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The determinant and the discriminant of a complete intersection of even dimension

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The determinant of the Galois action on the ℓ-adic cohomology of the middle degree of a proper smooth variety of even dimension defines a quadratic character of the absolute Galois group of the base field. In this article, we show that for a complete intersection of even dimension in a projective space, the character is computed via the square root of the discriminant of the defining polynomials of the variety.

Introduction

Let k be a field, $\bar{k}$ an algebraic closure of k and k^s the separable closure of k contained in $\bar{k}$. Let $\Gamma_k = \mathrm{Gal}(k^s/k) = \mathrm{Aut}_k(\bar{k})$.

Let X be a proper smooth variety of even dimension m over k. If ℓ is a prime number invertible in k, the ℓ-adic cohomology $V = H^m(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{m}{2}))$ defines an orthogonal representation of the absolute Galois group Γ_k. The determinant

$$\det V : \Gamma_k \to \{\pm 1\} \subset \mathbb{Q}_\ell^\times$$

is independent of the choice of ℓ (Corollary 2.2).

In this introduction we assume that the characteristic of k is not 2. Let $f_1, \ldots, f_r$ be homogeneous polynomials of $n+1$ variables of degrees $d_1, \ldots, d_r$ of coefficients in k. Let X be the intersection of r hypersurfaces defined by these polynomials in the projective space of dimension n. O. Benoist [1] studied the *discriminant of a complete intersection* and gave an explicit formula of its degree. The discriminant, here denoted by $\mathrm{disc}(f_1, \ldots, f_r)$, is a polynomial of the coefficients of $f_1, \ldots, f_r$, and is defined in [1] up to sign by requiring the property that X is smooth of dimension $n-r$ if and only if $\mathrm{disc}(f_1, \ldots, f_r) \neq 0$.

In this paper, we consider the value of the discriminant, not only its non-vanishing, when the dimension $n-r$ is even. In this case, there exists a unique choice of the sign of the discriminant such that the discriminant modulo 4 is a square (Theorem 2.3.1). Let us denote the discriminant defined in these steps by $\mathrm{disc}_\sigma(f_1, \ldots, f_r)$. We shall prove the following theorem (Theorem 2.3.2).

Theorem 0.1. *Assume that X is smooth of even dimension $m = n - r$. Then the quadratic character $\det V$ is defined by the square root of $\mathrm{disc}_\sigma(f_1, \ldots, f_r)$.*

In other words, the kernel of $\det V : \Gamma_k \to \{\pm 1\}$ is the subgroup of Γ_k corresponding to the field extension $k\big(\sqrt{\mathrm{disc}_\sigma(f_1, \ldots, f_r)}\big)/k$.

Let us outline the contents of this paper. In Section 1, we study the discriminant of a complete intersection. In Subsection 1.1, we recall the definition of the discriminant in [1]. In Subsection 1.4, we give a different calculation of the degree of the discriminant from that in [1]. For this purpose, we regard the projective toric variety X_A in [1] as a projective space bundle over the projective space (Lemma 1.6). We give a new explicit presentation of the degree, though we do not know the relation between our presentation (Lemma 1.9) and Benoist's formula [1, Théorèm 1.3].

In Section 2, we prove Theorem 0.1. We first recall the quadratic character of the absolute Galois group defined by the determinant of the ℓ-adic representation of the middle degree of a proper smooth variety defined over a field. In [8], T. Saito showed that, for a smooth hypersurface of even dimension, the character is computed via the square root of the discriminant of a defining polynomial of the hypersurface. We adapt his method to the case of a complete intersection of even dimension. By the same argument on the universal family as in the case of a hypersurface, the equality in Theorem 0.1.2 is true up to a sign of the discriminant. Then the sign is determined by a property of the discriminant modulo 4.

In Section 3, we give an explicit presentation of the discriminant of the complete intersection of two quadrics (Theorem 3.6). Let $n \geq 2$ be an integer. Let $F_1 = \sum_{0\leq i\leq j\leq n} C_{ij}^{(1)} X_i X_j$ and $F_2 = \sum_{0\leq i\leq j\leq n} C_{ij}^{(2)} X_i X_j$ be universal homogeneous polynomials of degree 2. We regard $t_1F_1 + t_2F_2$ as a quadratic form in variables $X_0, \ldots, X_n$ and denote its discriminant by $\mathrm{disc}(t_1F_2 + t_2F_2) \in \mathbb{Z}[t_1, t_2, (C_{ij}^{(l)})]$. Further we regard $\mathrm{disc}(t_1F_2 + t_2F_2)$ as a binary form in variables t_1, t_2 and denote its discriminant by $\mathrm{disc}(\mathrm{disc}(t_1F_1 + t_2F_2)) \in \mathbb{Z}[(C_{ij}^{(l)})]$.

Theorem 0.2. *The identities*

$$\mathrm{disc}(F_1, F_2) = \begin{cases} \mathrm{disc}(\mathrm{disc}(t_1F_1 + t_2F_2)) & \text{if } n \text{ is even} \\ 2^{-2(n+1)}\,\mathrm{disc}(\mathrm{disc}(t_1F_1 + t_2F_2)) & \text{if } n \text{ is odd} \end{cases}$$

hold up to sign.

The discriminant of a quadratic polynomial is the determinant of the symmetric matrix corresponding to the quadratic form (Example 3.2). Further, the discriminant of a binary polynomial is the determinant of the Sylvester matrix. Thus the above equalities give explicit presentations of the discriminant of a complete intersection of two quadrics.

The author was imformed by Takeshi Saito that Jean-Pierre Serre suggested him that the discriminant of a complete intersection of two quadrics should be given by the discriminants of a quadratic polynomial and a binary polynomial.

Finally, in Subsection 3.7, we give an application. The cohomology of the middle degree of the smooth complete intersection of even dimension $n-2$ of two quadrics in $\mathbb{P}^n_{\bar{k}}$ is generated by algebraic classes of linear subspaces. The group of all permutations of these linear subspaces preserving their intersection numbers is isomorphic to the Weyl group $W(D_{n+1})$ [7]. The action of the absolute Galois group Γ_k on the linear subspaces defines a homomorphism $\Gamma_k \to W(D_{n+1})$ unique up to conjugation. We show that the image of Γ_k is contained in the index two subgroup of $W(D_{n+1})$ if and only if the discriminant is a square (Corollary 3.8).

1. Discriminant

1.1. The universal family and the discriminant

We define the universal family of intersections of hypersurfaces and recall the set of singular intersections [1, 1.1]. We fix integers $0 \le r \le n$. We consider the polynomial ring $\mathbb{Z}[X_0, \ldots, X_n]$ and the free $\mathbb{Z}$-module $E = \bigoplus_{i=0}^n \mathbb{Z} \cdot X_i$. For an integer $d \ge 1$, we identify the d-th symmetric power $S^d E$ defined over $\mathbb{Z}$ with the free $\mathbb{Z}$-module of finite rank consisting of homogeneous polynomials of degree d in $\mathbb{Z}[X_0, \ldots, X_n]$. If $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ is a multi-index, we put $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n} \in \mathbb{Z}[X_0, \ldots, X_n]$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n$. The monomials X^α of degree $|\alpha| = d$ form a basis of $S^d E$.

We put $\mathbb{P}^n = \mathbb{P}(E) = \operatorname{Proj} \mathbb{Z}[X_0, \ldots, X_n]$ and fix integers $d_1, \ldots, d_r \ge 1$. We assume that $d_l \ge 2$ for an index l $(1 \le l \le r)$. Further we put $V = \bigoplus_{1 \le j \le r} S^{d_j} E$ and let $\mathbb{P}^\vee = \mathbb{P}(V^\vee) = \operatorname{Proj}(S^\bullet(V^\vee))$ be the projective space defined by the dual $V^\vee = \operatorname{Hom}(V, \mathbb{Z})$. Let $(C_\alpha^{(j)})_{|\alpha|=d_j}$ be the dual basis of $(S^{d_j} E)^\vee$ and define the universal polynomials $F_j = \sum_{|\alpha|=d_j} C_\alpha^{(j)} X^\alpha$. Then we define a closed subscheme $X \subset \mathbb{P}^n \times \mathbb{P}^\vee$ by the equations $F_1 = \cdots = F_r = 0$. This is the universal family of intersections of r hypersurfaces.

Let $\pi : X \subset \mathbb{P}^n \times \mathbb{P}^\vee \to \mathbb{P}^\vee$ be the second projection. Let k be an algebraically closed field and let $s : \operatorname{Spec} k \to \mathbb{P}^\vee$ be a geometric point. Then this s corresponds to a sequence of homogeneous polynomials $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ of coefficients in k. The geometric fiber X_s of π is isomorphic to the intersection of r hypersurfaces in $\mathbb{P}^n_k$ defined by the polynomials $f_1, \ldots, f_r$.

Let $\mathcal{J}$ be the ideal sheaf of $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^\vee}$ defined by all the $r \times r$ minor determinants of the Jacobian matrix

$$J(F_1, \ldots, F_r) = \left(\left(\frac{\partial F_j}{\partial X_i} \right)_{0 \leq i \leq n, 1 \leq j \leq r} \right)$$

of the universal polynomials $F_1, \ldots, F_r$. We define a closed subscheme $\Delta \subset X$ by the ideal sheaf $\mathcal{J} \cdot \mathcal{O}_X$. By the Jacobian criterion, the complement $U = X - \Delta$ is the maximum open subscheme of X on which the morphism $\pi : X \to \mathbb{P}^\vee$ is smooth of relative dimension $n - r$.

We define a closed subscheme $D \subset \mathbb{P}^\vee$ as the image $\pi(\Delta)$ with the reduced scheme structure. For an algebraically closed field k, the set of k-valued points $D(k)$ consists of the sequences of homogeneous polynomials $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ of coefficients in k such that the intersections $V((f_1, \ldots, f_r)) \subset \mathbb{P}^n_k$ are not smooth of dimension $n - r$.

By [1, Lemme 3.2], [1, Lemme 4.4 (i)], and [1, Corollaire 3.3], the base change $D_{\overline{\mathbb{Q}}}$ is an irreducible closed subscheme of codimension one in $\mathbb{P}^\vee_{\overline{\mathbb{Q}}}$. Thus there exists a geometrically irreducible homogeneous polynomial in $(C_I^{(j)})_{1 \leq j \leq r, |I| = d_j}$ of coefficients in $\mathbb{Z}$ uniquely defined up to sign, such that it defines the closed subscheme $D_{\mathbb{Q}} \subset \mathbb{P}^\vee_{\mathbb{Q}}$. We call this homogeneous polynomial defined up to sign the *discriminant of complete intersetions* and we denote it by $\operatorname{disc}(F_1, \ldots, F_r)$.

Proposition 1.2. *The reduced closed subscheme D in $\mathbb{P}^\vee$ is defined by the polynomial* $\operatorname{disc}(F_1, \ldots, F_r)$. *In particular, D is irreducible and of codimension one.*

Proof. By the definition of the discriminant, the closed subscheme D is defined by a polynomial $m \operatorname{disc}(F_1, \ldots, F_r)$ for some integer $m \neq 0$. By [1, Proposition 3.1], the base change $D_{\overline{\mathbb{F}}_p}$ for any prime p has strictly positive codimension in $\mathbb{P}^\vee_{\overline{\mathbb{F}}_p}$. Hence we have $m = \pm 1$. □

By specialization, the discriminant $\operatorname{disc}(f_1, \ldots, f_r)$ has a meaning for every homogeneous polynomials $f_1, \ldots, f_r$ in $n+1$ variables over a commutative ring R, and it satisfies the following smoothness criterion.

Proposition 1.3. *Let $f_1, \dots, f_r$ be homogeneous polynomials of degrees $d_1, \dots, d_r$ in $n+1$ variables of coefficients in a commutative ring R. Then, the discriminant* $\operatorname{disc}(f_1, \dots, f_r)$*, defined up to sign, is invertible in R if and only if the corresponding intersection $V((f_1, \dots, f_r))$ in $\mathbb{P}^n_R$ over R is smooth of relative dimension $n - r$.*

1.4. The degree of the discriminant

We can reduce the calculation of the degree of the discriminant on complete intersections of r hypersurfaces in $\mathbb{P}^n$ to that on hypersurfaces in a $\mathbb{P}^{r-1}$-bundle $T = \mathbb{P}(\mathcal{E}) = \operatorname{Proj} S^\bullet \mathcal{E}$ on $\mathbb{P}^n$ associated to a locally free $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{E} = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r)$.

We identify

$$\Gamma(T, \mathcal{O}_T(1)) = \Gamma(\mathbb{P}^n, \mathcal{E}) = \Gamma(\mathbb{P}^n, \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r)) = V.$$

Let $((S_{\alpha,1}, |\alpha| = d_1), \dots, (S_{\alpha,r}, |\alpha| = d_r))$ denote the basis $((X^\alpha, |\alpha| = d_1), \dots, (X^\alpha, |\alpha| = d_r))$ of $V = S^{d_1} E \oplus \cdots \oplus S^{d_r} E$. We consider the section

$$\begin{aligned} s &= \sum_{|\alpha|=d_1} C^{(1)}_\alpha S_{\alpha,1} + \cdots + \sum_{|\alpha|=d_r} C^{(r)}_\alpha S_{\alpha,r} \\ &\in V \otimes_{\mathbb{Z}} V^\vee = \Gamma(T \times \mathbb{P}^\vee, \mathcal{O}_T(1) \otimes \mathcal{O}_{\mathbb{P}^\vee}(1)). \end{aligned}$$

We define a closed subscheme Y of $T \times \mathbb{P}^\vee$ by the equation $s = 0$. Let $\psi : Y \subset T \times \mathbb{P}^\vee \to \mathbb{P}^\vee$ be the canonical map.

We put $N = \dim(V) - 1$ and $\mathbb{P}^N = \mathbb{P}(V) = \operatorname{Proj}(S^\bullet V)$. The projective space $\mathbb{P}^\vee = \mathbb{P}(V^\vee)$ is the dual of $\mathbb{P}^N$ parametrizing hyperplanes in $\mathbb{P}^N$.

Lemma 1.5. *The invertible sheaf $\mathcal{O}_T(1)$ is very ample relatively to* $\operatorname{Spec} \mathbb{Z}$*. More explicitly, the global sections $(S_{\alpha,1}, |\alpha| = d_1), \dots, (S_{\alpha,r}, |\alpha| = d_r)$ define a closed immersion $v : T \hookrightarrow \mathbb{P}^N = \mathbb{P}(V)$.*

Proof. For the sections $S_{\alpha,j}$, we define open sets $U_{\alpha,j} \subset T$ by $U_{\alpha,j} = \{x \in T \mid (S_{\alpha,j})_x \notin \mathfrak{m}_x \mathcal{O}_T(1)_x\}$.

Let $p : T \to \mathbb{P}^n$ denote the canonical map and $D_+(X_i) \subset \mathbb{P}^n$ $(0 \le i \le n)$ denote the fundamental open sets. Then we have

$$D_+(X_i) \cong \operatorname{Spec} \mathbb{Z}[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$$

where $x_k = \frac{X_k}{X_i}(0 \leq k \leq n, k \neq i)$. On the open subscheme $D_+(X_i)$, the section $X_i^{d_j} \in \Gamma(D_+(X_i), \mathcal{O}(d_j))$ gives the trivialization

$$\mathcal{O}(d_j)|_{D_+(X_i)} \cong \mathcal{O}_{D_+(X_i)}(1 \leq j \leq r).$$

Let T_j denote this generator $X_i^{d_j}$. Then we have an isomorphism

$$p^{-1}(D_+(X_i)) \cong D_+(X_i) \times \operatorname{Proj} \mathbb{Z}[T_1, \ldots, T_r] = D_+(X_i) \times \mathbb{P}_{\mathbb{Z}}^{r-1}.$$

For $0 \leq i \leq n$ and $1 \leq j \leq r$, we define multi-indices $\alpha(i,j) \in \mathbb{N}^{n+1}$ by $\alpha(i,j)_k = 0$ $(k \neq i)$ and $\alpha(i,j)_i = d_j$. Then we have

$$U_{\alpha(i,j),j} \cong D_+(X_i) \times D_+(T_j) \subset D_+(X_i) \times \mathbb{P}_{\mathbb{Z}}^{r-1}$$

for any fixed i. Thus the open sets $(U_{\alpha(i,j),j})_{0 \leq i \leq n, 1 \leq j \leq r}$ cover T.

We show that the global sections define a closed immersion $U_{\alpha(i,j),j} \to \mathbb{P}^N$ for each open set $U_{\alpha(i,j),j}$. We have an isomorphism

$$U_{\alpha(i,j),j} \cong \operatorname{Spec} \mathbb{Z}\left[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \frac{T_0}{T_j}, \ldots, \frac{T_r}{T_j}\right]$$

where $x_k = \frac{X_k}{X_i}$ as above. For each i and j, we define a ring homomorphism

$$\mathbb{Z}\left[(s_{\alpha,j'})_{1 \leq j' \leq r, \alpha \in \mathbb{N}^{n+1}, |\alpha| = d_{j'}, (\alpha, j') \neq (\alpha(i,j),j)}\right] \to \Gamma(U_{\alpha(i,j),j}, \mathcal{O}_T)$$

by the indeterminate $s_{\alpha,j'}$ mapping to the element $S_{\alpha,j'}/S_{\alpha(i,j),j}$. Then this morphism is surjective. In fact, by the isomorphism

$$\Gamma(U_{\alpha(i,j),j}, \mathcal{O}_T) \cong \mathbb{Z}\left[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \frac{T_0}{T_j}, \ldots, \frac{T_r}{T_j}\right],$$

the indeterminate $s_{\alpha(i,j'),j'}$ maps to $\frac{T_{j'}}{T_j}$. Further, if we define the multi-indices $\alpha(i,j,l) \in \mathbb{N}^{n+1}$ $(0 \leq l \leq n, l \neq i)$ by $\alpha(i,j,l)_k = 0$ $(k \neq i, l)$, $\alpha(i,j,l)_i = d_j - 1$, and $\alpha(i,j,k)_l = 1$, then the indeterminate $s_{\alpha(i,j,l),j}$ maps to x_l. □

Let k be an algebraically closed field. We consider T_k as a closed subscheme of $\mathbb{P}_k^N$ by the base change of the immersion v.

Now we recall the definition of the projective toric variety X_A introduced in [1]. We consider the finite set $A = \{Y_j X^\alpha\}_{1 \leq j \leq r, |\alpha| = d_j}$ of monomials in

$n+r+1$ variables $Y_1, \dots, Y_r, X_0, \dots, X_n$. Each monomial $Y_j X^\alpha$ in A defines the function

$$(k^\times)^{n+r+1} \to k^\times : (y_1, \dots, y_r, x_0, \dots, x_n) \mapsto y_j x^\alpha.$$

The variety $X_A \subset \mathbb{P}^N_k$ is defined to be the closure of the set

$$X^0_A := \{[y_i x^\alpha]_{1 \le j \le r, |\alpha| = d_j} \in \mathbb{P}^N_k : (y_1, \dots, y_r, x_0, \dots, x_n) \in (k^\times)^{n+r+1}\}.$$

Lemma 1.6. *The closed subscheme $T_k \subset \mathbb{P}^N_k$ is equal to the projective toric variety $X_A \subset \mathbb{P}^N_k$.*

Proof. The set X^0_A is an open dense subset of T_k by the definition of the embedding $v : T_k \hookrightarrow \mathbb{P}^N_k$. Since T_k is irreducible, we have $X_A = T_k$. □

Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^N_k}$ be the ideal sheaf defining T_k. Let $\mathcal{N} = (\mathcal{I}/\mathcal{I}^2)^\vee$ denote the normal sheaf and let $\mathbb{P}(\mathcal{N}) = \operatorname{Proj} S^\bullet \mathcal{N}$ denote the associated projective space bundle over T_k. By [2, Exposé XVII, 3.1, 5.1], the projective space bundle $\mathbb{P}(\mathcal{N})$ is canonically viewed as a closed subscheme in $\mathbb{P}^N_k \times \mathbb{P}^\vee_k$. We define $\varphi : \mathbb{P}(\mathcal{N}) \to \mathbb{P}^\vee_k$ by the composition $\mathbb{P}(\mathcal{N}) \subset \mathbb{P}^N_k \times \mathbb{P}^\vee_k \to \mathbb{P}^\vee_k$. We denote the reduced induced closed subscheme structure of the image of φ by $T^\vee_k$ and call it the *dual variety* of T_k (with respect to the immersion $v : T_k \hookrightarrow \mathbb{P}^N_k$).

Proposition 1.7. *We have an equality $D_k = T^\vee_k$ of the underlying sets of closed subschemes of $\mathbb{P}^\vee_k$.*

Proof. By [1, Proposition 3.1], we have $D_k = X^\vee_A = T^\vee_k$. Thus the assertion follows from Lemma 1.6. □

Thus the canonical morphism φ induces a morphism $\mathbb{P}(\mathcal{N}) \to D_{k,\mathrm{red}}$ to the maximum reduced subscheme of D_k. To compute the degree of the discriminant, we need the following.

Proposition 1.8. *Let k be an algebraically closed field. Assume that $n - r$ is even or $\operatorname{char} k \neq 2$. Then, the morphism $\mathbb{P}(\mathcal{N}) \to D_{k,\mathrm{red}}$ is birational.*

Proof. By Lemma 1.6 and [1, Lemme 4.5], for a general geometric point s in $D_{k,\mathrm{red}}$ the fiber Y_s of ψ has an ordinary quadratic singularity. Hence, by [2, Exposé XVII, Proposition 3.3], the morphism $\varphi : \mathbb{P}(\mathcal{N}) \to \mathbb{P}^\vee_k$ is generically unramified. Thus the assertion follows from [2, Exposé XVII, Proposition 3.5]. □

To compute the degree of the divisor D, we define a homogeneous polynomial $P(H,K) \in \mathbb{Z}[H,K]$ by

$$P(H,K) = (d_1 H - K)\cdots(d_r H - K).$$

We put $\bar{d} = d_1 \cdots d_r, \check{d}_i = \bar{d}/d_i$ for $i = 1,\ldots,r$ and $\check{d} = \check{d}_1 + \cdots + \check{d}_r$.

Lemma 1.9. *The degree of the discriminant is the coefficient of $h^n k^{r-1}$ of the element*

$$(1)\quad \bar{d}\cdot h^r \sum_{i=0}^{n-r} \binom{n+1}{i} (n-i) k^{n-1-i}(-h)^i + \check{d}\cdot h^{r-1} \sum_{i=0}^{n-r+1} \binom{n+1}{i} k^{n-i}(-h)^i$$

in the ring $\mathbb{Z}[h,k]/(h^{n+1}, P(h,k))$ with respect to the basis $(h^i k^j; i = 0,\ldots,n, j = 0,\ldots,r-1)$.

Proof. Let $k = \overline{\mathbb{Q}}$. The cycle class of $X_k \subset \mathbb{P}^n_k \times \mathbb{P}^\vee_k$ is given by

$$[X_k] = c_r(\mathcal{E}(1_{\mathbb{P}^\vee_k})) \in CH^r(\mathbb{P}^n_k \times \mathbb{P}^\vee_k)$$

and that of $\mathbb{P}(\mathcal{N}) \subset (T_k \times \mathbb{P}^\vee_k)_{X_k}$ is given by

$$[\mathbb{P}(\mathcal{N})] = c_n(\Omega^1_{\mathbb{P}^n_k/k}(1_{T_k}, 1_{\mathbb{P}^\vee_k})) \in CH^n((T_k \times \mathbb{P}^\vee_k)_{X_k}).$$

Hence, we have $[\mathbb{P}(\mathcal{N})] = c_r(\mathcal{E}(1_{\mathbb{P}^\vee_k})) \cap c_n(\Omega^1_{\mathbb{P}^n_k/k}(1_{T_k}, 1_{\mathbb{P}^\vee_k})) \in CH^{r+n}(T_k \times \mathbb{P}^\vee_k)$. Since the morphism $\mathbb{P}(\mathcal{N}) \to D_k$ is birational by Proposition 1.8, the class $[D_k] \in CH^1(\mathbb{P}^\vee_k)$ is the push-forward of $[\mathbb{P}(\mathcal{N})]$. Hence the degree of D_k is equal to the degree of the dimension 0-part

$$\{c(\mathcal{E}) \cap c(\Omega^1_{\mathbb{P}^n_k/k}(1_{T_k}))\}_{\dim 0} \in CH_0(T_k).$$

Let $h = [c_1(\mathcal{O}_{\mathbb{P}^n}(1))]$ and $k = [c_1(\mathcal{O}_T(1))]$ denote the classes of hyperplanes. Then, the Chow ring $CH^\bullet(T)$ is $\mathbb{Z}[h,k]/(h^{n+1}, P(h,k))$. For $i = 1,\ldots,r$, we define a homogeneous polynomial $P_i(H,K)$ of degree $i-1$ by requiring that $P(H,K) - (-K)^{r-i-1} P_i(H,K)$ is of degree $\leq r-i$ in K. Since

$$\begin{aligned} c(\mathcal{E}) \cdot c(\Omega_{\mathbb{P}^n}(1_T)) &= (1+d_1 h)\cdots(1+d_r h)\cdot(1-h+k)^{n+1}(1+k)^{-1} \\ &= \sum_{i=1}^{r} P_i(h,k)\cdot(1-h+k)^{n+1}, \end{aligned}$$

we obtain

$$\{c(\mathcal{E}) \cap c(\Omega_{\mathbb{P}^n}(1_T))\}_{\mathrm{dim}0} = (n+1)P_r(h,k)(k-h)^n + P_{r-1}(h,k)\cdot(k-h)^{n+1}.$$

Since

$$\begin{aligned} K\cdot P_r(H,K) &= \bar{d}\cdot H^r - P(H,K)\\ K^2\cdot P_{r-1}(H,K) &= \check{d}\cdot H^{r-1}K - \bar{k}\cdot H^r + P(H,K), \end{aligned}$$

the right hand side is equal to

$$\begin{aligned} &(n+1)\left(\bar{d}\cdot h^r\frac{(k-h)^n-(-h)^n}{k} + P_r(h,k)(-h)^n\right)\\ &+(\check{d}\cdot h^{r-1}k - \bar{d}\cdot h^r)\cdot\frac{(k-h)^{n+1}-\big((n+1)k(-h)^n+(-h)^{n+1}\big)}{k^2}\\ &+P_{r-1}(h,k)\cdot\big((n+1)k(-h)^n+(-h)^{n+1}\big)\\ =\;&\bar{d}\cdot h^r\Bigg((n+1)\frac{(k-h)^n-(-h)^n}{k}\\ &\qquad-\frac{(k-h)^{n+1}-\big((n+2)k(-h)^n+(-h)^{n+1}\big)}{k^2}\Bigg)\\ &+\check{d}\cdot h^{r-1}\frac{(k-h)^{n+1}-\big((n+1)k(-h)^n+(-h)^{n+1}\big)}{k}\\ &+(n+1)\big(P_r(h,k)+P_{r-1}(h,k)k\big)(-h)^n + P_{r-1}(h,k)(-h)^{n+1}. \end{aligned}$$

On the right hand side, the content of the big parantheses in the first line is

$$\begin{aligned} &(n+1)\sum_{i=0}^{n-1}\binom{n}{i}k^{n-1-i}(-h)^i - \sum_{i=0}^{n-1}\binom{n+1}{i}k^{n-1-i}(-h)^i\\ &=\sum_{i=0}^{n-1}\binom{n+1}{i}(n-i)k^{n-1-i}(-h)^i. \end{aligned}$$

Since $P_r(h,k)+P_{r-1}(h,k)k = \check{d}\cdot h^{r-1}$ and $h^{n+1}=0$, the sum of the remaining two lines is

$$\check{d}\cdot h^{r-1}\cdot\sum_{i=0}^{n}\binom{n+1}{i}k^{n-i}(-h)^i.$$

Since the dimension 0-part is the component generated by $h^n\cdot k^{r-1}$ of degree 1 with respect to the decomposition by the basis $(h^ik^j; i=0,\dots,n, j=1,\dots,r)$, the assertion follows. □

Corollary 1.10. *If $d_1 = \cdots = d_r = d$, the degree of D is*

$$(n-r+2)\binom{n+1}{r-1} d^{r-1}(d-1)^{n-r+1}. \tag{2}$$

If $r \geq 2$ and if $d_1 - c = d_2 = \cdots = d_r = d$, it is the sum of (2) and

$$\begin{aligned} & d^{r-2}\sum_{j=r}^{n+1}\binom{n+1}{j} c^{j-r+1}(d-1)^{n+1-j} \\ & + (n+1)d^{r-1}\sum_{j=r}^{n}\binom{n}{j} c^{j-r+1}(d-1)^{n-j} \\ & + (r-2)d^{r-1}\sum_{j=r}^{n} c^{j-r+1}\sum_{p=0}^{n-j}\binom{n-p}{j}(-1)^p(d-1)^{n-p-j}. \end{aligned} \tag{3}$$

If $d_1 = \cdots = d_r = d > 1$, the degree (2) of D is strictly positive.

Proof. We put $d_1 = d + c$. Then, we have an isomorphism

$$\mathbb{Z}[h,k]/\big(h^{n+1}, P(h,k)\big) \to \mathbb{Z}[h,l]/\big(h^{n+1}, l^{r-1}(l-ch)\big)$$

sending k to $l + dh$. Hence, the degree of D is the coefficient of $h^n l^{r-1}$ of the polynomial obtained by substituting $k = l + dh$ in (1). Since $\bar{d} = d^{r-1}(d+c)$ and $\check{d} = (r-1)d^{r-1} + (d+c)d^{r-2}$, after substituting $k = l + dh$ and $l^r = cl^{r-1}h$, we see that the coefficient of $h^n l^{r-1}$ is:

$$\begin{aligned} & d^{r-1}(d+c)\sum_{i=0}^{n-r}\binom{n+1}{i} \\ & \quad \times (n-i)\sum_{j=r-1}^{n-i-1}\binom{n-i-1}{j} c^{j-r-1}d^{n-i-j-1}(-1)^i \\ & + \big((r-1)d^{r-1} + (d+c)d^{r-2}\big) \\ & \quad \times \sum_{i=0}^{n-r+1}\binom{n+1}{i}\sum_{j=r-1}^{n-i}\binom{n-i}{j} c^{j-r+1}d^{n-i-j}(-1)^i. \end{aligned} \tag{4}$$

Since

$$
\begin{aligned}
&(d+c)\sum_{j=r-1}^{n-i-1}\binom{n-i-1}{j}c^{j-r+1}d^{n-i-j-1}\\
&=\binom{n-i-1}{r-1}d^{n+1-i-r}+\sum_{j=r}^{n-i}\binom{n-i}{j}c^{j-r+1}d^{n-i-j}
\end{aligned}
$$

and similarly for $(d+c)\sum_{j=r-1}^{n-i}\binom{n-i}{j}c^{j-r+1}d^{n-i-j}$, (4) is equal to

$$
\begin{aligned}
(5)\quad & d^r\cdot\sum_{i=0}^{n-r}\binom{n+1}{i}(n-i)\binom{n-i-1}{r-1}d^{n-r-i}(-1)^i\\
&+rd^{r-1}\cdot\sum_{i=0}^{n-r+1}\binom{n+1}{i}\binom{n-i}{r-1}d^{n+1-m-i}(-1)^i\\
&+d^{r-1}\sum_{j=r}^{n}c^{j-r+1}\sum_{i=0}^{n-j}\binom{n+1}{i}(n+r+1-i)\binom{n-i}{j}d^{n-i-j}(-1)^i\\
&+d^{r-2}\sum_{j=r}^{n+1}c^{j-r+1}\sum_{i=0}^{n+1-j}\binom{n+1}{i}\binom{n+1-i}{j}d^{n+1-i-j}(-1)^i.
\end{aligned}
$$

The sum of the first two lines in (5) is

$$
\begin{aligned}
&rd^{r-1}\sum_{i=0}^{n-r-1}\binom{n+1}{i}\binom{n+1-i}{r}d^{n-r+1-i}(-1)^i\\
&=rd^{r-1}\binom{n+1}{r}\sum_{i=0}^{n-r+1}\binom{n-r+1}{i}d^{n-r+1-i}(-1)^i\\
&=(n-r+2)\binom{n+1}{r-1}d^{r-1}(d-1)^{n-r+1}.
\end{aligned}
$$

Similarly, the last line in (5) is equal to

$$
d^{r-2}\sum_{j=r}^{n+1}\binom{n+1}{j}c^{j-r+1}(d-1)^{n+1-j}.
$$

Since

$$\begin{aligned}
&\binom{n+1}{i}(n+r+1-i)\binom{n-i}{j}\\
&=(j+1)\binom{n+1}{i}\binom{n+1-i}{j+1}+(r-2)\binom{n+1}{i}\binom{n-i}{j}\\
&=(n+1)\binom{n}{j}\binom{n-j}{i}+(r-2)\sum_{p=0}^{i}\binom{n-p}{i-p}\binom{n-j}{j},
\end{aligned}$$

similarly the third line in (5) is equal to

$$\begin{aligned}
&(n+1)d^{r-1}\sum_{j=r}^{n}\binom{n}{j}c^{j-r+1}\sum_{i=0}^{n-j}\binom{n-j}{i}d^{n-i-j}(-1)^i\\
&\quad+(r-2)d^{r-1}\sum_{j=r}^{n}c^{j-r+1}\sum_{p=0}^{n-j}\binom{n-p}{j}\sum_{i-p=0}^{n-j-p}\binom{n-j-p}{i-p}d^{n-i-j}(-1)^i\\
&=(n+1)d^{r-1}\sum_{j=r}^{n}\binom{n}{j}c^{j-r+1}(d-1)^{n-j}\\
&\quad+(r-2)d^{r-1}\sum_{j=r}^{n}c^{j-r+1}\sum_{p=0}^{n-j}\binom{n-p}{j}(-1)^p(d-1)^{n-j-p}.
\end{aligned}$$

□

Corollary 1.11. *If $n-r$ is even, the degree of D is even.*

Proof. If there exists at least 2 indices such that d_i is even, the integers $\bar{d}=d_1\cdots d_r$ and $\check{d}=\check{d}_1+\cdots+\check{d}_r$ are even.

We consider the case where there exists at most 1 index such that d_i is even. By the same argument as in the proof of Corollary 1.10, the congruences on d_i implies a congruence for the degree of D. Hence, if every d_i is congruent to 1, then the degree is even if $n-r$ is even by (2).

Assume there exists exactly 1 index such that d_i is even. We may assume that $i=1$, $d\equiv c\equiv 1 \pmod 2$ in (1.10). Then, (3) is congruent to $1+(n+1)+(r-2)(n-r+1) \pmod 2$, so is even if $n-r$ is even. □

1.12. The discriminant of a hypersurface

Let $r=1$ and fix a positive integers n and $d=d_1$. Let $F=F_1=\sum_{|\alpha|=d}C_\alpha X^\alpha$ denote the universal polynomial of degree d. We consider the resultant

$\mathrm{res}\left(\frac{\partial F_1}{\partial X_0}, \ldots, \frac{\partial F_1}{\partial X_n}\right)$ of partial derivatives of F. It is a homogeneous polynomial of degree $m = (n+1)(d-1)^n$ in $(C_\alpha)_{|\alpha|=d}$ with integral coefficients ([5, Chap.13, Section 1.A]). If we put

$$a(n,d) = \frac{(d-1)^{n+1} - (-1)^{n+1}}{d},$$

the greatest common divisor of the coefficients is $d^{a(n,d)}$ by [5, Chap.13.1.D, Proposition 1.7].

Definition 1.13. We call

$$\mathrm{disc}_d(F) = \frac{1}{d^{a(n,d)}} \,\mathrm{res}\left(\frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_n}\right)$$

the *divided discriminant* of F.

The relation between the discriminant of a complete intersection and the divided discriminant of a hypersurface is as follows.

Proposition 1.14. *If $r = 1$ and $d_1 = d$, then the discriminant* $\mathrm{disc}(F)$ *defined in Subsection 1.1 equals to* $\mathrm{disc}_d(F)$ *up to sign.*

Proof. The assertion follows from Proposition 1.3 and the smoothness criterion [8, Proposition 2.3] of the divided discriminant of a hypersurface. □

2. Determinant

Let S be a normal integral scheme over $\mathbb{Z}$ and $f : X \to S$ be a proper smooth morphism of relative even dimension n. For a prime number ℓ invertible in the function field of S, the cup-product defines a non-degenerate symmetric bilinear form on the smooth $\mathbb{Q}_\ell$-sheaf $R^n f_* \mathbb{Q}_\ell(\frac{n}{2})$ on $S[\frac{1}{\ell}]$. Hence the determinant defines a character $\pi_1(S[\frac{1}{\ell}])^{ab} \to \{\pm 1\} \subset \mathbb{Q}_\ell^\times$ of the fundamental group, which we denote by $[\det H^n_\ell(X)]$.

Lemma 2.1 ([8, Lemma 3.2]). *There exists a unique character*

$$[\det H^n(X)] : \pi_1(S)^{ab} \to \{\pm 1\}$$

such that, for every prime number ℓ invertible in the function field of S, the composition with the map $\pi_1(S[\frac{1}{\ell}])^{ab} \to \pi_1(S)^{ab}$ induced by the open immersion $S[\frac{1}{\ell}] \to S$ gives $[\det H^n_\ell(X)]$.

Corollary 2.2 ([8, Lemma 3.3]). *Let X be a proper smooth scheme of even dimension n over a field k. Then, for a prime number ℓ invertible in k, the character* $\det H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$ *of the absolute Galois group Γ_k is independent of ℓ.*

By applying Lemma 2.1 to the universal family of intersections of r hypersurfaces $\pi_U : X_U \to U$, we define $[\det H^{n-r}(X)] \in H^1(U, \mathbb{Z}/2\mathbb{Z})$. Let now k be a field and let $f_j \in S^{d_j}E \otimes k$ $(1 \le j \le r)$ be homogeneous polynomials of degrees $d_1, \dots, d_r$ which define a smooth complete intersection Y in $\mathbb{P}^n_k$. Then, the pull-back in $H^1(k, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\Gamma_k^{ab}, \mathbb{Z}/2\mathbb{Z})$ of $[\det H^{n-r}(X)]$ by the k-valued point of U corresponding to $f_1, \dots, f_r$ is given by the determinant of the orthogonal representation $H^{n-r}(Y_{\bar{k}}, \mathbb{Q}_\ell(\frac{n-r}{2}))$ for a prime number ℓ invertible in k.

Theorem 2.3. *Let $n \ge 1$ and $d_1, \dots, d_r \ge 1$ be integers. Assume that $n - r$ is even and that $d_j \ge 2$ for an index j $(1 \le j \le r)$.*

1. Let $m = \deg(\mathrm{disc}(F_1, \dots, F_r))$. Then there exists unique choice of sign of the polynomial $\mathrm{disc}(F_1, \dots, F_r)$ such that, there exist homogeneous polynomials $A \in S^{\frac{m}{2}}(V^\vee)$ and $B \in S^m(V^\vee)$ such that $\mathrm{disc}(F_1, \dots, F_r) = A^2 + 4B$.

We denote this polynomial by $\mathrm{disc}_\sigma(F_1, \dots, F_r)$.

2. The square roots of $\mathrm{disc}_\sigma(F_1, \dots, F_r)$ define a $\mathbb{Z}/2\mathbb{Z}$-torsor on $U_{\mathbb{Z}[\frac{1}{2}]}$. We denote by $[\mathrm{disc}_\sigma(F_1, \dots, F_r)]$ the class of this torsor in $H^1(U_{\mathbb{Z}[\frac{1}{2}]}, \mathbb{Z}/2\mathbb{Z})$. Then, we have

$$[\det H^{n-r}(X)] = [\mathrm{disc}_\sigma(F_1, \dots, F_r)]$$

in $H^1(U_{\mathbb{Z}[\frac{1}{2}]}, \mathbb{Z}/2\mathbb{Z})$.

Thus by a standard specialization argument, Theorem 2.3 implies Theorem 0.1.

Proof. We first show that the assertion 2 is true up to a sign by the same argument as in [8, Theorem 3.5], and next we prove the sign part and the assertion 1 by the argument in [8, Theorem 4.2].

The Kummer sequence gives an exact sequence

$$(6) \quad 0 \to \Gamma(U_{\frac{1}{2}}, \mathcal{O})^\times / (\Gamma(U_{\frac{1}{2}}, \mathcal{O})^\times)^2 \xrightarrow{\partial} H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Pic}(U_{\frac{1}{2}})[2] \to 0,$$

where we have written $U_{\frac{1}{2}}$ instead of $U_{\mathbb{Z}[\frac{1}{2}]}$, and $\mathrm{Pic}(U_{\frac{1}{2}})[2]$ denotes the subgroup of $\mathrm{Pic}(U_{\frac{1}{2}})$ killed by 2.

We compute $\Gamma(U_{\frac{1}{2}},\mathcal{O})^\times$ and $\mathrm{Pic}(U_{\frac{1}{2}})$. We have an exact sequence

$$(7)\qquad 0\to\Gamma(\mathbb{P}^\vee_{\frac{1}{2}},\mathcal{O})^\times\to\Gamma(U_{\frac{1}{2}},\mathcal{O})^\times\to\mathbb{Z}\to\mathrm{Pic}(\mathbb{P}^\vee_{\frac{1}{2}})\to\mathrm{Pic}(U_{\frac{1}{2}})\to 0.$$

The Picard group $\mathrm{Pic}(\mathbb{P}^\vee_{\frac{1}{2}})$ is canonically identified with $\mathbb{Z}$ by the generator $[\mathcal{O}(1)]$. Then, the map $\mathbb{Z}\to\mathrm{Pic}(\mathbb{P}^\vee_{\frac{1}{2}})$ is identified with the multiplication by the degree $m=\deg(\mathrm{disc}(F_1,\dots,F_r))>0$ since it sends 1 to the class $[\mathcal{O}(m)]$. Thus, we have

$$\Gamma(U_{\frac{1}{2}},\mathcal{O})^\times=\Gamma(\mathbb{P}^\vee_{\frac{1}{2}},\mathcal{O})^\times=\mathbb{Z}\left[\frac{1}{2}\right]^\times=\langle -1,2\rangle.$$

It also follows from (7) that $\mathrm{Pic}(U_{\frac{1}{2}})\cong\mathbb{Z}/m\mathbb{Z}$. Since m is even by Corollary 1.11, this shows $\mathrm{Pic}(U_{\frac{1}{2}})[2]\cong\mathbb{Z}/2\mathbb{Z}$. Thus, by (6), we have

$$H^1(U_{\frac{1}{2}},\mathbb{Z}/2\mathbb{Z})\cong\langle -1,2\rangle/\langle -1,2\rangle^2\oplus\mathrm{Pic}(U_{\frac{1}{2}})[2]\cong(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}.$$

Recall that D is an irreducible divisor in $\mathbb{P}^\vee$. Let $\bar{\xi}$ be a geometric generic point of D and let $I_{\bar{\xi}}$ denote the absolute Galois group of the fraction field of the strict henselization $\mathcal{O}_{\mathbb{P}^\vee,\bar{\xi}}$. Since the profinite group $I_{\bar{\xi}}$ is isomorphic to $\widehat{\mathbb{Z}}$, we have $\mathrm{Hom}(I_{\bar{\xi}},\mathbb{Z}/2\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z}$.

We recall that $\mathrm{disc}(F_1,\dots,F_r)$ is defined up to sign as the defining polynomial of D. Then the square roots of $\mathrm{disc}(F_1,\dots,F_r)$ defines a class of $\mathbb{Z}/2\mathbb{Z}$-torsor on $U_{\frac{1}{2}}$ up to sign. We denote by $[\pm\,\mathrm{disc}]$ this class of torsor in $H^1(U_{\frac{1}{2}},\mathbb{Z}/2\mathbb{Z})/\langle -1\rangle$.

Since we have $\sqrt{-1}\in\overline{\mathbb{Q}}\subset\mathcal{O}_{\mathbb{P}^\vee,\bar{\xi}}$, the restriction

$$H^1(U_{\frac{1}{2}},\mathbb{Z}/2\mathbb{Z})\to\mathrm{Hom}(I_{\bar{\xi}},\mathbb{Z}/2\mathbb{Z})$$

induces a map $H^1(U_{\frac{1}{2}},\mathbb{Z}/2\mathbb{Z})/\langle -1\rangle\to\mathrm{Hom}(I_{\bar{\xi}},\mathbb{Z}/2\mathbb{Z})$. We show that the images of $[\det H^{n-r}(X)]$ and $[\pm\,\mathrm{disc}]$ under this map are both the unique non-trivial element. For the latter $[\pm\,\mathrm{disc}]$, this follows from that $\mathrm{disc}(F_1,\dots,F_r)$ is the defining polynomial of the divisor D.

Let $\bar{\eta}$ denote the geometric generic point of $\mathrm{Spec}\,\mathcal{O}_{\mathbb{P}^\vee,\bar{\xi}}$. We show that the character $\det H^{n-r}(X_{\bar{\eta}},\mathbb{Q}_\ell)$ of $I_{\bar{\xi}}$ is the unique non-trivial character of order 2. By [1, Lemma 4.3.(ii)], the geometric fiber $X_{\bar{\xi}}$ has a unique singular point which is an ordinary quadratic singularity in $X_{\bar{\xi}}$. Hence, by the Picard-Lefschetz formula [2, Exposé XV, Théorème 3.4 (ii)], we have an exact sequence of ℓ-adic representation of the inertia group $I_{\bar{\xi}}$

$$(8) \qquad \begin{aligned} 0 &\to H^{n-r}(X_{\bar{\xi}}, \mathbb{Q}_\ell) \to H^{n-r}(X_{\bar{\eta}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell\left(\frac{n-r}{2}\right) \\ &\to H^{n-r+1}(X_{\bar{\xi}}, \mathbb{Q}_\ell) \to H^{n-r+1}(X_{\bar{\eta}}, \mathbb{Q}_\ell) \to 0. \end{aligned}$$

Further, since X is regular, the base change $X_{\mathcal{O}_{\mathbb{P}^\vee, \bar{\xi}}}$ to the strict henselization is also regular. Hence by [2, Exposé XV, Théorème 3.4 (iii)], the inertia group $I_{\bar{\xi}}$ acts on $\mathbb{Q}_\ell\left(\frac{n-r}{2}\right)$ via the unique non-trivial character $I_{\bar{\xi}} \to \{\pm 1\}$. Since $I_{\bar{\xi}}$ acts trivially on $H^{n-r+1}(X_{\bar{\xi}}, \mathbb{Q}_\ell)$ and on $H^{n-r}(X_{\bar{\xi}}, \mathbb{Q}_\ell)$, the map $\mathbb{Q}_\ell(\frac{n-r}{2}) \to H^{n-r+1}(X_{\bar{\xi}}, \mathbb{Q}_\ell)$ in (8) is the zero-map and hence the character $\det H^{n-r}(X_{\bar{\eta}}, \mathbb{Q}_\ell)$ of $I_{\bar{\xi}}$ is non-trivial.

The composition map

$$\Gamma(U_{\frac{1}{2}}, \mathcal{O})^\times / (\Gamma(U_{\frac{1}{2}}, \mathcal{O}^\times)^2 \to H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$$

is 0 since we have $\Gamma(U_{\frac{1}{2}}, \mathcal{O})^\times = \langle -1, 2\rangle$ and the strict henselization $\mathcal{O}_{\mathbb{P}^\vee, \bar{\xi}}$ contains $\overline{\mathbb{Q}}$ as a subfield. By (7) we thus have a map $\mathrm{Pic}(U_{\frac{1}{2}})[2] \to \mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$.

Since the images of $[\det H^{n-r}(X)]$ and $[\pm \mathrm{disc}]$ in $\mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$ are non-trivial, the map $\mathrm{Pic}(U_{\frac{1}{2}})[2] \to \mathrm{Hom}(I_{\bar{\xi}}, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism of groups of order 2. Further by (7), the difference $[\pm \mathrm{disc}] - [\det H^{n-r}(X)]$ is in the image of the map

$$\Gamma(U_{\frac{1}{2}}, \mathcal{O})^\times = \mathbb{Z}\left[\frac{1}{2}\right]^\times \to H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z})/\langle -1\rangle.$$

Therefore, $[\det H^{n-r}(X)]$ equals either $[\pm \mathrm{disc}]$ or $[\pm 2\,\mathrm{disc}]$. We show that the latter case is not possible.

Let ν be the generic point of the fiber $\mathbb{P}^\vee_{\mathbb{F}_2}$, and let K be the fraction field of the completion of the local ring $\mathcal{O}_{\mathbb{P}^\vee, \nu}$. Then, the character $[\det H^{n-r}(X)]$ induces an unramified character of the absolute Galois group Γ_K. On the other hand, the class $[\pm 2\,\mathrm{disc}]$ corresponds to a totally ramified quadratic extension of K. Hence we obtain $[\det H^{n-r}(X)] = [\pm \mathrm{disc}]$.

Hence there exists unique choice of the sign of the polynomial $\mathrm{disc}(F_1, \ldots, F_r)$ such that the $\mathbb{Z}/2\mathbb{Z}$-torsor defined by the square roots of the polynomial on $U_{\frac{1}{2}}$ is isomorphic to $[\det H^{n-r}(X)]$. We denote this polynomial by $\mathrm{disc}_\sigma(F_1, \ldots, F_r)$.

It remains to show that there exists a homogeneous polynomial A of degree $\frac{m}{2}$ such that $\mathrm{disc}_\sigma(F_1, \ldots, F_r) \equiv A^2 \pmod 4$. We use the following fact.

Lemma 2.4. *[8, Lemma 4.1] Let K be a complete discrete valuation field such that 2 is a uniformizer. Let $u \in \mathcal{O}_K^\times$ be a unit which is not a square and let L denote the quadratic extension $K(\sqrt{u})$.*

1. The extension L is unramified over K if and only if there exists a unit $v \in \mathcal{O}_K^\times$ such that $u \equiv v^2 \pmod 4$.

2. Assume that the extension L is unramified over K. Then, for every unit v satisfying $u \equiv v^2 \pmod 2$, we have $u \equiv v^2 \pmod 4$. Further, the corresponding residue field extension is given by the Artin-Schreier equation $t^2 + t = w$, where w is the image of $\frac{1}{4}(uv^{-2} - 1)$ in the residue field.

Let ν be the generic point of the fiber $\mathbb{P}^\vee_{\mathbb{F}_2}$, and let K be the fraction field of the completion of the local ring $\mathcal{O}_{\mathbb{P}^\vee,\nu}$. The residue field $F = \kappa(\nu)$ is the function field of $\mathbb{P}^\vee_{\mathbb{F}_2}$. Take a global section $A_1 \in \Gamma(\mathbb{P}^\vee, \mathcal{O}(\frac{m}{2}))$ not divisible by 2. Then the germ of A_1 generates the stalk $(\mathcal{O}(\frac{m}{2}))_\nu$. On the other hand, since the polynomial $\mathrm{disc}_\sigma(F_1, \ldots, F_r)$ is not divisible by 2, its germ generates the stalk $(\mathcal{O}(m))_\nu$. Hence the ratio $\mathrm{disc}_\sigma(F_1, \ldots, F_r)/A_1^2$ is a unit in $\mathcal{O}_{\mathbb{P}^\vee,\nu}$.

By Theorem 2.3.2 we have

$$[\det H^{n-r}(X)] = [\mathrm{disc}_\sigma(F_1, \ldots, F_r)]$$

in $H^1(U_{\mathbb{Z}[\frac{1}{2}]}, \mathbb{Z}/2\mathbb{Z})$. Since the class $[\det H^{n-r}(X)]$ is the restriction of a class of $H^1(U, \mathbb{Z}/2\mathbb{Z})$, the extension of K generated by the square root of $\mathrm{disc}_\sigma(F_1, \ldots, F_r)/A_1^2 \in K^\times$ is an unramified extension. Hence by Lemma 2.4.1, there exists a unit $v \in \mathcal{O}_K^\times$ such that $\mathrm{disc}_\sigma(F_1, \ldots, F_r) \equiv v^2 \cdot A_1^2 \pmod 4$.

We consider the germ $\bar{A} = v \cdot A_1 \pmod 2$ of the stalk of $\mathcal{O}_{\mathbb{P}^\vee_{\mathbb{F}_2}}(\frac{m}{2})$ at the generic point. Since its square is a germ of polynomial, the germ $\bar{A}$ has the same property and it defines a global section $\Gamma(\mathbb{P}^\vee_{\mathbb{F}_2}, \mathcal{O}(\frac{m}{2}))$. Let us choose a lifting $A \in \Gamma(\mathbb{P}^\vee, \mathcal{O}(\frac{m}{2}))$ of this section. Since we have $\mathrm{disc}_\sigma(F_1, \ldots, F_r)/A^2 \equiv 1 \pmod 2$, we get $\mathrm{disc}_\sigma(F_1, \ldots, F_r)/A^2 \equiv 1 \pmod 4$ by Lemma 2.4.2. Thus, $\mathrm{disc}_\sigma(F_1, \ldots, F_r) - A^2$ is divisible by 4 at ξ and hence divisible on $\mathbb{P}^\vee$. □

2.5. The determinant in characteristic 2

We denote $[B \cdot A^{-2}]$ by the class in $H^1(U_{\mathbb{F}_2}, \mathbb{Z}/2\mathbb{Z})$ defined by $t^2 + t = B \cdot A^{-2}$, and we denote again $[\det H^{n-r}(X)] \in H^1(U_{\mathbb{F}_2}, \mathbb{Z}/2\mathbb{Z})$ by the class of the pull-back of $[\det H^{n-r}(X)] \in H^1(U, \mathbb{Z}/2\mathbb{Z})$.

Theorem 2.6. *Let $n, d \geq 2$ be even numbers. Then we have*

$$[B \cdot A^{-2}] = [\det H^{n-r}(X)]$$

in $H^1(U_{\mathbb{F}_2}, \mathbb{Z}/2\mathbb{Z})$.

Proof. By Theorem 2.3, the pull-back of $[\det H^{n-r}(X)]$ in $H^1(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z})$ is defined by the square roots of

$$\mathrm{disc}_\sigma(F_1, \ldots, F_r) \in \Gamma(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z})^\times / (\Gamma(U_{\frac{1}{2}}, \mathbb{Z}/2\mathbb{Z})^\times)^2.$$

Since the polynomial $\mathrm{disc}_\sigma(F_1, \ldots, F_r)$ is not divisible by 2, the polynomial A is also not divisible by 2.

Let F be the function field of $\mathbb{P}^\vee_{\mathbb{F}_2}$ as in the proof of Theorem 2.3. Then the restriction map $H^1(U_{\mathbb{F}_2}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\Gamma_F^{ab}, \mathbb{Z}/2\mathbb{Z})$ is injective. By Theorem 2.3 and Lemma 2.4, the classes $[B \cdot A^{-2}]$ and $[\det H^{n-r}(X)]$ maps to the same element in $\mathrm{Hom}(\Gamma_F^{ab}, \mathbb{Z}/2\mathbb{Z})$, and hence we have $[B \cdot A^{-2}] = [\det H^{n-r}(X)]$ in $H^1(U_{\mathbb{F}_2}, \mathbb{Z}/2\mathbb{Z})$. □

3. The discriminant of the complete intersection of two quadrics

In this section, we give an explicit presentation of the discriminant of the complete intersection of two quadrics, by using the discriminant of a quadric and that of a binary form.

Let F denote the universal homogeneous polynomial of degree $d \geq 2$. Recall that the divided discriminant $\mathrm{disc}_d(F)$ of a hypersurface is defined in Definition 1.13.

Proposition 3.1. *Let $n \geq 1$ and $d \geq 2$ be integers. We assume that n is odd and define the sign $\epsilon(n, d) = \pm 1$ by*

$$\epsilon(n,d) = \begin{cases} (-1)^{\frac{d-1}{2}} & \text{if } d \text{ is odd} \\ (-1)^{\frac{d}{2}\frac{n+1}{2}} & \text{if } d \text{ is even.} \end{cases}$$

Then, we have

$$\mathrm{disc}_\sigma(F) = \epsilon(n,d) \cdot \mathrm{disc}_d(F).$$

Proof. By Proposition 1.14, the equality is true up to a sign. For the sign, the assertion follows from Theorem 2.3 and [8, Theorem 4.2]. □

Example 3.2 (Quadrics). Let $r = 1$ and $d = 2$. Let

$$F = \sum_{0 \leq i \leq j \leq n} C_{ij} X_i X_j \quad \text{and} \quad X = (X_0, \ldots, X_n).$$

Let $A \in M_{n+1}(S^\bullet((S^2E)^\vee))$ be the symmetric matrix such that $XA^tX = 2F$. Then the resultant of the partial derivatives is

$$\operatorname{res}\left(\frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_n}\right) = \det A.$$

We have $a(n, 2) = (1 - (-1)^{n+2})/2$. Thus we obtain

$$\operatorname{disc}_d(F) = \begin{cases} 2^{-1} \det A & \text{if } n-1 \text{ is odd} \\ \det A & \text{if } n-1 \text{ is even,} \end{cases} \tag{9}$$

$$\deg(\operatorname{disc}_d(F)) = n + 1. \tag{10}$$

Example 3.3 (Binary forms). Let $n = 1$ and $r = 1$. Let $F = C_0X_0^d + C_1X_0^{d-1}X_1 + \cdots + C_dX_1^d$ be the universal binary polynomial of degree $d \geq 2$. The divided discriminant $\operatorname{disc}_d(F)$ is a homogeneous polynomial in (C_i) of degree $m = 2d - 2$ and the sign $\epsilon(1, d)$ is $(-1)^{d(d-1)/2}$. It is well known that the discriminant $\operatorname{disc}_d(F)$ is explicitly presented by the determinant of the Sylvester matrix ([5, Ch.12,(1.30)]).

If the binary form F is decomposed as $F = \prod_{i=1}^d (u_iT_0 - v_iT_1)$, we have ([8, (5.1)])

$$\operatorname{disc}_d(F) = \prod_{i \neq j} (u_iv_j - u_jv_i). \tag{11}$$

Further we have $a(1, d) = d - 2$ and $\operatorname{disc}_d(F) = d^{-(d-2)} \operatorname{res}\left(\frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}\right)$.

We will use following properties of the resultant of two binary forms to calculate the discriminant of a binary form. Let l, $m \geq 1$ be integers and

$$\begin{aligned} G(t_0, t_1) &= a_0t_0^l + a_1t_0^{l-1}t_1 + \cdots + a_lt_1^l, \\ H(t_0, t_1) &= b_0t_0^m + b_1t_0^{m-1}t_1 + \cdots + b_mt_1^m \end{aligned} \tag{12}$$

be binary forms of degrees l and m over an algebraically closed field k. Further, let

$$g(t) = a_0 + a_1t + \cdots + a_lt^l, \quad h(t) = b_0 + b_1t + \cdots + b_mt^m \tag{13}$$

be polynomials in one variable corresponding to (12). They are of degrees at most l and m. Then the resultant $\mathrm{res}(G, H)$ of binary forms equals to the resultant $\mathrm{res}_{l,m}(g, h)$ of polynomials in one variable ([5, Ch12. p397]).

Let $x_1, \ldots, x_l$ be the roots of g and $y_1, \ldots, y_m$ be the roots of h. By [5, Ch12.(1.3)], if $a_l \neq 0$ and $b_m \neq 0$, we have the product formula

$$\mathrm{res}_{l,m}(g,h) = a_l^m b_m^l \prod_{i,j}(x_i - y_j). \tag{14}$$

By [5, Ch12. p400], if $l' \geq l$ then

$$\mathrm{res}_{l',m}(g,h) = b_m^{l'-l}\,\mathrm{res}_{l,m}(g,h). \tag{15}$$

3.4. Intersection of two quadrics

In this subsection, we consider the case $r = 2$ and $d_1 = d_2 = 2$. Then $V = \Gamma(\mathbb{P}^n, \mathcal{O}(2) \oplus \mathcal{O}(2))$ and we identify the dual $V^\vee$ with the module of pairs of quadratic forms over $\mathbb{Z}$.

Let k be an algebraically closed field. Let $(f_1, f_2) \in V_k^\vee = V^\vee \otimes k$ be pair of quadratic forms of coefficients in k and let $X_{(f_1,f_2)} = V((f_1, f_2)) \subset \mathbb{P}_k^n$ be the intersection of the two quadrics defined by f_1, f_2.

The following proposition is due to M. Reid.

Proposition 3.5. *[7, Proposition 2.1] Let k be an algebraically closed field of characteristic $\neq 2$. Let $(f_1, f_2) \in V_k^\vee$ be non-zero homogeneous polynomials of degree 2 with coefficients in k. Let $M_1, M_2 \in M_{n+1}(k)$ be symmetric matrices such that $XM_1{}^tX = 2f_1$ and $XM_2{}^tX = 2f_2$ where $X = (X_0, \ldots, X_n)$. Then the following two conditions are equivalent.*

1. The intersection $X_{(f_1,f_2)} = V((f_1, f_2))$ is smooth of dimension $n - 2$.

2. The binary form $\det(t_1M_1 + t_2M_2)$ is not identically zero, and has at most simple roots. In other words, if this binary form is decomposed as $\det(t_1M_1 + t_2M_2) = \prod_{i=1}^{n+1}(u_it_1 - v_it_2)$, we have $u_iv_j \neq u_jv_i$ for $0 \leq i,\ j \leq n,\ i \neq j$.

Let $F_1 = \sum_{0\leq i\leq j\leq n} C_{ij}^{(1)} X_iX_j$ and $F_2 = \sum_{0\leq i\leq j\leq n} C_{ij}^{(2)} X_iX_j$ be universal homogeneous polynomials of degree 2. Let $R = \mathbb{Z}[t_1, t_2]$ be the polynomial ring with variables t_1, t_2. We see $t_1F_1 + t_2F_2$ as a quadratic form with variables $X_0, \ldots, X_n$ and denote its divided discriminant by $\mathrm{disc}_d(t_1F_2 + t_2F_2) \in R[(C_{ij}^{(l)})]$. Further we see $\mathrm{disc}_d(t_1F_2 + t_2F_2)$ as a binary form with variables t_1, t_2 and denote its divided discriminant by $\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1 + t_2F_2)) \in \mathbb{Z}[(C_{ij}^{(l)})]$.

Theorem 3.6. *1. Let $n \geq 2$ be an even integer. Then*

$$\operatorname{disc}_\sigma(F_1, F_2) = (-1)^{\frac{n}{2}} \operatorname{disc}_d(\operatorname{disc}_d(t_1F_1 + t_2F_2)).$$

2. Let $n \geq 3$ be an odd integer. Then the equation

$$\operatorname{disc}(F_1, F_2) = 2^{-2(n+1)} \operatorname{disc}_d(\operatorname{disc}_d(t_1F_1 + t_2F_2))$$

holds up to sign.

Proof. Let k be an algebraically closed field of char $k \neq 2$. Let $(f_1, f_2) \in V_k^\vee$ be a pair of non-zero homogeneous polynomials of degree 2 and let $M_1, M_2 \in M_{n+1}(k)$ be corresponding symmetric matrices. By Example 3.2 (9) and Proposition 3.5, the closed subvariety $X_{(f_1,f_2)}$ in $\mathbb{P}^n_k$ defined by the zeros of the two polynomials f_1, f_2 is smooth of dimension $n-2$ if and only if the discriminant $\operatorname{disc}_d(t_1F_1 + t_2F_2)$ is not identically zero and has only simple roots. Further by Example 3.3 (11), this condition is equivalent to $\operatorname{disc}_d(\operatorname{disc}_d(t_1F_1 + t_2F_2)) \neq 0$. Hence we have the equality

$$V(\operatorname{disc}_d(\operatorname{disc}_d(t_1F_1 + t_2F_2)))_{\mathbb{Z}[\frac{1}{2}]} = D_{\mathbb{Z}[\frac{1}{2}]}$$

as subsets of $\mathbb{P}^\vee_{\mathbb{Z}[\frac{1}{2}]}$. By Example 3.3, the degrees of the two polynomials $\operatorname{disc}_d(\operatorname{disc}_d(t_1F_1 + t_2F_2))$ and $\operatorname{disc}(F_1, F_2)$ are both equal to $2n(n+1)$. Since the discriminant $\operatorname{disc}(F_1, F_2)$ is geometrically irreducible in characteristic 0 and the greatest common divisor of its coefficients is 1, the polynomial $\operatorname{disc}_d(\operatorname{disc}_d(t_1F_1 + t_2F_2))$ is the multiple by a non zero integer of $\operatorname{disc}(F_1, F_2)$. By the above equality as sets, for any prime $p \neq 2$ the polynomial $\operatorname{disc}_d(\operatorname{disc}_d(t_1F_1 + t_2F_2)) \pmod p$ is not identically zero. Thus there exists an integer $s \geq 0$ such that

$$2^s \operatorname{disc}(F_1, F_2) = \operatorname{disc}_d(\operatorname{disc}_d(t_1F_1 + t_2F_2)). \tag{16}$$

1. Assume that n is even. First we show $s = 0$. Let $\mathbb{P}' = \mathbb{P}((S^2E)^\vee)$ denote the space of quadrics in $\mathbb{P}^n$ and let $D' \subset \mathbb{P}'$ be the divisor defined by the discriminant of quadrics.

Let $k = \overline{\mathbb{F}}_2$. The pair (f_1, f_2) defines the line $l_{(f_1,f_2)} = \{t_1f_1 + t_2f_2\} \cong \mathbb{P}^1_k$ in the space $\mathbb{P}'_k$. The intersection $l_{(f_1,f_2)} \cap D'_k$ is isomorphic to the the hypersurface in the line $l_{(f_1,f_2)}$ defined by the binary form $\operatorname{disc}_d(t_1f_1 + t_2f_2)$. Hence by the smooth criterion of the discriminant, the value $\operatorname{disc}_d(\operatorname{disc}_d(t_1f_1 + t_2f_2))$ in k is not equals to zero if and only if $l_{(f_1,f_2)} \cap D'_k$ is smooth. Further, this is equivalent to that the line $l_{(f_1,f_2)}$ intersects with D'_k transversally.

By [2, Exposé XVIII, Théorème 2.5] and the assumption that n is even, there exists a *Lefschetz pencil* $l \subset \mathbb{P}'_k$. Further by [2, Exposé XVIII, Proposition 3.2.10], the pencil l intersects with D'_k transversally. We take quadratic forms f_1, f_2 corresponding to two different points on $l \subset \mathbb{P}'_k$. (In the above notation, we have $l = l_{(f_1,f_2)}$.) Then by (16), we have

$$2^s \operatorname{disc}(f_1, f_2) = \operatorname{disc}_d(\operatorname{disc}_d(t_1 f_2 + t_2 f_2)) \neq 0 \in k(= \overline{\mathbb{F}}_2),$$

so we get $s = 0$.

Next we calculate the sign. The degree of the polynomial

$$\operatorname{disc}_d(\operatorname{disc}_d(t_1 F_1 + t_2 F_2))$$

is $n+1$, and the sign is $\epsilon(1, n+1) = (-1)^{n/2}$. Thus the assertion follows from Proposition 3.1.

2. We assume that n is odd. We define a pair of quadratic forms $(f_1, f_2) \in V^\vee$ of coefficients in $\mathbb{Z}$ by

$$f_1 = \sum_{i=1}^{\frac{n-1}{2}} X_{2i-1} X_{2i} + X_n^2, \qquad f_2 = X_0^2 + \sum_{i=0}^{\frac{n-1}{2}} X_{2i} X_{2i+1}. \tag{17}$$

Let $X_{(f_1,f_2)} = V((f_1, f_2)) \subset \mathbb{P}^n_{\mathbb{Z}}$ be the intersection defined by f_1 and f_2.

First we show that its base extension $(X_{(f_1,f_2)})_{\overline{\mathbb{F}}_2}$ is smooth of dimension $n-2$. The Jacobian $J(f_1, f_2)$ of f_1, f_2 over $\overline{\mathbb{F}}_2$ is

$$\begin{pmatrix} 0 & X_2 & X_1 & \cdots & X_{2i-1} & X_{2i+2} & X_{2i+1} & X_{2i+4} & \cdots & X_{n-2} & 0 \\ X_1 & X_0 & X_3 & \cdots & X_{2i+1} & X_{2i} & X_{2i+3} & X_{2i+2} & \cdots & X_n & X_{n-1} \end{pmatrix}.$$

There exist following (2×2) minor matrices

$$\begin{pmatrix} 0 & X_1 \\ X_1 & X_3 \end{pmatrix}, \quad \begin{pmatrix} X_{2i-1} & X_{2i+1} \\ X_{2i+1} & X_{2i+3} \end{pmatrix} \big(1 \leq i \leq (n-3)/2\big), \tag{18}$$

$$\begin{pmatrix} X_{2i} & X_{2i+2} \\ X_{2i-2} & X_{2i} \end{pmatrix} \big(1 \leq i \leq (n-3)/2\big), \quad \begin{pmatrix} X_{n-1} & 0 \\ X_{n-3} & X_{n-1} \end{pmatrix}. \tag{19}$$

The determinants of (18) are

$$X_1^2,\ X_1 X_5 + X_3^2,\ \ldots,\ X_{2i-1} X_{2i+3} + X_{2i+1}^2,\ \ldots,\ X_{n-4} X_n + X_{n-2}^2, \tag{20}$$

and those of (19) are

$$X_2^2 + X_0 X_4,\ \ldots,\ X_{2i}^2 + X_{2i-2} X_{2i+2}^2,\ \ldots,\ X_{n-3}^2 + X_{n-1} X_{n-5},\ X_{n-1}^2. \tag{21}$$

The polynomials (20), (21), f_1, and f_2 do not have any non-trivial common root in $(\overline{\mathbb{F}}_2)^{n+1}$. In fact, by solving (20) from the left to the right we have $X_{2j+1} = 0$ for $0 \leq j \leq \frac{n-3}{2}$, and by solving (21) from the right to the left we have $X_{2j} = 0$ for $2 \leq j \leq \frac{n-1}{2}$. Further, by $f_1 = f_2 = 0$ we have $X_n = X_0 = 0$. Hence by the Jacobian criterion, the variety $(X_{(f_1,f_2)})_{\overline{\mathbb{F}}_2}$ is smooth of dimension $n-2$. By Proposition 1.3, we have $\mathrm{disc}_d(f_1, f_2) \equiv 1 \pmod 2$.

Next we calculate the integer $\mathrm{disc}_d(\mathrm{disc}_d(t_1 f_1 + t_2 f_2))$. The $(n+1) \times (n+1)$ symmetric matrices corresponding to the quadratic forms f_1, f_2 are

$$M_1 = \begin{pmatrix} 0 & & & & & & & & \\ & 0 & 1 & & & & & & O \\ & 1 & 0 & & & & & & \\ & & & 0 & 1 & & & & \\ & & & 1 & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & 1 & \\ & O & & & & & 1 & 0 & \\ & & & & & & & & 2 \end{pmatrix} \quad M_2 = \begin{pmatrix} 2 & 1 & & & & & & & O \\ 1 & 0 & & & & & & & \\ & & 0 & 1 & & & & & \\ & & 1 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & 1 & & \\ & O & & & & 1 & 0 & & \\ & & & & & & & 0 & 1 \\ & & & & & & & 1 & 0 \end{pmatrix}$$

Since these matrices are of even dimension, we have

$$\begin{aligned}
&\mathrm{disc}_d(t_1 f_1 + t_2 f_2) \\
&= \det(t_1 M_1 + t_2 M_2) \qquad \text{(by Example 3.2.(9))} \\
&= \det \begin{pmatrix} 2t_2 & t_2 & & & & & & & & O \\ t_2 & 0 & t_1 & & & & & & & \\ & t_1 & 0 & t_2 & & & & & & \\ & & t_2 & 0 & t_1 & & & & & \\ & & & t_1 & & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 0 & t_2 & & \\ & & & & & & t_2 & 0 & t_1 & \\ & O & & & & & & t_1 & 0 & t_2 \\ & & & & & & & & t_2 & 2t_1 \end{pmatrix} \\
&= (-1)^{\frac{n-1}{2}} 4 t_1^n t_2 + (-1)^{\frac{n+1}{2}} t_2^{n+1}.
\end{aligned}$$

Its partial derivatives are

$$\frac{\partial}{\partial t_1}\operatorname{disc}_d(t_1 f_1 + t_2 f_2) = (-1)^{\frac{n-1}{2}} 4n t_1^{n-1} t_2, \tag{22}$$

$$\frac{\partial}{\partial t_2}\operatorname{disc}_d(t_1 f_2 + t_2 f_2) = (-1)^{\frac{n-1}{2}} 4 t_1^{n} + (-1)^{\frac{n+1}{2}}(n+1) t_2^{n}. \tag{23}$$

Let

$$g_1 = (-1)^{\frac{n-1}{2}} 4n t^{n-1}, \qquad g_2 = (-1)^{\frac{n-1}{2}} 4 t^{n} + (-1)^{\frac{n+1}{2}}(n+1)$$

be the polynomials in one variable corresponding to (22) and (23). Then by Example 3.3, we have an equality of integers

$$\operatorname{res}\left(\frac{\partial}{\partial t_1}\operatorname{disc}_d(t_1 f_2 + t_2 f_2), \frac{\partial}{\partial t_2}\operatorname{disc}_d(t_1 f_1 + t_2 f_2)\right) = \operatorname{res}_{n,n}(g_1, g_2). \tag{24}$$

Further by Example 3.3 (15), we have

$$\operatorname{res}_{n,n}(g_1, g_2) = \left((-1)^{\frac{n-1}{2}} 4\right) \operatorname{res}_{n-1,n}(g_1, g_2). \tag{25}$$

The polynomial g_1 has 0 as $(n-1)$-multiple root. Let $y_1, \ldots, y_n \in \overline{\mathbb{Q}}$ be the roots of the polynomial g_2. Then, by Example 3.3 (14), we have

$$\operatorname{res}_{n-1,n}(g_1, g_2) = \left((-1)^{\frac{n-1}{2}} 4n\right)^{n} \left((-1)^{\frac{n-1}{2}} 4\right)^{n-1} \left(\prod_{j=1}^{n}(0 - y_j)\right)^{n-1}. \tag{26}$$

By the qualities (24), (25), (26), and $\prod_{j=1}^{n}(-y_j) = \frac{(-1)^{\frac{n+1}{2}}(n+1)}{(-1)^{\frac{n-1}{2}}4} = -\frac{n+1}{4}$, we obtain

$$\begin{aligned}&\operatorname{res}\left(\frac{\partial}{\partial t_1}\operatorname{disc}_d(t_1 f_2 + t_2 f_2), \frac{\partial}{\partial t_2}\operatorname{disc}_d(t_1 f_1 + t_2 f_2)\right)\\ &= 2^{2(n+1)} n^{n} (n+1)^{n-1}.\end{aligned}$$

Further, we have $a(0, n+1) = n-1$. By Definition 1.13, we obtain

$$\operatorname{disc}_d(\operatorname{disc}_d(t_1 f_1 + t_2 f_2)) = \frac{1}{(n+1)^{n-1}}\left(2^{2(n+1)} n^{n} (n+1)^{n-1}\right) = 2^{2(n+1)} n^{n}.$$

Thus we have

$$1 \equiv \operatorname{disc}(f_1, f_2) = 2^{-s}\operatorname{disc}_d(\operatorname{disc}_d(t_1 f_1 + t_2 f_2)) = 2^{-s}(2^{2(n+1)} n^{n}) \pmod 2.$$

Since the integer n is odd, we get $s = 2(n+1)$. □

3.7. Application

Let $n \geq 2$ be an even integer and k be a field of char $k \neq 2$. Let $X \subset \mathbb{P}^n_k$ be an $(n-2)$-dimensional smooth complete intersection of two quadrics defined by a pair of quadratic forms $(f_1, f_2) \in S^2E_k \oplus S^2E_k$. Then $H^{n-r}(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{n-r}{2}))$ is spanned by the classes of $\frac{n-r}{2}$-dimensional linear subspaces of $\mathbb{P}^n_{\bar{k}}$ contained in $X_{\bar{k}}$ ([7, Corollary 3.15], [2, Exposé XIX]). The group of $\mathbb{Z}$-lattice spanned by the classes of these linear subspaces permutationg them and preserving the intersection form is isomorphic to the Weyl group $W(D_{n+1})$ ([7, Theorem 3.14]).

The action of the absolute Galois group Γ_k on the linear subspaces defines a homomorphism

$$\Gamma_k \to W(D_{n+1}),$$

unique up to conjugation.

Corollary 3.8. *Assume that* char $k \neq 2$. *Then the composition*

$$\Gamma_k \to W(D_{n+1}) \to \{\pm 1\}$$

is given by the square root of $(-1)^{\frac{n}{2}} \operatorname{disc}_d(\operatorname{disc}_d(f_1, f_2))$.

Proof. The assertion follows from Theorem 2.3, Theorem 3.6.1, and specialization. □

Remark 3.9. The group of all permutations of the 27 lines on a smooth cubic surface preserving their intersection numbers is isomorphic to the Weyl group $W(E_6)$ ([6, Theorem 23.9]). In 1862, G. Salmon [9] studied the discriminant for a cubic surface in pentahedral normal form. A.-S. Elsenhanse and J. Jahnel showed that for a cubic surface in pentahedral normal form defined over a field k, the image of the morphism $\Gamma_k \to W(E_6)$ is included in the index two subgroup if and only if the Salmon discriminant is a square ([4, Theorem 2.12]). See also [8, Example 5.4]. Corollary 3.8 gives an analogue of their result.

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