Lifts of pseudo-Anosov homeomorphisms of nonorientable surfaces have vanishing SAF invariant

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We show that any pseudo-Anosov map that is a lift of a pseudo-Anosov homeomorphism of a nonorientable surface has vanishing SAF invariant. We also provide a criterion to certify that a pseudo-Anosov map is not such a lift.

1. Introduction

An *interval exchange transformation* is a bijection $f : [0, a) \to [0, a)$ with the property that there exist $0 = a_0 < a_1 < \cdots < a_n = a$ and $t_1, \ldots, t_n \in \mathbb{R}$ such that $f|_{[a_{i-1}, a_i)}(x) = x + t_i$ for $1 \leq i \leq n$. The *Sah-Arnoux-Fathi (SAF) invariant* of f takes values in $\mathbb{R} \wedge_\mathbb{Q} \mathbb{R}$, and it is defined as

$$SAF(f) = \sum_{i=1}^n (a_i - a_{i-1}) \wedge_\mathbb{Q} t_i.$$

Given a transversely orientable singular measured foliation $\mathcal{F}$ on an orientable surface, any transverse arc gives rise to an interval exchange transformation f by the first return map of the flow along the leaves of $\mathcal{F}$. It turns out that $SAF(f)$ is independent of the choice of the arc, hence one obtains the notion of the SAF invariant for foliations by setting $SAF(\mathcal{F}) = SAF(f)$. Note that if $\mathcal{F}$ is not transversely orientable or the surface is nonorientable, then the first return map is not an interval exchange transformation in the above sense, hence (at least in this way) $SAF(\mathcal{F})$ cannot be not defined.

A homeomorphism ψ of a surface is *pseudo-Anosov* if there is a number $\lambda > 1$, and a pair of transverse invariant singular measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ such that $\psi(\mathcal{F}^u) = \lambda \mathcal{F}^u$ and $\psi(\mathcal{F}^s) = \lambda^{-1} \mathcal{F}^s$. The number λ is called the stretch factor (or dilatation) of ψ. [7]

If $\mathcal{F}^u$ and $\mathcal{F}^s$ are transversely orientable, then we say that ψ is *orientable*. In this case, the surface is orientable, and $SAF(\mathcal{F}^u)$ and $SAF(\mathcal{F}^s)$ are defined up to scale. It turns out that $SAF(\mathcal{F}^u) = 0$ if and only if $SAF(\mathcal{F}^s) = 0$ [5, Lemma 2], and in this case we say that ψ has vanishing SAF invariant.

It is well-understood when the SAF invariant of foliations vanish in genus 2, and this was used for classifying Teichmüller curves in genus 2 by Calta [4] and McMullen [11]. In higher genera, the picture is more complicated. Pseudo-Anosov maps with vanishing SAF invariant (which do not exist in genus 2) have been constructed [2, 3, 5, 6] in genus 3 and up.

We say that a pseudo-Anosov homeomorphism $\tilde{\psi}$ of an orientable surface S is a *nonorientable lift* if

- S is the oriented double cover of a nonorientable surface N,

- and there is a pseudo-Anosov homeomorphism ψ of N such that $\tilde{\psi}$ is a lift of ψ.

The oriented double cover is unique: it corresponds to the index two subgroup of orientation-preserving loops in $\pi_1(N)$. The lift of a pseudo-Anosov homeomorphism is a pseudo-Anosov homeomorphism with same stretch factor, and the invariant foliations upstairs are the lifts of the foliations downstairs.

Theorem 1.1. *If an orientable pseudo-Anosov map is a nonorientable lift, then it has vanishing SAF invariant.*

This yields a new large collection of pseudo-Anosov maps with vanishing SAF invariant. It also explains why the Arnoux–Yoccoz [3] and Arnoux–Rauzy ([6, Section 4.1], [1]) examples have vanishing SAF invariant. We were unable to find a reference for the fact that these examples are nonorientable lifts, so we elaborate on this in Section 5.

The construction of the other known examples with vanishing SAF does not involve nonorientable surfaces, so the following question arises.

Question 1.2. *Which of the known examples of orientable pseudo-Anosov maps with vanishing SAF invariant are nonorientable lifts?*

Theorem 1.3. *Let $\tilde{\psi}$ be an orientable pseudo-Anosov homeomorphism of the closed orientable surface of genus g. Suppose $\tilde{\psi}$ is a nonorientable lift.*

Then the stretch factor λ is a root of a monic polynomial $p(x) \in \mathbb{Z}[x]$ of degree g whose constant coefficient is ± 1. Moreover, $p(x)$ is reciprocal mod 2.

A related result is the following theorem of Do and Schmidt [6, Theorem 1]: An orientable pseudo-Anosov map with stretch factor λ has vanishing SAF invariant if and only if the minimal polynomial of λ is not reciprocal.

As a corollary of Theorem 1.3, we obtain the following.

Corollary 1.4. *Orientable pseudo-Anosov maps with vanishing SAF invariant that are not nonorientable lifts include:*

- *the example for $q = 14$ in [2]*

- *the example in Remark 6 and several examples in Sections 4.2 and 4.3 in [6].*

There may be other examples in [2, 5, 6] where Theorem 1.3 applies. We only checked the examples where the minimal polynomial of λ was mentioned in the papers.

Theorem 1.1 gives a geometric reason for the vanishing of the SAF invariant: an orientation-reversing symmetry.

Question 1.5. *Is the vanishing of the SAF always a consequence of some symmetry?*

1.1. Remarks

Examples of pseudo-Anosov homeomorphisms of nonorientable surfaces are scarce in the literature. However, the general theory is the same as for orientable surfaces [7, 14]. Some examples are found in

- [3], where the surface is the thrice punctured projective plane;

- [12], where the surface is the connected sum of two Klein bottles;

- Section 5 of this paper, where the surface is any closed nonorientable surface of genus at least 4.

Penner's method [12] is in fact general enough to construct pseudo-Anosov mapping classes on every nonorientable surface that allows them [13].

The way in which "nonorientable lift" is defined may seem unnecessarily restrictive. Why not allow branched covers or higher degree covers of nonorientable surfaces? Any such covering would factor through the oriented double cover, so in fact no generality is lost.

2. Proof of the vanishing

Lemma 2.1. *[5, Lemma 2] An orientable pseudo-Anosov map with stretch factor λ has vanishing SAF invariant if and only if $\mathbb{Q}(\lambda) = \mathbb{Q}(\lambda + \frac{1}{\lambda})$.*

Note that $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda + \frac{1}{\lambda})$ is either 1 or 2.

Lemma 2.2. *$\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda + \frac{1}{\lambda}) = 2$ if and only if λ and $1/\lambda$ are Galois conjugates.*

Proof. λ and $1/\lambda$ are Galois conjugates if and only if there is an automorphism σ of $\mathbb{Q}(\lambda)$ such that $\sigma(\lambda) = 1/\lambda$. Note that σ acts trivially on $\mathbb{Q}(\lambda + \frac{1}{\lambda})$. Such a σ exists when $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda + \frac{1}{\lambda}) = 2$, but not if $\mathbb{Q}(\lambda) = \mathbb{Q}(\lambda + \frac{1}{\lambda})$.   □

We remark that a similar lemma with a similar proof appears as [6, Proposition 1], and it is used to prove [6, Theorem 1], which we mentioned in the introduction.

Proposition 2.3. *Let λ be the stretch factor of a pseudo-Anosov homeomorphism of a nonorientable surface. Then λ and $1/\lambda$ are not Galois conjugates.*

Proof. Denote the surface by N and the pseudo-Anosov map by ψ. Assume first that one of the invariant foliations of ψ is transversely orientable.

There is a degree 2 cover $S \to N$, where S is an orientable surface, and ψ lifts to an orientation-preserving pseudo-Anosov homeomorphism $\widetilde{\psi}$ of S whose invariant foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ are transversely orientable.

It is well-known that λ and $1/\lambda$ (or $-\lambda$ and $-1/\lambda$) are eigenvalues of $\widetilde{\psi}^* : H^1(S, \mathbb{R}) \to H^1(S, \mathbb{R})$, and every other eigenvalue η satisfies $1/\lambda < |\eta| < \lambda$ [10, Theorem 5.3]. The characteristic polynomial $\chi(\widetilde{\psi}^*)$ has integral coefficients, since $\widetilde{\psi}^*$ acts on $H^1(S, \mathbb{Z})$.

Let $h : S \to S$ be the orientation-reversing deck transformation. We have $H^1(S, \mathbb{R}) = W^+ \oplus W^-$, where W^+ and W^- are the ± 1-eigenspaces of h^*. Since h commutes with $\widetilde{\psi}$, the subspaces W^+ and W^- are invariant under $\widetilde{\psi}^*$. In particular, $\chi(\widetilde{\psi}^*) = \chi(\widetilde{\psi}^*|W^+)\chi(\widetilde{\psi}^*|W^-)$, and the polynomials on the right have integral coefficients.

Note that $\mathcal{F}^u$ and $\mathcal{F}^s$ are represented by 1-forms $\omega^u, \omega^s \in H^1(S, \mathbb{R})$. We have $h(\mathcal{F}^u) = \pm\mathcal{F}^u$ and $h(\mathcal{F}^s) = \pm\mathcal{F}^s$, where one of the signs is positive, the other one is negative. Hence $\omega^u \in W^+$ and $\omega^s \in W^-$ or the other way around. In particular, $\pm\lambda$ and $\pm 1/\lambda$ are roots of different factors of $\chi(\widetilde{\psi}^*)$.

This finishes the proof for the case when one of the invariant foliations of ψ is transversely orientable.

If none of the invariant foliations of ψ are transversely orientable, then let ψ' be the lift of ψ to the branched double cover orienting one of the foliations. Since ψ' is still supported on a nonorientable surface, but has an orientable invariant foliation, the previous case of the proof applies. Note also that the stretch factors of ψ and ψ' equal. This completes the proof of the second case. $\qquad\square$

Theorem 1.1 is a corollary of Lemmas 2.1 and 2.2 and Proposition 2.3.

3. A geometric point of view

One may also desire a geometric proof of Theorem 1.1 that uses the definition of the SAF invariant. We sketch such an argument below.

Let ψ be a pseudo-Anosov homeomorphism of a nonorientable surface. Assume that its unstable foliation $\mathcal{F}^u$ is orientable. Take a one-sided simple closed curve c transverse to $\mathcal{F}^u$. Normalize the measure on $\mathcal{F}^u$ so that c has measure $1/2$. The flow of $\mathcal{F}^u$ induces an interval exchange map f on the boundary of the Möbius band neighborhood of c. Note that the intervals come in pairs that are interchanged by f.

When lifted to the oriented double cover, c lifts to a curve of measure 1, and the induced interval exchange is defined by the formula

$$\tilde{f}(x) = f(x) + 1/2 \pmod 1.$$

It is straightforward to check that $SAF(\tilde{f}) = 0$ using the fact that the terms in the sum come in pairs such as $l_i \wedge (t_i \pm 1/2) + l_i \wedge (-t_i \pm 1/2)$ and this causes cancellations.

4. The certificate

A monic degree n polynomial $p(x) = x^n + \cdots + a_{n-1}x + a_n$ is *reciprocal* if $p(x) = x^n p(1/x)/a_n$. We will use the following well-known result for the field with two elements.

Lemma 4.1. *[9, Theorem 8.14] Let $f : V \to V$ be a linear transformation of the vector space V over a field K. If f preserves a non-degenerate bilinear form, then the characteristic polynomial $\chi(f)$ is reciprocal.*

Proposition 4.2. *Let ϕ be a homeomorphism of the closed nonorientable surface N of genus g. The characteristic polynomial $p(x)$ of $\phi^* : H^1(N, \mathbb{Z}) \to H^1(N, \mathbb{Z})$ is reciprocal mod 2.*

Proof. Recall that $H_1(N, \mathbb{Z}) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$, so $H^1(N, \mathbb{Z}) \cong \mathbb{Z}^{g-1}$ and $H^1(N, \mathbb{Z}_2) \cong \mathbb{Z}_2^g$. The cup product on $H^1(N, \mathbb{Z}_2)$ is a non-degenerate bilinear form [8, Example 3.8].

Consider the mod 2 reduction $r : H^1(N, \mathbb{Z}) \to H^1(N, \mathbb{Z}_2)$. The image $\mathrm{im}(r) \subset H^1(N, \mathbb{Z}_2)$ has codimension 1. It is ϕ^*-invariant and $\phi^*|\mathrm{im}(r)$ can be described by the same matrix as $\phi^*|H^1(N, \mathbb{Z})$. Since V has codimension 1, the characteristic polynomial of $\phi^*|H^1(N, \mathbb{Z}_2)$ is $(x + 1)p(x)$ or $xp(x)$ mod 2, but the latter is not possible since ϕ^* is invertible. According to Lemma 4.1, $p(x)(x + 1)$ is reciprocal mod 2 and hence the same holds for $p(x)$.   □

Proof of Theorem 1.3. Suppose $\tilde{\psi}$ is a lift of the pseudo-Anosov map $\psi : N \to N$, where N is the closed genus $g + 1$ nonorientable surface. The orientable invariant foliation of ψ is represented by an element of $H^1(N, \mathbb{R})$ and it is an eigenvector of $\psi^*|H^1(N, \mathbb{R})$ with eigenvalue $\pm\lambda$ if it is the unstable foliation and $\pm 1/\lambda$ if it is the stable foliation.

Let $p(x) \in \mathbb{Z}[x]$ be the characteristic polynomial of $\psi^*|H^1(N, \mathbb{R})$. Note that λ is a root of $p(x)$, $p(-x)$, $x^g p(1/x)$ or $x^g p(-1/x)$. The statement of the theorem now follows from Proposition 4.2.   □

5. The Arnoux–Yoccoz and Arnoux–Rauzy examples as nonorientable lifts

In this section we define pseudo-Anosov homeomorphisms of nonorientable surfaces whose lifts by the oriented double cover are the well-known pseudo-Anosov homeomorphisms of orientable surfaces defined by Arnoux and Yoccoz [3]. This justifies the statement made in the introduction that the Arnoux–Yoccoz examples are nonorientable lifts.

Let M be a Möbius band with meridian γ. Define a measured foliation on M transverse to γ such that the length of γ is 1. (See Figure 5.1, where M arises from identifying the vertical sides of the rectangle by a flip, and leaves of the foliation are vertical lines.) Fix some $g \geq 3$, and divide the boundary of M into intervals of lengths $\alpha, \alpha, \alpha^2, \alpha^2, \ldots, \alpha^g, \alpha^g$, where α is the unique root of the polynomial $x^g + x^{g-1} + \cdots + x - 1$ lying in the interval $[0, 1]$. Identification of pairs of intervals of the same length by translations gives rise to a singular measured foliation $\mathcal{F}$ of N_g, the closed nonorientable surface of genus $g + 1$ whose oriented double cover is S_g, the closed orientable surface of genus g. The curve γ is one-sided and is transverse to $\mathcal{F}$.
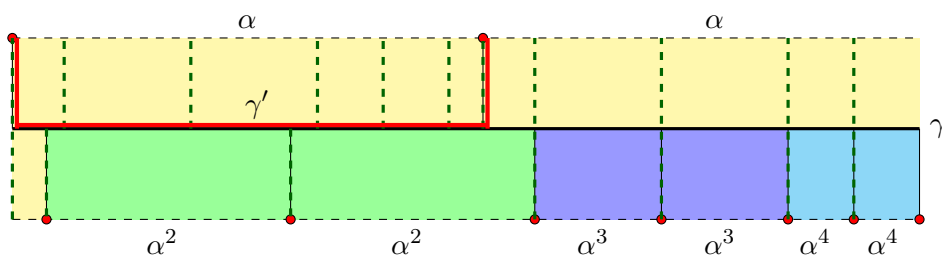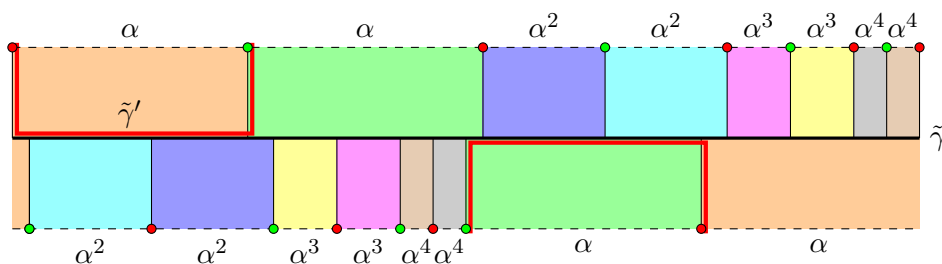
Figure 5.1: The nonorientable surface.

Figure 5.2: The oriented double cover.

Consider another transverse curve γ' that is obtained by a small perturbation of the U-shaped curve shown on Figure 5.1. By following the leaves of $\mathcal{F}$ emanating from the unique singularity until they hit γ', one can see that the intervals appearing along γ' have lengths $\alpha^2, \alpha^2, \alpha^3, \alpha^3, \ldots, \alpha^{g+1}, \alpha^{g+1}$, in this order. It follows that there is a homeomorphism of γ to γ' that brings intervals to intervals by shrinking by a factor of α, and this map extends to the homeomorphism ψ of the surface such that $\psi(\mathcal{F}) = \alpha\mathcal{F}$.

We claim that the lift of ψ to the oriented double cover is the Arnoux–Yoccoz example on S_g. First notice that $\mathcal{F}$ is transversely orientable, hence so is the lift $\tilde{\mathcal{F}}$. The curve γ lifts to a two-sided curve $\tilde{\gamma}$ transverse to $\tilde{\mathcal{F}}$ whose measure is 2. The induced interval exchange transformation is the transformation T in [3] (up to scaling by 2). See Figure 5.2.

The lift $\tilde{\psi}$ maps $\tilde{\gamma}$ to $\tilde{\gamma}'$, so the interval exchanges f and f' induced by the two curves are conjugates by scaling by α. Note also that f' is the same as the restriction of f to the initial subinterval of length 2α. Arnoux and Yoccoz use exactly this restriction to construct their homeomorphism. Hence their example coincides with $\tilde{\psi}$.

Before Arnoux and Yoccoz gives the above mentioned examples in their paper, they produce a pseudo-Anosov homeomorphism of the thrice punctured projective plane. The genus 3 example in the Arnoux–Yoccoz family is actually a lift of this map. Arnoux and Rauzy has given many more examples on the thrice punctured projective plane. The lifts of these examples to S_3 are the Arnoux–Rauzy examples mentioned in [6, Section 4.1], hence they are also nonorientable lifts.

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