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On symplectic periods for inner forms of GL_n

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In this paper we study the question of determining when an irreducible admissible representation of $\mathrm{GL}_n(D)$ admits a symplectic model, that is when such a representation has a linear functional invariant under $\mathrm{Sp}_n(D)$, where D is a quaternion division algebra over a non-archimedean local field k and $\mathrm{Sp}_n(D)$ is the unique non-split inner form of the symplectic group $\mathrm{Sp}_{2n}(k)$. We show that if a representation has a symplectic model it is necessarily unique. For $\mathrm{GL}_2(D)$ we completely classify those representations which have a symplectic model. Globally, we show that if a discrete automorphic representation of $\mathrm{GL}_n(D_{\mathbb{A}})$ has a non-zero period for $\mathrm{Sp}_n(D_{\mathbb{A}})$, then its Jacquet-Langlands lift also has a non-zero symplectic period. A somewhat striking difference between distinction question for $\mathrm{GL}_{2n}(k)$, and $\mathrm{GL}_n(D)$(with respect to $\mathrm{Sp}_{2n}(k)$ and $\mathrm{Sp}_n(D)$ resp.) is that there are supercuspidal representations of $\mathrm{GL}_n(D)$ which are distinguished by $\mathrm{Sp}_n(D)$. The paper ends by formulating a general question classifying all unitary distinguished representations of $\mathrm{GL}_n(D)$, and proving a part of the local conjectures through a global conjecture.

1. Introduction

Let G be a group and H a subgroup of G. We recall that a complex representation π of G is said to be H-distinguished if

$$\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0,$$

where $\mathbb{C}$ denotes the trivial representation of H. When $G = \mathrm{GL}_{2n}(k)$, and $H = \mathrm{Sp}_{2n}(k)$, such representations of $\mathrm{GL}_{2n}(k)$ are said to have a symplectic model. When k is a non-archimedean local field of characteristic 0, and π is an irreducible admissible complex representation of $\mathrm{GL}_{2n}(k)$, this question has been extensively studied by several authors starting with the work of M.

J. Heumos and S. Rallis in [5]. A rather complete classification of $\mathrm{Sp}_{2n}(k)$-distinguished unitary representations of $\mathrm{GL}_{2n}(k)$ is due to O. Offen and E. Sayag [12].

When F is a number field, the analogous global question is framed in terms of the non-vanishing of certain periods of automorphic forms f on $G(F) \setminus G(\mathbb{A}_F)$, where $\mathbb{A}_F$ is the ring of adèles of F, given by

$$\int_{H(F)\setminus H(\mathbb{A}_F)} f(h)dh.$$

This question has been settled in [10, 11] and, in fact, Offen and Sayag treat some aspects of the local questions via global methods.

In this paper we study the irreducible admissible representations of $\mathrm{GL}_n(D)$ which are $\mathrm{Sp}_n(D)$-distinguished, where $\mathrm{Sp}_n(D)$ is an inner form of $\mathrm{Sp}_{2n}(k)$ constructed using the unique quaternion division algebra D over k (we will define this more precisely in Section 2). We proceed to state the main results of this paper.

Theorem 1.1. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(D)$. Then*

$$\dim \mathrm{Hom}_{\mathrm{Sp}_n(D)}(\pi, \mathbb{C}) \leq 1.$$

The following theorem gives a partial answer to the question on distinction of a supercuspidal representation of $\mathrm{GL}_n(D)$ by $\mathrm{Sp}_n(D)$.

Theorem 1.2. *Let π be a supercuspidal representation of $\mathrm{GL}_n(D)$ with Langlands parameter $\sigma_\pi = \sigma \otimes \mathrm{sp}_r$ where σ is an irreducible representation of the Weil group W_k, and sp_r is the r-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ part of the Weil-Deligne group W_k'. Then if r is odd, π is not distinguished by $\mathrm{Sp}_n(D)$.*

In Section 5, we have constructed explicit examples of supercuspidal representations of $\mathrm{GL}_n(D)$ which are distinguished by $\mathrm{Sp}_n(D)$ for any odd $n \geq 1$. In Section 7 we prove a complete classification of discrete series representations of $\mathrm{GL}_n(D)$ which are distinguished by $\mathrm{Sp}_n(D)$ assuming globalization of locally distinguished representations to globally distinguished representations together with a natural global conjecture on distinction of automorphic representations of $\mathrm{GL}_n(D)$ by $\mathrm{Sp}_n(D)$.

Here is a global theorem which is a simple consequence of Offen and Sayag's work.

Theorem 1.3. *Let D be a quaternion division algebra over F and $D_{\mathbb{A}} = D \otimes_F \mathbb{A}_F$. Let Π be an automorphic representation of $\mathrm{GL}_n(D_{\mathbb{A}})$ which appears in the discrete spectrum of $\mathrm{GL}_n(D_{\mathbb{A}})$ and has non-vanishing period integral on $\mathrm{Sp}_n(D) \setminus \mathrm{Sp}_n(D_{\mathbb{A}})$. Let $\mathrm{JL}(\Pi)$ be the Jacquet-Langlands lift of Π. Then the representation $\mathrm{JL}(\Pi)$ of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ has non-vanishing period integral on $\mathrm{Sp}_{2n}(F) \setminus \mathrm{Sp}_{2n}(\mathbb{A}_F)$.*

We now briefly describe the organization of this paper. In Section 2, we set up notation and give definitions. In this section we define the inner forms of a symplectic group over a local field k. In Section 3, we prove the uniqueness of the symplectic model for irreducible representations of $\mathrm{GL}_n(D)$. In section 4, we are able to completely analyze the question of distinction of subquotients of principal series representations of $\mathrm{GL}_2(D)$ by $\mathrm{Sp}_2(D)$ via Mackey theory. In Section 5, we construct examples of supercuspidal representations of $\mathrm{GL}_n(D)$ which are distinguished by $\mathrm{Sp}_n(D)$. In Section 6, we prove that non-vanishing of symplectic period of an irreducible discrete spectrum automorphic representation of $\mathrm{GL}_n(D_{\mathbb{A}})$ is preserved under the Jacquet-Langlands correspondence. In this section, we partially analyze distinction problem for supercuspidal representations of $\mathrm{GL}_n(D)$.

The paper ends by formulating a general question classifying all unitary distinguished representations of $\mathrm{GL}_n(D)$, and proving a part of the local conjectures through a global conjecture.

2. Notation and definitions

Let k be a non-archimedean local field of characteristic zero, and let D be the unique quaternion division algebra over k. We denote the reduced trace and reduced norm maps on D by $T_{D/k}$ and $N_{D/k}$ respectively. Let τ be the involution on D defined by $x \to \overline{x} = T_{D/k}(x) - x$.
For $n \in \mathbb{N}$, let

$$V_n = e_1 D \oplus \cdots \oplus e_n D$$

be a right D-vector space of dimension n.

Definition 2.1. We define a Hermitian form on V_n by

1) $(e_i, e_{n-j+1}) = \delta_{ij}$ for $i = 1, 2, \ldots, n$;
2) $(v, v') = \tau(v', v)$;
3) $(vx, v'x') = \tau(x)(v, v')x'$, for $v, v' \in V_n, x, x' \in D$.

Let $\mathrm{Sp}_n(D)$ be the group of isometries of the Hermitian form $(\cdot,\cdot)$. The group $\mathrm{Sp}_n(D)$ is the unique non-split inner form of the group $\mathrm{Sp}_{2n}(k)$. Clearly $\mathrm{Sp}_n(D) \subset \mathrm{GL}_n(D)$. The group $\mathrm{Sp}_n(D)$ can also be defined as

$$\mathrm{Sp}_n(D) = \left\{A \in \mathrm{GL}_n(D) |\ AJ\,{}^t\bar{A} = J\right\},$$

where ${}^t\bar{A} = (\bar{a}_{ji})$ for $A = (a_{ij})$ and

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & \cdot & & & \\ & \cdot & & & & \\ 1 & & & & & \end{pmatrix}$$

For a right D-vector space V, let $\mathrm{GL}_D(V)$ be the group of all invertible D-linear transformations on V. Similarly, let $\mathrm{Sp}_D(V)$ be the group of all invertible D-linear transformations on V which preserve the above defined Hermitian form on V. Let ν denote the character of $\mathrm{GL}_n(D)$ which is the absolute value of the reduced norm on the group $\mathrm{GL}_n(D)$. For any p-adic group G, let δ_G denote the modular character of G. We denote the trivial representation of any group by $\mathbb{C}$. For any representation π, we will denote its contragredient representation by $\hat{\pi}$.

3. Uniqueness of symplectic models

In this section we will show that for an irreducible representation π of $\mathrm{GL}_n(D)$, $\dim \mathrm{Hom}_{\mathrm{Sp}_n(D)}(\pi, \mathbb{C}) \leq 1$. This result is due to M. J. Heumos and S. Rallis [5] when D is replaced by a local field k. Our proof is a straightforward adaptation of their methods. We first need a result from [17] which gives the realization of the contragredient representation of an irreducible representation of $\mathrm{GL}_n(D)$.

Theorem 3.1. *Let D be the quaternion division algebra over k, $x \to \overline{x} = T_{D/k}(x) - x$ be the canonical anti-automorphism of order 2 on D. Let $G = \mathrm{GL}_n(D)$, and let $\sigma : G \to G$ be the automorphism of G given by $\sigma(g) = J\left({}^t\bar{g}^{-1}\right)J$, where $\bar{g} = (\overline{g}_{ij})$ and J is the anti-diagonal matrix with all entries 1. Let π be an irreducible admissible representation of $\mathrm{GL}_n(D)$ and π^σ be the representation defined by $\pi^\sigma(g) = \pi(\sigma(g))$. Then $\pi^\sigma = \hat{\pi}$, where $\hat{\pi}$ is the contragredient of π.*

Let k be a local field of characteristic different from 2, $\bar{k}$ the algebraic closure of k and M (resp. $\bar{M}$) denote the set of $n \times n$ matrices with coefficients in k (respectively $\bar{k}$). Let σ denote an anti-automorphism on $\bar{M}$ of order 2. We will record two lemmas from [5] below.

Lemma 3.2 (Lemma 2.2.1 of [5]). *For any $A \in \mathrm{GL}_n(k)$, there exists a polynomial $f \in \bar{k}[t]$ such that $f(A)^2 = A$.*

Proposition 3.3 (Proposition 2.2.2 of [5]). *For any $A \in \mathrm{GL}_n(\bar{k})$, there exists $U, V \in \mathrm{GL}_n(\bar{k})$ such that $\sigma(U) = U, \sigma(V) = V^{-1}$ and $A = UV$.*

Set $A^J = J\,{}^t\bar{A}J$ for $A \in \mathrm{GL}_n(D)$. Then $A \to A^J$ is an anti-involution on $\mathrm{GL}_n(D)$ of order 2. By Proposition 3.3, over an algebraically closed field, there exist $U, V \in GL_{2n}(\bar{k})$, such that $V^J = V^{-1}, U^J = U$ and $A = UV$. Then $A^J = V^J U^J = V^{-1}U = V^{-1}AV^{-1}$. Since $V \in \mathrm{Sp}_{2n}(\bar{k})$ if and only if $V \in \mathrm{GL}_{2n}(\bar{k})$ and $V^J = V^{-1}$, A^J and A lie in the same double cosets over algebraic closure.

The next result shows that A and A^J lie in the same double coset of $\mathrm{Sp}_n(D)$ in $\mathrm{GL}_n(D)$. Let us first recall a theorem due to Kneser and Bruhat-Tits.

Theorem 3.4. *Let G be any semi-simple simply connected group over p-adic field k. Then $H^1(k, G) = 0$.*

The theorem above will be used in conjunction with our modification of Lemma 2.3.3 [5] given below.

Proposition 3.5. *Let D be a quaternion division algebra over a local field k of characteristic zero. Let $A \in \mathrm{GL}_n(D)$. Then there exist $P_1, P_2 \in \mathrm{Sp}_n(D)$, such that $A^J = P_1AP_2$.*

Proof. Consider the set

$$V(A) = \{(P_1, P_2) \in \mathrm{Sp}_n(D) \times \mathrm{Sp}_n(D) | A^J = P_1AP_2\}.$$

The assertion contained in the proposition is equivalent to saying that $V(A)$ is non-empty. Clearly $V(A)$ is an algebraic subset of $\mathrm{Sp}_{2n}(\bar{k}) \times \mathrm{Sp}_{2n}(\bar{k})$. Note that $A \cap A\mathrm{Sp}_n(D)A^{-1}$ is the subgroup of $\mathrm{GL}_n(D)$ which leaves the symplectic form associated with the matrix $J' = {}^t\bar{A}JA^{-1}$ invariant. Denote the group $\mathrm{Sp}_n(D) \cap A\mathrm{Sp}_n(D)A^{-1}$ by $\mathrm{Sp}(J, J')$. Now consider the right action of

$\mathrm{Sp}(J,J')$ on $V(A)$ by $R\,(P_1,P_2) = \big(P_1R^{-1}, A^{-1}RAP_2\big)$. Since

$$\begin{aligned} &P_1R^{-1}AA^{-1}RAP_2 = P_1AP_2 = A^J,\\ &(P_1R^{-1}, A^{-1}RAP_2) = R(P_1,P_2) \in V(A), \end{aligned}$$

we have,

$$\begin{aligned} R(P_1,P_2) &= (P_1R^{-1}, A^{-1}RAP_2),\\ S(R(P_1,P_2)) &= (P_1R^{-1}S^{-1}, A^{-1}SAA^{-1}RAP_2),\\ &= (P_1R^{-1}S^{-1}, A^{-1}SRAP_2) \end{aligned}$$

for $R,S \in \mathrm{Sp}(J,J')$ and $(P_1,P_2) \in V(A)$, verifying that we do indeed have an action. We check that this action is fixed point free. This is because if $R(P_1,P_2) = (P_1,P_2)$ for $R \in \mathrm{Sp}(J,J')$ and $(P_1,P_2) \in V(A)$, then $P_1R^{-1} = P_1$ which gives $R = 1$.

We next check that the action is transitive. For this let $P = (P_1,P_2)$ and $Q = (Q_1,Q_2)$ be two points in $V(A)$. We need to prove that there exists $R \in \mathrm{Sp}(J,J')$ such that $RP = Q$, that is, that $R(P_1,P_2) = (Q_1,Q_2)$, or equivalently that

$$(P_1R^{-1}, A^{-1}RAP_2) = (Q_1,Q_2).$$

Let $R = Q_1^{-1}P_1 \in \mathrm{Sp}_n(D)$ then $P_1R^{-1} = Q_1$. With this choice of R

$$A^{-1}RAP_2 = A^{-1}Q_1^{-1}P_1AP_2 = A^{-1}Q_1^{-1}Q_1AQ_2 = Q_2.$$

In the second equality we have used the definition of $V(A)$ because of which $A^J = P_1AP_2 = Q_1AQ_2$. Also $P_1AP_2 = Q_1AQ_2$ gives

$$R = Q_1^{-1}P_1 = AQ_2P_2^{-1}A^{-1} \in A\mathrm{Sp}_n(D)A^{-1}.$$

Hence, $R \in \mathrm{Sp}(J,J')$ which shows that the action of $\mathrm{Sp}(J,J')$ on $V(A)$ is transitive. Therefore $V(A)$ is a right principal homogeneous space for the group $\mathrm{Sp}(J,J')$.

Klyachko proved that over an algebraically closed field, $\mathrm{Sp}(J,J')$ is an extension of a product of symplectic groups by a unipotent group. Therefore, over a general field, $\mathrm{Sp}(J,J')$ is an extension of a form of a product of symplectic groups by a unipotent group, that is, there exists an exact sequence of algebraic groups of the form

$$1 \to U \to \mathrm{Sp}(J, J') \to S \to 1,$$

with S, a form of a product of symplectic groups. Therefore we get the following exact sequence of Galois cohomology sets:

$$H^1(k, U) \to H^1(k, \mathrm{Sp}(J, J')) \to H^1(k, S).$$

It is well-known that $H^1(k, U) = 0$ for any unipotent group U over a field of characteristic zero [18]. Since by Theorem 3.4, $H^1(k, S) = 0$, the exact sequence above gives $H^1(k, \mathrm{Sp}(J, J')) = 0$. Since $V(A)$ is a principal homogeneous for $\mathrm{Sp}(J, J')$ and $H^1(k, \mathrm{Sp}(J, J')) = 0$, it follows that $V(A)(k) \neq \emptyset$, proving the proposition. □

We recall the following result from [14].

Lemma 3.6. *Let G be an l-group and H be a closed subgroup of G such that G/H carries a G-invariant measure. Suppose $x \to \bar{x}$ is an anti-automorphism of G which leaves H invariant and acts trivially on those distributions on G which are H bi-invariant. Then for any smooth irreducible representation π of G,* $\dim \mathrm{Hom}_H(\pi, \mathbb{C}) \cdot \dim \mathrm{Hom}_H(\hat{\pi}, \mathbb{C}) \leqslant 1$.

Corollary 3.7. *Let $G = \mathrm{GL}_n(D), H = \mathrm{Sp}_n(D)$, and let i be the anti-automorphism on G given by $A \to {}^J A^{-1}$ Then for any smooth irreducible representation π of G,* $\dim \mathrm{Hom}_{\mathrm{Sp}_n(D)}(\pi, \mathbb{C}) \cdot \dim \mathrm{Hom}_{\mathrm{Sp}_n(D)}(\hat{\pi}, \mathbb{C}) \leqslant 1$.

Proof. The hypotheses of Lemma 3.6 follow from Proposition 3.5 by standard methods in Gelfand-Kazhdan theory. Hence, the corollary is an immediate consequence of Lemma 3.6. □

We are now in a position to prove the main theorem of this section.

Theorem 3.8. *Let π be an irreducible admissible representation of* $\mathrm{GL}_n(D)$. *Then* $\dim \mathrm{Hom}_{\mathrm{Sp}_n(D)}(\pi, \mathbb{C}) \leqslant 1$.

Proof. Let (π_1, V) be the representation defined by $\pi_1(g) = \pi({}^J g^{-1})$. Let $\lambda \in \mathrm{Hom}_{\mathrm{Sp}_n(D)}(\pi_1, \mathbb{C})$. Then $\lambda(\pi_1(g)v) = \lambda(v)$ which gives $\lambda(\pi({}^J g^{-1})v) = \lambda(v)$. Since H is invariant under $g \to {}^J g^{-1}$, $\lambda(\pi(g)v) = \lambda(v)$ for $g \in H$, so $\lambda \in \mathrm{Hom}_{\mathrm{Sp}_n(D)}(V, \mathbb{C})$. The other inclusion follows similarly. Hence,

$$\dim \mathrm{Hom}_{\mathrm{Sp}_n(D)}(\pi, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{Sp}_n(D)}(\pi_1, \mathbb{C}).$$

Now the result follows from Theorem 3.1 and above corollary. □

4. Local theory

The aim of this section is to analyze the principal series representations of $\mathrm{GL}_2(D)$ which have a symplectic model. This can be easily done by the usual Mackey theory which is what we do here.

4.1. Orbits and Mackey theory

Let H and P be two closed subgroups of a group G and let (σ, W) be a smooth representation of P. We assume that G and H are unimodular. Also, assume that $H\backslash G/P$ has only two elements, that is, the natural action of H on G/P has two orbits, which we will call O_1 and O_2.

Assume without loss of generality that the orbit O_1 of H through eP is closed and the orbit O_2 is open. Let H_1 be the stabilizer in H of the element eP in G/P, then $H_1 = P \cap H$. Choose an element x in G such that the coset xP lies in O_2. Then $H_2 = \mathrm{Stab}_H(xP) = H \cap xPx^{-1}$. Therefore, $O_1 \simeq H/H_1$ and $O_2 \simeq H/H_2$. Using Mackey theory we obtain an exact sequence of H-representations:

$$0 \to \mathrm{ind}_{H_2}^{H} \sigma_2 \to \mathrm{Ind}_P^G \sigma|_H \to \mathrm{Ind}_{H_1}^{H} \sigma_1 \to 0,$$

where

$$\sigma_1(h) = (\delta_P/\delta_{H_1})^{1/2}\, \sigma(h) \text{ for } h \in H_1,$$

and

$$\sigma_2(h) = (\delta_P/\delta_{H_2})^{1/2}\, \sigma(h) \text{ for } h \in H_2.$$

The question of the existence of an H-invariant linear form for π can thus be addressed by studying H-invariant linear forms for representations of H induced from its subgroups.

Now we apply the Mackey theory discussed above to the our situation for $G = \mathrm{GL}_2(D), H = \mathrm{Sp}_2(D)$ and a parabolic subgroup P of $\mathrm{GL}_2(D)$.

Let V be a 2-dimensional Hermitian right D-vector space with a basis $\{e_1, e_2\}$ of V with $(e_1, e_1) = (e_2, e_2) = 0$ and $(e_1, e_2) = 1$. Let X be the set of all 1-dimensional D-subspaces of V. The group $G = \mathrm{GL}_D(V)$ acts naturally on V, and induces a transitive action on X, realizing X as homogeneous space for G. Then the stabilizer of a line W in G is a parabolic subgroup P of G, with $X \simeq G/P$. Using the above basis, $\mathrm{GL}_D(V)$ can be identified with $\mathrm{GL}_2(D)$. For $W = \langle e_1 \rangle$, P is the parabolic subgroup consisting of upper triangular matrices in $\mathrm{GL}_2(D)$. As we have a Hermitian structure on V, $H = \mathrm{Sp}_D(V) \subset \mathrm{GL}_D(V)$.

We want to understand the space $H\backslash G/P$. This space can be seen as the orbit space of H on the flag variety X. This action has two orbits. One of them, say O_1, consists of all 1-dimensional isotropic subspaces of V and the other, say O_2 consists of all 1-dimensional anisotropic subspaces of V. Here, the one dimensional subspace generated by a vector v is called isotropic if $(v, v) = 0$; otherwise, it is called anisotropic. The fact that $\mathrm{Sp}_D(V)$ acts transitively on O_1 and O_2 follows from Witt's theorem [8, page 6, §9], together with the well known theorem that the reduced norm $N_{D/k} : D^\times \to k^\times$ is surjective, and as a result if a vector $v \in V$ is anisotropic, we can assume that in the line $\langle v\rangle = \langle v \cdot D\rangle$ generated by v, there exists a vector v' such that $(v', v') = 1$.

It is easily seen that the stabilizer of the line $\langle e_1\rangle$ in $\mathrm{Sp}_D(V)$ is

$$P_H = \left\{ \begin{pmatrix} a & b \\ 0 & \bar{a}^{-1} \end{pmatrix} \mid a \in D^\times, b \in D, a\bar{b} + b\bar{a} = 0 \right\}.$$

Now we consider the line $\langle e_1 + e_2\rangle$ inside O_2. To calculate the stabilizer of this line in $\mathrm{Sp}_D(V)$, note that if an isometry of V stabilizes the line generated by $e_1 + e_2$, it also stabilizes its orthogonal complement which is the line generated by $e_1 - e_2$. Hence, the stabilizer of the line $\langle e_1 + e_2\rangle$ in $\mathrm{Sp}_D(V)$ stabilizes the orthogonal decomposition of V as

$$V = \langle e_1 + e_2\rangle \oplus \langle e_1 - e_2\rangle,$$

and also acts on the vectors $\langle e_1 + e_2\rangle$ and $\langle e_1 - e_2\rangle$ by scalars. Thus the stabilizer in $\mathrm{Sp}_D(V)$ of the line $\langle e_1 + e_2\rangle$ is $D^1 \times D^1$ sitting in a natural way in the Levi $D^\times \times D^\times$ of the parabolic P in $\mathrm{GL}_2(D)$. Here D^1 is the subgroup of $D^\times$ consisting of reduced norm 1 elements in $D^\times$.

Now consider the principal series representation $\pi = \sigma_1 \times \sigma_2 := \mathrm{Ind}_P^{\mathrm{GL}_2(D)} \sigma$ of $\mathrm{GL}_2(D)$, where $\sigma = \sigma_1 \otimes \sigma_2$ is an irreducible representation of $D^\times \times D^\times$. We analyze the restriction of π to $\mathrm{Sp}_2(D)$. By Mackey theory, we get the following exact sequence of $\mathrm{Sp}_2(D)$ representations

$$\begin{aligned} 0 &\to \mathrm{ind}_{D^1\times D^1}^{\mathrm{Sp}_2(D)}[(\sigma_1 \otimes \sigma_2)\,|_{D^1\times D^1}] \\ &\to \pi \to \mathrm{Ind}_{P_H}^{\mathrm{Sp}_2(D)} \nu^{1/2}[(\sigma_1 \otimes \sigma_2)\,|_{M_H}] \to 0. \end{aligned} \tag{4.1}$$

Here ν is the character on P_H given by

$$\nu \left[\begin{pmatrix} a & b \\ 0 & \bar{a}^{-1} \end{pmatrix} \right] = \left| N_{D/k}(a) \right|.$$

Suppose π has a nonzero $\mathrm{Sp}_2(D)$-invariant linear form. Then one of the representations in the above exact sequence,

$$\text{(4.2)} \qquad \mathrm{ind}_{D^1\times D^1}^{\mathrm{Sp}_2(D)}[(\sigma_1\otimes\sigma_2)\,|_{D^1\times D^1}] \quad \text{or} \quad \mathrm{Ind}_{P_H}^{\mathrm{Sp}_2(D)}\nu^{1/2}[(\sigma_1\otimes\sigma_2)\,|_{M_H}],$$

must have an $\mathrm{Sp}_2(D)$-invariant form. First, consider the case when

$$\mathrm{Hom}_{\mathrm{Sp}_2(D)}(\mathrm{Ind}_{P_H}^{\mathrm{Sp}_2(D)}\nu^{1/2}[(\sigma_1\otimes\sigma_2)\,|_{M_H}],\mathbb{C})\neq 0.$$

Since H/P_H is compact, by Frobenius reciprocity, this is equivalent to

$$\mathrm{Hom}_{M_H}(\nu^{1/2}\,(\sigma_1\otimes\sigma_2)\,,\nu^{3/2})\neq 0.$$

Since $M_H=\{\left(d,\bar{d}^{-1}\right)|d\in D^\times\}\simeq\Delta(D^\times\times D^\times)$, we have

$$\mathrm{Hom}_{D^\times}((\sigma_1\otimes\hat{\sigma_2})\,,\nu)\neq 0,$$

and hence

$$\text{(4.3)} \qquad \mathrm{Hom}_{D^\times}\,(\sigma_1,\sigma_2\otimes\nu)\neq 0,$$

or

$$\sigma_1\simeq\nu\otimes\sigma_2.$$

Now assume that

$$\mathrm{Hom}_{\mathrm{Sp}_2(D)}(\mathrm{ind}_{D^1\times D^1}^{\mathrm{Sp}_2(D)}[(\sigma_1\otimes\sigma_2)\,|_{D^1\times D^1}],\mathbb{C})\neq 0.$$

Then by Frobenius reciprocity, this is equivalent to

$$\text{(4.4)} \qquad \mathrm{Hom}_{D^1\times D^1}((\sigma_1\otimes\sigma_2),\mathbb{C})\neq 0.$$

Lemma 4.1. *Let (σ,V) be a finite dimensional irreducible representation of $D^\times$ with $\mathrm{Hom}_{D^1}(V,\mathbb{C})\neq 0$. Then σ is one dimensional.*

Proof. By a theorem due to Matsushima [9], D^1 is the commutator subgroup of $D^\times$. Since D^1 is a normal subgroup of $D^\times$, $V^{D^1}\neq\{0\}$ is invariant under $D^\times$ and so by the irreducibility of V, $V=V^{D^1}$. Since (σ,V) is an irreducible representation of $D^\times$, on which D^1 operates trivially, (σ,V) as a representation of $D^\times/D^1$ is also irreducible. Since $D^\times/D^1$ is abelian, σ must be one dimensional. □

From the analysis above, we deduce that if the representation

$$\pi = \sigma_1 \times \sigma_2 := \mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1 \otimes \sigma_2)$$

has an $\mathrm{Sp}_2(D)$-invariant linear form, then either

1) $\sigma_1 \simeq \sigma_2 \otimes \nu$, or
2) both σ_1 and σ_2 are 1-dimensional representations of $D^\times$, hence are of the form $\sigma_1 = \chi_1 \circ N_{D/k}$, $\sigma_2 = \chi_2 \circ N_{D/k}$ for characters $\chi_i : k^\times \to \mathbb{C}^\times$.

Further, we note that the closed orbit for the action of $\mathrm{Sp}_2(D)$ on $P \backslash \mathrm{GL}_2(D)$ contributes to a $\mathrm{Sp}_2(D)$-invariant form in the first case above, whereas it is the open orbit which contributes to a $\mathrm{Sp}_2(D)$-invariant linear form in the second case. Since the part of the representation supported on the closed orbit arises as a quotient of π, we find that in the first case π must have a $\mathrm{Sp}_2(D)$-invariant linear form.

If $\dim(\sigma_1 \otimes \sigma_2) > 1$, then the open orbit cannot contribute to an $\mathrm{Sp}_2(D)$-invariant linear form, and therefore we conclude that if $\dim(\sigma_1 \otimes \sigma_2) > 1$, then $\pi = \sigma_1 \times \sigma_2$ has an $\mathrm{Sp}_2(D)$-invariant form if and only if $\sigma_1 = \sigma_2 \otimes \nu$. Observe that if π has an $\mathrm{Sp}_2(D)$-invariant linear form, and is irreducible, then by an analogue of a theorem of Gelfand-Kazhdan [4] due to Raghuram [17], $\hat{\pi}$ too has an $\mathrm{Sp}_2(D)$-invariant linear form. However, if $\pi = \sigma_1 \times \sigma_2$, and π is irreducible, then $\hat{\pi} = \hat{\sigma}_1 \times \hat{\sigma}_2$, and if $\sigma_1 \simeq \sigma_2 \otimes \nu$, we get $\hat{\sigma}_1 \simeq \hat{\sigma}_2 \otimes \nu^{-1}$. This means by our analysis above that the representation $\hat{\sigma}_1 \times \hat{\sigma}_2$ of $\mathrm{GL}_2(D)$ does not carry an $\mathrm{Sp}_2(D)$-invariant linear form. Therefore, we conclude that if $\sigma_1 \simeq \sigma_2 \otimes \nu$, then $\pi = \sigma_1 \times \sigma_2$ must be reducible, which is one part of the following theorem of Tadic [20].

Theorem 4.2. *(Tadic) Let σ_1 and σ_2 be two irreducible representations of $D^\times$. Let $\pi = \mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1 \otimes \sigma_2)$ be the corresponding principal series representation of $\mathrm{GL}_2(D)$. Assume dim$(\sigma_1 \otimes \sigma_2) > 1$. Then π is reducible if and only if $\sigma_1 \simeq \sigma_2 \otimes \nu^{\pm 1}$. If π is reducible then it has length two. Assuming $\sigma_1 = \sigma_2 \otimes \nu$, we have the following non-split exact sequence:*

$$0 \to \mathrm{St}(\pi) \to \pi \to \mathrm{Sp}(\pi) \to 0,$$

where $\mathrm{St}(\pi)$ is a discrete series representation called a generalized Steinberg representation of $\mathrm{GL}_2(D)$ and $\mathrm{Sp}(\pi)$ is called a Speh representation of $\mathrm{GL}_2(D)$.

If $\dim(\sigma_1 \otimes \sigma_2) = 1$*, then* $\pi = \sigma_1 \times \sigma_2$ *is reducible if and only if* $\sigma_1 \simeq \sigma_2 \otimes \nu^{\pm 2}$*. If* $\sigma_1 = \sigma_2 \otimes \nu^2$*,* π *has a one dimensional quotient, and the submodule is a twist of the Steinberg representation of* $\mathrm{GL}_2(D)$*.*

In the exact sequence of $\mathrm{GL}_2(D)$-modules

$$0 \to \mathrm{Sp}(\sigma_1) \to \mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1\nu^{-1/2} \otimes \sigma_1\nu^{1/2}) \to \mathrm{St}(\sigma_1) \to 0,$$

and assuming that $\dim(\sigma_1) > 1$, we know by our previous analysis that representation $\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1\nu^{-1/2} \otimes \sigma_1\nu^{1/2})$ does not have an $\mathrm{Sp}_2(D)$-invariant linear form. Therefore, from the exact sequence above, it is clear that $\mathrm{St}(\sigma_1)$ also does not have an $\mathrm{Sp}_2(D)$-invariant linear form.

On the other hand, we know that $\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1\nu^{1/2} \otimes \sigma_1\nu^{-1/2})$ does have an $\mathrm{Sp}_2(D)$-invariant linear form, and $\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1\nu^{1/2} \otimes \sigma_1\nu^{-1/2})$ fits in the following exact sequence:

$$0 \to \mathrm{St}(\sigma_1) \to \mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1\nu^{1/2} \otimes \sigma_1\nu^{-1/2}) \to \mathrm{Sp}(\sigma_1) \to 0.$$

Since we have already concluded that $\mathrm{St}(\sigma_1)$ does not have an $\mathrm{Sp}_2(D)$-invariant linear form and since $\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1\nu^{1/2} \otimes \sigma_1\nu^{-1/2})$ has a $\mathrm{Sp}_2(D)$-invariant linear form, we conclude that $\mathrm{Sp}(\sigma_1)$ must have an $\mathrm{Sp}_2(D)$-invariant linear form.

Having completed the analysis of $\mathrm{Sp}_2(D)$-invariant linear forms on representations $\pi = \sigma_1 \times \sigma_2$ with $\dim(\sigma_1 \otimes \sigma_2) > 1$, we turn our attention to the case when σ_1 and σ_2 are both one dimensional representations of $D^\times$. In this case, the part of π supported on the open orbit, which is a submodule of π, contributes to an $\mathrm{Sp}_2(D)$-invariant linear form. Suppose that $\sigma_1 \neq \sigma_2 \otimes \nu$, as otherwise there is an $\mathrm{Sp}_2(D)$-invariant linear form arising from the closed orbit.

Since the part of π supported on the open orbit, that is,

$$\mathrm{ind}_{D^1 \times D^1}^{\mathrm{Sp}_2(D)}(\sigma_1 \otimes \sigma_2),$$

is a submodule of π, it is not obvious that an $\mathrm{Sp}_2(D)$-invariant linear form on $\mathrm{ind}_{D^1 \times D^1}^{\mathrm{Sp}_2(D)}(\sigma_1 \otimes \sigma_2)$ will extend to an $\mathrm{Sp}_2(D)$-invariant linear form on π. For this, as in [14], we need to ensure that

$$\mathrm{Ext}^1_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_{P_H}^{\mathrm{Sp}_2(D)}\nu^{1/2}(\sigma_1 \otimes \sigma_2)|_{M_H}, \mathbb{C}] = 0.$$

For proving this, we recall the notion of the Euler-Poincaré pairing between two finite length representations of any reductive group G, defined by

$$\mathrm{EP}_G[\pi_1, \pi_2] = \sum_{i=0}^{r(G)} (-1)^i \dim \mathrm{Ext}^i_G[\pi_1, \pi_2],$$

where $r(G)$ is the split rank of G which for $\mathrm{Sp}_2(D)$ is 1. Therefore, for $\mathrm{Sp}_2(D)$,

$$\mathrm{EP}_{\mathrm{Sp}_2(D)}[\pi_1, \pi_2] = \dim \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\pi_1, \pi_2] - \dim \mathrm{Ext}^1_{\mathrm{Sp}_2(D)}[\pi_1, \pi_2].$$

By a well known theorem of [19], $\mathrm{EP}_G[\pi_1, \pi_2] = 0$ if π_1 is a (not necessarily irreducible) principal series representation of G. Therefore, we find that

$$\mathrm{EP}_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_{P_H}^{\mathrm{Sp}_2(D)} \nu^{1/2} \left(\sigma_1 \otimes \sigma_2\right), \mathbb{C})] = 0,$$

and so

$$\begin{aligned} &\dim \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_{P_H}^{\mathrm{Sp}_2(D)} \nu^{1/2} \left(\sigma_1 \otimes \sigma_2\right), \mathbb{C})] \\ &= \dim \mathrm{Ext}^1_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_{P_H}^{\mathrm{Sp}_2(D)} \nu^{1/2} (\sigma_1 \otimes \sigma_2), \mathbb{C}]. \end{aligned}$$

Since we are assuming that $\sigma_1 \neq \sigma_2 \otimes \nu$,

$$\dim \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_{P_H}^{\mathrm{Sp}_2(D)} \nu^{1/2} \left(\sigma_1 \otimes \sigma_2\right), \mathbb{C})] = 0.$$

Therefore we conclude that

$$\mathrm{Ext}^1_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_{P_H}^{\mathrm{Sp}_2(D)} \nu^{1/2} (\sigma_1 \otimes \sigma_2), \mathbb{C}] = 0.$$

As a result, we now have proved that if σ_1 and σ_2 are one dimensional representations of $D^\times$, with $\sigma_1 \neq \sigma_2 \otimes \nu$, then $\pi = \sigma_1 \times \sigma_2$ has a $\mathrm{Sp}_2(D)$-invariant linear form.

We have proved most of the following theorem, which we will now complete.

Theorem 4.3. *The only subquotients of a principal series representation* $\pi = \sigma_1 \times \sigma_2 := \mathrm{Ind}_P^{\mathrm{GL}_2(D)} \left(\sigma_1 \otimes \sigma_2\right)$ *of* $\mathrm{GL}_2(D)$ *which have a* $\mathrm{Sp}_2(D)$*- invariant linear form are the following.*

1) *When* $\dim\left(\sigma_1 \otimes \sigma_2\right) > 1$*, the unique irreducible quotient of the principal series representation* $\mathrm{Ind}_P^{\mathrm{GL}_2(D)} (\sigma\nu^{1/2} \otimes \sigma\nu^{-1/2})$ *denoted by* $\mathrm{Sp}(\sigma)$*.*

2) *When* $\dim(\sigma_1) = \dim(\sigma_2) = 1$, *any of the irreducible principal series representations* $\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1 \otimes \sigma_2)$, *whenever* $\sigma_1 \neq \sigma_2 \otimes \nu^{\pm 2}$.

3) *When* $\dim(\sigma_1) = \dim(\sigma_2) = 1$, *and* $\sigma_1 = \sigma_2 \otimes \nu^2$, *the principal series representation* $\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\sigma_1 \nu \otimes \sigma_1 \nu^{-1})$ *fits in the following exact sequence:*

$$0 \to \mathrm{St} \otimes \chi \to \mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\chi\nu \otimes \chi\nu^{-1}) \to \mathbb{C}_\chi \to 0,$$

where $\mathbb{C}_\chi$ *is the one dimensional representation of* $\mathrm{GL}_2(D)$ *on which* $\mathrm{GL}_2(D)$ *operates by the character* $\chi \circ N_{D/k}$, $N_{D/k}$ *is the reduced norm map and* St *is the Steinberg representation of* $\mathrm{GL}_2(D)$. *The only subquotient of* $\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\chi\nu \otimes \chi\nu^{-1})$ *having* $\mathrm{Sp}_2(D)$*-invariant linear form is* $\mathbb{C}_\chi$.

Proof. The only part of this theorem not shown by the arguments above is that

$$\mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{St}, \mathbb{C}] = 0,$$

where St is the Steinberg representation of $\mathrm{GL}_2(D)$, an irreducible admissible representation of $\mathrm{GL}_2(D)$ fitting in the exact sequence

$$0 \to \mathrm{St} \to \mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\nu \otimes \nu^{-1}) \to \mathbb{C} \to 0.$$

Applying $\mathrm{Hom}_{\mathrm{Sp}_2(D)}[-, \mathbb{C}]$ to this exact sequence, we have:

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathbb{C}, \mathbb{C}] &\to \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\nu \otimes \nu^{-1}), \mathbb{C}] \\ &\to \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{St}, \mathbb{C}] \to \mathrm{Ext}^1_{\mathrm{Sp}_2(D)}[\mathbb{C}, \mathbb{C}] \to \cdots . \end{aligned}$$

However, it is easy to see that $\mathrm{Ext}^1_{\mathrm{Sp}_2(D)}[\mathbb{C}, \mathbb{C}] = 0$. Therefore, we have a short exact sequence

$$0 \to \mathbb{C} \to \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\nu \otimes \nu^{-1}), \mathbb{C}] \to \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{St}, \mathbb{C}] \to 0.$$

Therefore, if $\mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{St}, \mathbb{C}] \neq 0$, $\dim \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\nu \otimes \nu^{-1}), \mathbb{C}] \geqslant 2$. However, by the analysis with Mackey theory done in this section, we know that $\dim \mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\nu \otimes \nu^{-1}), \mathbb{C}] = 1$. Thus we have proved that

$$\mathrm{Hom}_{\mathrm{Sp}_2(D)}[\mathrm{St}, \mathbb{C}] = 0.$$

□

Remark 4.4. Let $\pi = \chi_1 \times \chi_2 := \mathrm{Ind}_P^{\mathrm{GL}_2(D)}(\chi_1 \otimes \chi_2)$ for characters χ_1 and χ_2 of $D^\times$ which arise from the characters χ_1 and χ_2 of $k^\times$ via the reduced norm map from $D^\times$ to $k^\times$, with $\chi_1\chi_2^{-1} \neq \nu^{\pm 2}$. Then the representation π of $\mathrm{GL}_2(D)$ is distinguished by $\mathrm{Sp}_2(D)$. In this case, $\mathrm{JL}(\pi)$ is the irreducible principal series representation $\mathrm{JL}(\pi) = \mathrm{Ind}_P^{\mathrm{GL}_4(k)}(\chi_1\mathrm{St}_2 \otimes \chi_2\mathrm{St}_2)$ of $\mathrm{GL}_4(k)$ where St_2 denotes the Steinberg representation of $\mathrm{GL}_2(k)$. Since $\mathrm{JL}(\pi)$ is a generic representation of $\mathrm{GL}_4(k)$, it is not distinguished by $\mathrm{Sp}_4(k)$. Thus Jacquet-Langlands correspondence between representations of $\mathrm{GL}_2(D)$ and $\mathrm{GL}_4(k)$ does not always preserve distinction.

5. Explicit examples of distinguished supercuspidal representations

In this section we construct examples of supercuspidal representations of $\mathrm{GL}_n(D)$ which are distinguished by $\mathrm{Sp}_n(D)$ for any odd $n \geq 1$. These examples are due to Dipendra Prasad.

Recall that $\mathcal{O}_D$ is the maximal compact subring of D with π_D a uniformizing parameter of $\mathcal{O}_D$, and $\mathcal{O}_D/\langle \pi_D \mathcal{O}_D \rangle \simeq \mathbb{F}_{q^2}$ where $\mathbb{F}_q$ is the residue field of k. The anti-automorphism $x \to \bar{x}$ of D preserve $\mathcal{O}_D$ and acts as the Galois involution of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$.

Recall also that we have defined $\mathrm{Sp}_n(D)$ to be the subgroup of $\mathrm{GL}_n(D)$ by:

$$\mathrm{Sp}_n(D) = \left\{ A \in \mathrm{GL}_n(D) |\ AJ\ {}^t\bar{A} = J \right\},$$

where ${}^t\bar{A} = (\bar{a}_{ji})$ for $A = (a_{ij})$ and

$$J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & \cdot & & & \\ \cdot & & & & \\ 1 & & & & \end{pmatrix}.$$

It follows that $\mathrm{Sp}_n(\mathcal{O}_D) \subset \mathrm{GL}_n(\mathcal{O}_D)$, and taking the reduction of these compact groups modulo π_D, we have:

$$\mathrm{U}_n(\mathbb{F}_q) \hookrightarrow \mathrm{GL}_n(\mathbb{F}_{q^2}),$$

where U_n is defined using the Hermitian form

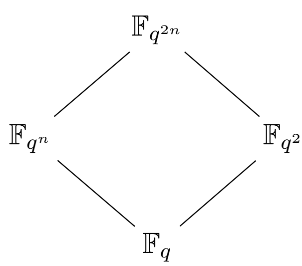
$$J = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & \cdot & & & \\ & \cdot & & & & \\ 1 & & & & & \end{pmatrix}.$$

Proposition 5.1. *Let π_{00} be an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, n an odd integer, and $\pi_0 = \mathrm{BC}(\pi_{00})$ be the base change of π_{00} to $\mathrm{GL}_n(\mathbb{F}_{q^2})$. Using the reduction mod $\pi_D : \mathrm{GL}_n(\mathcal{O}_D) \to \mathrm{GL}_n(\mathbb{F}_{q^2})$, we can lift π_0 to an irreducible representation of $\mathrm{GL}_n(\mathcal{O}_D)$ to be denoted by π_0 again. Let χ be a character of $k^\times$ which matches with the central character of π_0 on $\mathcal{O}_k^\times$. Then*

$$\pi = \mathrm{ind}_{k^\times \mathrm{GL}_n(\mathcal{O}_D)}^{\mathrm{GL}_n(D)}(\chi \cdot \pi_0)$$

is an irreducible supercuspidal representation of $\mathrm{GL}_n(D)$ which is distinguished by $\mathrm{Sp}_n(D)$.

Proof. The fact that π is an irreducible supercuspidal representation of $\mathrm{GL}_n(D)$ is a well-known fact about compact induction valid in a great generality once we have checked that $\pi_0 = \mathrm{BC}(\pi_{00})$ is a cuspidal representation. This assertion on $\mathrm{GL}_n(\mathbb{F}_{q^2})$ follows from the fact that n is odd in which case we have a diagram of fields:



In particular,

$$\mathrm{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_q) = \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \times \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q).$$

Thus given a character $\chi_{00} : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$ whose Galois conjugates are distinct (and which therefore gives rise to the cuspidal representation π_{00} of $\mathrm{GL}_n(\mathbb{F}_q)$), the character $\chi_0 : \mathbb{F}_{q^{2n}}^\times \to \mathbb{C}^\times$ obtained from χ_{00} using the norm map: $\mathbb{F}_{q^{2n}}^\times \to \mathbb{F}_{q^n}^\times$, has exactly n distinct Galois conjugates, therefore χ_0 gives rise to a cuspidal representation π_0 of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ which is the base change of the representation π_{00} of $\mathrm{GL}_n(\mathbb{F}_q)$.

The distinction of π by $\mathrm{Sp}_n(D)$ follows from the earlier observation that reduction mod π_D of the inclusion $\mathrm{Sp}_n(\mathcal{O}_D) \subset \mathrm{GL}_n(\mathcal{O}_D)$ is

$$\mathrm{U}_n(\mathbb{F}_q) \hookrightarrow \mathrm{GL}_n(\mathbb{F}_{q^2}),$$

together with Theorem 2 of [15] that irreducible representations of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ which are base change from $\mathrm{GL}_n(\mathbb{F}_q)$ are distinguished by $\mathrm{U}_n(\mathbb{F}_q)$. □

Remark 5.2. 1) The Langlands parameter of the irreducible representation $\pi = \mathrm{ind}_{k^\times \mathrm{GL}_n(\mathcal{O}_D)}^{\mathrm{GL}_n(D)}(\pi_0)$ is of the form $\sigma = \sigma_0 \otimes \mathrm{sp}_2$ where σ_0 is the Langlands parameter of the supercuspidal representation of $\mathrm{GL}_n(k)$ compactly induced from the representation $\chi \cdot \pi_{00}$ of $k^\times \mathrm{GL}_n(\mathcal{O}_k)$, and sp_2 is the 2-dimensional natural representation of the $\mathrm{SL}_2(\mathbb{C})$ part of the Weil-Deligne group $W_k' = W_k \times \mathrm{SL}_2(\mathbb{C})$ of k.

2) If, on the other hand, the cuspidal representation π_0 of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ is not obtained by base change from $\mathrm{GL}_n(\mathbb{F}_q)$ then the Langlands parameter of such a π is that of the cuspidal representation of $\mathrm{GL}_{2n}(k)$ which is obtained by compact induction of the representation of $k^\times \mathrm{GL}_{2n}(\mathcal{O}_k)$ which is χ on $k^\times$, and on $\mathrm{GL}_{2n}(\mathcal{O}_k)$ it corresponds to a representation of $\mathrm{GL}_{2n}(\mathbb{F}_q)$ which is the automorphic induction of the representation π_{00} of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ (and which is cuspidal since we are assuming that the representation π_0 of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ is not a base change for $\mathrm{GL}_n(\mathbb{F}_q)$).

6. Using Jacquet-Langlands correspondence

In the rest of the paper, we will formulate certain conjectures — and prove some parts of them — on distinction of representations of $\mathrm{GL}_n(D)$ by $\mathrm{Sp}_n(D)$ both locally and globally. For doing this, we will use Jacquet-Langlands correspondence to transfer representations of $\mathrm{GL}_n(D)$ to $\mathrm{GL}_{2n}(k)$. An essential component of our discussion will therefore revolve around the Jacquet-Langlands correspondence which we recall initially matched discrete series representations of $\mathrm{GL}_{nd}(k)$ and $\mathrm{GL}_n(D_k)$ where k is any local field, D_k a central division algebra over k of dimension d^2; there was also the global Jacquet-Langlands correspondence matching cuspidal automorphic representations on the two groups with appropriate local conditions. In the questions on symplectic periods, we must deal with representations which are not discrete series locally, and are non-cuspidal globally, and for this reason we must appeal to relatively recent results of Badulescu which completes these earlier works on Jacquet-Langlands correspondence. For much of this section, we refer to [1].

Let D_k be a central division algebra over a local field k of dimension d^2 over k. An irreducible admissible representation π of $\mathrm{GL}_{nd}(k)$ is said to be d-compatible if χ_π, the character of π, takes non-zero value at a regular element g_0 of $\mathrm{GL}_{nd}(k)$ which has the same characteristic polynomial as an element g_0' on $\mathrm{GL}_n(D_k)$. As an example — which will be of use to us later — the reducible principal series representation of $\mathrm{GL}_{2n}(k)$ which is $\tau\nu^{1/2} \times \tau\nu^{-1/2}$ where τ is a cuspidal representation of $\mathrm{GL}_n(k)$, has the Langlands quotient with parameter $\tau\nu^{1/2} \oplus \tau\nu^{-1/2}$, and a generic discrete series as a sub-module. Both these representations of $\mathrm{GL}_{2n}(k)$ are 2-compatible.

It is a theorem of Badulescu that given a d-compatible irreducible unitary representation π of $\mathrm{GL}_{nd}(k)$, there exists a unique irreducible unitary representation π' of $\mathrm{GL}_n(D_k)$ such that,

$$\chi_\pi(g) = \epsilon\chi_{\pi'}(g'),$$

where $\epsilon = \pm 1$, and $g \in \mathrm{GL}_{nd}(k), g' \in \mathrm{GL}_n(D_k)$ are any two elements with the same characteristic polynomials.

This theorem of Badulescu defines a map, which is denoted by him by $|\mathbf{LJ}|$ from the set of irreducible unitary d-compatible representations of $\mathrm{GL}_{nd}(k)$ to the set of all irreducible unitary representations of $\mathrm{GL}_n(D_k)$. The map $|\mathbf{LJ}|$ is the inverse to the usual Jacquet-Langlands map defined on the set of irreducible discrete series representations of $\mathrm{GL}_n(D_k)$ to the set of irreducible discrete series representations of $\mathrm{GL}_{nd}(k)$. In the example above of the reducible principal series representation of $\mathrm{GL}_{2n}(k)$ which is $\tau\nu^{1/2} \times \tau\nu^{-1/2}$ where τ is a cuspidal representation of $\mathrm{GL}_n(k)$, both the sub-quotients of this principal series are unitary representations, and for both of them for D_k the quaternion division algebra, $|\mathbf{LJ}|$ is the same discrete series representation of $\mathrm{GL}_n(D_k)$ which corresponds by the usual Jacquet-Langlands correspondence to the discrete series component of the representation $\tau\nu^{1/2} \times \tau\nu^{-1/2}$ of $\mathrm{GL}_{2n}(k)$.

Now, let F be a number field and D central division algebra over F of dimension d^2. Let $\mathbb{A}_F$ be the ring of adèles of F. For a place v of F, let $D_v = D \otimes_F F_v$ and $D_\mathbb{A} = D \otimes_F \mathbb{A}_F$. The index of D_v is denoted by d_v, and the places v of F where $d_v \neq 1$ are called the set of ramified places for D.

Denote by DS_{nd} the set of discrete automorphic representations of $\mathrm{GL}_{nd}(\mathbb{A}_F)$ and DS_n' the set of discrete automorphic representations of $\mathrm{GL}_n(D_\mathbb{A})$. The global correspondence between the discrete spectrum of a general linear group and its inner form is defined and proved in Badulescu [1] and Badulescu-Renard [2].

Theorem 6.1 (Jacquet-Langlands, ..., Badulescu, Badulescu-Renard). *There exits a unique injective map* $\mathrm{JL} : DS'_n \to DS_{nd}$ *such that for all* $\Pi' = \otimes \Pi'_v \in DS'_n$, *we have* $\mathrm{JL}(\Pi')_v = \Pi'_v$ *for all places* $v \notin V$ *where* V *is the set of places of* F *where* D *does not split completely. For every* $v \in V$, $\mathrm{JL}(\Pi')_v$ *is* d_v*-compatible and we have* $|\mathbf{LJ}|_v(\mathrm{JL}(\Pi')_v) = \Pi'_v$. *The image of* JL *consists of all those representations in the discrete spectrum of* $\mathrm{GL}_{nd}(\mathbb{A}_F)$ *which are* d_v*-compatible at all the places* $v \in V$. *Moreover, either* $\mathrm{JL}(\Pi')_v$, *or its Zelevinsky involution has the same Langlands parameter as* Π'_v.

In [11], Offen studied the symplectic periods on the discrete automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}_F)$. For an automorphic form f in the discrete spectrum of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, consider the period integral

$$\int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A}_F)} f(h)dh.$$

We say that an irreducible, discrete automorphic representation Π of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ is $\mathrm{Sp}_{2n}(\mathbb{A}_F)$-distinguished if the above period integral is not identically zero on the space of Π. We now recall a result from [13] that we will use in this section.

Theorem 6.2. *Let* F *be a number field and let* $\Pi = \otimes'_v \Pi_v$ *be an irreducible automorphic representation of* $\mathrm{GL}_{2n}(\mathbb{A}_F)$ *in the discrete spectrum. Then the following are equivalent:*

1) Π *is* $\mathrm{Sp}_{2n}(\mathbb{A}_F)$*-distinguished,*

2) Π_v *is* $\mathrm{Sp}_{2n}(F_v)$*-distinguished for all places* v *of* F,

3) Π_{v_0} *is* $\mathrm{Sp}_{2n}(F_{v_0})$*-distinguished for some finite place* v_0 *of* F,

Jacquet and Rallis have shown in [6], that the symplectic period vanishes for a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, that is

$$\int_{\mathrm{Sp}_{2n}(F)\backslash \mathrm{Sp}_{2n}(\mathbb{A}_F)} f(h)dh = 0.$$

In the next theorem, in the spirit of Jacquet-Rallis result mentioned above, we prove that those cuspidal automorphic representations Π of $\mathrm{GL}_n(D_{\mathbb{A}})$ for which $\mathrm{JL}(\Pi)$ is a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$, have vanishing symplectic periods.

Theorem 6.3. *Suppose that* Π *is a cuspidal automorphic representation of* $\mathrm{GL}_n(D_{\mathbb{A}})$ *whose Jacquet-Langlands lift* $\mathrm{JL}(\Pi)$ *to* $\mathrm{GL}_{2n}(\mathbb{A}_F)$ *is cuspidal then the symplectic period integrals of* Π *vanish identically.*

Proof. Assume if possible that Π has a non-zero symplectic period. Then Π_v has a non-zero symplectic period for all places v of F. The representations $\mathrm{JL}(\Pi)$ and Π are the same at all places v of F where D splits and therefore by the Theorem 3.2.2 of [5], Π_v is not generic for any v where D splits. Since a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ is globally generic, the local representations Π_v are locally generic for all v, which gives a contradiction. □

Theorem 6.4. *If* Π *is an automorphic representation of* $\mathrm{GL}_n(D_{\mathbb{A}})$ *which appears in the discrete spectrum, and is distinguished by* $\mathrm{Sp}_n(D_{\mathbb{A}})$ *then* $\mathrm{JL}(\Pi)$*, which is an automorphic representation of* $\mathrm{GL}_{2n}(\mathbb{A}_F)$*, is globally distinguished by* $\mathrm{Sp}_{2n}(\mathbb{A}_F)$.

Proof. If Π is $\mathrm{Sp}_n(D_{\mathbb{A}})$-distinguished, then it is locally distinguished at all places v of F. Also we know that D splits at almost all places of F so $\Pi_v = \mathrm{JL}(\Pi)_v$ at almost all places of F. By Theorem 6.2, global distinction of Jacquet-Langland lift $\mathrm{JL}(\Pi)$ is a consequence of local distinction at any place v of F which we know. □

Remark 6.5. If Π is a global automorphic representations of $\mathrm{GL}_2(D_{\mathbb{A}})$ which is distinguished by $\mathrm{Sp}_2(D_{\mathbb{A}})$ with a local component $\Pi_v = \chi_1 \times \chi_2$, a representation of $\mathrm{GL}_2(D_v)$ for characters $\chi_1, \chi_2 : D_v^\times \to \mathbb{C}^\times$, then $\mathrm{JL}(\Pi)$, an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$, must be distinguished by $\mathrm{Sp}_4(\mathbb{A}_F)$ by Theorem 6.4. Since $\mathrm{JL}(\Pi_v) = \chi_1\mathrm{St}_2 \times \chi_2\mathrm{St}_2$ as a representation of $\mathrm{GL}_4(k_v)$, this seems to be in contradiction to the fact that $\mathrm{JL}(\Pi)$ is globally distinguished by $\mathrm{Sp}_4(\mathbb{A}_F)$. The source of this apparent contradiction is the fact that in this case, $\mathrm{JL}(\Pi)_v = \chi_1 \times \chi_2$ as a representation of $\mathrm{GL}_4(k_v)$ (and this is allowed by theorem 6.1).

A supercuspidal representation of $\mathrm{GL}_{2n}(k)$ is not distinguished by $\mathrm{Sp}_{2n}(k)$. The situation in the case of $\mathrm{GL}_n(D)$ is different since in the previous section we gave an example of distinguished supercuspidal representations due to Dipendra Prasad. The following theorem gives a partial answer to the question on distinction of a supercuspidal representation of $\mathrm{GL}_n(D)$ by $\mathrm{Sp}_n(D)$.

Theorem 6.6. *Let π_v be a supercuspidal representation of $\mathrm{GL}_n(D_v)$ with Langlands parameter $\sigma_{\pi_v} = \sigma \otimes \mathrm{sp}_r$ where σ is an irreducible representation of the Weil-group W_k, and sp_r is the r-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ part of the Weil-Deligne group W'_k. Then if r is odd, π_v is not distinguished by $\mathrm{Sp}_n(D_v)$.*

Proof. Assuming r is odd, we prove that π_v is not distinguished by $\mathrm{Sp}_n(D_v)$. Using a theorem of [16], we globalize π_v to be globally distinguished automorphic representation Π of $\mathrm{GL}_n(D_{\mathbb{A}})$ where D is a global quaternion division algebra over a number field F such that $F_v = k$, and $D \otimes F_v = D_v$.

Using the Jacquet-Langlands correspondence, Theorem 6.1, we get an automorphic representation $\mathrm{JL}(\Pi)$ of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ which is locally distinguished by $\mathrm{Sp}_{2n}(F_w)$ at all places w of F where D splits. By a theorem of Offen-Sayag, $\mathrm{JL}(\Pi)$ is globally distinguished by $\mathrm{Sp}_{2n}(\mathbb{A}_F)$. By Theorem 6.1, $\mathrm{JL}(\Pi)_v$ is one of the following

1) $\mathrm{JL}(\Pi)_v = \mathrm{JL}(\Pi_v)$, a discrete series representation, or
2) $\mathrm{JL}(\Pi)_v =$ a Speh representation with Langlands parameter

$$\sigma \otimes (\nu^{(r-1)/2} \oplus \nu^{(r-3)/2} \oplus \cdots \oplus \nu^{-(r-1)/2}).$$

The first choice being a discrete series representation, in particular generic, is never distinguished by $\mathrm{Sp}_{2n}(F_v)$. The fact that the second choice is also not distinguished by $\mathrm{Sp}_{2n}(F_v)$ uses that r is odd, and is consequence of a theorem of Offen-Sayag about them. □

Remark 6.7. The only place we used supercuspidality of the representation π_v of $\mathrm{GL}_n(D_v)$ with Langlands parameter $\sigma_{\pi_v} = \sigma \otimes \mathrm{sp}_r$ where σ is an irreducible representation of the Weil-group W_k, and sp_r is the r-dimensional irreducible representation of the $\mathrm{SL}_2(\mathbb{C})$ part of the Weil-Deligne group W'_k is in the globalization theorem of [16]. If we grant ourselves such a globalization theorem for discrete series too, then we have the same conclusion as in the theorem.

The theorem below together with local analysis done in Section 4 completes the distinction problem for $\mathrm{GL}_2(D)$.

Theorem 6.8. *No discrete series representation of $\mathrm{GL}_2(D_v)$ is distinguished by $\mathrm{Sp}_2(D_v)$.*

Proof. By our local analysis, we know this already for those discrete series representations of $\mathrm{GL}_2(D_v)$ which are not supercuspidal. By the previous theorem, we also know that no supercuspidal representation of $\mathrm{GL}_2(D_v)$ is distinguished by $\mathrm{Sp}_2(D_v)$ as long as its Langlands parameter is not of the form $\sigma_\pi = \sigma \otimes \mathrm{sp}_r$ where $r = 2, 4$. By Proposition 7.3, such Langlands parameter correspond to non-supercuspidal discrete series representations of $\mathrm{GL}_2(D_v)$, completing the proof of theorem. □

7. Conjectures on distinction

The following conjectures have been proposed by Dipendra Prasad.

Conjecture 7.1. 1) *Let k be a non-archimedean local field, and D_k the unique quaternion division algebra over k. Then an irreducible discrete series representation π of $\mathrm{GL}_n(D_k)$ is distinguished by $\mathrm{Sp}_n(D_k)$ if and only if π is supercuspidal and the Langlands parameter σ_π of π is of the form $\sigma_\pi = \tau \otimes \mathrm{sp}_r$ where τ is irreducible representation of W_k and sp_r is the r-dimensional irreducible representation of the $\mathrm{SL}_2(\mathbb{C})$ part of the Weil-Deligne group $W'_k = W_k \times \mathrm{SL}_2(\mathbb{C})$ for r even. By Proposition 7.3 below, such a parameter σ_π defines a supercuspidal representation of $\mathrm{GL}_n(D_k)$ if and only if $r = 2$, and n is odd. (This is precisely the case in which we constructed in Section 5 a supercuspidal representation of $\mathrm{GL}_n(D_k)$ which is distinguished by $\mathrm{Sp}_n(D_k)$.)*

2) *We follow the notation of Offen-Sayag in Theorem 1 of [12], to recall that the unitary representations of $\mathrm{GL}_{2n}(k)$ which are distinguished by $\mathrm{Sp}_{2n}(k)$ are of the form*

$$\sigma_1 \times \cdots \times \sigma_t \times \tau_{t+1} \times \cdots \times \tau_{t+s},$$

where σ_i are the Speh representations $U(\delta_i, 2m_i)$ for discrete series representations δ_i of $\mathrm{GL}_{r_i}(k)$, and τ_i are complementary series representations $\pi(U(\delta_i, 2m_i), \alpha_i)$ with $|\alpha_i| < 1/2$. We suggest that unitary representations of $\mathrm{GL}_n(D_k)$ distinguished by $\mathrm{Sp}_n(D_k)$ are exactly those representations of $\mathrm{GL}_n(D_k)$ which are of the form

$$\pi = \sigma_1 \times \cdots \times \sigma_t \times \tau_{t+1} \times \cdots \times \tau_{t+s} \times \mu_{t+s+1} \times \cdots \times \mu_{t+s+r},$$

where

(a) *the parameter σ_π of π is relevant for $\mathrm{GL}_n(D_k)$, that is, all irreducible subrepresentations of σ_π have even dimension,*

(b) *the representations σ_i and τ_i are as in the theorem of Offen-Sayag recalled above,*

(c) *μ_i are supercuspidal representations of* $\mathrm{GL}_{m_i}(D_k)$ *as in Part (1) of the conjecture.*

3) *Let D be a quaternion division algebra over a number field F, and $D_{\mathbb{A}} = D \otimes_F \mathbb{A}_F$. A global automorphic representation Π of* $\mathrm{GL}_n(D_{\mathbb{A}})$ *is distinguished by* $\mathrm{Sp}_n(D_{\mathbb{A}})$ *if and only if* $\mathrm{JL}(\Pi)$ *as an automorphic representation of* $\mathrm{GL}_{2n}(\mathbb{A}_F)$ *is distinguished by* $\mathrm{Sp}_{2n}(\mathbb{A}_F)$.

Proposition 7.2. *The global conjecture in Part (3) above implies the local conjecture in Part (1).*

Proof. Let D be a quaternion division algebra over a number field F, v_0 a finite place of F with $k = F_{v_0}$, and $D_k = D_{v_0} = D \otimes_F F_{v_0}$. To prove the proposition, note that a discrete series representation π of $\mathrm{GL}_n(D_{v_0})$ with parameter $\tau \otimes \mathrm{sp}_r$ with r odd is not distinguished by $\mathrm{Sp}_n(D_{v_0})$ as follows from Theorem 6.6 and Remark 6.7 following it (which assumes validity of the globalization theorem of [16] for discrete series representations).

Now we prove that a non-cuspidal discrete series representation π of $\mathrm{GL}_n(D_{v_0})$ with parameter $\tau \otimes \mathrm{sp}_r$ with r even are not distinguished by $\mathrm{Sp}_n(D_{v_0})$. Again we will grant ourselves an automorphic representation Π of $\mathrm{GL}_n(D_{\mathbb{A}})$ which is globally distinguished by $\mathrm{Sp}_n(D_{\mathbb{A}})$. By the Jacquet-Langlands transfer, we get a representation $\mathrm{JL}(\Pi)$ of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ which is distinguished by $\mathrm{Sp}_{2n}(\mathbb{A}_F)$, and therefore by the theorem of Offen-Sayag, $\mathrm{JL}(\Pi)$ is in the residual spectrum with the Moeglin-Waldspurger type, $\mathrm{JL}(\Pi) = \Sigma \otimes \mathrm{sp}_d$, where Σ is a cuspidal automorphic representation of $\mathrm{GL}_r(\mathbb{A}_F)$ for some integer r, and d is a certain even integer; here the notation $\Sigma \otimes \mathrm{sp}_d$ is supposed to denote a certain Speh representation. The only option for d in our case is $d = r$, and $\Sigma_{v_0} = \tau$. By Proposition 7.3 below, we get a contradiction to π being a non-cuspidal discrete series representation of $\mathrm{GL}_n(D_{v_0})$.

Finally, we prove that if we have a cuspidal representation π of $\mathrm{GL}_n(D_{v_0})$ with parameter $\tau \otimes \mathrm{sp}_r$ with r even, so $r = 2$, and $\dim \tau = n$ odd, then π is distinguished by $\mathrm{Sp}_n(D_{v_0})$.

Construct an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ whose local component at the place v_0 of F has Langlands parameter τ with $\dim \tau = n$. Since τ is an irreducible representation of the Weil group, we are considering supercuspidal representation of $\mathrm{GL}_n(F_{v_0})$, and therefore globalization is possible (and is automatically cuspidal). We moreover assume in this globalization that the global automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is supercuspidal at all places of F where D is not split. By Moeglin-Waldspurger, this gives an automorphic representation say Π of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ in the residual spectrum, which by the theorems of Offen and Sayag is distinguished by $\mathrm{Sp}_{2n}(\mathbb{A}_F)$. By Theorem 6.1, Π can be lifted to $\mathrm{GL}_n(D_{\mathbb{A}})$, which by our global conjecture Part (3) above is globally distinguished by $\mathrm{Sp}_n(D_{\mathbb{A}})$, and therefore locally distinguished at every place of F. It remains to make sure that in this lifted representation to $\mathrm{GL}_n(D_{\mathbb{A}})$, the local representation obtained for $\mathrm{GL}_n(D_{v_0})$ is the cuspidal representation π that we started with, with parameter $\tau \otimes \mathrm{sp}_2$; this is so by Theorem 6.1. □

The following proposition used in Theorem 6.8 as well as Proposition 7.2 is due to Deligne-Kazhdan-Vigneras [3], Theorem B.2.b.1, as well as Badulescu, Proposition 3.7 of [1].

Proposition 7.3. *A discrete series representation of* $\mathrm{GL}_n(D_v)$*, where* D_v *is an arbitrary division algebra over the local field* F_v*, with parameter* $\tau \otimes \mathrm{sp}_r$ *is a cuspidal representation of* $\mathrm{GL}_n(D_v)$ *if and only if* $(r, n) = 1$.

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