundary points) and radial (curves between a boundary and an interior point), which both have trivial conformal modulus, and thus their Loewner driving term is given by a Brownian motion without a drift. Soon afterwards Lawler, Schramm and Werner introduced a generalization [13] for domains with several marked points and the driving process drift having a very particular and elegant dependence on their conformal moduli. While including several fundamental cases, this process does not cover all the important situations, and it was quickly realized that one should also look at more general SLEs, weighted by partition functions and having more complicated drifts [1, 4, 8, 12, 15, 22].

In this paper we are concerned with a particular case of SLEs in a domain with 4 marked boundary points connected in pairs by two non-intersecting SLE curves. Such arrangement corresponds to the wired-free-wired-free boundary conditions in the underlying FK model. The marked boundary points can be connected in two ways, and conditioning on one of those we obtain the hypergeometric SLE, cf. [16, 21].

1.1. FK Ising model on $\mathbb{Z}^2$

Let $\mathbb{L}^\bullet$ and $\mathbb{L}^\circ$ be the even and odd sublattices of the square lattice $\mathbb{Z}^2$, respectively, that is, the sum of the x and y coordinates is even or odd on $\mathbb{L}^\bullet$ and $\mathbb{L}^\circ$, respectively. The lattices $\mathbb{L}^\bullet$ and $\mathbb{L}^\circ$ are both square lattices with a lattice mesh $\sqrt{2}$. The medial lattice $\mathbb{L}^\diamond$ is formed by the midpoints of edges of $\mathbb{L}^\bullet$ (or equivalently of $\mathbb{L}^\circ$) which then are connected with edges by going around each face of $\mathbb{L}^\bullet$. The graph $\mathbb{L}^\diamond$ is also a square lattice. The modified medial lattice $\mathbb{L}^\spadesuit$ is the square–octagon lattice which we get by replacing all vertices of $\mathbb{L}^\diamond$ by a small square. See the introduction of [10] for more information.

We call the octagons white or black, if their centers are in $\mathbb{L}^\bullet$ and $\mathbb{L}^\circ$, respectively. Those faces of $\mathbb{L}^\spadesuit$ that are squares are called small squares.



(a) A discrete domain with four marked boundary points. Marked points are the cusp points in the picture. The boundary of the discrete domain consists of four "admissible paths" (boundaries of chains of octagons and small squares alternating) on $\mathbb{L}^\spadesuit$.

(b) In the 4 marked boundary points setting, a configuration of (the loop representation) of the FK Ising model consists of two interfaces and a number of loops. If a, b, c, d are as on the figure (a), then the present configuration belongs to the internal arc configuration event $(a \frown d, c \frown b)$.

Figure 1: Discrete domain with 4 marked boundary points and a loop configuration on it.

Consider a domain Ω whose boundary is the outer boundary of a simple closed chain of faces where the octagons and the small squares are alternating. We assume that the "boundary conditions" change at 4 "marked" points, that is, the chain consist of exactly 4 open monochromatic chains of octagons and small squares. See Figure 1. The marked points are denoted by a, b, c, d in general and we can assume they are the joint vertices of a black octagon, a white octagon and a small square in the chain and thus on the boundary of the domain. Assume that a, b, c, d are counterclockwise ordered on the boundary and that the octagons next to the boundary arc $[ab]$ are white. Then necessarily the octagons next to $[cd]$ are white as well and the ones next to $[bc]$ or $[da]$ are black.

Let $G^{\spadesuit} \subset \mathbb{L}^{\spadesuit}$ be the graph which contains the vertices $(V(\mathbb{L}^{\spadesuit}) \cap \Omega) \cup \{a, b, c, d\}$ and all the edges contained in Ω. Let us consider a loop configuration which in the present case contains 2 open paths that both connect $\{a, c\}$ to $\{b, d\}$ and a number of closed loops. We assume that the configuration is dense on $G^{\spadesuit}$, in the sense that it covers all the vertices, and that the paths are simple and mutually disjoint. See Figure 1.

Define a probability measure on the dense simple loop configurations of $G^{\spadesuit}$ by requiring that the probability of a loop configuration is proportional to

$$\sqrt{2}^{(\#\text{ of loops})}. \tag{1}$$

Notice that the number of open paths doesn't enter this formula, since there are always 2 such paths. The model is called the loop representation of the critical FK Ising model (Fortuin–Kasteleyn random cluster model with the parameter values that corresponds to the critical Ising model).

1.1.1. The motivation for the FK random cluster model. The (spin) Ising model is a model for ferromagnetic substance. The Ising model configuration is a field of ± 1 random variables, one on each vertex of a graph, and their probability law is given by the Boltzmann distribution of an energy functional with a nearest neighbor interaction. A parameter β determines the strength of the interaction. The Fortuin–Kasteleyn random cluster model (FK model) is a percolation-type model with two parameters $p \in [0, 1]$ and $q \geq 0$. The FK model configuration is a random subset of the set of edges of a graph. These edges are called open and the edge in the complement are called closed. A connected component (of vertices) in that random graph is called a cluster. In the FK model, the probability of the configuration is proportional to $q^{(\#\text{ of clusters})}\, p^{(\#\text{ of open edges})}\, (1-p)^{(\#\text{ of closed edges})}$.

The FK Ising model is a particular case $q = 2$ of the FK model, with other values of q leading to the FK representations of the q-state Potts model. The spin Ising model and the FK Ising model are connected by the Edwards–Sokal coupling, that is, there exists a random field on the vertices and edges such that the marginal distribution of the random field on the vertices is the spin Ising model and the marginal distribution of the random field on the edges is the FK Ising model. For instance, spin correlations can be expressed in terms of connection probabilities using this coupling. See for example [5] for more information.

We consider only the case $q = 2$ with the critical parameter $p = \frac{\sqrt{2}}{\sqrt{2}+1}$ in this article. Also we consider only the square lattice $\mathbb{Z}^2$, although, this assumption could be relaxed following [3].

The loop representation of the random cluster configuration is a dense set of loops such that no loop intersects any open or dual-open edges (the loops are the boundaries of primal and dual clusters). The choice of critical parameter for the FK Ising random cluster model leads to the weight (1) for the loop representation.

1.2. Setting and notation for the scaling limit

1.2.1. Discrete setting, conditional measure and the scaling limit. For some $\delta > 0$, $(\Omega_\delta, a_\delta, b_\delta, c_\delta, d_\delta)$ be a simply connected discrete domain with four marked boundary points and lattice mesh $\delta > 0$, that is, the boundary of Ω_δ is a path on $\delta\mathbb{L}^\spadesuit$ with properties given above. We assume that the boundary arcs $\alpha_1 = [a_\delta b_\delta]$, $\alpha_2 = [b_\delta c_\delta]$, $\alpha_3 = [c_\delta, d_\delta]$ and $\alpha_4 = [d_\delta a_\delta]$ are simple lattice paths on the modified medial lattice $\delta\mathbb{L}^\spadesuit$ such that the first and last edges are edges between two octagons and $\alpha_1, \alpha_2^\leftarrow, \alpha_3, \alpha_4^\leftarrow$, where $\alpha^\leftarrow$ denotes the reversal of α, have white octagons and small squares to their left and black octagons and white squares on their right.

Let G_δ be the graph on $\delta\mathbb{L}^\bullet$ corresponding to Ω_δ and consider the FK model with wired boundary conditions on $[b_\delta c_\delta]$ and $[d_\delta a_\delta]$. Define also an enhanced graph $\hat{G}_\delta$ where we add the external arc pattern $(a_\delta \smile b_\delta, c_\delta \smile d_\delta)$ in the sense that the wired arcs are counted to be in the same component and in the weight (1), if the interface starting at a_δ ends at b_δ, then it is counted as a closed loop.

It turns out, that on the enhanced graph $\hat{G}_\delta$, discrete versions of the Cauchy–Riemann equations are valid for a discrete observable that generalizes the one introduced in [19]. Thus that choice of the boundary conditions leads to holomorphic scaling limit of the observable and allows the scaling limit to be identified explicitly. This observable was originally proposed to deduce convergence of the FK Ising interface tree to the branching SLE, which was eventually done in in [10]. This required the (far from easy) stability of boundary conditions in the limit, which was justified in [3, see Remark 6.3], where the observable first appeared in print and was used to derive the crossing probabilities.

There are two interfaces γ and γ^* starting at a_δ and c_δ respectively. Denote by $\mathbb{P}_\delta$ the probability law of γ and by $\mathbb{P}_\delta^+$ the measure $\mathbb{P}_\delta$ conditional to the fact that γ ends to d_δ.

We assume that the sequence of domains $(\Omega_\delta, a_\delta, b_\delta, c_\delta, d_\delta)$ converges to (Ω, a, b, c, d) in the Carathéodory sense, that is, the conformal maps from a reference domain $(\mathbb{H}, 0, x, 1, \infty)$ onto $(\Omega_\delta, a_\delta, b_\delta, c_\delta, d_\delta)$ (The marked points are mapped on the marked points in the given order. The map exists for a unique $x \in (0,1)$ for each δ.) converge uniformly on compact subsets of $\mathbb{H}$ to the conformal map from a reference domain $(\mathbb{H}, 0, x, 1, \infty)$ onto (Ω, a, b, c, d).

The scaling limits $\mathbb{P} = \lim_{\delta\to 0} \mathbb{P}_\delta$ and $\mathbb{P}^+ = \lim_{\delta\to 0} \mathbb{P}_\delta^+$ are considered below.

Notice that those results that we use from [3, 10] don't require any particular regularity from the boundary of Ω_δ.

1.2.2. Conformal transformation to the upper half-plane. It is useful to describe the probability laws in the upper-half plane (or in another fixed reference domain). We apply a conformal transformation such that the points $a_\delta, b_\delta, c_\delta, d_\delta$ are mapped to points $U_0^\delta, V_0^\delta, W_0^\delta, \infty$ respectively. Then $U_0^\delta < V_0^\delta < W_0^\delta$. We choose the conformal transformations such that, as $\delta \to 0$, these points tend to some points $U_0 < V_0 < W_0$.

We will consider simple curves starting at U_0 as Loewner evolutions. See [9, 14, 20] for basic definitions of half-plane capacity, Loewner equation etc. In particular, we assume that the curves are parametrized by the half-plane capacity. The driving process is denoted by U_t and three other marked points are V_t, W_t and ∞. In particular, V_t and W_t satisfy the Loewner equation driven by U_t. Then also $U_t < V_t < W_t$. Auxiliary processes are defined by setting

$$X_t = V_t - U_t, \qquad Y_t = W_t - V_t.$$

1.3. The hypergeometric SLE($\frac{16}{3}$)

The hypergeometric SLE (the name, which we abbreviate to hSLE, was proposed in [21], but such and more general processes appeared earlier in [4, 16, 23]) with parameter value $\kappa = \frac{16}{3}$ is defined by letting the driving process satisfy the stochastic differential

$$\mathrm{d}U_t = \frac{4}{\sqrt{3}}\mathrm{d}B_t + \left(-\frac{2}{X_t} + \frac{2}{X_t+Y_t} - \frac{4}{3}\frac{Y_t\left(-1+\sqrt{1+\frac{X_t}{Y_t}}\right)}{X_t(X_t+Y_t)} \right)\mathrm{d}t. \tag{2}$$

Note that the third term inside the brackets is equal to $-\frac{16}{3}\frac{F'(z)}{F(z)}\frac{1-z}{s}$ evaluated at $z=\frac{X_t}{X_t+Y_t}$ and $s = X_t+Y_t$, where $F(z)$ is the hypergeometric function ${}_2F_1(\frac{3}{4},\frac{1}{4};\frac{3}{2};z)$. Here B_t is a standard one-dimensional Brownian motion.

1.4. The main result

By the results of [10], the scaling limit $\mathbb{P} = \lim_{\delta\to 0}\mathbb{P}_\delta$ is equal to a certain $\mathrm{SLE}[\frac{16}{3}, Z]$ process, that is, an $\mathrm{SLE}(\frac{16}{3})$ process whose drift is given by a partition function Z. That result of [10] extends the convergence of FK Ising chordal interface to the chordal $\mathrm{SLE}(\frac{16}{3})$ shown in [2] and its proof uses the generalized martingale observable considered already in [3]. We show in the current paper, that this process when its law is weighted by the generalized martingale observable is in fact a hypergeometric SLE. The topology of the convergence is given by the weak convergence of probability measures on the metric space of continuous functions. We use that result to prove the following theorem.

Theorem 1.1. *The sequence $\mathbb{P}^+_\delta$ converges, in the same topology as above, to $\mathbb{P}^+$ which is the law of a hypergeometric SLE($\frac{16}{3}$).*

See also Section 2.5 below for description of the scaling limit of the joint law of the pair (γ,γ^*).

2. Scaling limit of FK Ising model interface as hyperbolic SLE

2.1. Discrete martingale observable

In the random cluster model we take the boundary conditions which are free–wired–free–wired and they change across the edges corresponding to $a_\delta, b_\delta, c_\delta, d_\delta$. There are interfaces starting and ending to these points. Due to topological (as well as parity) reasons, the interface starting at a_δ has to end at b_δ or d_δ. We denote these two mutually exclusive events as $(a_\delta \frown$

$b_\delta, c_\delta \frown d_\delta)$ and $(a_\delta \frown d_\delta, c_\delta \frown b_\delta)$, respectively, and we call them internal arc patterns.

We consider the quantity

$$M_t^\delta = \mathbb{P}_\delta(\,(a_\delta \frown d_\delta, c_\delta \frown b_\delta)\,|\,\mathcal{F}_t) \tag{3}$$

which we call an observable. Here $\mathcal{F}_t$ is the σ-algebra generated by $\gamma(s)$, $s \in [0,t]$. Since M_t^δ is a conditional expected value of a random variable with respect to $\mathcal{F}_t$, the process $(M_t^\delta)_{t\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and the probability measure $\mathbb{P}_\delta$.

2.2. Scaling limit of the observable

In [3], see also Section 4.4.1 of [10] (the article whose notation we are using here), it was shown that the observables M_t^δ converge to a scaling limit M_t. It has an explicit formula

$$M_t = \sqrt{1 + \frac{Y_t}{X_t}} - \sqrt{\frac{Y_t}{X_t}}. \tag{4}$$

The mode of convergence is given by the following result:

Proposition 2.1. *For each $\varepsilon > 0$ and $T > 0$, there exists an event E and $\delta_0 > 0$ such that the following holds. If $\delta \leq \delta_0$, then $\mathbb{P}_\delta(E) > 1 - \varepsilon$ and*

$$\sup_E \sup_{t\in[0,T]} |M_t^\delta - M_t| \leq \varepsilon.$$

This result and the martingale property of $(M_t^\delta)_{t\geq 0}$ imply the following.

Proposition 2.2. *The process $(M_t)_{t\geq 0}$ is a martingale with respect to filtration $(\mathcal{F}_t)_{t\geq 0}$ and the probability measure $\mathbb{P}$.*

Namely, let $s < t$ and let f be any continuous, bounded random variable which is measurable with respect to $\mathcal{F}_s$. Then $\mathbb{E}_\delta(M_t^\delta f) = \mathbb{E}_\delta(M_s^\delta f)$ by the martingale property of the discrete observable. By the triangle inequality

$$\begin{aligned} |\mathbb{E}(M_t f) - \mathbb{E}(M_s f)| \leq{}& |\mathbb{E}(M_t f) - \mathbb{E}_\delta(M_t f)| + |\mathbb{E}(M_s f) - \mathbb{E}_\delta(M_s f)| \\ &+ |\mathbb{E}_\delta((M_t - M_t^\delta) f)| + |\mathbb{E}_\delta((M_s - M_s^\delta) f)|. \end{aligned}$$

First and second term tend to zero as $\delta \to 0$ by the weak convergence of probability measures. The third and fourth term also tend to zero by Proposition 2.1, since

$$|\mathbb{E}_\delta((M_t - M_t^\delta)f)| \leq 2\,\mathbb{E}_\delta(\mathbb{1}_{E^c}|f|) + \sup_E \sup_{t\in[0,T]} |M_t^\delta - M_t|\,\mathbb{E}_\delta(|f|).$$

2.3. Weighting by a martingale

We weight the probability measure by the martingale M_t/M_0 (the process is stopped upon the martingale hitting 0 or 1, i.e. when X_t or Y_t hit zero).

Denote the event $(a_\delta \frown d_\delta, c_\delta \frown b_\delta)$ by A. Then by properties of conditional expected values

$$\mathbb{E}(fM_t) = \mathbb{E}(f\,\mathbb{E}(\mathbb{1}_A \,|\, \mathcal{F}_t)) = \mathbb{E}(\mathbb{E}(f\,\mathbb{1}_A \,|\, \mathcal{F}_t)) = E(f\,\mathbb{1}_A)$$

for any $\mathcal{F}_t$-measurable bounded random variable f. Thus the probability measure $\mathbb{P}$ weighted by M_t/M_0 can be interpreted as to be conditioned by the event $(a_\delta \frown d_\delta, c_\delta \frown b_\delta)$ and thus equals to $\mathbb{P}^+$.

2.3.1. Girsanov's theorem. Suppose that N_t is a martingale such that

$$M_t = \exp\left(N_t - \frac{1}{2}\langle N\rangle_t\right)$$

Then by Itô's lemma, M_t and N_t satisfy the identity

$$N_t = N_0 + \int_0^t \frac{\mathrm{d}M_s}{M_s}$$

which can be used for defining N_t for any positive martingale M_t.

Under the probability measure weighted by the martingale M_t/M_0, it holds that the process

$$B_t - \langle B, N\rangle_t \tag{5}$$

is a standard Brownian motion by Girsanov's theorem (see for instance [6], Section 2.12). Thus if we have a Loewner evolution whose driving process is

$$U_t = U_0 + \sqrt{\kappa}B_t + D_t$$

where D_t is the drift of U_t in the sense that D_t is a bounded variation process, then the driving process can be written as

$$U_t = U_0 + \sqrt{\kappa}\hat{B}_t + D_t + \Delta_t$$

where $\hat{B}_t$ is a standard Brownian motion under the weighted probability measure. Here

$$\Delta_t = \sqrt{\kappa}\langle B, N\rangle_t$$

by (5).

2.4. The driving process conditioned on the internal arc configuration

Remember that by results of [10], see in particular Section 5.5.2 therein, M_t satisfies a stochastic differential equation which is written in the integral form as

$$M_t = M_0 + \int_0^t \frac{1}{2\sqrt{3}} \frac{(1-M_s^2)^3}{Y_s\, M_s(M_s^2+1)} \mathrm{d}B_s$$

and the driving function U_t satisfies

$$\mathrm{d}U_t = -\frac{4}{\sqrt{3}}\mathrm{d}B_t + \left(\frac{2}{X_t} - \frac{1}{3}\frac{(3M_t^4+2M_t^2+1)(1-M_t^2)^2}{Y_t M_t^2(M_t^2+1)^2}\right)\mathrm{d}t.$$

These results are based on the holomorphic observables and a martingale characterization given formulas of the type (4) for a pair of martingales. For more information, see [10].

Consequently by the considerations of Section 2.3.1, for a process $(\hat{B}_t)$ which is a Brownian motion under the measure $\mathbb{P}^+$ (the one weighted by (M_t/M_0)), it holds that

$$\begin{aligned}\mathrm{d}U_t &= -\frac{4}{\sqrt{3}}\mathrm{d}\hat{B}_t + \left(\frac{2}{X_t} - \frac{1}{3}\frac{(3M_t^4+2M_t^2+1)(1-M_t^2)^2}{Y_t M_t^2(M_t^2+1)^2}\right.\\ &\qquad\qquad \left. - \frac{2}{3}\frac{(1-M_s^2)^3}{Y_s\, M_s^2(M_s^2+1)}\right)\mathrm{d}t\\ &= -\frac{4}{\sqrt{3}}\mathrm{d}\hat{B}_t + \left(\frac{2}{X_t} - \frac{1}{3}\frac{(M_t^4+2M_t^2+3)(1-M_t^2)^2}{Y_t M_t^2(M_t^2+1)^2}\right)\mathrm{d}t.\end{aligned}$$

The rightmost term on the first line is $\sqrt{\kappa}\,\mathrm{d}\langle B, N\rangle_t$. By plugging in the expression (4) gives after some algebra

$$\mathrm{d}U_t = -\frac{4}{\sqrt{3}}\mathrm{d}\hat{B}_t + \left(\frac{2}{X_t} - \frac{2}{X_t+Y_t} - \frac{4}{3}\frac{Y_t\left(2+\sqrt{1+\frac{X_t}{Y_t}}\right)}{X_t(X_t+Y_t)}\right)\mathrm{d}t$$

which is equivalent to (2). Thus it follows that $\mathbb{P}^+$ is the law of a hypergeometric SLE($\frac{16}{3}$).

For another result which also uses the convergence of a sequence of probability measures and a sequence of martingales (each with respect to a corresponding probability measure) to derive the convergence of a sequence of weighted probability measures, see [7].

2.5. Joint law of the pair of interfaces in the arc configuration

The scaling limit of the joint law of the interfaces from a_δ to d_δ and c_δ to b_δ under the probability measure conditioned on the event $(a_\delta \frown d_\delta, c_\delta \frown b_\delta)$ can be characterized in the following way.

Consider the scaling limit of the pair of interfaces in the conditioned arc configuration after conformal transformation to the upper half-plane and let the curves be γ_1 and γ_2 such that γ_1 and γ_2 start at U_0 and W_0, respectively. Parametrize the curves γ_1 and γ_2 in some way. For instance, use the half-plane capacity seen from ∞ or V_0 as parametrization for γ_1 and γ_2, respectively. Then define $\mathcal{F}_{s,t}$ to be the σ-algebra generated by $\gamma_1(q)$, $q \in [0,s]$, and $\gamma_2(r)$, $r \in [0,t]$.

By the same argument that says that the marginal law of γ_1 is the hSLE, we see that conditionally on $\mathcal{F}_{s,t}$, the marginal law of γ_1 is the hSLE. Degenerate versions of these statements give that (i) the pair (γ_1, γ_2) can be sampled by sampling first γ_j, $j = 1$ or 2, as hSLE in $\mathbb{H}$ and then sampling γ_{3-j} in H, where H is the component of $\gamma_{3-j}(0) + \mathrm{i}\,(0+)$ in $\mathbb{H} \setminus \gamma_j(0,\infty)$, as an independent chordal SLE and (ii) that a similar conditional version holds (i.e. conditional on $\mathcal{F}_{s,t}$, the pair can be sampled as an hSLE and an independent chordal SLE).

2.6. On the topology of convergence of the interfaces

The topology of convergence of random curves used in [2, 9, 10] is given by the weak convergence of probability measures on the set of capacity-parametrized curves with the uniform norm. Once we have this type of convergence for the "auxiliary" sequence $\mathbb{P}_\delta$ which is shown in [10], we can apply Proposition 2.1 to show that $\mathbb{P}_\delta^+$ converges weakly. Namely, for any $\delta > 0$ and $t > 0$ and any continuous, bounded, t-measurable f

$$(6)\quad |\mathbb{E}_\delta(M_t^\delta f) - \mathbb{E}(M_t f)| \leq |\mathbb{E}_\delta(M_t^\delta f) - \mathbb{E}_\delta(M_t f)| + |\mathbb{E}_\delta(M_t f) - \mathbb{E}(M_t f)|.$$

The second term on the right goes to zero by the weak convergence and the first term by Proposition 2.1 by the same argument as we gave in Section 2.2.

This topology of convergence extends to the pair of interfaces.

Notice that alternatively we could use the fact that we do have the estimates of [11] also for the sequence weighted probability measures, since the weighting factor is bounded and the estimates hold for the non-weighted sequence.

3. Comparison to a similar result on percolation

Let us compare the previous case of FK Ising model to that of the critical site percolation model on triangular lattice.

Consider the site percolation model on the triangular lattice

$$\delta L_{\mathrm{tri}} = \{\delta(j + k e^{\mathrm{i}\pi/3}) \,:\, j, k \in \mathbb{Z}\}.$$

It was shown in [18], that the interface of this model in the chordal setup converges to SLE(6).

Note that the existence of an open percolation crossing from $[a_\delta b_\delta]$ to $[c_\delta d_\delta]$ in Ω_δ is exactly the event of an internal arc pattern $(a_\delta \frown d_\delta, c_\delta \frown b_\delta)$ of interfaces. A central result in [18] is that the probability of such a crossing event is given by Cardy's formula

$$\lim_{\delta\to 0} \mathbb{P}_\delta((a_\delta \frown d_\delta, c_\delta \frown b_\delta)) = C \left(\frac{X_0}{X_0+Y_0}\right)^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \frac{X_0}{X_0+Y_0}\right)$$

holds, where $X_0 = V_0 - U_0$ and $Y_0 = W_0 - V_0$ with the notation used above and C is a constant, whose exact value we don't need below.

It follows then that

$$M_t = \left(\frac{X_t}{X_t+Y_t}\right)^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \frac{X_t}{X_t+Y_t}\right)$$

is a martingale for the scaling limit for $t \le \tau$ where τ is the time when the quadrilateral degenerates (the interface hits $[bc]$ or $[cd]$). Since the interface (exploration process from a to d) converges to the chordal SLE(6),

$$\mathrm{d}U_t = \sqrt{6}\,\mathrm{d}B_t, \qquad \mathrm{d}X_t = -\sqrt{6}\,\mathrm{d}B_t + \frac{2}{X_t}, \qquad \partial_t Y_t = \frac{2}{X_t+Y_t} - \frac{2}{X_t}.$$

Thus it follows that if $\mathrm{d}N_t = \frac{\mathrm{d}M_t}{M_t}$

$$\sqrt{6}\,\mathrm{d}\langle B, N\rangle_t = -\frac{2\left(\frac{Y_t}{X_t+Y_t}\right)^{\frac{1}{3}}}{X_t\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};\frac{X_t}{X_t+Y_t}\right)}\mathrm{d}t$$

Thus for the process $\hat{B}_t = B_t - \langle B, N\rangle_t$ which is a Brownian motion under the probability measure weighted by the martingale M_t/M_0, the driving process U_t satisfies

$$\mathrm{d}U_t = \sqrt{6}\mathrm{d}\hat{B}_t - \frac{2\left(\frac{Y_t}{X_t+Y_t}\right)^{\frac{1}{3}}}{X_t\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{4}{3};\frac{X_t}{X_t+Y_t}\right)}\mathrm{d}t$$

which shows that the Loewner evolution is the hypergeometric SLE(6).

4. Discussion

We do expect a similar result for other values of the q parameter, namely that similar processes to those studied here are fact SLE($\kappa, \kappa - 6$) processes in the 4-point fused setting (analogous to [10]) and hypergeometric SLE(κ) processes in the 4-point setting when conditioned on an arc pattern. The missing link, as for the chordal interface convergence, is the discrete holomorphicity of the observables – so far they have only been shown to satisfy half of Cauchy-Riemann equations.

The approach we took gives the hypergeometric SLE as a product of a calculation based on the holomorphic observables, the Itô calculus of semi-martingales as well as the Girsanov theorem. A different approach based on the uniqueness of the hypergeometric SLE and the convergence of the interfaces in the chordal setup is taken in [21].

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