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Deformed cohomology of flag varieties

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This paper introduces a two-parameter deformation of the cohomology of generalized flag varieties. One special case is the Belkale-Kumar deformation (used to study eigencones of Lie groups). Another picks out intersections of Schubert varieties that behave nicely under projections. Our construction yields a new proof that the Belkale-Kumar product is well-defined. This proof is shorter and more elementary than earlier proofs.

1. Introduction

In 2006, P. Belkale and S. Kumar [1] introduced a new product structure on the cohomology of generalized flag varieties. They used this deformed product to obtain a maximally efficient solution to the Horn problem in general Lie type (generalizing the famous Horn problem on eigenvalues of sums of Hermitian matrices). The irredundancy of this solution was proved in 2010 by N. Ressayre [7]. More recently this product has been used to further study eigencones of compact connected Lie groups [2, 9] and the representation theory of (semisimple parts of) Levi subgroups [3].

This paper introduces a more general product that has the Belkale-Kumar product as a specialization. Another specialization identifies intersections of Schubert varieties with nice projection properties. From our general construction, we obtain a new and significantly easier proof that the Belkale-Kumar product is well-defined.

Let G be a complex connected reductive Lie group. Choose Borel and opposite Borel subgroups B, B_- and maximal torus $T = B \cap B_-$. Let W denote the Weyl group $N_G(T)/T$. For $w \in W$, we denote the Coxeter length of w by $l(w)$. Fix a parabolic subgroup $B \subseteq P \subset G$. Let W_P denote the associated parabolic subgroup of W, and W^P denote the set of minimal length coset representatives of W/W_P. For $w \in W^P$, the *Schubert variety* $X_w = \overline{B_- wP/P} \subseteq G/P$ has codimension $l(w)$. The Poincaré duals $\{\sigma_w\}$ of the Schubert varieties form an additive basis of the cohomology ring

$H^\star(G/P)$. That is,

$$\sigma_u \smile \sigma_v = \sum_w c_{u,v}^w \sigma_w,$$

where $c_{u,v}^w \in \mathbb{Z}_{\geq 0}$ is a *Schubert structure constant.* (In the case $G = \mathrm{GL}_n(\mathbb{C})$ and P is maximal, these structure constants are the Littlewood-Richardson coefficients.) Let $w^\vee = w_0 w w_0^P$, where w_0, w_0^P are the longest elements of W, W_P, respectively. The number $c_{u,v}^w$ is nonzero exactly when generic translates of $X_u, X_v, X_{w^\vee}$ intersect in a finite nonzero number of points; in that case, $c_{u,v}^w$ counts the number of such points.

Let $\mathrm{Inv}(W^P)$ denote the set of simple roots that are inverted by some element of W^P. For each $\alpha \in \mathrm{Inv}(W^P)$, we introduce a complex variable t_α and a positive rational variable s_α. For a positive root β, let $n_{\alpha\beta}$ denote the multiplicity of α in the simple root expansion of β and define

$$t^\beta = \prod_{\alpha \in \mathrm{Inv}(W^P)} t_\alpha^{n_{\alpha\beta}^{s_\alpha}}.$$

Then define $F_w(t,s)$ to be the product of the t^β over all positive roots β that are inverted by w. Let A denote the algebraic closure of the ring $\mathbb{C}(t_\alpha : \alpha \in \mathrm{Inv}(W^P))$ of rational functions. We define a product on $H^\star(G/P) \otimes A$ by

$$\sigma_u \star_{t,s} \sigma_v = \sum_w \frac{F_w(t,s)}{F_u(t,s)F_v(t,s)} c_{u,v}^w \sigma_w. \tag{1.1}$$

We recover the Belkale-Kumar product $\odot_t$ as the specialization $\star_{t,1}$. (This is immediate from the description of $\odot_t$ in [4].) Most interest has been in the further specialization $\odot_0 = \star_{0,1}$ given by evaluating each t_α to 0.

Theorem 1. *The product $\star_{t,s}$ is well-defined, commutative, and associative. In particular, $F_u(t,s)F_v(t,s)$ divides $F_w(t,s)$ in A whenever the Schubert structure constant $c_{u,v}^w$ is nonzero.*

Corollary 2 ([1, 4]). *The Belkale-Kumar product is well-defined.*

Proof. This follows from Theorem 1 by $\odot_t = \star_{t,1}$. □

Corollary 2 was proved by P. Belkale–S. Kumar [1] using geometric invariant theory (specifically a Hilbert-Mumford criterion for semistability) and by S. Evens–W. Graham [4] using relative Lie algebra cohomology.

In contrast, our proof of Theorem 1 (and hence of Corollary 2) uses only straightforward analysis of the tangent spaces to Schubert varieties.

This paper is structured as follows. In Section 2, we prove Theorem 1. In Section 3 we study the limit of $\star_{t,s}$ as $s \to 0$, and describe its geometric significance. As a corollary, we obtain an independent and completely elementary proof of Corollary 2 in the case $G = \mathrm{GL}_n(\mathbb{C})$.

2. Proof of Theorem 1

Assuming $\star_{t,s}$ is well-defined, we show it is commutative and associative. Commutativity is clear. For associativity, observe that

$$\begin{aligned}(\sigma_u \star_{t,s} \sigma_v) \star_{t,s} \sigma_w &= \sigma_w \star_{t,s} \sum_x \frac{F_x}{F_u F_v} c_{u,v}^x \sigma_x \\ &= \sum_{x,y} \frac{F_x}{F_u F_v} \frac{F_y}{F_w F_x} c_{u,v}^x c_{w,x}^y \sigma_y \\ &= \sum_{x,y} \frac{F_y}{F_u F_v F_w} c_{u,v}^x c_{w,x}^y \sigma_y,\end{aligned}$$

while similarly

$$\sigma_u \star_{t,s} (\sigma_v \star_{t,s} \sigma_w) = \sum_{x,y} \frac{F_y}{F_u F_v F_w} c_{v,w}^x c_{u,x}^y \sigma_y.$$

Associativity then follows immediately from that of the ordinary cup product.

We now prove $\star_{t,s}$ is well-defined. Let $w_1, w_2, w_3 \in W^P$. Then $c_{w_1,w_2}^{w_3^\vee}(G/P) = c_{w_1,w_2}^{w_3^\vee}(G/B)$. In particular, since $c_{w_1,w_2}^{w_3^\vee}(G/P) \neq 0$ implies $c_{w_1,w_2}^{w_3^\vee}(G/B) \neq 0$, it suffices to assume that $c_{w_1,w_2}^{w_3^\vee}(G/B) \neq 0$ and to show that $F_{w_1} F_{w_2}$ divides $F_{w_3^\vee}$.

Most of the facts described below are well-known. We learned some of these ideas from [10], where they appear with further details. Our proof is heavily indebted to work of K. Purbhoo, in particular for the key idea that filters give rise to B-stable subspaces. Indeed, much of our proof could be replaced by an appeal to [6, Theorem 1]. We however give a self-contained exposition.

Claim 3. *If $c_{w_1,w_2}^{w_3^\vee}(G/B) \neq 0$, then for generic $b_i \in B$,*

$$T_{eB}(G/B) = \bigoplus_{i=1}^{3} \frac{T_{eB}(G/B)}{b_i T_{eB}(w_i^{-1} X_{w_i})}.$$

Proof. By Kleiman transversality [5], $c_{w_1,w_2}^{w_3^\vee}(G/B) \neq 0$ implies that the intersection $\bigcap_{i=1}^3 g_i X_{w_i}$ is transverse and nonempty for generic $g_i \in G$. Therefore the intersection $\bigcap_{i=1}^3 g_i w_i^{-1} X_{w_i}$ is transverse at eB for some $g_i \in G$. Now $e \in g_i w_i^{-1} B_- w_i B$ implies $g_i \in B w_i^{-1} B_- w_i$, so $g_i = b_i w_i^{-1} b_i^- w$ for some $b_i \in B$ and $b_i^- \in B_-$. Hence,

$$g_i w_i^{-1} B_- w_i B = b_i w_i^{-1} B_- w_i B$$

and

$$g_i w_i^{-1} X_{w_i} = b_i w_i^{-1} X_{w_i}.$$

Therefore $\bigcap_{i=1}^3 b_i w_i^{-1} X_{w_i}$ is transverse at eB for some, and hence for generic, $b_i \in B$. Since $T_{eB}(b_i w_i^{-1} X_{w_i}) = b_i \cdot T_{eB}(w_i^{-1} X_{w_i})$, the claim follows. □

Fix generic $b_i \in B$ and let

$$I_i = \frac{T_{eB}(G/B)}{T_{eB}(w_i^{-1} X_{w_i})}.$$

Let $\Phi = \Phi^+ \sqcup \Phi^-$ denote a partition of the roots of G into positives and negatives, so that the positive root spaces correspond to infinitesimal curves through the opposite Borel B_-. (This convention is more convenient in regard to our definition of Schubert varieties as B_--orbit closures.) We will use the usual poset structure on Φ^+, that is $\beta \leq \gamma$ if and only if $\gamma - \beta$ is a nonnegative integral combination of positive roots. We have

$$T_{eB}(G/B) = \bigoplus_{\beta \in \Phi^+} \mathfrak{g}_\beta \qquad \text{and} \qquad I_i = \bigoplus_{\beta \in \Phi^+ \cap w_i^{-1} \Phi^-} \mathfrak{g}_\beta,$$

where $\mathfrak{g}_\beta$ denotes the root space corresponding to the root β.

Recall that a *filter* (or *upset*) of a poset is a subset $\mathcal{J}$ such that if $x \in \mathcal{J}$ and $x \leq y$, then $y \in \mathcal{J}$. For $\mathcal{J}$ a filter in Φ^+, let $J = \bigoplus_{\beta \in \mathcal{J}} \mathfrak{g}_\beta \subseteq T_{eB}(G/B)$. Since $\mathcal{J}$ is a filter, $b_i \cdot J = J$.

Let $|w_i|_{\mathcal{J}}$ denote the number of $\beta \in \mathcal{J}$ such that $w_i\beta \in \Phi^-$. Suppose

$$|w_1|_{\mathcal{J}} + |w_2|_{\mathcal{J}} + |w_3|_{\mathcal{J}} > |\mathcal{J}|.$$

For $S \subseteq T_{eB}(G/B)$, we write S/J for the image of S under the projection $T_{eB}(G/B) \to T_{eB}(G/B)/J$. Then

$$\begin{aligned}
\dim\,(b_1 \cdot I_1 + b_2 \cdot I_2 + b_3 \cdot I_3)/J &\leq \dim\, b_1 \cdot I_1/J + b_2 \cdot I_2/J + b_3 \cdot I_3/J \\
&\leq \dim\, I_1/J + I_2/J + I_3/J \\
&\leq \sum_{i=1}^{3} \dim I_i - \sum_{i=1}^{3} \dim I_i \cap J \\
&< \dim T_{eB}(G/B) - \dim J \\
&= \dim T_{eB}(G/B)/J,
\end{aligned}$$

so by Claim 3 we must have $|w_1|_{\mathcal{J}} + |w_2|_{\mathcal{J}} + |w_3|_{\mathcal{J}} \leq |\mathcal{J}|$, i.e., $|w_1|_{\mathcal{J}} + |w_2|_{\mathcal{J}} \leq |w_3^\vee|_{\mathcal{J}}$.

Fix a simple root α. Let $\mathcal{J}_{\alpha,k}$ be the set of roots of Φ^+ that use α at least k times in their expansion into simple roots. Each $\mathcal{J}_{\alpha,k}$ is a filter in Φ^+. By the above, we have $|w_1|_{\mathcal{J}_{\alpha,k}} + |w_2|_{\mathcal{J}_{\alpha,k}} \leq |w_3^\vee|_{\mathcal{J}_{\alpha,k}}$ for all α and all k. Hence the degree of t_α in $F_{w_3^\vee}$ is at least the degree of t_α in $F_{w_1}F_{w_2}$, so $F_{w_1}F_{w_2}$ divides $F_{w_3^\vee}$. □

3. The limit $s \to 0$

We write $\star_t$ for the limit of $\star_{t,s}$ as $s \to 0$. In this section, we give an independent and elementary proof that $\star_t$ is a well-defined associative and commutative product. We then interpret $\star_0$ geometrically.

Let $S_w(t)$ denote the limit of $F_w(t,s)$ as $s \to 0$. The product $\star_t$ on $H^\star(G/P) \otimes A$ may then be defined by replacing each $F(t,s)$ by $S(t)$ in Equation (1.1).

For $G = \mathrm{GL}_n(\mathbb{C})$, we will show that $\star_t$ coincides with the Belkale-Kumar product $\odot_t$, while for maximal parabolics in general type $\star_t$ coincides instead with the ordinary cup product. In general it is distinct from both.

Theorem 4. *The product $\star_t$ is well-defined, commutative, and associative.*

Proof. Assuming $\star_t$ is well-defined, commutativity is clear, while associativity is proved exactly as in the proof of Theorem 1.

We now prove $\star_t$ is well-defined. For a simple root $\alpha \in \mathrm{Inv}(W^P)$, let P_α denote the maximal parabolic subgroup of G with $\mathrm{Inv}(W^{P_\alpha}) = \{\alpha\}$. Define a projection $\pi_\alpha : G/P \to G/P_\alpha$ by $\pi_\alpha(gP) = gP_\alpha$. Observe that π_α is G-equivariant.

For $w \in W^P$, let w_α denote the minimal length coset representative of wW_{P_α}. Then π_α maps X_w onto X_{w_α}.

Claim 5. *If $c_{u,v}^w(G/P) \neq 0$, then for each α, $l(u_\alpha) + l(v_\alpha) \leq l(w_\alpha)$.*

Proof. If $l(u_\alpha) + l(v_\alpha) > l(w_\alpha)$, then for dimension reasons, generic translates of X_{u_α}, X_{v_α}, $X_{(w^\vee)_\alpha}$ have empty intersection in G/P_α.

Since $c_{u,v}^w \neq 0$, for general $(g_1, g_2, g_3) \in G^3$, there is a point $gP \in g_1X_u \cap g_2X_v \cap g_3X_{w^\vee} \subseteq G/P$. This implies $\pi_\alpha(gP) \in g_1X_{u_\alpha} \cap g_2X_{v_\alpha} \cap g_3X_{(w^\vee)_\alpha} \subseteq G/P_\alpha$. In particular this latter intersection is nonempty, so $l(u_\alpha) + l(v_\alpha) \leq l(w_\alpha)$. □

The degree of t_α in $S_w(t)$ is exactly the number of positive roots β inverted by w that use α in their simple root expansion. This number is $l(w_\alpha)$. Therefore, the degree of t_α in $\frac{S_w(t)}{S_u(t)S_v(t)}$ is $l(w_\alpha) - l(u_\alpha) - l(v_\alpha)$.

Let $u, v, w \in W^P$ with $c_{u,v}^w(G/P) \neq 0$. Then by Claim 5, $l(w_\alpha) - l(u_\alpha) - l(v_\alpha) \geq 0$ for all α, and so $S_u(t)S_v(t)$ divides $S_w(t)$ as desired. □

As a corollary, we obtain the following special case of Corollary 2.

Corollary 6. *For $G = \mathrm{GL}_n(\mathbb{C})$, the Belkale-Kumar product $\odot_t$ is well-defined.*

Proof. For $\mathrm{GL}_n(\mathbb{C})$, we always have $n_{\alpha\beta} \leq 1$, so $F(t, s) = S(t)$ and $\odot_t = \star_{t,s}$. □

For any $Q \supset P$ and $w \in W^P$, there is a unique parabolic decomposition $w = w'w''$, where $w' \in W^Q$ and $w'' \in W^P \cap W_Q$. Suppose $c_{u,v}^{w^\vee} \neq 0$. We say that the triple $(u, v, w) \in (W^P)^3$ is *Q-factoring* if $g_1X_{u'} \cap g_2X_{v'} \cap g_3X_{w'}$ is zero-dimensional for general $g_i \in G$ (or equivalently if $g_1X_{u''} \cap g_2X_{v''} \cap g_3X_{w''}$ has expected dimension zero).

Let $a_{u,v}^w := \frac{S_w(0)}{S_u(0)S_v(0)} c_{u,v}^w$ denote the structure constants of the ring $(H^\star(G/P), \star_0)$.

Proposition 7.

$$a_{u,v}^w = \begin{cases} c_{u,v}^w & \text{if } (u, v, w^\vee) \text{ is } Q\text{-factoring for every } Q \supset P, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is trivial if $c_{u,v}^w = 0$, so assume it is positive. Suppose $(u, v, w^\vee)$ is not Q-factoring for some $Q \supset P$. We may assume that Q is a maximal parabolic P_α for some simple root α. Then $l(w') > l(u') + l(v')$. Therefore t_α has positive degree in $\frac{S_w(t)}{S_u(t)S_v(t)}$, whence $\frac{S_w(0)}{S_u(0)S_v(0)} = 0$. □

Remark 8. Triples (u, v, w) that are P_α-factoring for some fixed collection of maximal parabolics P_α may be picked out by taking the limit of $\star_{t,s}$ as the corresponding $s_\alpha \to 0$, and then setting $t = 0$ and other $s_\alpha = 1$.

Remark 9. It was observed independently by N. Ressayre [8, Theorem 2] and E. Richmond [11, Theorem 1.1] that the numbers $a_{u,v}^w$ factor as $c_{u',v'}^{w'} c_{u'',v''}^{w''}$. Iterating this factorization for every maximal $P_\alpha \supset P$, we obtain a factorization of $a_{u,v}^w$ as a product of Schubert structure constants $c_{x,y}^z$ on maximal parabolic quotients G/P_α.

N. Ressayre [8] and E. Richmond [11] also note that (u, v, w) is Q-factoring for each $Q \supset P$ when (u, v, w) is *Levi-movable* in the sense of [1, Definition 4]. Therefore $\star_0$ may be thought of as 'less-degenerate' than $\odot_0$, since a generally smaller collection of Schubert structure constants is set to 0.

Example 10. Let $G = \mathrm{SO}_9(\mathbb{C})$ and P be the parabolic with $\mathrm{Inv}(W^P) = \{\alpha_2, \alpha_4\}$ (where α_4 is the short root). Of the 8271 nonzero Schubert structure constants for $H^\star(G/P)$, 807 are nonzero for the deformation $\star_0$. Of these only 597 represent Levi-movable triples and so are nonzero in the Belkale-Kumar deformation $\odot_0$. An example of one of the 210 nonzero $a_{u,v}^w$ coefficients not coming from a Levi-movable triple is $a_{1324,1\bar{2}34}^{3\bar{2}14} = 1$. (Here we identify W with the group of signed permutations on four letters).

Of the 193116 nonzero Schubert structure constants for $H^\star(G/B)$, only 2439 are nonzero for $\star_0$. Of these, 2103 arise from Levi-movable triples. □

Example 11. Let $G = \mathrm{Sp}_{12}(\mathbb{C})$ and P be the parabolic with $\mathrm{Inv}(W^P) = \{\alpha_4\}$ (where α_6 is the long root). There are 99105 nonzero Schubert structure constants for $H^\star(G/P)$. Since P is maximal, these are all nonzero for the deformation $\star_0$. However only 7962 are nonzero for the Belkale-Kumar deformation $\odot_0$. □

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