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Higher decay inequalities for multilinear oscillatory integrals

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In this paper we establish sharp estimates (up to logarithmic losses) for the multilinear oscillatory integral operator studied by Phong, Stein, and Sturm [17] and Carbery and Wright [3] on any product $\prod_{j=1}^{d} L^{p_j}(\mathbb{R})$ with each $p_j \geq 2$, extending the known results outside the previously-studied range $\sum_{j=1}^{d} p_j^{-1} = d-1$. Our theorem assumes a second-order nondegeneracy condition of Varčenko type, and as a corollary reproduces a variant of Varčenko's theorem and implies Fourier decay estimates for measures of smooth density on degenerate hypersurfaces in $\mathbb{R}^d$.

1. Introduction

Let $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and let $\phi(\boldsymbol{x})$ be real analytic on some neighborhood of the origin $\mathbf{0} \in \mathbb{R}^d$. Fix a smooth cutoff function χ compactly supported in that neighborhood, and consider the multilinear functional

$$\Lambda(\boldsymbol{f}) = \int_{\mathbb{R}^d} e^{i\lambda\phi(\boldsymbol{x})} \chi(\boldsymbol{x}) \prod_{j=1}^{d} f_j(x_j) d\boldsymbol{x}, \tag{1.1}$$

where $\boldsymbol{f} = (f_1, \ldots, f_d)$ is any d-tuple of locally integrable functions on $\mathbb{R}$. The purpose of this article is to study the asymptotic behavior in the real parameter λ as $|\lambda| \to \infty$ of the norm of Λ when viewed as a linear functional on $\prod_{j=1}^{d} L^{p_j}(\mathbb{R})$.

Bilinear variants of this form have a long history in harmonic analysis in connection with the study of Fourier integral operators and Radon-like transforms (see, e.g., Greenleaf and Uhlmann [11] and Seeger [19]). In the

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1990s, Phong and Stein initiated the study of these oscillatory integrals as a subject in its own right [13]. Their program focused primarily on weighted and unweighted $L^2 \times L^2$ estimates [14–16], as the L^2 case was most directly connected to the earlier FIO roots. In this setting, the undamped bilinear case with real analytic phase was ultimately settled in [16], with the transition to C^∞ phases being later accomplished by Rychkov [18] and Greenblatt [8]. These works demonstrated the primary role of the (reduced) Newton polyhedron of the phase ϕ, which had also been identified as a key object in Varčenko's study of scalar oscillatory integrals some twenty years earlier [20]. To define the Newton polyhedron, expand $\phi(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$ near the origin, where $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, and define the Taylor support of ϕ by $\operatorname{supp}(\phi) = \{\boldsymbol{\alpha} : c_{\boldsymbol{\alpha}} \neq 0\}$. Let $\mathbb{R}_{\geq}$ denote the nonnegative real numbers. The Newton polyhedron of ϕ, denoted by $\mathcal{N}(\phi)$, is defined to be the convex hull of

$$\bigcup_{\boldsymbol{\alpha} \in \operatorname{supp}(\phi)} \left(\boldsymbol{\alpha} + \mathbb{R}^d_{\geq} \right),$$

and the Newton distance d_ϕ of ϕ is defined to be the minimum over all t such that $(t, \ldots, t) \in \mathcal{N}(\phi)$. In the specific case of the form (1.1), modulating each function f_j by a function of the form $e^{-i\lambda\phi_j(x_j)}$, it can be easily seen that terms in the power series of ϕ which depend on only one coordinate function do not affect the norm of Λ on $\prod_{j=1}^d L^{p_j}(\mathbb{R})$, so it will be assumed without loss of generality that every $\boldsymbol{\alpha} \in \operatorname{supp}(\phi)$ has at least two strictly positive components. After removing all such single-variable terms in the Taylor support of ϕ, the resulting Newton polyhedron corresponds to the object known as the reduced Newton polyhedron in other contexts.

The success of the program of Phong and Stein to establish $L^2 \times L^2$ estimates for (1.1) prompted generalizations and extensions to a variety of higher-dimensional settings, including results of Carbery, Christ, and Wright [1] as well as Carbery and Wright [3]. The most natural extension of the work of Phong and Stein to higher dimensions turned out to be (1.1) itself, which was studied by Phong, Stein, and Sturm [17] and Carbery and Wright [3]. The main theorem of Phong, Stein, and Sturm which is most closely related to the present work is as follows:

Theorem B ([17]). Let $\alpha^{(1)}, \ldots, \alpha^{(K)} \in \mathbb{N}^d \setminus \{0\}$ be K given vertices, and let $S \in \mathbb{R}[x_1, \ldots, x_d]$ be any polynomial of degree n_S. Set

$$D(\alpha^{(1)}, \ldots, \alpha^{(K)}) = \left\{ x \in U : |S^{(\alpha^{(k)})}(x)| > 1, \quad 1 \leq k \leq K \right\}.$$

Let $N^*(\alpha^{(1)},\dots,\alpha^{(K)})$ be the reduced Newton polyhedron generated by the vertices $\alpha^{(k)}$, i.e., the Newton polyhedron generated by those vertices $\alpha^{(k)}$ with at least two strictly positive components. Then for any algebraic domain $D \subset D(\alpha^{(1)},\dots,\alpha^{(K)})$ and any $\alpha \in N^*(\alpha^{(1)},\dots,\alpha^{(K)})$, we have

$$\left| \int_D e^{i\lambda S(\boldsymbol{x})} \prod_{j=1}^{d} f_j(x_j) d\boldsymbol{x} \right| \leq C|\lambda|^{-\frac{1}{|\alpha|}} \ln^{d-\frac{1}{2}}(2+|\lambda|) \prod_{j=1}^{d} \|f_j\|_{p_j}, \quad d \geq 2. \tag{1.2}$$

Here χ_D is the characteristic function of the algebraic domain D, $\lambda \neq 0$ is any real number, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and $\frac{1}{p_j'} = 1 - \frac{1}{p_j} = \frac{\alpha_j}{|\alpha|}$. The constant C depends only on n_S, $|\alpha|$, and the so-called type of D.

Phong, Stein, and Sturm's purpose in proving Theorem B was to establish a robust stability result for the multilinear form (1.1), focusing on the role of the Newton polyhedron. In the present paper, our main result, Theorem 1.1, focuses on a somewhat different question inspired by Theorem B concerning the possible range of exponents p_j and the effect of this range on the decay as a function of $|\lambda|$. It is certainly the case that the exponent $-\frac{1}{|\alpha|}$ of $|\lambda|$ which appears in (1.2) is sharp for the particular choice of exponents p_j given by that theorem. In that sense, Theorem B cannot be improved (in this sense, Theorem B surpasses Theorem 1.1 below in the case $\frac{1}{p_j'} = \frac{\alpha_j}{|\alpha|}$ since we do not investigate the stability of the constant analogous to C as a function of the phase). The main contribution of Theorem 1.1 is that when the exponents p_j are taken to be strictly larger than the exponents in (1.2), the resulting norm decays at a strictly faster rate as a function of $|\lambda|$ for generic phases (meaning those that satisfy the nondegeneracy condition (1.5) below). We also note that the geometry of the domain of integration plays no major role in our result because we do not prove stability of constants.

By studying (1.1) in the large-exponent regime as we will here, the decay in λ of the form (1.1) is generally of a higher order than in the inequality (1.2). However, this extra decay brings with it additional difficulties not encountered in [17] or [3], which make it necessary to introduce certain auxiliary nondegeneracy assumptions that were not previously needed. The formulation we have chosen is essentially a second-order version of the so-called Varčenko hypothesis [20]. Let $\mathcal{F}(\phi)$ denote the set of compact faces of $\mathcal{N}(\phi)$. In particular, the set of zero-dimensional faces $\mathcal{V}(\phi) \subset \mathcal{F}(\phi)$ is the collection of vertices of $\mathcal{N}(\phi)$. For each $F \in \mathcal{F}(\phi)$, define the polynomial ϕ_F by

$$\phi_F(\boldsymbol{x}) = \sum_{\alpha \in F \cap \mathbb{N}^d} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Varčenko's original nondegeneracy condition (the Varčenko hypothesis) can be phrased as: for all $F \in \mathcal{F}(\phi)$,

$$\bigcap_{1 \le i \le d} \{\boldsymbol{x} : \partial_i \phi_F(\boldsymbol{x}) = 0\} \subset \bigcup_{1 \le j \le d} \{\boldsymbol{x} : x_j = 0\}. \tag{1.3}$$

Under this hypothesis, Varčenko showed:

Theorem ([20]). Let ϕ be a real analytic function defined in a sufficiently small neighborhood of the origin satisfying (1.3). Let $\ell - 1$ denote the smallest dimension over all faces of $\mathcal{N}(\phi)$ containing $\boldsymbol{\nu} = (\nu, \ldots, \nu)$, where $\nu = d_\phi$ is the Newton distance of ϕ. Then

$$\left| \int_{\mathbb{R}^d} e^{i\lambda\phi(\boldsymbol{x})} \chi(\boldsymbol{x}) d\boldsymbol{x} \right| \lesssim |\lambda|^{-\frac{1}{\nu}} \ln^{d-\ell}(2 + |\lambda|). \tag{1.4}$$

The nondegeneracy condition which proves to be most useful for our present purposes is a second-order Varčenko-type nondegeneracy condition: we assume that for all $F \in \mathcal{F}(\phi)$,

$$\bigcap_{i \neq j} \{\boldsymbol{x} : \partial_i \partial_j \phi_F(\boldsymbol{x}) = 0\} \subset \bigcup_{1 \le j \le d} \{\boldsymbol{x} : x_j = 0\}. \tag{1.5}$$

In other words, we assume that, for any $F \in \mathcal{F}(\phi)$, any point at which all off-diagonal terms of the Hessian matrix $\nabla^2 \phi_F$ simultaneously vanish must belong to a coordinate hyperplane. Note that Varčenko's original hypothesis does not require the Newton polytope and the *reduced* Newton polytope to coincide for the phase ϕ as we have already assumed. This small distinction makes (1.3) and (1.5) slightly less similar than appearances suggest; in particular, neither implies the other.

Our principal result is as follows:

Theorem 1.1. *Suppose ϕ is real analytic and satisfies* (1.5)*. Let $p_j \in [2, \infty]$ for $1 \le j \le d$. If the support of χ is contained in a sufficiently small neighborhood of* $\mathbf{0}$*, then for any real number $\nu > 2$,*

$$|\Lambda(\boldsymbol{f})| \lesssim |\lambda|^{-\frac{1}{\nu}} \ln^m(2 + |\lambda|) \prod_{j=1}^{d} \|f_j\|_{p_j} \tag{1.6}$$

for some implicit constant independent of $\boldsymbol{f}$ *and some* $m \geq 0$ *if and only if*

$$\frac{\boldsymbol{\nu}}{\boldsymbol{p}'} = \left(\frac{\nu}{p_1'}, \dots, \frac{\nu}{p_d'}\right) \in \mathcal{N}(\phi), \tag{1.7}$$

where p' *denotes the conjugate of* p*.*

It should be noted that the full result of Theorem 1.1 requires at least some version of the nondegeneracy condition (1.5). For example, the phase $\phi(\boldsymbol{x}) = (\sum_{j=1}^{d} x_j)^k$ for $k \geq 3$ yields a multilinear functional Λ whose norm decays no faster than $|\lambda|^{-\frac{1}{k}}$ regardless of the choice of exponents p_j. This can be seen by simply testing on smooth, nonnegative functions f_j compactly supported near the origin. Phong, Stein, and Sturm did not study this phase in any regime in which decay greater than $|\lambda|^{-\frac{1}{k}}$ might have otherwise been expected, so no nondegeneracy hypotheses were necessary there. In our case, ϕ fails to satisfy (1.5) because all second derivatives happen to vanish on the hyperplane $x_1 + \cdots + x_d = 0$. However, this means that when 1.5 is satisfied, the best decay exponent provided by Theorem 1.1 will generally be higher than that of Theorem B and will match the exponent from Varčenko's Theorem. An example of this type is the phase $\phi(x_1, x_2) = x_1^{2k+1} x_2 + x_1 x_2^{2k+1}$ for any integer $k \geq 1$. Both Theorem B and Theorem 1.1 give that the norm of the associated linear functional Λ on $L^2 \times L^2$ decays like $|\lambda|^{-\frac{1}{2k+2}}$. However, Theorem 1.1 also predicts that this functional has a norm decaying like $|\lambda|^{-\frac{1}{k+1}}$ on $L^\infty \times L^\infty$, which is a case outside the Phong-Stein-Sturm regime $\sum_j \frac{1}{p_j'} = 1$ and an exponent which agrees with the decay rate of the corresponding scalar oscillatory integral (i.e., Varčenko's Theorem).

The reader may also note that Theorem 1.1 falls within the general framework of Christ, Li, Tao and Thiele [2], who investigated more general multilinear integrals of the form

$$\left| \int_{\mathbb{R}^d} e^{i\lambda\phi(\boldsymbol{x})} \chi(\boldsymbol{x}) \prod_{j=1}^{J} f_j(\pi_j(\boldsymbol{x})) d\boldsymbol{x} \right| \leq C|\lambda|^{-\delta} \prod_{j=1}^{J} \|f_j\|_{p_j}. \tag{1.8}$$

Here $J \in \mathbb{N}$ is any positive integer and each $\pi_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ is a surjective linear transformation with $1 \leq d_j \leq d-1$. The main focus of [2] is to explore general conditions on the phase ϕ and the transformations π_j to ensure decay estimates (1.8) hold for some $\delta > 0$ and some exponents $(p_1, \dots, p_d)$. In contrast, Theorem 1.1 deals with a very specific choice of projections π_j and deals with the question of finding sharp decay exponents δ. The methods used here also differ significantly from the methods of [2].

For readers interested in the exponent of the logarithmic factor, our proof provides a value of m which can be calculated easily from the geometry of the Newton polyhedron: $m = 0$ if $\frac{\boldsymbol{\nu}}{\boldsymbol{p}'}$ is an interior point of $\mathcal{N}(\phi)$, and $m = d - \ell$ if the face of lowest dimension containing $\frac{\boldsymbol{\nu}}{\boldsymbol{p}'}$ has dimension $(\ell - 1)$. The value of m may not be sharp in general, but is sharp when all $p_j = \infty$. In particular, Theorem 1.1 recovers the classical result of Varčenko under the modified hypothesis (1.5):

Corollary 1.2. Let ϕ be the same as above and let $(\ell - 1)$ denote the smallest dimension over all faces of $\mathcal{N}(\phi)$ containing $\boldsymbol{\nu} = (\nu, \ldots, \nu)$, where $\nu = d_\phi$ is the Newton distance of ϕ. Then

$$|\Lambda(\boldsymbol{f})| \lesssim |\lambda|^{-\frac{1}{\nu}} \ln^{d-\ell}(2 + |\lambda|) \prod_{j=1}^{d} \|f_j\|_\infty.$$

When $\chi(\mathbf{0}) \neq 0$, the power of the log term is also sharp.

The usefulness of this functional variant of Varčenko's theorem becomes apparent when the functions f_j are taken to be complex exponentials. Fixing $f_j(x_j) = e^{i\xi_j x_j}$ and setting $\lambda = \xi_{d+1}$, the above corollary together with standard nonstationary phase estimates implies sharp estimates for the Fourier decay of measures of smooth density on the surface $(\boldsymbol{x}, \phi(\boldsymbol{x}))$; that is, for $\boldsymbol{\xi} \in \mathbb{R}^{d+1}$,

$$\left| \int e^{i\boldsymbol{\xi} \cdot (\boldsymbol{x}, \phi(\boldsymbol{x}))} \chi(\boldsymbol{x}) d\boldsymbol{x} \right| \lesssim \|\boldsymbol{\xi}\|_2^{-\frac{1}{d_\phi}} \ln^{d-\ell}(2 + |\xi_{d+1}|).$$

The strategy we employ to prove Theorem 1.1 can be sketched as follows. After a series of reductions, we decompose the support of χ into boxes of the form $Q_{\boldsymbol{\epsilon}} := [\epsilon_1, 8\epsilon_1] \times \cdots \times [\epsilon_d, 8\epsilon_d]$ for $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_d) \in (0,1)^d$. On each such box, the nondegeneracy hypothesis (1.5) makes it possible to establish a uniform lower bound of the form

$$\inf_{\boldsymbol{x} \in Q_{\boldsymbol{\epsilon}}} \max_{i \neq j} |x_i x_j \partial_i \partial_j \phi(x)| \gtrsim \max_{\boldsymbol{\alpha} \in \mathcal{N}(\phi)} \epsilon_1^{\alpha_1} \cdots \epsilon_d^{\alpha_d} \tag{1.9}$$

(see Lemma 3.1 and Corollary 3.3). The proof of this inequality is accomplished in Section 4 and is a consequence of the fact that for any box $Q_{\boldsymbol{\epsilon}}$, there is always a face $F \in \mathcal{N}(\phi)$ such that ϕ_F strongly dominates the rest of the terms of the Taylor series at all points of the box. With these bounds, Section 6 obtains the sharp estimate for $\Lambda(\boldsymbol{f})$ assuming $\boldsymbol{f}$ is supported on a single box $Q_{\boldsymbol{\epsilon}}$ (see Lemma 2.1) by coupling the aforementioned lower bound with the usual operator van der Corput estimates of Phong and Stein [15]. The passage from individual boxes $Q_{\boldsymbol{\epsilon}}$ to the entire multilinear functional (1.1) is then fairly direct because the norms of the individual pieces sum without any serious difficulty. The precise calculation of the sum is contained in Lemma 2.2, which is proved in Section 7. In terms of the layout of the full proof of Theorem 1.1, Section 2 reduces Theorem 1.1 to Lemmas 2.1 and 2.2. Section 3 records and proves some technical results needed for the proofs of these two lemmas as well as the related Corollary 3.3, whose proof appears in Section 5.

Before we begin, it is perhaps worth noting explicitly that the methods we will use are quite different than the familiar resolution of singularities arguments found, for example, in the work of Greenblatt [9, 10], Collins, Greenleaf, and Pramanik [4], or the second and third authors [12, 21, 22]. In particular, at no point do we use any nonlinear coordinate changes or make any detailed analyses of the nature of any algebraic singularities. The only somewhat delicate part of the proof is the establishment of the (effectively sharp) quantitative lower bound (1.9) (which is the content of Lemma 3.1). To think of this in another way, our methods rely on detailed quantitative analysis of rescalings on each $Q_{\boldsymbol{\epsilon}}$ rather than on more abstract monomializations or changes of variables. We expect this alternative approach to find use in future applications well, as it is in many cases more stable under perturbation.

2. Reduction of (1.6) to Lemmas 2.1 and 2.2

In this section, we will prove Theorem 1.1 using Lemma 2.1 and Lemma 2.2. After that, we will also demonstrate that the estimate (1.6) is optimal up to possibly the exponent of the logarithm.

We begin with basic notation: given two vectors $\boldsymbol{x} = (x_1, \dots, x_d)$, $\boldsymbol{y} = (y_1, \dots, y_d) \in \mathbb{R}^d$, and a scalar c, we define

- $|\boldsymbol{x}| = |x_1| + \cdots + |x_d|$, so that $|\boldsymbol{x}| = x_1 + \cdots + x_d$ if $\boldsymbol{x} \in \mathbb{R}^d_{\geq}$,
- $\boldsymbol{xy} = (x_1y_1, \dots, x_dy_d)$,
- $\boldsymbol{x}^{\boldsymbol{y}} = x_1^{y_1} \cdots x_d^{y_d}$,
- $c^{\boldsymbol{x}} = (c^{x_1}, \dots, c^{x_d})$,
- $\boldsymbol{c} = (c, \dots, c)$, and
- $\frac{\boldsymbol{x}}{\boldsymbol{y}} = \left(\frac{x_1}{y_1}, \dots, \frac{x_d}{y_d}\right)$,

assuming in each case that the right-hand side makes sense. We also use the standard notation $\partial^{\boldsymbol{\alpha}}$ for multiindices $\boldsymbol{\alpha}$, i.e., $\partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. Lastly, when $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_d)$, where each ϵ_i is an integer power of 2, we use $Q_{\boldsymbol{\epsilon}}$ to denote the box

$$Q_{\boldsymbol{\epsilon}} = [\epsilon_1, 8\epsilon_1] \times \cdots \times [\epsilon_d, 8\epsilon_d].$$

Suppose $Q_{\boldsymbol{\epsilon}}$ is any fixed box within a sufficiently small neighborhood of the origin and let $\chi_{\boldsymbol{\epsilon}}$ be any smooth function supported on $Q_{\boldsymbol{\epsilon}}$ with

$$|\partial^{\boldsymbol{k}} \chi_{\boldsymbol{\epsilon}}(\boldsymbol{x})| \leq C_{\boldsymbol{k}} \boldsymbol{\epsilon}^{-\boldsymbol{k}}, \quad \forall\, \boldsymbol{k} \in \mathbb{N}^d, \tag{2.10}$$

where the constants $C_{\boldsymbol{k}}$ are fixed uniformly for all boxes $Q_{\boldsymbol{\epsilon}}$. Fixing $\chi = \chi_{\boldsymbol{\epsilon}}$, our main multilinear functional (1.1) takes the form

$$\Lambda_{\boldsymbol{\epsilon}}(\boldsymbol{f}) = \int_{Q_{\boldsymbol{\epsilon}}} e^{i\lambda\phi(\boldsymbol{x})} \chi_{\boldsymbol{\epsilon}}(\boldsymbol{x}) \prod_{j=1}^{d} f_j(x_j) d\boldsymbol{x}.$$

The basic lemma on which the proof rests is as follows:

Lemma 2.1. *Assume ϕ satisfies (1.5) and $Q_{\boldsymbol{\epsilon}}$ is contained in a sufficiently small neighborhood of the origin. For fixed exponents $(p_1, \ldots, p_d) \in [2, \infty]^d$,*

$$|\Lambda_{\boldsymbol{\epsilon}}(\boldsymbol{f})| \lesssim \min_{\boldsymbol{\alpha} \in \mathcal{N}(\phi)} \{ |\lambda \boldsymbol{\epsilon}^{\boldsymbol{\alpha}}|^{-\frac{1}{2}} \boldsymbol{\epsilon}^{\frac{\boldsymbol{1}}{\boldsymbol{p}'}},\ \boldsymbol{\epsilon}^{\frac{\boldsymbol{1}}{\boldsymbol{p}'}} \} \prod_{1 \leq j \leq d} \|f_j\|_{p_j} \tag{2.11}$$

with an implicit constant that is independent of the functions $\boldsymbol{f}$ and the box $Q_{\boldsymbol{\epsilon}}$.

We will refer to the estimate of the above lemma as a single-box estimate. Its proof is given in Section 6. To sum these estimates over all boxes, we use Lemma 2.2 (proved in Section 7):

Lemma 2.2. *Suppose $\boldsymbol{\gamma} \in \mathbb{R}^d$ belongs to the reduced Newton polyhedron $\mathcal{N}(\phi)$ and let $F \subset \mathcal{N}(\phi)$ be the face of lowest dimension containing $\boldsymbol{\gamma} \in \mathbb{R}^d_{\geq}$ (or $F = \mathcal{N}(\phi)$ when $\boldsymbol{\gamma}$ is an interior point). If $\boldsymbol{z} = \nu^{-1}\boldsymbol{\gamma}$ for some $\nu > 2$, then for all $\lambda \geq 2$,*

$$\sum_{j_1, \ldots, j_d = 0}^{\infty} \min_{\substack{N \in \{0, 1/2\}, \\ \boldsymbol{\alpha} \in \mathcal{N}(\phi)}} \{ \lambda^{-N} 2^{\langle N\boldsymbol{\alpha} - \boldsymbol{z}, \boldsymbol{j} \rangle} \} \lesssim \lambda^{-\frac{1}{\nu}} \ln^{d-\ell}(\lambda),$$

where $\ell = \min\{\dim F + 1, d\}$.

Assuming that Lemmas 2.1 and 2.2 have been established, the proof of Theorem 1.1 is fairly immediate. We write the original cutoff function χ of (1.1) as a sum of functions, each supported on a distinct orthant of $\mathbb{R}^d$:

$$\chi(\boldsymbol{x}) = \sum_{sign \in \{+,-\}^d} \chi_{sign}(\boldsymbol{x}) \quad \text{for all } \prod_{1 \le j \le d} x_j \neq 0,$$

where χ_{sign} is the restriction of χ to the orthant corresponding to $sign \in \{+,-\}^d$. By the triangle inequality, it suffices to prove (1.6) for each χ_{sign}. Without loss of generality, it suffices to assume χ is restricted to the first orthant. Multiplying by a standard smooth partition of unity adapted to dyadic boxes in $\mathbb{R}^d$ (where by dyadic box we mean a box of the form $[2^{-j_1+1}, 2^{-j_1+2}] \times \cdots \times [2^{-j_d+1}, 2^{-j_d+2}]$ for $(j_1, \ldots, j_d) \in \mathbb{Z}^d$), one can write

$$\chi(\boldsymbol{x}) = \sum_{\boldsymbol{\epsilon}} \chi_{\boldsymbol{\epsilon}}(\boldsymbol{x}),$$

where each $\chi_{\boldsymbol{\epsilon}}$ is a smooth function supported in a corresponding box $Q_{\boldsymbol{\epsilon}}$ and satisfying (2.10). Let $\boldsymbol{p} \in [2, \infty]^d$. By the triangle inequality and Lemma 2.1,

$$|\Lambda(\boldsymbol{f})| \lesssim \sum_{\boldsymbol{\epsilon} = 2^{-\boldsymbol{j}},\ \boldsymbol{j} \in \mathbb{N}^d} \min_{\boldsymbol{\alpha} \in \mathcal{N}(\phi)} \{ |\lambda \boldsymbol{\epsilon}^{\boldsymbol{\alpha}}|^{-\frac{1}{2}} \boldsymbol{\epsilon}^{\frac{\boldsymbol{1}}{\boldsymbol{p}'}},\ \boldsymbol{\epsilon}^{\frac{\boldsymbol{1}}{\boldsymbol{p}'}} \} \prod_{1 \le k \le d} \|f_k\|_{p_k}.$$

If $\nu > 2$ is such that $\frac{\boldsymbol{\nu}}{\boldsymbol{p}'} \in \mathcal{N}(\phi)$, then Lemma 2.2 can be applied to estimate the sum of the series on the right-hand side using $\boldsymbol{z} = \frac{\boldsymbol{1}}{\boldsymbol{p}'}$. In particular, (1.6) follows $m = 0$ with if $\frac{\boldsymbol{\nu}}{\boldsymbol{p}'}$ is an interior point of $\mathcal{N}(\phi)$ and $m = d - \ell$ if the face of lowest dimension containing $\frac{\boldsymbol{\nu}}{\boldsymbol{p}'}$ itself has dimension $(\ell - 1)$.

Before taking up the work of proving Lemmas 2.1 and 2.2, we first pause to show that the estimate (1.6) is sharp up to a logarithmic factor. For convenience, let us define the dual polyhedron $\mathcal{N}^*(\phi)$ of $\mathcal{N}(\phi)$ by

$$\mathcal{N}^*(\phi) = \{ \boldsymbol{w} \in \mathbb{R}^d_{\geq} : \langle \boldsymbol{\alpha}, \boldsymbol{w} \rangle \ge 1 \ \text{ for all } \ \boldsymbol{\alpha} \in \mathcal{N}(\phi) \}. \tag{2.12}$$

The double dual $\mathcal{N}^{**}(\phi)$ can easily be checked to equal $\mathcal{N}(\phi)$. Likewise, it is not difficult to see that for any $\boldsymbol{w} \in \mathcal{N}^*(\phi)$, there is a constant $\delta > 0$, depending on ϕ but independent of λ, such that for all λ sufficiently large,

$$|\lambda \phi(\boldsymbol{x})| \le 10^{-10} \text{ provided } |x_j| \le \delta |\lambda|^{-w_j},\ j = 1, \ldots, d.$$

If each f_j is taken to equal the characteristic function of $[-\delta|\lambda|^{-w_j}, \delta|\lambda|^{-w_j}]$ and if the cutoff function χ is smooth and nonvanishing at the origin, then

$$|\Lambda(\boldsymbol{f})| \sim \|\boldsymbol{f}\|_1 = 2^d \delta^d |\lambda|^{-\langle \boldsymbol{1}, \boldsymbol{w} \rangle} \sim |\lambda|^{-\langle \boldsymbol{1}, \boldsymbol{w} \rangle}$$

for all λ sufficiently large. Then the estimate (1.6) implies

$$|\lambda|^{-\langle \mathbf{1},\boldsymbol{w}\rangle} \lesssim \ln^m(2+|\lambda|)|\lambda|^{-\frac{1}{\nu}}|\lambda|^{-\langle \frac{\mathbf{1}}{\boldsymbol{p}},\boldsymbol{w}\rangle}.$$

Letting $|\lambda| \to \infty$ implies

$$\left\langle \frac{\boldsymbol{\nu}}{\boldsymbol{p}'}, \boldsymbol{w} \right\rangle \geq 1$$

for all $\boldsymbol{w} \in \mathcal{N}^*(\phi)$. Consequently, $\frac{\boldsymbol{\nu}}{\boldsymbol{p}'} \in \mathcal{N}^{**}(\phi) = \mathcal{N}(\phi)$.

3. Statements of technical lemmas

In this section, we prove several technical lemmas that are needed for the proof of Lemma 2.1. To quantify the nondegeneracy condition (1.5), for any subset S of $\mathbb{R}^d$, we define

$$\|\phi\|_{V(S)} = \inf_{\boldsymbol{x}\in S} \max_{i\neq j} |x_i x_j \partial_i \partial_j \phi(\boldsymbol{x})|.$$

The quantitative lower bound for nondegeneracy that we use is as follows:

Lemma 3.1. *Let ϕ be real analytic near the origin and suppose that it satisfies the nondegeneracy condition* (1.5). *Then there is a neighborhood U of $\mathbf{0}$ and a positive constant K such that for all $Q_{\boldsymbol{\epsilon}} \subset U$*

$$\|\phi\|_{V(Q_{\boldsymbol{\epsilon}})} \geq K\boldsymbol{\epsilon}^{\boldsymbol{\alpha}}, \quad \text{for all} \quad \boldsymbol{\alpha} \in \mathcal{V}(\phi). \tag{3.13}$$

Lemma 3.1 is a variation of a corresponding key lemma from the first author's PhD Thesis [5]. The proof, which is similar to the proof in [5], is given in Section 4. In addition to this lower bound, it is also necessary to have control from above on sufficiently many derivatives of ϕ. A suitable inequality of this sort is the following:

Lemma 3.2. *There is a neighborhood U of $\mathbf{0}$ and a constant K' such that for all $\boldsymbol{k} \in \{0,1,2,3\}^d$ and all $Q_{\boldsymbol{\epsilon}} \subset U$,*

$$\sup_{\boldsymbol{x}\in 2Q_{\boldsymbol{\epsilon}}} |\boldsymbol{x}^{\boldsymbol{k}} \partial^{\boldsymbol{k}} \phi(\boldsymbol{x})| \leq K' \max_{\boldsymbol{\alpha}\in\mathcal{V}(\phi)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}}. \tag{3.14}$$

Unlike Lemma 3.1, the proof of Lemma 3.2 is extremely simple and follows, for example, from the analyticity of the function ϕ. Grouping terms of the Taylor series appropriately, one can write ϕ as a finite sum

$$\phi(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathcal{V}(\phi)} \boldsymbol{x}^{\boldsymbol{\alpha}} \varphi_{\boldsymbol{\alpha}}(\boldsymbol{x})$$

where each $\varphi_{\boldsymbol{\alpha}}$ is nonvanishing at the origin and the sum ranges over those $\boldsymbol{\alpha}$ which are vertices of the reduced Newton polyhedron $\mathcal{N}(\phi)$. (One way to achieve this decomposition is to chose any function Φ mapping each multiindex in the Taylor support of ϕ to a vertex of $\mathcal{N}(\phi)$ such that $\beta \in \Phi(\beta) + \mathbb{R}^d_{\geq}$ for all β and then let $\varphi_{\boldsymbol{\alpha}}$ be the sum of the terms in the Taylor series of ϕ over the set of multiindices $\Phi^{-1}(\boldsymbol{\alpha})$.) Using this decomposition, the inequality (3.14) follows immediately from the product rule for differentiation. Now let U be as in Lemma 3.1 and Lemma 3.2. By coupling these two lemmas with the Mean Value Theorem, we will then prove a slightly stronger version of Lemma 3.1:

Corollary 3.3. There exists $N \in \mathbb{N}$ depending on K, K' and ϕ, such that the following holds: Each $Q_{\boldsymbol{\epsilon}} \subset U$ can be partitioned into a collection of 2^{dN} congruent boxes $Q_{\boldsymbol{\epsilon},l}$ for $1 \leq l \leq 2^{dN}$ such that for each $Q_{\boldsymbol{\epsilon},l}$ and for all $\boldsymbol{\alpha} \in \mathcal{V}(\phi)$, there is a pair of indices (i,j), $i \neq j$, such that

$$\inf_{\boldsymbol{x} \in 2Q_{\boldsymbol{\epsilon},l}} |x_i x_j \partial_i \partial_j \phi(\boldsymbol{x})| \geq \frac{K}{2^{20}} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}}. \tag{3.15}$$

The main analytic tool to be employed is the following Operator van der Corput Lemma due to Phong and Stein [14, 15]. The proof can be found throughout the literature; see, for example, [7].

Lemma 3.4. *Let $\chi(x,y)$ be a smooth function supported in a box with dimensions $\delta_1 \times \delta_2$ such that $|\partial_y^l \chi(x,y)| \leq C_1 \delta_2^{-l}$ for $l = 0,1,2$ and some $C_1 > 0$. Let $\mu > 0$ and $S(x,y)$ be a smooth function s.t. for all (x,y) in the support of χ,*

$$C_2 \mu \leq |\partial_x \partial_y S(x,y)| \leq C_3 \mu \quad \textit{and}$$
$$|\partial_x \partial_y^l S(x,y)| \leq C_3 \mu \delta_2^{-l} \quad \textit{for } l = 1,2\,.$$

Then the operator defined by

$$T_\lambda f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi(x,y) f(y) dy$$

satisfies

$$\|T_\lambda f\|_2 \le C|\lambda\mu|^{-\frac{1}{2}}\|f\|_2, \tag{3.16}$$

where the constant C depends on C_1, C_2, and C_3, but is independent of μ, λ and other information of the phase S.

Lemma 3.1 and Corollary 3.3 are proved in Sections 4 and 5, respectively. In Section 6, we will combine the technical lemmas stated above to prove Lemma 2.1.

4. Proof of Lemma 3.1

As mentioned in the previous section, Lemma 3.1 and the proof to be presented now are both closely based on earlier work of the first author in his PhD thesis [5, 6]. For the rest of the section, we write $\phi = P_m + R_m$, where P_m is the degree m Taylor polynomial at the origin and R_m is the remainder. The integer m is chosen so that $\mathcal{N}(\phi) = \mathcal{N}(P_m)$; such an m always exists because the Newton polyhedron has finitely many vertices. Write $P_m(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|\le m} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$ and $R_m(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|=m} h_{\boldsymbol{\alpha}}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}}$ for some real analytic functions $h_{\boldsymbol{\alpha}}$. For each $1 \le i \ne j \le d$ we can write $x_i x_j \partial_i \partial_j \phi(\boldsymbol{x})$ as

$$x_i x_j \partial_i \partial_j \phi(\boldsymbol{x}) = \sum_{|\boldsymbol{\alpha}|\le m} c'_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} + \sum_{|\boldsymbol{\alpha}|=m} h'_{\boldsymbol{\alpha}}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}}, \tag{4.17}$$

where $c'_{\boldsymbol{\alpha}} = \alpha_i \alpha_j c_{\boldsymbol{\alpha}}$, and $h'_{\boldsymbol{\alpha}}$ depends on i and j. For each compact $F \subset \mathcal{N}(\phi)$, we further decompose the sum over $|\boldsymbol{\alpha}| \le m$ in (4.17) into terms in F and its complement, respectively, i.e.,

$$\sum_{|\boldsymbol{\alpha}|\le m} c'_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}\in F} c'_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} + \sum_{\substack{\boldsymbol{\alpha}\notin F \\ |\alpha|\le m}} c'_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}. \tag{4.18}$$

The goal is to show for all $\boldsymbol{x}$ small enough, one may choose F and $1 \le i \ne j \le d$ so that (4.17) is dominated by the sum over $\boldsymbol{\alpha} \in F$ and all remaining terms are of a perturbative quality.

4.1. Scaling calculations

The difficulty of dividing the sum (4.17) into finitely many terms on a compact face F of $N(\phi)$ and a remainder term of sufficiently small magnitude comes when on some box $Q_{\boldsymbol{\epsilon}}$, there are $\boldsymbol{\alpha} \in N(\phi)$ such that $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}} \approx \boldsymbol{\epsilon}^{\boldsymbol{\beta}} = \max_{\boldsymbol{\alpha}' \in \mathcal{N}(\phi)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}'}$ but $\boldsymbol{\alpha}$ does not belong to the specified face F. In such cases, however, we show that $\boldsymbol{\alpha}$ must effectively belong to a face of lower codimension which contains the specified face. We begin the process of making this idea precise with Proposition 4.1:

Proposition 4.1. *Let $\boldsymbol{\epsilon} \in (0,1)^d$ and let $\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^n$ be multiindices which are linearly independent as vectors in $\mathbb{R}^d$. Suppose that there is a multiindex $\boldsymbol{\beta}$ and a positive $K < 1$ such that for all $1 \le k \le n$, $K\boldsymbol{\epsilon}^{\boldsymbol{\beta}} \le \boldsymbol{\epsilon}^{\boldsymbol{\alpha}^k} \le \boldsymbol{\epsilon}^{\boldsymbol{\beta}}$. Then there is some $b \in (0,1)$ depending only on $\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^n$ and K such that for some $\boldsymbol{y} \in [b, b^{-1}]^d$ and all $1 \le k \le n$ we have*

$$\boldsymbol{y}^{\boldsymbol{\alpha}^k} = \boldsymbol{\epsilon}^{\boldsymbol{\alpha}^k - \boldsymbol{\beta}}. \tag{4.19}$$

Consequently, if $\boldsymbol{\alpha} = \sum_k \lambda_k \boldsymbol{\alpha}^k$ for $\sum_k \lambda_k = 1$, then

$$\boldsymbol{y}^{\boldsymbol{\alpha}} = \boldsymbol{\epsilon}^{\boldsymbol{\alpha} - \boldsymbol{\beta}}. \tag{4.20}$$

Proof. Let A be the $n \times d$ matrix with rows $\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^n$. Without loss of generality, assume that the first n columns of A are linearly independent. Let $\boldsymbol{v} \in \mathbb{R}^n$ be the vector whose k-th coordinate is given by $v_k = \log_2(\boldsymbol{\epsilon}^{\boldsymbol{\alpha}^k - \boldsymbol{\beta}})$, and consider the equation $\tilde{A}\boldsymbol{u} = \boldsymbol{v}$, where $\tilde{A} = (\alpha_i^j)_{1 \le i,j \le n}$ is the leftmost $n \times n$ minor of A. Since $\tilde{A}$ has full rank, we can invert $\tilde{A}$ and write $\boldsymbol{u} = \tilde{A}^{-1}\boldsymbol{v}$. Fixing $\rho = \|\tilde{A}^{-1}\|_{\infty \to \infty}$, we have that

$$\|\boldsymbol{u}\|_\infty \le \rho \|\boldsymbol{v}\|_\infty.$$

Therefore $-\|\boldsymbol{v}\|_\infty \rho \le u_k \le \|\boldsymbol{v}\|_\infty \rho$ for all k. However, $\|\boldsymbol{v}\|_\infty \le |\log_2 K|$, so

$$K^\rho \le 2^{-\|\boldsymbol{v}\|_\infty \rho} \le 2^{u_k} \le 2^{\|\boldsymbol{v}\|_\infty \rho} \le K^{-\rho}.$$

Hence, letting $b = K^\rho \in (0,1)$, we see that the vector $\boldsymbol{y} \in [b, b^{-1}]^d$ defined by $y_k = 2^{u_k}$ for $1 \le k \le n$ and $y_k = 1$ otherwise satisfies the system of equations (4.19). Finally, (4.20) follows from writing $\boldsymbol{\alpha} - \boldsymbol{\beta} = \sum_k \lambda_k (\boldsymbol{\alpha}^k - \boldsymbol{\beta})$. □

One should think of $\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^n$ as vertices of the reduced Newton polyhedron $\mathcal{N}(\phi)$ such that each $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}^\ell}$ nearly belongs to a compact face F and is very close or equal to $\boldsymbol{\epsilon}^{\boldsymbol{\beta}} = \max_{\boldsymbol{\alpha} \in \mathcal{N}(\phi)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}}$ in the sense that there is some $K > 0$ such that $K\boldsymbol{\epsilon}^{\boldsymbol{\beta}} \le \boldsymbol{\epsilon}^{\boldsymbol{\alpha}^\ell} \le \boldsymbol{\epsilon}^{\boldsymbol{\beta}}$. In general, the set of such points need not be linearly independent, but the proposition can always be applied to some maximal linearly independent subset. The vector $\boldsymbol{y}$ then essentially dictates how one can rescale coordinates in such a way that the approximate equality of the $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}^\ell}$ becomes exact.

4.2. The main result required for Lemma 3.1

Let $K_0, \ldots, K_{d-1} \in (0,1)$ be fixed small constants to be determined later. For any d-tuple $\boldsymbol{\epsilon} \in (0,1)^d$, we say that $\boldsymbol{\epsilon}$ is n-dominated when there is a n-dimensional compact face $F \subset \mathcal{N}(P_m)$ such that for all $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ in F and all multiindices $\boldsymbol{\alpha} \in \mathcal{N}(P_m) \setminus F$,

$$\boldsymbol{\epsilon}^{\boldsymbol{\beta}} = \boldsymbol{\epsilon}^{\boldsymbol{\beta}'} \quad \text{and} \quad \boldsymbol{\epsilon}^{\boldsymbol{\alpha}} \leq K_n \boldsymbol{\epsilon}^{\boldsymbol{\beta}}.$$

The property of n-domination is extremely useful (when it holds) to estimate the size of ϕ and its derivatives on the box $Q_{\boldsymbol{\epsilon}}$ via the decomposition (4.18) since it gives comparability of all the terms arising from face F and further shows relative smallness of the remaining terms not from F.

Unfortunately, not all $\boldsymbol{\epsilon}$ are n-dominated for some n. In general, for a given $\boldsymbol{\epsilon}$, the set of multiindices $\boldsymbol{\beta} \in \mathcal{N}(P_m)$ such that $\boldsymbol{\epsilon}^{\boldsymbol{\beta}} = \max_{\boldsymbol{\alpha}' \in \mathcal{N}(\phi)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}'}$ must lie in *some* compact face F of $\mathcal{N}(\phi)$ with dimension n for some $n < d$. If $\boldsymbol{\epsilon}$ fails to be n-dominated for n equal to the dimension of this face F, then there must exist some $\boldsymbol{\alpha} \notin F$ such that

$$K_n \boldsymbol{\epsilon}^{\boldsymbol{\beta}} = K_n \max_{\boldsymbol{\alpha}' \in \mathcal{N}(P_m)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}'} \leq \boldsymbol{\epsilon}^{\boldsymbol{\alpha}} \leq \max_{\boldsymbol{\alpha}' \in \mathcal{N}(P_m)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}'} = \boldsymbol{\epsilon}^{\boldsymbol{\beta}}. \tag{4.21}$$

In this case, let $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$ be lattice points in F which are vertices of a nondegenerate n-dimensional simplex and let $\boldsymbol{\alpha}^{n+2}$ be any multiindex satisfying (4.21) in the place of $\boldsymbol{\alpha}$. First observe that $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$ must in fact be linearly independent. This is true because the simplex they generate must be contained in the linear subspace spanned by $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$ as well as in F. Since F is contained in an affine subspace not containing the origin, the intersection of F with the span of $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$ must lie in some set of codimension at least one inside the span. If the dimension of the span of $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$ is n or smaller, then the dimension of the intersection will be too small to contain a nondegenerate n-dimensional simplex. To apply Proposition 4.1, it suffices to demonstrate that $\boldsymbol{\alpha}^{n+2}$ does not lie in the linear span of $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$. Unfortunately, once again, this is not always the case. We can say, however, that for those $\boldsymbol{\epsilon}$ such that $\boldsymbol{\epsilon}^{\boldsymbol{\beta}}$ is constant for all $\boldsymbol{\beta}$ in a compact face F of $\mathcal{N}(P_m)$, for all multiindices $\boldsymbol{\alpha} \in \mathcal{N}(P_m)$ which belong to the linear span of these $\boldsymbol{\beta}$ but not the face F itself, the quantity $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}}$ is $o(\boldsymbol{\epsilon}^{\boldsymbol{\beta}})$. To see this, write $\boldsymbol{\alpha}^{n+2} = c_1 \boldsymbol{\alpha}^1 + \cdots + c_{n+1} \boldsymbol{\alpha}^{n+1}$ for unique constants $c_1, \ldots, c_{n+1}$. Since $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}^j}$ is independent of j, we have that $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}^{n+2}} = (\boldsymbol{\epsilon}^{\boldsymbol{\alpha}^1})^{c_1+\cdots+c_{n+1}}$. The quantity $c_1 + \cdots + c_{n+1}$ depends only on $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+2}$ and is greater than 1 since we may assume that $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}^{n+2}} < \boldsymbol{\epsilon}^{\boldsymbol{\alpha}^1} \leq 1$. Moreover, the mapping $\boldsymbol{\alpha}^{n+2} \mapsto c_1 + \cdots + c_{n+1}$ is linear and therefore is given by taking an inner product of $\boldsymbol{\alpha}^{n+2}$ with some vector $\boldsymbol{w} \in \mathbb{R}^d$ which is unique up to the addition of vectors perpendicular to the span of $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$. In particular, because $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$ all lie in a common compact face F, we may assume that $\boldsymbol{w}$ has nonnegative entries. Fixing any such choice of $\boldsymbol{w}$ for this compact face F, the observation that $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}} = o(\boldsymbol{\epsilon}^{\boldsymbol{\beta}})$ follows because there must be a positive δ such that $\langle \boldsymbol{\alpha}, \boldsymbol{w} \rangle \geq 1 + \delta$ for all multiindices $\boldsymbol{\alpha} \in \mathcal{N}(P_m)$ which do not lie on F (which follows because the set of multiindices is discrete).

The conclusion of this line of reasoning is that for all $\boldsymbol{\epsilon}$ sufficiently small, it must either be the case that $\boldsymbol{\epsilon}$ is n-dominated for some $n \leq d-1$ or that the multiindices $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+2}$ identified above are linearly independent and satisfy the hypotheses of Proposition 4.1 when $\boldsymbol{\beta}$ is taken to equal any one of $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$. Using the vector $\boldsymbol{y}$ from the conclusion of Proposition 4.1 to rescale $\boldsymbol{\epsilon}$, it follows that the multiindices $\boldsymbol{\beta}$ in $\mathcal{N}(P_m)$ maximizing $(\frac{\boldsymbol{\epsilon}}{\boldsymbol{y}})^{\boldsymbol{\beta}}$ must contain the original maximizers of $\boldsymbol{\epsilon}^{\boldsymbol{\beta}}$ (i.e., $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{n+1}$ and all other multiindices in the original face F) *and* the new multiindex $\boldsymbol{\alpha}^{n+2}$. Iterating this process, it follows that for every $\boldsymbol{\epsilon} = (2^{-j_1}, \ldots, 2^{-j_d})$ sufficiently small, there is an $\boldsymbol{\epsilon}' \in (0,1)^d$ (i.e., not necessarily having powers of two for coordinates) such that $\boldsymbol{\epsilon}'$ is n'-dominated for some $n' \in \{0, \ldots, d-1\}$ and such that $\frac{\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}'}$ has coordinates bounded uniformly above and below by constants depending only on $\mathcal{N}(P_m)$ and $K_0, \ldots, K_{n'}$.

We can now finish the proof of the main lemma. Consider once again the sums (4.17) and (4.18). Fix any dyadic d-tuple $\boldsymbol{\epsilon} = (2^{-j_1}, \ldots, 2^{-j_d})$ and let $\boldsymbol{\epsilon}'$ be the n-dominated d-tuple identified above which is close to $\boldsymbol{\epsilon}$. Let $\boldsymbol{\beta}$ be any vertex in the dominant face F associated to $\boldsymbol{\epsilon}'$. If we define coordinates $\boldsymbol{z} \in \mathbb{R}^d$ so that $\boldsymbol{x} = \boldsymbol{\epsilon}'\boldsymbol{z}$ for all $\boldsymbol{x} \in Q_{\boldsymbol{\epsilon}}$, then

$$\left| x_i x_j \partial_i \partial_j \phi(\boldsymbol{x}) - \boldsymbol{\epsilon}'^{\boldsymbol{\beta}} \sum_{\boldsymbol{\alpha} \in F} c'_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \right| \leq K_n \boldsymbol{\epsilon}'^{\boldsymbol{\beta}} \sum_{\boldsymbol{\alpha} \notin F} \left| c'_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \right| + C \sum_{|\boldsymbol{\alpha}| = m} \left| \boldsymbol{\epsilon}'^{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \right|$$

where the constant C depends only on the functions $h_{\boldsymbol{\alpha}}$. If we assume that $\boldsymbol{\epsilon}$ is sufficiently small (or equivalently, that the cutoff function χ of (1.1) is supported sufficiently near the origin) depending on $K_0, \ldots, K_{n-1}$ and ϕ, we may assume that

$$C \sum_{|\boldsymbol{\alpha}|=m} \left| \boldsymbol{\epsilon}'^{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \right| \leq \frac{1}{3} \max_{i \neq j} \left| \boldsymbol{\epsilon}'^{\boldsymbol{\beta}} \sum_{\boldsymbol{\alpha} \in F} c'_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \right|$$

for every $\boldsymbol{z}$ such that $\boldsymbol{\epsilon}'\boldsymbol{z} \in Q_{\boldsymbol{\epsilon}}$ since by induction the coordinates of $\boldsymbol{z}$ are bounded away from 0 and ∞, which means by the nondegeneracy hypothesis (1.5) that

$$\max_{i \neq j} \left| \sum_{\boldsymbol{\alpha} \in F} c'_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \right|$$

is bounded below uniformly in $\boldsymbol{z}$ by a constant that depends only on ϕ and $K_0, \ldots, K_{n-1}$. Likewise, if K_n is chosen sufficiently small depending on $K_0, \ldots, K_{n-1}$ and ϕ, we may also assume that

$$K_n \sum_{\boldsymbol{\alpha} \notin F} \left| c'_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \right| \leq \frac{1}{3} \max_{i \neq j} \left| \sum_{\boldsymbol{\alpha} \in F} c'_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \right|,$$

which finally implies that

$$\max_{i \neq j} |x_i x_j \partial_i \partial_j \phi(\boldsymbol{x})| \geq \frac{1}{3} \boldsymbol{\epsilon}'^{\boldsymbol{\beta}} \left| \sum_{\boldsymbol{\alpha} \in F} c'_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \right| \gtrsim \boldsymbol{\epsilon}'^{\boldsymbol{\beta}}$$

uniformly for all $\boldsymbol{x} \in Q_{\boldsymbol{\epsilon}}$ with some constant that depends only on ϕ and $K_0, \ldots, K_n$. Since $\boldsymbol{\epsilon}'^{\boldsymbol{\beta}}$ dominates $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \mathcal{V}(\phi)$ and the coordinates of $\frac{\boldsymbol{\epsilon}}{\boldsymbol{\epsilon}'}$ are bounded above and below, Lemma 3.1 follows.

5. Proof of Corollary 3.3

Recall that we may assume that $Q_{\boldsymbol{\epsilon}}$ is in the positive orthant. Decompose $Q_{\boldsymbol{\epsilon}}$ into 2^{dN} congruent boxes $Q_{\boldsymbol{\epsilon},l}$ of dimensions $2^{-N}\boldsymbol{\epsilon}$, with N to be determined momentarily. Lemma 3.1 guarantees for each l there exist $\boldsymbol{x} \in Q_{\boldsymbol{\epsilon},l}$ and a pair (i,j) such that $|x_i x_j \partial_i \partial_j \phi(\boldsymbol{x})| \geq K \boldsymbol{\epsilon}^{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \mathcal{V}(\phi)$. Let $\boldsymbol{y}$ be a point in $2Q_{\boldsymbol{\epsilon},l}$, i.e., the box whose center is the same as $Q_{\boldsymbol{\epsilon},l}$ and whose side lengths have increased by a factor of two. By the Mean Value Theorem and Lemma 3.2,

$$\epsilon_i \epsilon_j |\partial_i \partial_j \phi(\boldsymbol{y}) - \partial_i \partial_j \phi(\boldsymbol{x})| \leq \sum_{k=1}^{d} \frac{16 \epsilon_i \epsilon_j \epsilon_k}{N} |\partial_i \partial_j \partial_k \phi(\tilde{\boldsymbol{y}}_k)| \leq \frac{C}{N} \max_{\boldsymbol{\alpha} \in \mathcal{V}(\phi)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}}$$

for some constant C depending on K' defined in Lemma 3.2. Now

$$\epsilon_i \epsilon_j |\partial_i \partial_j \phi(\boldsymbol{x})| \geq \frac{1}{64} x_i x_j |\partial_i \partial_j \phi(\boldsymbol{x})| \geq \frac{K}{64} \max_{\boldsymbol{\alpha} \in \mathcal{V}(\phi)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}}$$

since $\epsilon_k \leq x_k \leq 8\epsilon_k$ for $1 \leq k \leq d$. By choosing N large enough (independent of $\boldsymbol{\epsilon}$), one can conclude that

$$\epsilon_i \epsilon_j |\partial_i \partial_j \phi(\boldsymbol{y})| \geq \frac{K}{128} \max_{\boldsymbol{\alpha} \in \mathcal{V}(\phi)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}},$$

meaning that

$$y_i y_j |\partial_i \partial_j \phi(\boldsymbol{y})| \geq \frac{1}{64} \frac{K}{128} \max_{\boldsymbol{\alpha} \in \mathcal{V}(\phi)} \boldsymbol{\epsilon}^{\boldsymbol{\alpha}}.$$

It is clear that (3.15) is a consequence of this estimate.

6. Proof of Lemma 2.1

To prove Lemma 2.1, let us first assume that $\boldsymbol{\epsilon}$ is fixed but arbitrary. For this $\boldsymbol{\epsilon}$, let $\boldsymbol{\beta} \in \mathcal{V}(\phi)$ be any vertex which maximizes $\boldsymbol{\epsilon}^{\boldsymbol{\alpha}}$ as $\boldsymbol{\alpha}$ ranges over $\mathcal{V}(\phi)$. To prove Lemma 2.1, it suffices to show

$$|\Lambda_{\boldsymbol{\epsilon}}(\boldsymbol{f})| \lesssim \min\{|\lambda \boldsymbol{\epsilon}^{\boldsymbol{\beta}}|^{-\frac{1}{2}} \boldsymbol{\epsilon}^{\frac{\boldsymbol{1}}{\boldsymbol{p}'}},\ \boldsymbol{\epsilon}^{\frac{\boldsymbol{1}}{\boldsymbol{p}'}}\} \prod_{1 \leq j \leq d} \|f_j\|_{p_j}. \tag{6.22}$$

To prove this, let $\{Q_{\boldsymbol{\epsilon},l}\}_{1 \leq l \leq 2^{Nd}}$ be the decomposition of $Q_{\boldsymbol{\epsilon}}$ from Corollary 3.3. Using a smooth partition of unity adapted to this decomposition, we may write $\chi_{\boldsymbol{\epsilon}}$ as a sum over cutoff functions $\chi_{\boldsymbol{\epsilon},l}$ which are smooth, supported in $2Q_{\boldsymbol{\epsilon},l}$ and satisfy (2.10) for possibly new constants C_k (and we define $\Lambda_{\boldsymbol{\epsilon},l}$ to be the multilinear form (1.1) with χ replaced by $\chi_{\boldsymbol{\epsilon},l}$). By Corollary 3.3, for each $Q_{\boldsymbol{\epsilon},l}$ there is a fixed pair (i,j) such that

$$\inf_{\boldsymbol{x} \in 2Q_{\boldsymbol{\epsilon},l}} |x_i x_j \partial_i \partial_j \phi(\boldsymbol{x})| \geq \frac{K}{2^{20}} \boldsymbol{\epsilon}^{\boldsymbol{\beta}}.$$

Notice also that by Lemma 3.2, we have for all $\boldsymbol{x} \in 2Q_{\boldsymbol{\epsilon},l}$ and $b = 1,2,3$ that

$$|x_i x_j^b \partial_i \partial_j^b \phi(\boldsymbol{x})| \leq K_2 \boldsymbol{\epsilon}^{\boldsymbol{\beta}}$$

for some constant K_2 independent of $\boldsymbol{\epsilon}$. By Fubini's Theorem,

$$\Lambda_{\boldsymbol{\epsilon},l}(\boldsymbol{f}) = \int_{\mathbb{R}^{d-2}} \left(\int_{\mathbb{R}^2} e^{i\lambda\phi(\boldsymbol{x})} f_i(x_i) f_j(x_j) \chi_{\boldsymbol{\epsilon},l}(\boldsymbol{x}) dx_i dx_j \right) \prod_{k \neq i,j} f_k(x_k) d\boldsymbol{x}' \tag{6.23}$$

where $\boldsymbol{x}'$ is the vector in $\mathbb{R}^{d-2}$ whose coordinates equal x_k for all $k \neq i, j$. Applying Lemma 3.4 to the inner integral and then integrating over the remaining variables $\boldsymbol{x}'$ gives

$$|\Lambda_{\boldsymbol{\epsilon},l}(\boldsymbol{f})| \lesssim |\lambda \boldsymbol{\epsilon}^{\boldsymbol{\beta}} \epsilon_i^{-1} \epsilon_j^{-1}|^{-\frac{1}{2}} \|f_i\|_2 \|f_j\|_2 \prod_{k \neq i,j} \|f_k\|_1. \tag{6.24}$$

Hölder's inequality and the assumption $p_j \geq 2$ for all j yield

$$|\Lambda_{\boldsymbol{\epsilon},l}(\boldsymbol{f})| \lesssim |\lambda \boldsymbol{\epsilon}^{\boldsymbol{\beta}} \epsilon_i^{-1} \epsilon_j^{-1}|^{-\frac{1}{2}} |\epsilon_i|^{\frac{1}{2} - \frac{1}{p_i}} \|f_i\|_{p_i} |\epsilon_j|^{\frac{1}{2} - \frac{1}{p_j}} \|f_j\|_{p_j} \prod_{k \neq i,j} |\epsilon_k|^{1 - \frac{1}{p_k}} \|f_k\|_{p_k},$$

that is,

$$|\Lambda_{\boldsymbol{\epsilon},l}(\boldsymbol{f})| \lesssim |\lambda \boldsymbol{\epsilon}^{\boldsymbol{\beta}}|^{-\frac{1}{2}} \prod_{1 \leq k \leq d} |\epsilon_k|^{1 - \frac{1}{p_k}} \|f_k\|_{p_k}. \tag{6.25}$$

Alternatively, estimating $\Lambda_{\boldsymbol{\epsilon},l}$ by a simple application of the triangle inequality rather than Lemma 3.4 yields (after an application Hölder's inequality in the same way that it was used for (6.25))

$$|\Lambda_{\boldsymbol{\epsilon},l}(\boldsymbol{f})| \lesssim \prod_{1 \leq j \leq d} |\epsilon_j|^{1 - \frac{1}{p_j}} \|f_j\|_{p_j}. \tag{6.26}$$

Summing (6.25) and (6.26) over l and using whichever inequality has the smaller right-hand side gives (6.22).

7. Proof of Lemma 2.2

Finally we come to the proof of Lemma 2.2, which is essentially an elementary calculation to sum the estimates (2.11) over all $Q_{\boldsymbol{\epsilon}}$. Recall that the setup of the lemma begins with a $\boldsymbol{\gamma} \in \mathbb{R}^d_{\geq}$ (in our case, $\boldsymbol{\gamma} = \frac{\boldsymbol{\nu}}{\boldsymbol{p}'}$) that belongs to $\mathcal{N}(\phi)$. If $\boldsymbol{\gamma}$ is not an interior point, then $F \subset \mathcal{N}(\phi)$ is the face of lowest dimension that contains $\boldsymbol{\gamma}$. We let ℓ equal 1 plus the dimension of F in this case; for interior points $\boldsymbol{\gamma}$, we let $\ell = d$. Fixing $\boldsymbol{z} = \frac{\boldsymbol{\gamma}}{\boldsymbol{\nu}}$, we must show for all $\lambda \geq 2$ that

$$\sum_{j_1,\ldots,j_d=0}^{\infty} \min_{\substack{N\in\{0,1/2\},\\ \boldsymbol{\alpha}\in\mathcal{N}(\phi)}} \{\lambda^{-N} 2^{\langle N\boldsymbol{\alpha}-\boldsymbol{z},\boldsymbol{j}\rangle}\} \lesssim \lambda^{-\frac{1}{\nu}} \ln^{d-\ell}(\lambda).$$

It suffices to consider the sum over indices j_i which are of at most logarithmic size for all $1 \leq i \leq d$, since for any fixed i,

$$\sum_{j_1=0}^{\infty} \cdots \sum_{j_i \geq \log_2(\lambda)/\gamma_i} \cdots \sum_{j_d=0}^{\infty} 2^{-\langle \boldsymbol{z},\boldsymbol{j}\rangle} \lesssim \lambda^{-\frac{z_i}{\gamma_i}} = \lambda^{-\frac{1}{\nu}}.$$

In other words, it suffices to show that

$$\sum_{j_1=0}^{\lfloor \log_2(\lambda)/\gamma_1 \rfloor} \cdots \sum_{j_d=0}^{\lfloor \log_2(\lambda)/\gamma_d \rfloor} \min_{\substack{N\in\{0,1/2\},\\ \boldsymbol{\alpha}\in\mathcal{N}(\phi)}} \{\lambda^{-N} 2^{\langle N\boldsymbol{\alpha}-\boldsymbol{z},\boldsymbol{j}\rangle}\} \lesssim \lambda^{-\frac{1}{\nu}} \ln^{d-\ell}(\lambda). \tag{7.27}$$

Comparing the sum over $j_1, \ldots, j_d$ to an integral over $\boldsymbol{x} = (x_1, \ldots, x_d)$, it follows that the left-hand side of (7.27) is bounded above by a fixed constant times

$$\int_0^{\log_2(\lambda)/\gamma_1} \cdots \int_0^{\log_2(\lambda)/\gamma_d} \min_{\substack{N\in\{0,1/2\},\\ \boldsymbol{\alpha}\in\mathcal{N}(\phi)}} \{\lambda^{-N} 2^{\langle N\boldsymbol{\alpha}-\boldsymbol{z},\boldsymbol{x}\rangle}\} d\boldsymbol{x}. \tag{7.28}$$

For any compact face F, if $F \ni \boldsymbol{\gamma}$ is of dimension $(\ell - 1)$, then there are linearly independent $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^\ell \in F$ whose convex hull contains $\boldsymbol{\gamma}$. It follows that

$$\boldsymbol{\gamma} = \sum_{i=1}^{\ell} \lambda_i \boldsymbol{\alpha}^i \tag{7.29}$$

for nonnegative coefficients λ_i that sum to 1. Moreover, if F is the face of minimal dimension containing $\boldsymbol{\gamma}$, then none of the λ_i will equal zero. On the other hand, if $\boldsymbol{\gamma}$ is an interior point, then there must be some nondegenerate $(d-1)$-dimensional simplex which lies in some hyperplane not containing the origin, which is contained in $\mathcal{N}(\phi)$, and which itself contains $\boldsymbol{\gamma}$; this implies that (7.29) will still hold for positive λ_i summing to one when $\ell = d$ and $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^d$ are taken as the vertices of this nondegenerate simplex.

For $1 \le i \le \ell$ let $\theta_i = 2\frac{\lambda_i}{\nu}$ and further define $\theta_0 = 1 - \frac{2}{\nu}$. Because $\nu > 2$, all θ_i are positive and their sum is 1. Moreover, it is easy to check that

$$\theta_0(-\boldsymbol{z}) + \sum_{i=1}^{\ell} \theta_i \Big(\frac{\boldsymbol{\alpha}^i}{2} - \boldsymbol{z} \Big) = \mathbf{0}. \tag{7.30}$$

Restricting the minimum over all $\boldsymbol{\alpha} \in \mathcal{N}(\phi)$ in (7.28) specifically to the $\boldsymbol{\alpha}^i$ chosen above, the integral (7.28) must be bounded above by

$$\int_0^{\log_2(\lambda)/\gamma_1} \cdots \int_0^{\log_2(\lambda)/\gamma_d} \min_{1 \le i \le \ell} \{ 2^{-\langle \boldsymbol{z}, \boldsymbol{x} \rangle}, \lambda^{-\frac{1}{2}} 2^{\langle \frac{\boldsymbol{\alpha}^i}{2} - \boldsymbol{z}, \boldsymbol{x} \rangle} \} d\boldsymbol{x}. \tag{7.31}$$

If we take $\mathbf{e}_1, \ldots, \mathbf{e}_d$ to be the standard basis vectors of $\mathbb{R}^d$, then when $\ell < d$, we may assume without loss of generality that $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^\ell, \mathbf{e}_{\ell+1}, \ldots, \mathbf{e}_d$ are linearly independent. We define an invertible matrix A by

$$A\boldsymbol{\alpha}^i = \mathbf{e}_i \text{ for } 1 \le i \le \ell, \qquad A\mathbf{e}_i = \mathbf{e}_i \text{ for } \ell < i \le d.$$

Note that $\langle A^T \boldsymbol{x}, \boldsymbol{\alpha}^i \rangle = x_i$ for $1 \le i \le \ell$. Let

$$R = \{ \boldsymbol{y} \in \mathbb{R}^d : 0 \le \langle A^T \boldsymbol{y}, \mathbf{e}_j \rangle \le \ln(\lambda)/\gamma_j \text{ for all } 1 \le j \le d \}.$$

Now apply the change of variables $\boldsymbol{x} = \frac{1}{\ln 2} A^T \boldsymbol{y}$; up to a factor depending only on A, the integral (7.31) equals

$$\int_R \min_{1 \le i \le \ell} \{ e^{-\langle A\boldsymbol{z}, \boldsymbol{y} \rangle}, \lambda^{-\frac{1}{2}} e^{\langle A(\frac{\boldsymbol{\alpha}^i}{2} - \boldsymbol{z}), \boldsymbol{y} \rangle} \} d\boldsymbol{y}. \tag{7.32}$$

First integrating over directions $\ell < i \le d$,

$$\int_{\substack{0 \le \langle A^T \boldsymbol{y}, \mathbf{e}_i \rangle \le \ln(\lambda)/\gamma_i \\ \ell < i \le d}} dy_{\ell+1} \cdots dy_d \lesssim \ln^{d-\ell}(\lambda).$$

We can therefore bound (7.32) above by

$$\ln^{d-\ell}(\lambda) \int_{\mathbb{R}^\ell} \min_{1 \le i \le \ell} \{ e^{-\langle A\boldsymbol{z}, \boldsymbol{y} \rangle}, \lambda^{-\frac{1}{2}} e^{\langle A(\frac{\boldsymbol{\alpha}^i}{2} - \boldsymbol{z}), \boldsymbol{y} \rangle} \} dy_1 \cdots dy_\ell. \tag{7.33}$$

Since $A\boldsymbol{\alpha}^i = \mathbf{e}_i$ and $\sum_{i=1}^{\ell} \lambda_i = 1$, we see

$$\langle A\boldsymbol{z}, \ln(\lambda)\mathbf{1} \rangle = \frac{1}{\nu} \ln(\lambda) \sum_{i=1}^{\ell} \lambda_i \langle A\boldsymbol{\alpha}^i, \mathbf{1} \rangle = \frac{1}{\nu} \ln(\lambda).$$

Exponentiating, we obtain $e^{-\langle A\boldsymbol{z},\ln(\lambda)\mathbf{1}\rangle} = \lambda^{-\frac{1}{\nu}}$. This calculation inspires a second change of variables $y \to y + \ln(\lambda)\mathbf{1}$; after the change, $e^{-\langle A\boldsymbol{z},\boldsymbol{y}\rangle}$ becomes $\lambda^{-\frac{1}{\nu}} e^{-\langle A\boldsymbol{z},\boldsymbol{y}\rangle}$ and $\lambda^{-\frac{1}{2}} e^{\left\langle A(\frac{\boldsymbol{\alpha}^i}{2}-\boldsymbol{z}),\boldsymbol{y}\right\rangle}$ becomes $\lambda^{-\frac{1}{\nu}} e^{\left\langle A(\frac{\boldsymbol{\alpha}^i}{2}-\boldsymbol{z}),\boldsymbol{y}\right\rangle}$ (since $\mathbf{e}_i = A\boldsymbol{\alpha}^i$). We then factor out $\lambda^{-\frac{1}{\nu}}$ and bound (7.33) above by $\lambda^{-\frac{1}{\nu}} \ln^{d-\ell}(\lambda)$ times

$$\int_{\mathbb{R}^\ell} \min_{1\le i\le \ell} \{e^{-\langle A\boldsymbol{z},\boldsymbol{y}\rangle}, e^{\langle A(\frac{\boldsymbol{\alpha}^i}{2}-\boldsymbol{z}),\boldsymbol{y}\rangle}\} dy_1 \cdots dy_\ell.$$

By (7.30), for any $\boldsymbol{y} \in \mathbb{R}^d$, it must be the case that

$$\min_{1\le i\le \ell} \left\{ -\langle A\boldsymbol{z},\boldsymbol{y}\rangle, \left\langle A\left(\frac{\boldsymbol{\alpha}^i}{2} - \boldsymbol{z}\right), \boldsymbol{y}\right\rangle \right\} \le 0.$$

Assuming that $\boldsymbol{y}$ is a nonzero vector lying in the span of $\mathbf{e}_1, \ldots, \mathbf{e}_\ell$, it must moreover be true that this minimum is not zero. Were this not the case, it would follow from the definition of A that there would exist a vector $\boldsymbol{x}$ which is a nonzero linear combination of $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^\ell$ such that

$$0 = -\langle \boldsymbol{z}, \boldsymbol{x}\rangle = \left\langle \frac{\boldsymbol{\alpha}^i}{2} - \boldsymbol{z}, \boldsymbol{x} \right\rangle \text{ for all } i = 1, \ldots, \ell.$$

However, since the vectors $\boldsymbol{\alpha}^i$ are linearly independent, $\langle \boldsymbol{\alpha}^i, \boldsymbol{x}\rangle = 0$ for some x in the span of the $\boldsymbol{\alpha}^i$ implies $\boldsymbol{x} = 0$. Therefore, by homogeneity and compactness of the unit sphere, there must exist a constant constant $c = c(\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^\ell, \boldsymbol{z}) > 0$ such that

$$\min_{1\le i\le \ell} \left\{ -\langle A\boldsymbol{z},\boldsymbol{y}\rangle, \left\langle A\left(\frac{\boldsymbol{\alpha}^i}{2} - \boldsymbol{z}\right), \boldsymbol{y}\right\rangle \right\} < -c\|\boldsymbol{y}\|_2$$

for all $\boldsymbol{y}$ in the span of $\mathbf{e}_1, \ldots, \mathbf{e}_\ell$. After a polar change of variables, we can bound (7.33) by a constant independent of λ times

$$\lambda^{-\frac{1}{\nu}} \ln^{d-\ell}(\lambda) \int_0^\infty e^{-cr} dr \lesssim \lambda^{-\frac{1}{\nu}} \ln^{d-\ell}(\lambda).$$

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