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A unified approach to partial and mock theta functions

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The theta functions

$$\sum_{n\in\mathbb{Z}} \psi(n)n^{\nu}e^{2\pi i n^2 z},$$

with ψ a Dirichlet character and $\nu = 0, 1$, have played an important role in the theory of holomorphic modular forms and modular L-functions. A partial theta function is defined by a sum over part of the integer lattice, such as $\sum_{n>0}\psi(n)n^{\nu}e^{2\pi i n^2 z}$. Such sums typically fail to have modular properties. We give an analytic construction which unifies these partial theta functions with the mock theta functions introduced by Ramanujan.

1. Introduction and main result

Shimura [19] studied the theta functions

$$\theta(\psi, \nu; z) = \sum_{n\in\mathbb{Z}} \psi(n)n^{\nu}e(n^2 z)$$

where ψ is a primitive Dirichlet character of conductor r satisfying $\psi(-1) = (-1)^{\nu}$ and $\nu = 0$ or 1 with $e(z) := e^{2\pi i z}$ and z in the upper half plane. He proved (Proposition 2.2 of [19]) that this theta series is a holomorphic modular form of weight $1/2 + \nu$ on $\Gamma_0(4r^2)$ with Nebentypus ψ when $\nu = 0$ and $\psi \cdot \chi_{-4}$ when $\nu = 1$ with χ_{-4} the nontrivial Dirichlet character modulo 4. These theta series have played a significant role in the development of modular forms and in particular the development of half integral weight modular forms and modular L-functions.

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A partial theta function is a function

$$\theta^*(\psi, \nu; z) := \sum_{n\geq 0} \psi(n) n^\nu e\left(n^2 z\right)$$

where ψ and ν are as above, but $\psi(-1) = (-1)^{\nu+1}$. While these functions are not modular, they arise in many varied contexts. For example, the arise in the theory of quantum invariants of 3-manifolds (see, for instance, the works of Lawrence-Zagier and Hikami [8, 9] and Section 4) and Vassiliev knot invariants [21, 22]. They often arise in enumerative combinatorics. For example, the partial theta function $\theta^*\left(\chi_{-4}, 0, \frac{1}{2}z\right)$ arises in the study of unimodal sequences. Specifically, the generating function for the number of unimodal sequences of weight n, denoted $u(n)$, is given by

$$\sum_{n=0}^{\infty} u(n)q^n = \frac{1}{(1-q)^2(1-q^2)^2\cdots} \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{1}{2}n(n+1)}$$

see Corollary 2.5.3 of [20] and the sum on the right hand side is easily seen to be related to the partial theta function. Additionally, they arise in the study of partition ranks [5] and cranks. For example, the number of partitions with crank 0, denoted $M(0, n)$, is given by

$$\sum_{n=0}^{\infty} M(0,n)q^n = \frac{1}{(1-q)(1-q^2)\cdots} \sum_{n\geq 0} (-1)^{n-1} q^{\frac{1}{2}n(n-1)}(1-q^n)$$

and the number of partitions with rank equal to 0, denoted $N(0, n)$, is given by

$$\sum_{n=0}^{\infty} N(0,n)q^n = \frac{1}{(1-q)(1-q^2)\cdots} \sum_{n\geq 0} (-1)^{n-1} q^{\frac{1}{2}n(3n-1)}(1-q^n).$$

It is straightforward to relate the sums in these series to a sum of $\theta^*\left(\chi_{-4}, 0, \frac{m}{2}z\right)$ for integers m.

Let $q = e^{2\pi i z}$ and $\left(\frac{\cdot}{\cdot}\right)$ be the Kronecker symbol. We give an analytic construction of a single function that equals the partial theta function

$$\psi(q) := \sum_{n\geq 1} \left(\frac{-12}{n}\right) q^{\frac{n^2-1}{24}}$$

in the lower half plane and equals Ramanujan's mock theta function $f(q)$ in the upper half plane. Ramanujan's mock theta function is defined for $|q| < 1$

by the q-hypergeometric series

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{1.1}$$

where $(x)_n = (x;q)_n := \prod_{j=0}^{n-1}(1-xq^j)$. This mock theta function (see Section 4 for a definition) is celebrated for its connection with the rank statistic for partitions (see the surveys of Ono [12] and Zagier [23] and the references therein).

Define

$$a_k(s) := \sum_{m\geq 0} \left(\frac{\pi}{12k}\right)^{2m+\frac{1}{2}} \frac{1}{\Gamma\left(m+\frac{3}{2}\right)} \frac{1}{s^{m+1}}$$

and

$$\Phi_{d,k}(z) := \frac{1}{2\pi i}\int_{|s|=r} \frac{a_k(s)e^{23s}}{1-\zeta_{2k}^d q e^{24s}} ds$$

where r is taken sufficiently small so that $\left|\log\left(\zeta_{2k}^d q\right)\right| \gg r$ and the integral converges. Furthermore, define

$$\omega_{h,k} := \exp\left(\pi i s(h,k)\right). \tag{1.2}$$

Here we follow the standard notation for Dedekind sums, namely

$$s(h,k) := \sum_{\mu \pmod{k}} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),$$

with the sawtooth function defined as

$$((x)) := \begin{cases} x-\lfloor x\rfloor - \frac{1}{2} & \text{if } x\in\mathbb{R}\setminus\mathbb{Z},\\ 0 & \text{if } x\in\mathbb{Z}.\end{cases}$$

The following is our main theorem.

Theorem 1.1. *Let $q = e^{2\pi i z}$. The function*

$$F(z) := 1 + \pi\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{k} \times \sum_{d \pmod{2k}} \omega_{-d,2k}\exp\left(2\pi i\left(-\frac{d}{8}\left(1+(-1)^k\right)+\frac{d}{2k}+z\right)\right)\Phi_{d,k}(z)$$

converges for $z \in \mathbb{H}$ *and* $z \in \mathbb{H}^- = \{z : Im(z) < 0\}$. *Moreover,*

$$F(z) = \begin{cases} f(q) & z \in \mathbb{H}, \\ 2\psi(q^{-1}) & z \in \mathbb{H}^-. \end{cases}$$

Remark. According to our definition, $q\psi(q^{24})$ is a partial theta function. Similar theorems exist for partial theta functions with nontrivial character ψ. This theorem explains that partial theta functions may be constructed as lower half plane analogous of the mock theta functions.

Remark. It is an open problem to interpret the meaning of this series for $z \in \mathbb{Q}$. Such an interpretation is related to Zagier's notion of a quantum modular form [24].

Theorem 1.1 relies on the construction of a Maass-Poincaré series for the mock theta function and the "expansion of zero" principle of Rademacher [16] (see, for instance, Chapter IX of Lehner's book on Discontinuous Groups [11]). Rademacher proved an exact formula for $p(n)$, the number of partitions of n. Using his formula he found an extension of the generating function to the lower half plane. Rademacher conjectured and later proved [17] that each of the Fourier coefficients of the function in the lower half plane is zero. Rademacher's conjecture was proved independently by Petersson [13]. Such expansions were noticed earlier by Poincaré. See his memoir on Fuchsian groups [14] or the english translation of Poincaré's paper by Stillwell [15] (p. 204). Extensions of the "expansion of zero" principle were obtained by Lehner [10]. Additionally, Knopp [6] wrote about this principle in connection with Eichler cohomology. The perspective of Knopp's work is relevant here when one makes the connection between mock theta functions and their completions (see Section 4). We do not address this connection here, but we hope to take it up in future work.

Remark. There are at least four ways to see a connections between partial theta functions and mock theta functions. In Section 4 we explain a way this connection arises from q-hypergeometric series (see also [2] and the references therein). An asymptotic relation is discussed in works of Lawrence-Zagier [9] and Zwegers [26]. Finally, a relation using the Mordell integral explains the connection (see the work of Chern and the author [4]). Each of these perspectives has its own advantages, the advantage here is that, since the construction passes through Poincaré series, the construction generalizes to any weight and group. The asymptotic approach has the same benefit.

In Section 2 we recall two constructions for the modular completion of Ramanujan's mock theta function $f(q)$. In Section 3 we give the proof of Theorem 1.1. Finally, in Section 4 we discuss some of the relationships between partial theta functions, mock theta functions, and WRT invariants of Seifert manifolds.

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2. Preliminaries

In this section we describe two "completions" of Ramanujan's mock theta function $f(q)$. The first is due to Zwegers [25]. The second is due to Bringmann and Ono [3] and relies on the construction of a certain Poincaré series for $f(q)$.

In Zwegers notation [23, 25] let

$$h_3(z) = q^{-\frac{1}{24}} f(q)$$

and set

$$R_3(z) := \frac{i}{\sqrt{3}} \int_{-\bar{z}}^{\infty} \frac{g_3(\tau)}{\sqrt{(\tau+z)/i}} d\tau$$

where

$$g_3(z) = \sum_{n \equiv 1 \pmod 6} n q^{\frac{n^2}{24}} = \sum_{n=1}^{\infty} \left(\frac{-12}{n}\right) n q^{\frac{n^2}{24}}.$$

Applying the straightforward calculation

$$\int_{-\bar{z}}^{i\infty} \frac{e^{2\pi i \tau \frac{n^2}{24}}}{\sqrt{-i(\tau+z)}} d\tau = i \left(\frac{12}{\pi}\right)^{\frac{1}{2}} n^{-1} \Gamma\left(\frac{1}{2}, \frac{\pi n^2 y}{6}\right) q^{-\frac{n^2}{24}},$$

where $\Gamma(\alpha, x) := \int_x^{\infty} e^{-t} t^{\alpha} \frac{dt}{t}$ is the incomplete Gamma function, we may rewrite R_3 as

$$R_3(z) = -2 \sum_{n=1}^{\infty} \left(\frac{-12}{n}\right) \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}, \frac{\pi n^2 y}{6}\right) q^{\frac{-n^2}{24}}. \tag{2.1}$$

Then the corrected function

$$\widehat{h_3}(z) = h_3(z) + R_3(z)$$

is a weight $\frac{1}{2}$ harmonic Maass form with respect to $\Gamma(2)$ (see [23] page 07).

By work of Bringmann and Ono [3] we may write $f(q)$ as a Poincaré series. Define the Kloosterman-like sum by

$$A_k(n) = \sum_{x \pmod{k}} \omega_{-x,k} \cdot e\left(\frac{nx}{k}\right)$$

where the sum is over those x relatively prime to k and $e(x) := e^{2\pi i x}$.

Theorem 2.1 (Bringmann-Ono, Theorem 3.2 and Section 5 of [3]). *In the notation above we have* $\widehat{h_3}(z) = P_h(z) + P_{nh}(z)$ *where*

$$P_h(z) := q^{-\frac{1}{24}} + \sum_{n=1}^{\infty} \alpha(n) q^{n-\frac{1}{24}}$$
$$P_{nh}(z) := -\pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}, \frac{\pi y}{6}\right) q^{-\frac{1}{24}} + \sum_{n=-\infty}^{0} \gamma_y(n) q^{n-\frac{1}{24}}$$

where

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k}) I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right)$$

and

$$\gamma_y(-n) = \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}, \frac{\pi\,|24n+1|\,y}{6}\right) \frac{\pi}{(24n+1)^{\frac{1}{4}}}$$
$$\times \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor}}{k} A_{2k}\left(-n - \frac{k(1+(-1)^k)}{4}\right) J_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n+1}}{12k}\right).$$

Remark. The function P_h is a holomorphic function while the function P_{nh} is a non-holomorphic function. This explains the subscripts.

From this we deduce the following lemma.

Lemma 2.2. *Define*

$$\widetilde{\alpha}(n) := \frac{\pi}{(24n+1)^{\frac{1}{4}}} \times \sum_{k\geq 1} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor}}{k} A_{2k}\left(-n - \frac{k(1+(-1)^k)}{4}\right) J_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n+1}}{12k}\right).$$

For $\widetilde{\alpha}(0) = 1$ *and for* $\ell = \frac{n^2-1}{24} > 0$ *we have*

$$-2\left(\frac{-12}{n}\right) = \widetilde{\alpha}(\ell).$$

Furthermore, $\widetilde{\alpha}(\ell) = 0$ *in all remaining cases.*

Proof. From the definition of $\widehat{h_3}$ and Theorem 2.1 we have

$$q^{\frac{1}{24}} R_3(z) = -\pi^{-\frac{1}{2}}\Gamma(\frac{1}{2}, \frac{\pi y}{6}) + \sum_{n=0}^{\infty} \gamma_y(-n) q^{-n}.$$

Using (2.1) and the series expansion of $\gamma_y(n)$ we see that $\widetilde{\alpha}(0) = 1$ and for $\ell > 0$ we have $\gamma_y(-\ell) = 0$ unless $\ell = \frac{n^2-1}{24}$ for some $n \equiv 1, 5 \pmod 6$, that is $24\ell + 1 = n^2$ for some $n \geq 1$. For such ℓ we have $-2\left(\frac{-12}{n}\right) = \widetilde{\alpha}(\ell)$. □

3. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. As mentioned in the introduction the construction relies on the expansion of zero principle. We follow the treatment in Chapter IX of [11].

Throughout this section, for $c \in \mathbb{N}$ let $\zeta_c := e^{2\pi i \frac{1}{c}}$ be a root of unity and $e(\alpha) := e^{2\pi i \alpha}$. We begin by showing that $f(q)$ equals $F(q)$ for $|q| < 1$. We apply Theorem 2.1 and switch the order of summation to obtain

$$f(q) = 1 + \pi \sum_{n=1}^{\infty} (24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \times I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) e\left(n(x+iy)\right)$$

$$= 1 + \pi \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor}}{k} \sum_{n \geq 1} \frac{A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{(24n-1)^{\frac{1}{4}}} \times I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) e\left(n(x+iy)\right).$$

Remark. The order we have switched to is in some sense more natural. To obtain the Fourier expansion, one completes the switch in the other direction. For instance see the proof of Theorem 3.2 of [3].

Next we insert the sum defining A_k to obtain

$$f(q) = 1 + \pi \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor}}{k} \sum_{d \pmod{2k}} \omega_{-d,2k} e\left(-\frac{d(1+(-1)^k)}{8}\right) \times \sum_{n \geq 1} \frac{e\left(n\left(\frac{d}{2k} + x + iy\right)\right)}{(24n-1)^{\frac{1}{4}}} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right).$$

We begin by rewriting the function

$$S(\zeta_{2k}^d q) := \sum_{n \geq 1} \frac{e\left(n\left(\frac{d}{2k} + x + iy\right)\right)}{(24n-1)^{\frac{1}{4}}} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right).$$

Later we will continue this function for values with $y < 0$. Write $\tilde{q} := \zeta_{2k}^d q$ so that

$$S(\tilde{q}) = \sum_{n \geq 1} \frac{1}{(24n-1)^{\frac{1}{4}}} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) \tilde{q}^n.$$

and set

$$B_k(24t-1) := \frac{1}{(24t-1)^{\frac{1}{4}}} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24t-1}}{12k}\right).$$

Lemma 3.1. *In the notation above, we have*

$$B_k(t) = \sum_{m \geq 0} \frac{b_m(k)}{m!} t^m$$

with $b_m(k) = \left(\frac{\pi}{12k}\right)^{2m+\frac{1}{2}} \frac{1}{\Gamma\left(m+\frac{3}{2}\right)}$.

Proof. The proof follows from the fact that $I_{\frac{1}{2}}(x) = i^{-\frac{1}{2}} J_{\frac{1}{2}}(ix)$ and the Taylor expansion of $J_{\frac{1}{2}}(ix)$ is given by $\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma\left(m+\frac{3}{2}\right)} \left(\frac{i}{2}x\right)^{2m+\frac{1}{2}}$. Hence

$$I_{\frac{1}{2}}\left(\frac{\pi}{12k}\sqrt{t}\right) = t^{\frac{1}{4}} \sum_{m\geq 0} \left(\frac{\pi}{12k}\right)^{2m+\frac{1}{2}} \frac{1}{m!\Gamma\left(m+\frac{3}{2}\right)} t^m.$$

□

A standard calculation gives

$$B_k(t) = \frac{1}{2\pi i} \int_{|s|=r} e^{st} \sum_{m\geq 0} \frac{b_m(k)}{s^{m+1}} ds$$

where r may be taken to sufficiently small. We have $a_k(s) = \sum_{m\geq 0} \frac{b_m(k)}{s^{m+1}}$. By the previous lemma this series is absolutely convergent for all s. We have

$$\begin{aligned} S(\tilde{q}) &= \sum_{n\geq 1} B_k(24n-1)(q\zeta_{2k}^d)^n \\ &= \frac{1}{2\pi i} \int_{|s|=r} \sum_{n\geq 1} e^{s(24n-1)} (q\zeta_{2k}^d)^n a_k(s) ds \\ &= \frac{1}{2\pi i} \zeta_{2k}^d q \int_{|s|=r} \frac{a_k(s) e^{23s} ds}{1-\zeta_{2k}^d q e^{24s}}. \end{aligned} \tag{3.1}$$

We point out that r can be taken to be small enough so that $\left|e\left(x+iy\right)e^{24s}\right| < 1$.

Thus we have established the first part of the Theorem, namely

$$f(q) = 1 + \pi \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor}}{k} \times \sum_{d \pmod{2k}} \omega_{-d,2k} e\left(-\frac{d}{8}(1+(-1)^k) + \frac{d}{2k} + z\right) \Phi_{d,k}(z)$$

where we recall that

$$\Phi_{d,k}(z) = \frac{1}{2\pi i} \int_{|s|=r} \frac{a_k(s) e^{23s}}{1-\zeta_{2k}^d q e^{24s}} ds.$$

To prove the second part of the theorem we return to the the integral in (3.1). This integral converges so long as $-\frac{1}{24}\log(\zeta_{2k}^d q)$ is outside the circle we integrate over, that is so long as $\left|\log(\zeta_{2k}^d q)\right| > 24r$. When $\zeta_{2k}^d q \neq 1$ we can find r small enough so that this integral converges. In particular $\Phi_{d,k}(z)$ is regular in the entire complex sphere except at $\zeta_{2k}^d q = 1$.

Now we will construct the Fourier expansion in the region where $|q| > 1$. As above, with $\tilde{q} := \zeta_{2k}^d q$, we take r sufficiently small so that $\left|\tilde{q}e^{24s}\right| > 1$ and obtain

$$\begin{aligned}
\tilde{q}\Phi_{d,k}(z) &= \sum_{n\geq 0} B_k(24n-1)(q\zeta_{2k}^d)^n \\
&= \sum_{m\geq 0} b_m(k)\frac{\tilde{q}}{2\pi i}\int_{|s|=r}\frac{e^{23s}ds}{s^{m+1}(1-\tilde{q}e^{24s})} \\
&= -\sum_{m\geq 0} b_m(k)\int_{|s|=r}\frac{ds}{s^{m+1}}\sum_{n=0}^{\infty}(\tilde{q})^{-n}e^{-(24n+1)s} \\
&= -\sum_{m\geq 0} b_m(k)\sum_{n=0}^{\infty}(\tilde{q})^{-n}\int_{|s|=r}\frac{ds}{s^{m+1}}e^{-(24n+1)s} \\
&= -\sum_{n=0}^{\infty}\zeta_{2k}^{-dn}q^{-n}\sum_{m\geq 0}\frac{b_m(k)}{m!}(-24n-1)^m \\
&= -\sum_{n=0}^{\infty}\zeta_{2k}^{-dn}q^{-n}\frac{(-1)^{\frac{1}{2}}i^{-\frac{1}{2}}}{(-(24n+1))^{\frac{1}{4}}}J_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n+1}}{12k}\right)
\end{aligned}$$

where we have used

$$I_{\frac{1}{2}}(ix) = i^{-\frac{1}{2}}J_{\frac{1}{2}}(-x) = i^{-\frac{1}{2}}(-1)^{\frac{1}{2}}J_{\frac{1}{2}}(x).$$

The Fourier expansion of $F(q)$ in the *lower half-plane* becomes

$$\begin{aligned}
F(q) = 1 &- \pi\sum_{k=1}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{k}\sum_{d \pmod{2k}}\omega_{-d,2k}e\left(-\frac{d(1+(-1)^k)}{8}\right) \\
&\times\sum_{n=0}^{\infty}\zeta_{2k}^{-dn}q^{-n}(24n+1)^{-\frac{1}{4}}J_{\frac{1}{2}}\left(\frac{\pi}{12k}\sqrt{24n+1}\right) \\
= 1 &- \pi\sum_{k=1}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{k}\sum_{n=0}^{\infty}\frac{A_{2k}\left(-n-\frac{k(1+(-1)^k)}{4}\right)}{(24n+1)^{\frac{1}{4}}}J_{\frac{1}{2}}\left(\frac{\pi}{12k}\sqrt{24n+1}\right)q^{-n} \\
= 1 &- \sum_{n\geq 0} q^{-n}\widetilde{\alpha}(n)
\end{aligned}$$

where the last equality follows from switching the order of summation. Also, we note that convergence of the series $\tilde{\alpha}$ follows from Theorem 4.1 of [3]. Applying Lemma 2.2 we may deduce that for $|q| > 1$ we have $F(q) = 2\sum_{n\geq 1}\left(\frac{-12}{n}\right) q^{-\frac{n^2-1}{24}}$.

Remark. Note the convergence of the sum defining $F(q)$ follows from the above calculations.

4. Relationship to other works

A relationship between partial theta functions and mock theta functions has been observed in the work of Lawrence-Zagier [9] and Zwegers [26]. The connection has been further expounded upon in Hikami's work [8] and the paper of Bringmann, Folsom, and the author [2]. In this section we describe the framework laid out in [9]. It remains an open question in what way the construction of this work might shed light on the applications those authors had in mind.

4.1. Mock theta functions and q-hypergeometric series

Following Zagier [23], a mock theta function is a q-series $H(q) = \sum_{n=0}^{\infty} a_n q^n$ such that there exists a rational number λ and a unary theta function of weight $3/2$, $g(z) = \sum_{n>0} b_n q^n$, such that setting $q = e^{2\pi iz}$, then $h(z) = q^{\lambda}H(q) + g^*(z)$ non-holomorphic modular form of weight $1/2$, where

$$g^*(z) = \int_{-\overline{z}}^{i\infty} \frac{\overline{g(-\overline{\tau})}d\tau}{\sqrt{\tau+z}}.$$

The theta function g is called the *shadow* of the mock theta function H.

For instance, the shadow of the mock theta function $f(q)$ is proportional to $g_f(z) := \sum_{n\in\mathbb{Z}}\left(\frac{-12}{n}\right) nq^{\frac{n^2}{24}}$. Thus the failure of $f(q)$ to satisfy a modular transformation is related to the failure of the Eichler integral $\int_{-\overline{z}}^{i\infty} \frac{g_f(\tau)d\tau}{\sqrt{\tau+z}}$ to be modular.

Ramanujan's third order mock theta function, defined in (1.1), may also be defined for $|q| < 1$ by the q-hypergeometric series

$$f_2(q) := 1 + \sum_{n\geq 1} \frac{(-1)^n q^n}{(1+q)(1+q^2)\cdots(1+q^n)}.$$

That is for $|q| < 1$

$$f_2(q) = f(q)$$

see, for instance, [8] (4.20) or the introduction of [2].

A striking property of the q-hypergeometric series defining $f(q)$ and $f_2(q)$ is that they are equal for $|q| < 1$, but are not equal for $|q| > 1$. Namely, for $|q| > 1$ we have (see [8])

$$f_2(q) = 1 + \sum_{n \geq 1} \frac{(-1)^n q^{-\frac{n(n+1)}{2}}}{(1+q^{-1})\cdots(1+q^{-n})} = 2\sum_{n\geq 1}\left(\frac{-12}{n}\right) q^{-\frac{n^2-1}{24}} = 2\psi(q^{-1}). \tag{4.1}$$

On the other hand, for $|q| > 1$ (see [1])

$$f(q) = 1 + \sum_{n \geq 1} \frac{q^{-n}}{(1+q^{-1})^2\cdots(1+q^{-n})^2} = 2\psi(q^{-1}) - \frac{1}{(-q^{-1};q^{-1})_\infty^2}\sum_{n\geq 0}(-1)^n q^{-\frac{n(n+1)}{2}} \tag{4.2}$$

The existence of $2\psi(q)$ in each of these identities is consistent with Theorem 1.1. However, the existence of the extra term in the identity for $f(q)$ remains mysterious.

Remark. See [2] for details and many more examples of q-hypergeometric series which are equal to mock theta functions in the domain $|q| < 1$ and not equal but related to partial theta functions in the domain $|q| > 1$.

4.2. WRT-invariants

Partial theta functions arise in the computation of topological invariants. See, for instance, [9, 21, 22]. For instance, for each root of unity ξ the Witten-Reshetikhin-Turaev (WRT) invariant associated to the Poincaré homology sphere M is an element $W(\xi) \in \mathbb{Z}[\xi]$. The Poincaré homology sphere is the quotient space $X = \mathrm{SO}(3)/\Gamma$ where Γ is the rotational symmetry group of the icosahedron. Thus, X is a 3-manifold X which has the same homology as a 3-sphere, namely $H_0(X,\mathbb{Z}) = H_3(X,\mathbb{Z}) = \mathbb{Z}$ and $H_n(X,\mathbb{Z}) = \{0\}$ for all other n.

Let

$$A_{\pm}(q) := \sum_{\substack{n>0 \\ n\equiv\pm1 \pmod 5}} \left(\frac{12}{n}\right) q^{(n^2-1)/120}.$$

Lawrence and Zagier (Theorem 1 and the remark preceding (20) of [9]) proved that the radial limit as $q \to \xi$ (for $|q| < 1$) of $1 - A_{\pm}(q)$ agrees with $W(\xi)$. The q-series $A_+(q)$ and $A_-(q)$ are related to the partial theta function

$$\phi(q) := \sum_{n>0} \left(\frac{12}{n}\right) \epsilon(n) q^{\frac{n^2}{120}},$$

where ϵ is the nontrivial Dirichlet character of modulus 5, via the identity $q^{\frac{1}{120}}(A_-(q) + A_+(q)) = \sum_{n>0} \mathrm{Re}\left(\left(\frac{12}{n}\right)\epsilon(n)\right) q^{\frac{n^2}{120}}$.

Lawrence and Zagier explain that the asymptotics of $\phi(q)$ toward roots of unity are equal, up to a constant, to the asymptotics toward roots of unity of the Eichler integral

$$\Theta^*(z) := \int_{-\bar{z}}^{i\infty} \frac{\Theta(\tau)d\tau}{\sqrt{\tau + z}}$$

where $\Theta(z) = \sum_{n\in\mathbb{Z}} n\left(\frac{12}{n}\right)\epsilon(n)q^{\frac{n^2}{120}}$ is a weight 3/2 theta function.

Additionally, Zwegers (see Section 5 of [9]) demonstrated a curious connection between one of Ramanujan's mock theta functions and the functions $A_{\pm}(q)$. The q-hypergeometric series

$$\Phi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_n(q^4;q^5)_n}.$$

is a mock theta function. As in Section 4.1, the series defining $\Phi(q)$ converges not only for $|q| < 1$, but also for $|q| > 1$. Let

$$\Phi^*(q) := -1 - \sum_{n=0}^{\infty} \frac{q^{5n+1}}{(q;q^5)_n(q^4;q^5)_n} = \Phi(1/q).$$

Then Zwegers [9] established

$$\begin{aligned} -\Phi^*(q) &= A_+(q) - \frac{1}{(q^4;q^5)_\infty(q;q^5)_\infty} F_+(q) \\ &= A_-(q) + \frac{1}{(q^4;q^5)_\infty(q;q^5)_\infty} F_-(q) \end{aligned}$$

where $F_{\pm}(q) := \sum_{\pm(n-\frac{1}{2})>0} (-1)^n q^{(5n^2-n)/2}$.

These q-series identities are analogous to (4.1) and (4.2) and establish a deep relationship between the WRT invariants, the asymptotics of partial theta functions, and the asymptotics of a mock theta function. Moreover, Zwegers conjectured a stronger relationship concerning the asymptotics of mock and partial theta functions.

To explain this conjecture we turn to Ramanujan's "definition" of a mock theta function. In his final letter to Hardy, Ramanujan gave examples of what he called mock theta functions. Following Zwegers's slight rephrasing [25], Ramanujan "defined"' a mock theta function as a function F of the complex variable q, defined by a q-hypergeometric series, which converges for $|q| < 1$ and satisfies the following

1) infinitely many roots of unity are exponential singularities
2) for every root of unity ξ there is a theta-function $\vartheta_\xi(q)$ such that the difference $F(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially,
3) there is no theta function that works for all ξ, i.e. F is not the sum of two functions one of which is a theta function and the other a function which is bounded at all roots of unity.

Remark. Ramanujan referred to sums, products, and quotients of series of the form $\sum_{n\in\mathbb{Z}} \epsilon^n q^{an^2+bn}$ with $a, b \in \mathbb{Q}$ and $\epsilon = \pm 1$ as theta functions.

Conjecture 4.1 (Zwegers [25]). *Let f be Ramanujan's third order mock theta function defined above. If ξ is a root of unity where f is bounded (as $q \to \xi$ radially inside the unit circle), for example $\xi = 1$, then f is C^∞ over the line radially through ξ. If ξ is a root of unity where f (or f_2) is not bounded, for example, $\xi = -1$, then the asymptotic expansion of the bounded term in condition (2) in the "definition" of a mock theta function is the same as the asymptotic expansion of f_2 as $q \to \xi$ radially outside the unit circle.*

Remark. Zwegers states this conjecture for a different mock theta function of Ramanujan. However, the same should follow for f.

The work of Lawrence and Zagier [9] shows that the limits must agree. It would be interesting to use the function F from Theorem 1.1 of this paper to prove this conjecture.

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