

The triangulated hull of periodic complexes

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In the terms of a ‘periodic derived category’, we describe explicitly how the orbit category of the bounded derived category of an algebra with respect to a power of the shift functor embeds in its triangulated hull. We obtain a large class of algebras whose orbit categories are strictly smaller than their triangulated hulls and a realization of the phenomenon that an automorphism need not induce the identity functor on the associated orbit category.

1. Introduction

Given a category \mathbb{T} with an automorphism F , an obvious gadget for studying the associated orbits is the *orbit category* \mathbb{T}/F . This has the objects of \mathbb{T} and morphism spaces given by the coproducts

$$\mathbb{T}/F(X, Y) = \coprod_{i \in \mathbb{Z}} \mathbb{T}(X, F^i Y)$$

with the natural composition, ensuring that objects in the same F -orbit become isomorphic. In order for this to be a very useful tool, however, it may be vital that the orbit category somehow resembles the original category, and so it is only reasonable to ask under what conditions certain properties of \mathbb{T} are inherited by \mathbb{T}/F . For instance, when does a triangulated structure on the former induce one on the latter such that the canonical $\mathbb{T} \rightarrow \mathbb{T}/F$ becomes a triangle functor? It seems hopeless to aim for an answer in the generality of arbitrary triangulated categories, the evident obstacle being the fact that it is not clear how to define cones in the orbit category, since a morphism $X \rightarrow Y$ will typically be represented by some

$$X \rightarrow \coprod_{i=1}^k F^{l_i} Y.$$

Fortunately, the introduction of the *cluster category* $D^b(\mathbf{mod} \Lambda)/\mathbb{S} \circ \Sigma^{-2}$ in [5] about a decade ago motivated Keller to devise a *triangulated hull* for certain orbit categories of derived categories in [17]. As the name suggests, this comes with a universal property and allows us to reformulate the above question as ‘when does the orbit category coincide with its triangulated hull?’

The drawback is that this construction might appear difficult to grasp, and attempting calculations can be daunting. We seek to mend this, albeit in a specific setting. To be more precise, when $\text{gldim } \Lambda < \infty$ the rather lenient hypotheses of Keller are satisfied by the automorphism Σ^n of $D^b(\mathbf{mod} \Lambda)$, and thus ensure that the orbit category $D^b(\mathbf{mod} \Lambda)/\Sigma^n$ embeds in a triangulated hull. Our goal is to develop a concrete realization of this a priori rather abstract embedding, in order to explore when the orbit category is triangulated. Notice that certain instances of this setup have already attracted considerable attention. Prominently, if Λ is hereditary and $n = 2$ we get the *root category* introduced by Happel in [13] and studied further in [20].

The framework for our hands-on description of said embedding is the abelian category $C_n(\mathbf{mod} \Lambda)$ of n -periodic differential complexes and chain maps over Λ , together with the homotopy category $K_n(\mathbf{mod} \Lambda)$ and derived category $D_n(\mathbf{mod} \Lambda)$, both of whom are triangulated. Our exposition will focus on the case $n = 1$, that is to say modules over the algebra of dual numbers $\Lambda[t]/(t^2)$. These were considered already in the monograph of Cartan and Eilenberg [7] as ‘modules with differentiation’, and were employed as a means to prove theorems in commutative algebra under the name ‘differential modules’ in work by Avramov, Buchweitz and Iyengar [3], from which we shall adopt much of our notation. While the study of periodic complexes is certainly of independent interest, we are primarily motivated by their applications. More explicitly, for each n the link from the orbit category $D^b(\mathbf{mod} \Lambda)/\Sigma^n$ is given by *compression of complexes*, i.e. associating to

$$0 \rightarrow X^0 \rightarrow X^1 \rightarrow \dots \rightarrow X^{l-1} \rightarrow X^l \rightarrow 0$$

the n -periodic

$$\dots \rightarrow \coprod_{i \equiv 1 (n)} X^i \rightarrow \coprod_{i \equiv 2 (n)} X^i \rightarrow \dots \rightarrow \coprod_{i \equiv 0 (n)} X^i \rightarrow \coprod_{i \equiv 1 (n)} X^i \rightarrow \dots$$

with the obvious differentials. Our most basal result then states the following.

Theorem (See 4.3). *Compression of complexes, considered as a functor*

$$\Delta_n: D^b(\text{mod } \Lambda)/\Sigma^n \rightarrow D_n(\text{mod } \Lambda),$$

is precisely the embedding of the orbit category into its triangulated hull.

A realization of the triangulated hull as the n -periodic derived category was discussed independently in [23], but the reader with our simple minded approach might miss some details. For one the full faithfulness of the restricted localization functor $K_n(\text{proj } \Lambda) \rightarrow D_n(\text{mod } \Lambda)$, which follows from our Lemma 3.8 and Proposition 3.10, is hardly automatic and in fact fails if $\text{gldim } \Lambda = \infty$ (Observation 3.12). Upon learning how elusive bimodule isomorphisms can be in this business (Lemma 5.8) one could also call for a more detailed account of the use of the universal property of the ‘generalized n -root category’ of [10].

In accordance with our objective, the upshot of Theorem 4.3 is the facilitation of computations. For one it is not difficult to demonstrate, in purely homological terms, that Δ_n is dense whenever Λ is iterated tilted. Once Theorem 4.3 is established it will thus follow that $D^b(\text{mod } \Lambda)/\Sigma^n$ is triangulated for such algebras (Proposition 5.3). This is a weak version of [17, Theorem 1], which not only includes the piecewise hereditary algebras, but also deals with a much larger class of automorphisms than the powers of Σ . However, an orbit category need not coincide with its triangulated hull, and examples of this behavior were exhibited already in the fundamental paper of Keller. In fact, to the best of the author’s knowledge, for arbitrary automorphisms F adhering to Keller’s premises, all known examples of algebras Λ with the property that $D^b(\text{mod } \Lambda)/F$ coincides with its triangulated hull, are piecewise hereditary. We are thus naturally led to investigate to what extent this condition is necessary, and a modest start could be answering if the piecewise hereditary algebras are the only ones for which $D^b(\text{mod } \Lambda)/\Sigma^n$ is triangulated. In the context of τ_2 -finite algebras and their *generalized cluster categories* (see [1]), the analogous question was raised by Amiot and Oppermann in [2], and similar to Theorem 7.1 therein, we obtain the following.

Theorem (See 5.5). *If Λ is a non-triangular algebra, then $D^b(\text{mod } \Lambda)/\Sigma^n$ is strictly smaller than its triangulated hull.*

The cluster category is famously a 2-Calabi–Yau triangulated category, meaning that its Serre functor \mathbb{S} and the square of its shift functor are isomorphic (as triangle functors). In other words, the composition $\mathbb{S} \circ \Sigma^{-2}$ becomes naturally isomorphic to the identity on $D^b(\text{mod } \Lambda)/\mathbb{S} \circ \Sigma^{-2}$. As the

latter is a most prominent orbit category, and since it does look plausible at first glance, one could be led to believe that this always happens, i.e. that an automorphism F on a category T always induces the identity on T/F . The following might thus be surprising.

Proposition (See 5.6). *The identity functor and Σ^n are not in general isomorphic on the orbit category $D^b(\text{mod } \Lambda)/\Sigma^n$.*

This oddity, which was in fact noted also in [19], can be attributed to a sign, and consequently we must avoid characteristic 2 and even n in order to find instances. Then, however, our description of the context as that of n -periodic complexes allows us to demonstrate plainly how the phenomenon materializes.

Overview and conventions

Section 2 starts with recalling a few well established concepts regarding modules over differential graded (DG) categories, together with some minor observations of our own, all of which will be employed in the sequel (Section 4). We then review the central construction of the triangulated hull, and note that the crucial step of ‘enhancing’ $D^b(\text{mod } \Lambda)/\Sigma^n$ is in fact not as involved as for orbit categories with respect to arbitrary automorphisms. Lastly we explain how to view DG modules over a DG algebra as graded modules over a related graded algebra. In Section 3 we are concerned with differential modules, or more generally periodic complexes, and their homotopy and derived categories. The key notion of compression of differential complexes is introduced, as well as the tensor product of a differential module with a complex along with its fundamental properties. We proceed by developing a comprehensive theory of resolutions of differential modules, including the above mentioned Proposition 3.10 with its notably constructive proof, which allows us to identify the n -periodic derived category with $K_n(\text{proj } \Lambda)$ and to explain how compression yields an embedding of the orbit category $D^b(\text{mod } \Lambda)/\Sigma^n$ into $D_n(\text{mod } \Lambda)$. Theorem 4.3 is proved in Section 4, and put to use in Section 5. After the foreknown Proposition 5.3 is attained, the investigation goes in the opposite direction, in the sense that we arrive at Theorem 5.5. The latter is collected as evidence in favor of a conjecture which we briefly discuss in the light of [6, 14]. Lastly we turn to the phenomenon of Σ^n failing to induce the identity functor on $D^b(\text{mod } \Lambda)/\Sigma^n$. A simple is provided in some detail before Lemma 5.8 presents an intrinsic reason for this behavior.

Throughout, proofs and detailed constructions will be formulated only in terms of differential modules and the category $D^b(\text{mod } \Lambda)/\Sigma$, but we stress that this is solely for cosmetic reasons and that our results extend in obvious ways to n -periodic complexes and the category $D^b(\text{mod } \Lambda)/\Sigma^n$ for each positive integer n .

2. Preliminaries

We let \mathbb{k} be a field, and by an algebra we mean a finite dimensional \mathbb{k} -algebra. Moreover we restrict to algebras of finite global dimension. A module is a right module, and if Λ and Λ' are algebras then on a Λ - Λ' -bimodule, Λ acts from the left while Λ' acts from the right. $\text{Mod } \Lambda$ is the category of Λ -modules with $\text{mod } \Lambda$ ($\text{proj } \Lambda$) as its subcategory of finitely generated (projective) modules. When Π is a graded algebra, $\text{Gr } \Pi$ ($\text{gr } \Pi$) is the category of graded (finitely generated) Π -modules.

2.1. DG modules

For a more comprehensive introduction and further details, see e.g. Keller's [15, 18]. Recall that a (*right*) *DG module* over a DG category \mathcal{A} is a DG functor $\mathcal{A}^{\text{op}} \rightarrow \text{Dif}(\mathbb{k})$ where $\text{Dif}(\mathbb{k})$ is the DG category of chain complexes over \mathbb{k} , and that the DG modules over \mathcal{A} form a DG category again, denoted by $\text{dg-mod } \mathcal{A}$. Associated to the latter is the additive category $\mathcal{C}(\mathcal{A}) = Z^0(\text{dg-mod } \mathcal{A})$ which carries an exact structure where

$$0 \rightarrow L \xrightarrow{m} M \xrightarrow{p} N \rightarrow 0$$

is a conflation if there is some $s \in \text{dg-mod } \mathcal{A}(N, M)^0$ such that $ps = 1_N$ (equivalently, there is some $r \in \text{dg-mod } \mathcal{A}(M, L)^0$ such that $rm = 1_L$). This generalizes the ‘degreewise split’ exact structure that can be imposed on any category of chain complexes and, indeed, $\mathcal{C}(\mathcal{A})$ is Frobenius with these conflations. The homotopy category $\mathcal{K}(\mathcal{A}) = H^0(\text{dg-mod } \mathcal{A})$ coincides with the stable category associated to $\mathcal{C}(\mathcal{A})$ and is hence triangulated. Finally the derived category $D(\mathcal{A})$ is obtained by formally inverting the class of quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$.

The *Yoneda embedding*

$$\iota: \mathcal{A} \rightarrow \text{dg-mod } \mathcal{A}$$

given by $X \mapsto X^\wedge = \mathcal{A}(-, X)$ identifies \mathcal{A} with the subcategory of its module category consisting of the representables and, by virtue of being a DG

functor, restricts to a functor $Z^0(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A})$. If the image of the latter is stable under extensions and shifts, then $H^0(\mathcal{A})$ becomes a triangulated subcategory of $\mathcal{K}(\mathcal{A})$ and \mathcal{A} is called *pre-triangulated*. The *pre-triangulated hull* of \mathcal{A} is the full DG subcategory $\mathcal{A}^{\text{pre-tr}}$ of $\text{dg-mod } \mathcal{A}$ whose objects are the DG \mathcal{A} -modules obtained by taking the closure in $\mathcal{C}(\mathcal{A})$ of the representables under extensions, shifts and summands. The pre-triangulated hull comes with a universal property, making it functorial and left adjoint to the inclusion of the category of pre-triangulated DG categories in the category of DG categories. Our first observation is that the pre-triangulated hull determines the entire module category.

Lemma 2.1. *If \mathcal{A} and \mathcal{A}' are DG categories such that $\mathcal{A}^{\text{pre-tr}} \cong \mathcal{A}'^{\text{pre-tr}}$, then there is a DG equivalence $\text{dg-mod } \mathcal{A} \cong \text{dg-mod } \mathcal{A}'$.*

Proof. It is sufficient, as well as more elegant, to check that the categories of left DG modules are equivalent. Denoting by $(\mathcal{A}, \mathcal{A}')$ the DG functors $\mathcal{A} \rightarrow \mathcal{A}'$, the left adjointness of the pre-triangulated hull reads $(\mathcal{A}, \mathcal{P}) \cong (\mathcal{A}^{\text{pre-tr}}, \mathcal{P})$ for each pre-triangulated \mathcal{P} . In particular, since $\text{Dif}(\mathbb{k})$ is pre-triangulated, we find that

$$\begin{aligned} \text{dg-mod } \mathcal{A}^{\text{op}} &= (\mathcal{A}, \text{Dif}(\mathbb{k})) \\ &\cong (\mathcal{A}^{\text{pre-tr}}, \text{Dif}(\mathbb{k})) \\ &\cong (\mathcal{A}'^{\text{pre-tr}}, \text{Dif}(\mathbb{k})) \\ &\cong (\mathcal{A}', \text{Dif}(\mathbb{k})) \\ &= \text{dg-mod } \mathcal{A}'^{\text{op}}. \end{aligned}$$

□

Given a DG functor $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{A}'$, denote by $\mathcal{R}_{\mathcal{F}}: \text{dg-mod } \mathcal{A}' \rightarrow \text{dg-mod } \mathcal{A}$ the associated DG restriction functor. $\mathcal{R}_{\mathcal{F}}$ admits a left adjoint

$$\mathcal{J}_{\mathcal{F}}: \text{dg-mod } \mathcal{A} \rightarrow \text{dg-mod } \mathcal{A}'$$

called the DG induction functor (see e.g. [9, Section 14]) whose definition will be recalled in the proof of the following observation.

Lemma 2.2. *If a DG functor $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{A}'$ is fully faithful, then so is the DG induction functor $\mathcal{J}_{\mathcal{F}}: \text{dg-mod } \mathcal{A} \rightarrow \text{dg-mod } \mathcal{A}'$.*

Proof. Since $(\mathcal{J}_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ is an adjoint pair, it suffices to show that these DG functors compose to the identity on $\text{dg-mod } \mathcal{A}$. As \mathcal{F} is fully faithful we

identify \mathcal{A} with its essential image in \mathcal{A}' . By definition, for $Y \in \mathcal{A}'$ and $M \in \text{dg-mod } \mathcal{A}$ we have

$$\mathcal{J}_{\mathcal{F}}(M)(Y) = M \otimes_{\mathcal{A}} (\mathcal{R}_{\mathcal{F}}(\mathcal{A}'(Y, -))) = M \otimes_{\mathcal{A}} \mathcal{A}'(Y, -)$$

where only objects from \mathcal{A} are allowed in the last argument. More explicitly,

$$\mathcal{J}_{\mathcal{F}}(M)(Y) = \left(\coprod_{X \in \mathcal{A}} M(X) \otimes_{\mathcal{A}(X, X)} \mathcal{A}'(Y, X) \right) / \sim$$

where \sim is what one might expect: To each $f \in \mathcal{A}(X, X')$ we associate the two induced maps $f^*: M(X') \rightarrow M(X)$ and $f_*: \mathcal{A}'(Y, X) \rightarrow \mathcal{A}'(Y, X')$. The relation is then given by $f^*(u) \otimes v \sim u \otimes f_*(v)$ for each $u \in M(X')$ and $v \in \mathcal{A}'(Y, X)$. We claim that if $Y \in \mathcal{A}$, then $\mathcal{J}_{\mathcal{F}}(M)(Y) \cong M(Y)$, which is sufficient. Denoting by $u \otimes v$ the simple tensors, the crucial observation is that

$$u \otimes v = u \otimes v_*(1_Y) = v^*(u) \otimes 1_Y$$

for each $u \in M(X)$ and $v \in \mathcal{A}(Y, X)$. Therefore we can identify

$$\mathcal{J}_{\mathcal{F}}(M)(Y) \cong \{v^*(u) \in M(Y) \mid u \in M(X), X \in \mathcal{A}, v \in \mathcal{A}(Y, X)\} = M(Y)$$

where the last equality is clear (choose $X = Y$ and $v = 1_Y$). □

Lemma 2.3. *The DG induction functor restricts to $\mathcal{J}_{\mathcal{F}}: \mathcal{A}^{\text{pre-tr}} \rightarrow \mathcal{A}'^{\text{pre-tr}}$.*

Proof. Suppose the DG \mathcal{A} -module M appears in a conflation

$$0 \rightarrow X^\wedge \rightarrow M \rightarrow Y^\wedge \rightarrow 0$$

in $\mathcal{C}(\mathcal{A})$, with $X, Y \in \mathcal{A}$. One easily checks that induction preserves representables in the sense that $\mathcal{J}_{\mathcal{F}}(X^\wedge) \cong (\mathcal{F}X)^\wedge$. Hence the induced sequence in $\mathcal{C}(\mathcal{A}')$ is

$$0 \rightarrow (\mathcal{F}X)^\wedge \rightarrow \mathcal{J}_{\mathcal{F}}(M) \rightarrow (\mathcal{F}Y)^\wedge \rightarrow 0,$$

which is a conflation again. Further, for each $U \in \mathcal{A}$, the shift of U^\wedge appears in a conflation

$$0 \rightarrow U^\wedge \rightarrow V^\wedge \rightarrow \Sigma(U^\wedge) \rightarrow 0$$

for some $V \in \mathcal{A}$, which induces the conflation

$$0 \rightarrow (\mathcal{F}U)^\wedge \rightarrow (\mathcal{F}V)^\wedge \rightarrow \mathcal{J}_{\mathcal{F}}(\Sigma(U^\wedge)) \rightarrow 0$$

in $\mathcal{C}(\mathcal{A}')$. Thus $\mathcal{J}_{\mathcal{F}}(\Sigma(U^\wedge))$, appearing as an admissible cokernel of $(\mathcal{F}U)^\wedge$ into a projective-injective, is isomorphic to $\Sigma((\mathcal{F}U)^\wedge)$. In other words the objects of $\mathcal{A}^{\text{pre-tr}}$, i.e. summands of iterated extensions and shifts in $\mathcal{C}(\mathcal{A})$ of representables, are sent by $\mathcal{J}_{\mathcal{F}}$ to $\mathcal{A}'^{\text{pre-tr}}$. \square

Remark. The restriction of the DG induction $\mathcal{J}_{\mathcal{F}}: \mathcal{A}^{\text{pre-tr}} \rightarrow \mathcal{A}'^{\text{pre-tr}}$ is nothing but $\mathcal{F}^{\text{pre-tr}}$, i.e. the functor ‘pre-triangulated hull’ applied to \mathcal{F} . Indeed, restricting to pre-triangulated hulls, the fact that induction preserves representables means precisely that

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota} & \mathcal{A}^{\text{pre-tr}} \\ \downarrow \mathcal{F} & & \downarrow \mathcal{J}_{\mathcal{F}} \\ \mathcal{A}' & \xrightarrow{\iota} & \mathcal{A}'^{\text{pre-tr}} \end{array}$$

is commutative. Since the functoriality of the pre-triangulated hull comes from its universal property, the claim follows.

2.2. The triangulated hull

For an algebra Λ and positive integer n , the automorphism $\Sigma^n \cong - \otimes_{\Lambda} \Sigma^n \Lambda$ of $\mathcal{D}^b(\text{mod } \Lambda)$ clearly meets the requirements of [17] that ensure the existence of a *triangulated hull* of the orbit category $\mathcal{D}^b(\text{mod } \Lambda)/\Sigma^n$, that is to say, the data of a triangulated category $\mathcal{D}_{\Sigma^n}(\text{mod } \Lambda)$ and an embedding

$$\mathcal{D}^b(\text{mod } \Lambda)/\Sigma^n \rightarrow \mathcal{D}_{\Sigma^n}(\text{mod } \Lambda)$$

with a certain universal property. To construct $\mathcal{D}_{\Sigma^n}(\text{mod } \Lambda)$, recall from [4] that a *DG enhancement* of a (triangulated) category \mathbb{T} is a (pre-triangulated) DG category \mathcal{T} with a (triangle) equivalence $\mathbb{T} \cong H^0(\mathcal{T})$. The upshot of lifting to DG categories is that canonical choices then become available, for instance, cones are famously functorial in ‘enhanced’ triangulated categories. What is more, a DG enhancement of the orbit category will unveil a canonical triangulated hull. Since $\text{gldim } \Lambda < \infty$, the DG category $\text{per } \Lambda$ of perfect complexes over Λ enhances its bounded derived category, and Σ^n clearly lifts to a DG equivalence, also denoted by Σ^n , on $\text{per } \Lambda$. This allows us to form the orbit category $\mathcal{B}_{\Sigma^n} = \text{per } \Lambda/\Sigma^n$ which is inherently a DG category and gives the desired enhancement, namely

$$\mathcal{D}^b(\text{mod } \Lambda)/\Sigma^n \cong H^0(\mathcal{B}_{\Sigma^n}).$$

The alluded to canonical choice of triangulated hull is thus

$$D_{\Sigma^n}(\text{mod } \Lambda) = \text{thick}(\mathcal{B}_{\Sigma^n}) \subset D(\mathcal{B}_{\Sigma^n}),$$

i.e. the triangulated subcategory of the derived category of \mathcal{B}_{Σ^n} generated by the representable functors, and the embedding of the orbit category is simply Yoneda

$$H^0(\mathcal{B}_{\Sigma^n}) \rightarrow D(\mathcal{B}_{\Sigma^n})$$

which obviously factors through $D_{\Sigma^n}(\text{mod } \Lambda)$.

Remark. The above is a special case of a broader definition by Keller. In general, any automorphism F which is a *standard equivalence*, i.e. isomorphic to

$$- \otimes_{\Lambda}^L Z: D^b(\text{mod } \Lambda) \rightarrow D^b(\text{mod } \Lambda)$$

for a complex Z of Λ - Λ -bimodules, lifts to a DG functor $\mathcal{F}: \text{per } \Lambda \rightarrow \text{per } \Lambda$ in an obvious way. However, the latter might not be an equivalence, and so in order to enhance $D^b(\text{mod } \Lambda)/F$ we must invoke the *DG orbit category*. That is, the DG category $\mathcal{B}'_{\mathcal{F}}$ with the objects of $\text{per } \Lambda$ and morphism spaces of the form

$$\mathcal{B}'_{\mathcal{F}}(X, Y) = \text{colim}_p \coprod_{k \geq 0} \text{per } \Lambda(\mathcal{F}^k X, \mathcal{F}^p Y).$$

Its merit is of course that, when F satisfies certain mild technical hypotheses, there is an equivalence of categories $D^b(\text{mod } \Lambda)/F \cong H^0(\mathcal{B}'_{\mathcal{F}})$. We are thus in the same situation as above, and it is clear how to obtain the triangulated hull $D_F(\text{mod } \Lambda)$. It is also straightforward to verify that the above \mathcal{B}_{Σ^n} is a special case of the DG orbit category. Indeed, whenever F lifts to a DG equivalence \mathcal{F} on $\text{per } \Lambda$, the orbit category $\mathcal{B}_{\mathcal{F}} = \text{per } \Lambda/\mathcal{F}$ exists and is naturally a DG category again. Moreover,

$$\coprod_{k \geq 0} \text{per } \Lambda(\mathcal{F}^k X, \mathcal{F}^p Y) \cong \coprod_{k \geq -p} \text{per } \Lambda(\mathcal{F}^k X, Y)$$

in this case, and hence the directed system defining $\mathcal{B}'_{\mathcal{F}}(X, Y)$ is the sequence

$$\coprod_{k \geq 0} \text{per } \Lambda(\mathcal{F}^k X, Y) \rightarrow \coprod_{k \geq -1} \text{per } \Lambda(\mathcal{F}^k X, Y) \rightarrow \coprod_{k \geq -2} \text{per } \Lambda(\mathcal{F}^k X, Y) \rightarrow \dots$$

in which each morphism is the canonical split mono. It follows that the colimit $\mathcal{B}'_{\mathcal{F}}(X, Y)$ is nothing but the coproduct $\mathcal{B}_{\mathcal{F}}(X, Y)$.

Remark. One can construct the triangulated hull also when $\text{gldim } \Lambda$ is infinite. The enhancement of $\mathbf{D}^b(\text{mod } \Lambda)$ is then the DG version of $\mathbf{C}^{-,b}(\text{proj } \Lambda)$, but otherwise the construction carries over verbatim. Similarly one can handle orbit categories of $\mathbf{D}(\text{Mod } \Lambda)$ using the enhancement by homotopically projective complexes (see [16]).

2.3. DG modules as graded modules

Although the following construction appears to complicate matters, it will prove useful in Section 4. For a DG algebra \mathcal{A} let $\mathcal{A}\langle\varepsilon\rangle$ be the graded algebra with $|\varepsilon| = 1$, and $\mathcal{A}(\varepsilon)$ the graded algebra obtained as the quotient of $\mathcal{A}\langle\varepsilon\rangle$ modulo the relations $\varepsilon^2 = 0$ and $\partial_{\mathcal{A}}(a) + \varepsilon a - (-1)^{|a|}a\varepsilon = 0$.

Lemma 2.4. *$\text{Gr } \mathcal{A}(\varepsilon)$ and $\mathbf{C}(\mathcal{A})$ are equivalent categories.*

Proof. Let $M \in \mathbf{C}(\mathcal{A})$, i.e. a complex

$$\dots \rightarrow M^{i-1} \xrightarrow{\partial_M} M^i \xrightarrow{\partial_M} M^{i+1} \rightarrow \dots$$

with an \mathcal{A} -action satisfying Leibniz’s rule

$$\partial_M(ma) = \partial_M(m)a + (-1)^{|m|}m\partial_{\mathcal{A}}(a).$$

Associate to M the graded $\mathcal{A}(\varepsilon)$ -module $(M^i)_i$ with action $m \cdot a = ma$ for $a \in \mathcal{A}$, and $m \cdot \varepsilon = (-1)^{|m|}\partial_M(m)$. Putting $\alpha = \partial_{\mathcal{A}}(a) + \varepsilon a - (-1)^{|a|}a\varepsilon$ gives

$$m \cdot \alpha = m\partial_{\mathcal{A}}(a) + (-1)^{|m|}\partial_M(m)a - (-1)^{|m|}\partial_M(ma) = 0,$$

so this action is indeed well-defined. On the other hand, take $N \in \text{Gr } \mathcal{A}(\varepsilon)$, that is $N = (N^i)_i$ with an $\mathcal{A}\langle\varepsilon\rangle$ -action satisfying $n\varepsilon^2 = 0$ and moreover $n(\partial_{\mathcal{A}}(a) + \varepsilon a - (-1)^{|a|}a\varepsilon) = 0$. Associate to N the DG \mathcal{A} -module given by the complex

$$\dots \rightarrow N^{i-1} \xrightarrow{(-1)^{i-1}\varepsilon} N^i \xrightarrow{(-1)^i\varepsilon} N^{i+1} \rightarrow \dots$$

with \mathcal{A} -action given by $n \cdot a = na$. This action satisfies Leibniz’s rule, since

$$\partial_N(n \cdot a) = (-1)^{|na|}na\varepsilon$$

while

$$\partial_N(n) \cdot a + (-1)^{|n|}n \cdot \partial_{\mathcal{A}}(a) = (-1)^{|n|}n\varepsilon a + (-1)^{|n|}n\partial_{\mathcal{A}}(a),$$

and the difference between the latter two expressions is

$$n((-1)^{|a|}a\varepsilon - \varepsilon a - \partial_{\mathcal{A}}(a)) = 0.$$

Leaving morphisms unaltered, the above two assignments clearly give mutually inverse equivalences between the categories in question. \square

3. Differential modules

This section introduces the context in which we will describe the orbit category $D^b(\text{mod } \Lambda)/\Sigma^n$ and its triangulated hull for an algebra Λ . Our exposition focuses on $n = 1$, but we point out the general versions of important notions and results along the way. At these points the reader might find it instructive to fill in details.

3.1. Triangulated structure

To an additive category \mathbf{A} one can associate $\mathbf{C}_1(\mathbf{A})$, i.e. the category of 1-periodic complexes and chain maps in \mathbf{A} . In other words, the objects of $\mathbf{C}_1(\mathbf{A})$ are pairs (M, ε_M) with *underlying object* $M \in \mathbf{A}$ and *differential* $\varepsilon_M \in \mathbf{A}(M, M)$ squaring to zero, and its morphisms are those between the underlying objects that commute with the differentials involved. In [22] it is shown that if \mathbf{A} is Frobenius exact, then so is $\mathbf{C}_1(\mathbf{A})$. This means that if we call the sequence

$$0 \rightarrow (L, \varepsilon_L) \rightarrow (M, \varepsilon_M) \rightarrow (N, \varepsilon_N) \rightarrow 0$$

in $\mathbf{C}_1(\mathbf{A})$ a conflation whenever it admits a splitting in \mathbf{A} , then $\mathbf{C}_1(\mathbf{A})$ becomes Frobenius. The injective envelope of (M, ε_M) is the middle term in the conflation

$$(M, \varepsilon_M) \xrightarrow{\begin{pmatrix} \varepsilon_M \\ 1_M \end{pmatrix}} \left(M \oplus M, \begin{pmatrix} 0 & 0 \\ 1_M & 0 \end{pmatrix} \right) \xrightarrow{(1_M \quad -\varepsilon_M)} (M, -\varepsilon_M),$$

and each projective-injective object is of this form. Hence the stable category $\mathbf{K}_1(\mathbf{A})$ is triangulated with suspension Σ acting by $(M, \varepsilon_M) \mapsto (M, -\varepsilon_M)$ on objects and trivially on morphisms. The *mapping cone* of a morphism

$f: (M, \varepsilon_M) \rightarrow (N, \varepsilon_N)$ is the object

$$C_f = \left(M \oplus N, \begin{pmatrix} -\varepsilon_M & 0 \\ f & \varepsilon_N \end{pmatrix} \right),$$

and it is straightforward to verify that the standard triangle associated to f is

$$M \xrightarrow{f} N \rightarrow C_f \rightarrow \Sigma M$$

with the canonical morphisms, as well as the fact that each conflation embeds in a triangle. Equivalently, if we say that the above f is *null-homotopic* whenever there is some $s \in \mathbf{A}(M, N)$ such that $f = s\varepsilon_M + \varepsilon_N s$, then $\mathbf{K}_1(\mathbf{A})$ is the *1-periodic homotopy category*. When \mathbf{A} is abelian, the *homology* of (M, ε_M) is the quotient $\text{Ker}(\varepsilon_M)/\text{Im}(\varepsilon_M)$, giving rise to a homological functor $H: \mathbf{K}_1(\mathbf{A}) \rightarrow \mathbf{A}$. The class S of quasi-isomorphisms in $\mathbf{K}_1(\mathbf{A})$ is thus a multiplicative system compatible with the triangulation, which means that the *1-periodic derived category*

$$\mathbf{D}_1(\mathbf{A}) = S^{-1} \mathbf{K}_1(\mathbf{A}),$$

carries a triangulated structure such that the localization functor $\mathbf{K}_1(\mathbf{A}) \rightarrow \mathbf{D}_1(\mathbf{A})$ is a triangle functor. Further, e.g. by [11], $\mathbf{D}_1(\mathbf{A})$ admits a calculus of roofs.

Our primary engagement is with the case $\mathbf{A} = \text{mod } \Lambda$. Evidently, denoting by

$$\Lambda[\varepsilon] = \Lambda[t]/(t^2)$$

the algebra of dual numbers, $\mathbf{C}_1(\text{mod } \Lambda)$ is precisely $\text{mod } \Lambda[\varepsilon]$. We hence refer to the objects of $\mathbf{C}_1(\text{mod } \Lambda)$ as *differential modules* and its morphisms as being $\Lambda[\varepsilon]$ -linear.

Remark. For each positive integer n , the category $\mathbf{C}_n(\mathbf{A})$ of n -periodic complexes and chain maps in \mathbf{A} is Frobenius with respect to the ‘degree-wise split’ exact structure. Hence the n -periodic homotopy category $\mathbf{K}_n(\mathbf{A})$, obtained as the stabilization of $\mathbf{C}_n(\mathbf{A})$, and its localization $\mathbf{D}_n(\mathbf{A})$ at quasi-isomorphisms are both triangulated. Moreover, $\mathbf{C}_n(\text{mod } \Lambda)$ is nothing but the finitely generated modules over $\Lambda \otimes_{\mathbb{k}} I_n$ where I_n is the selfinjective Nakayama algebra with n vertices and Loewy length 2.

3.2. Compression and tensor products

Since $\Lambda[\varepsilon]$ is graded with $|\varepsilon| = 1$ and a graded module is nothing but a complex over Λ , there is a forgetful functor

$$\Delta: \mathbf{C}^b(\text{mod } \Lambda) \rightarrow \mathbf{C}_1(\text{mod } \Lambda).$$

Explicitly, forgetting about the grading on the complex

$$C = 0 \rightarrow C^0 \xrightarrow{\partial^0} C^1 \rightarrow \dots \rightarrow C^{l-1} \xrightarrow{\partial^{l-1}} C^l \rightarrow 0$$

results in the differential module

$$\Delta C = (\Delta C, \varepsilon_{\Delta C}) = \left(\prod_{i=0}^l C^i, \prod_{i=0}^{l-1} \partial^i \right).$$

With respect to the obvious decomposition, the differential $\varepsilon_{\Delta C}$ is clearly the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \partial^0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \partial^1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \partial^{l-1} & 0 \end{pmatrix}.$$

In [3] the authors coin the descriptive term *compression* for this functor. Observe that the procedure of compression also gives rise to triangle functors

$$\Delta: \mathbf{K}^b(\text{mod } \Lambda) \rightarrow \mathbf{K}_1(\text{mod } \Lambda) \quad \text{and} \quad \Delta: \mathbf{D}^b(\text{mod } \Lambda) \rightarrow \mathbf{D}_1(\text{mod } \Lambda)$$

as it commutes with suspensions and cones, and preserves quasi-isomorphisms. Naturally, the objects in the essential image of Δ are referred to as *gradable*.

For each algebra Λ' and each complex

$$X = 0 \rightarrow X^0 \xrightarrow{\partial^0} X^1 \rightarrow \dots \rightarrow X^{l-1} \xrightarrow{\partial^{l-1}} X^l \rightarrow 0$$

of Λ - Λ' -bimodules, the *tensor product* of a differential Λ -module (M, ε_M) with X is the differential Λ' -module

$$M \boxtimes_{\Lambda} X = \left(\prod_{i=0}^l (M \otimes_{\Lambda} X^i), m \otimes x \mapsto m \otimes \partial(x) + (-1)^{|x|} \varepsilon_M(m) \otimes x \right).$$

This construction, together with the obvious action on morphisms, gives a functor

$$- \boxtimes_{\Lambda} X: \mathbf{C}_1(\text{mod } \Lambda) \rightarrow \mathbf{C}_1(\text{mod } \Lambda').$$

The tensor product commutes with compression in the sense that when Y is a complex of right Λ -modules, there is a canonical isomorphism

$$(\Delta Y) \boxtimes_{\Lambda} X \cong \Delta(Y \otimes_{\Lambda} X)$$

of differential Λ' -modules. Further, if Λ'' is also an algebra and Z is a bounded complex of Λ' - Λ'' -bimodules, there is a canonical isomorphism

$$(3.1) \quad M \boxtimes_{\Lambda} (X \otimes_{\Lambda'} Z) \cong (M \boxtimes_{\Lambda} X) \boxtimes_{\Lambda'} Z$$

of differential Λ'' -modules. What is more, when X satisfies the expected hypotheses (i.e. when each X^i is flat as Λ -module), then $- \boxtimes_{\Lambda} X$ preserves quasi-isomorphisms and hence defines a functor $\mathbf{D}_1(\text{mod } \Lambda) \rightarrow \mathbf{D}_1(\text{mod } \Lambda')$ making the diagram

$$(3.2) \quad \begin{array}{ccc} \mathbf{D}^b(\text{mod } \Lambda) & \xrightarrow{- \otimes_{\Lambda} X} & \mathbf{D}^b(\text{mod } \Lambda') \\ \downarrow \Delta & & \downarrow \Delta \\ \mathbf{D}_1(\text{mod } \Lambda) & \xrightarrow{- \boxtimes_{\Lambda} X} & \mathbf{D}_1(\text{mod } \Lambda') \end{array}$$

commutative. The following fundamental property will be exploited in Section 5.

Lemma 3.3. *If X gives rise to an equivalence*

$$- \otimes_{\Lambda} X: \mathbf{D}^b(\text{mod } \Lambda) \rightarrow \mathbf{D}^b(\text{mod } \Lambda'),$$

then also

$$- \boxtimes_{\Lambda} X: \mathbf{D}_1(\text{mod } \Lambda) \rightarrow \mathbf{D}_1(\text{mod } \Lambda').$$

is invertible.

Proof. A quasi-inverse of $- \otimes_{\Lambda} X$ is

$$\mathbf{RHom}_{\Lambda'}(X, -) \cong - \otimes_{\Lambda'} Y,$$

where Y is a Λ' -projective resolution of $\mathbf{RHom}_{\Lambda'}(X, \Lambda')$. We claim that $- \boxtimes_{\Lambda'} Y$ is a quasi-inverse of $- \boxtimes_{\Lambda} X$. In order to show that the two compose to

the identity on $D_1(\text{mod } \Lambda)$, observe first that the stalk complex Λ serves as a one-sided tensor unit in the sense that for each differential Λ -module M there is an isomorphism

$$M \boxtimes_{\Lambda} \Lambda \cong M$$

in $C_1(\text{mod } \Lambda)$, natural in M . Further, quasi-isomorphic (bounded) complexes clearly give rise to naturally isomorphic tensor product functors. Combining these observations with the isomorphisms

$$X \otimes_{\Lambda'} Y \cong \mathbf{RHom}_{\Lambda'}(X, X) \cong \Lambda$$

in $D^b(\text{mod } \Lambda)$, it follows from (3.1) that there are natural isomorphisms

$$(M \boxtimes_{\Lambda} X) \boxtimes_{\Lambda'} Y \cong M \boxtimes_{\Lambda} (X \otimes_{\Lambda'} Y) \cong M \boxtimes_{\Lambda} \Lambda = M$$

in $D_1(\text{mod } \Lambda)$. A similar argument shows that the reversed composition is isomorphic to the identity on $D_1(\text{mod } \Lambda')$. \square

3.3. Resolutions

Using the terminology of [3], a *projective flag* in a differential module (P, ε_P) is a decomposition of the underlying module $P = P_l \oplus P_{l-1} \oplus \dots \oplus P_0$ where each $P_i \in \text{proj } \Lambda$, with respect to which ε_P is of the form

$$(3.4) \quad \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \partial^{l,l-1} & 0 & 0 & \dots & 0 & 0 \\ \partial^{l,l-2} & \partial^{l-1,l-2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial^{l,1} & \partial^{l-1,1} & \partial^{l-2,1} & \dots & 0 & 0 \\ \partial^{l,0} & \partial^{l-1,0} & \partial^{l-2,0} & \dots & \partial^{1,0} & 0 \end{pmatrix}.$$

Not only are the differential modules admitting projective flags homologically most viable (e.g. Lemma 3.8 and [3, Proposition 2.4]), they also provide resolutions in the following sense.

Lemma 3.5. *Each differential module $(M, \varepsilon_M) \in C_1(\text{mod } \Lambda)$ is quasi-isomorphic to one admitting a projective flag.*

Proof. Choose projective resolutions

$$0 \rightarrow X_{l'} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow \text{Im}(\varepsilon_M) \rightarrow 0$$

and

$$0 \rightarrow Y_{l''} \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 \rightarrow H(M) \rightarrow 0$$

in $\text{mod } \Lambda$. Applying the Horseshoe Lemma, first to

$$0 \rightarrow \text{Im}(\varepsilon_M) \rightarrow \text{Ker}(\varepsilon_M) \rightarrow H(M) \rightarrow 0$$

and then to

$$0 \rightarrow \text{Ker}(\varepsilon_M) \rightarrow M \rightarrow \text{Im}(\varepsilon_M) \rightarrow 0,$$

produces a projective resolution

$$0 \rightarrow P_l \xrightarrow{\partial_l} P_{l-1} \rightarrow \cdots \rightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \rightarrow 0$$

of M in $\text{mod } \Lambda$, in which $P_i = X_i \oplus Y_i \oplus X_i$. By construction, equipping each P_i with the endomorphism

$$\hat{\varepsilon}_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1_{X_i} & 0 & 0 \end{pmatrix}$$

turns the latter resolution into a sequence of $\Lambda[\varepsilon]$ -linear morphisms. Introducing the sign $\varepsilon_i = (-1)^i \hat{\varepsilon}_i$ results in the diagram

$$(3.6) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & P_l & \xrightarrow{\partial_l} & P_{l-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & M & \longrightarrow & 0 \\ & & \downarrow \varepsilon_l & & \downarrow \varepsilon_{l-1} & & & & \downarrow \varepsilon_1 & & \downarrow \varepsilon_0 & & \downarrow \varepsilon_M & & \\ 0 & \longrightarrow & P_l & \xrightarrow{\partial_l} & P_{l-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & M & \longrightarrow & 0 \end{array}$$

in which the rows are exact and each square is anti-commutative, except for the rightmost one which is commutative. With respect to the indicated

decomposition of the Λ -module $\mathfrak{p}M = P_l \oplus P_{l-1} \oplus \cdots \oplus P_0$, define

$$\varepsilon_{\mathfrak{p}M} = \begin{pmatrix} \varepsilon_l & 0 & 0 & \cdots & 0 & 0 \\ \partial_l & \varepsilon_{l-1} & 0 & \cdots & 0 & 0 \\ 0 & \partial_{l-1} & \varepsilon_{l-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \varepsilon_1 & 0 \\ 0 & 0 & 0 & \cdots & \partial_1 & \varepsilon_0 \end{pmatrix}.$$

Now $\varepsilon_{\mathfrak{p}M}$ squares to zero because of the introduced sign, and the assignment

$$(p_l, p_{l-1}, \dots, p_0) \mapsto \partial_0(p_0)$$

defines a $\Lambda[\varepsilon]$ -linear morphism $f: (\mathfrak{p}M, \varepsilon_{\mathfrak{p}M}) \rightarrow (M, \varepsilon_M)$ by commutativity of the rightmost square in (3.6). With this diagram in mind it is straightforward to verify that f is a quasi-isomorphism, which suffices since $\varepsilon_{\mathfrak{p}M}$ is of the required form. \square

We say that a differential Λ -module K is *homotopically projective* if

$$\text{Hom}_{\mathbf{K}_1(\Lambda)}(K, N) = 0$$

for each acyclic N . The class of homotopically projectives is closed under extensions and hence constitutes a triangulated subcategory

$$\mathbf{K}_1^{\text{hp}}(\text{mod } \Lambda) \subset \mathbf{K}_1(\text{mod } \Lambda).$$

It is easy to check that each $(Q, 0)$ with $Q \in \text{proj } \Lambda$ is homotopically projective, from which it follows that the homotopically projectives encompass the differential modules admitting projective flags. Indeed, each object in the latter class can be obtained as an iterated extension of differential modules with underlying projective module and vanishing differential. To show this, note that the differential ε_P in (3.4) on the underlying $P = P_l \oplus P_{l-1} \oplus \cdots \oplus P_0$ restricts to a differential, also denoted by ε_P , on each summand of P of the form $P_i \oplus P_{i-1} \oplus \cdots \oplus P_0$ for $i = 0, \dots, l$. Moreover these differentials fit in a filtration

$$(3.7) \quad (P_0, 0) \subset (P_1 \oplus P_0, \varepsilon_P) \subset \cdots \subset (P_{l-1} \oplus \cdots \oplus P_0, \varepsilon_P) \subset (P, \varepsilon_P)$$

in $\mathbf{C}_1(\text{mod } \Lambda)$ with the property that each filtration factor has vanishing differential. Hence, iterating from the canonical conflation

$$0 \rightarrow (P_0, 0) \rightarrow (P_1 \oplus P_0, \varepsilon_P) \rightarrow (P_1, 0) \rightarrow 0$$

we reach each term of (3.7), including (P, ε_P) itself, as an extension

$$0 \rightarrow (P_{i-1} \oplus \cdots \oplus P_0, \varepsilon_P) \rightarrow (P_i \oplus \cdots \oplus P_0, \varepsilon_P) \rightarrow (P_i, 0) \rightarrow 0.$$

Lemma 3.8. *If P is a homotopically projective differential Λ -module, then the localization functor induces an isomorphism*

$$\text{Hom}_{\mathbf{K}_1(\Lambda)}(P, M) \cong \text{Hom}_{\mathbf{D}_1(\Lambda)}(P, M)$$

for each M .

Proof. First, observe that if $f \in \text{Hom}_{\mathbf{K}_1(\Lambda)}(P, M)$ maps to the zero morphism in $\mathbf{D}_1(\text{mod } \Lambda)$, then it factors through some acyclic N and hence vanishes already in $\mathbf{K}_1(\text{mod } \Lambda)$ since $\text{Hom}_{\mathbf{K}_1(\Lambda)}(P, N) = 0$. Second, in the calculus of roofs a morphism $P \rightarrow M$ in $\mathbf{D}_1(\text{mod } \Lambda)$ is represented by a diagram

$$\begin{array}{ccc} & X & \\ q \swarrow & & \searrow f \\ P & & M \end{array}$$

with q a quasi-isomorphism. In the triangle

$$X \xrightarrow{q} P \rightarrow C_q \rightarrow \Sigma X$$

the middle morphism must be zero, as C_q is acyclic. Thus q is a split epimorphism and there is some \hat{q} ensuring commutativity of both squares in

$$\begin{array}{ccccc} & & P & & \\ & & \hat{q} \swarrow & \searrow 1 & \\ & X & & & P \\ q \swarrow & & \searrow & \swarrow & \searrow f\hat{q} \\ P & & 1 & & f & & M \end{array}$$

Hence the roofs

$$\begin{array}{ccc} & X & \\ q \swarrow & & \searrow f \\ P & & M \end{array} \quad \text{and} \quad \begin{array}{ccc} & P & \\ 1 \swarrow & & \searrow f\hat{q} \\ P & & M \end{array}$$

are equivalent and it follows that the left hand roof lies in the image of the localization functor, since the right hand one clearly does. \square

Note that the incredibly convenient attribute of homotopically projectives expressed in Lemma 3.8 is a 1-periodic analog of the essential property enjoyed by projective resolutions of differential complexes. Lemma 3.5 hence indicates that, up to homotopy, the class of differential modules admitting a projective flag should coincide with the class of homotopically projectives.

Proposition 3.9. *The class of homotopically projectives in $K_1(\text{mod } \Lambda)$ is, up to isomorphism, precisely the class of differential modules admitting projective flags. Moreover, restriction of the localization functor gives a triangle equivalence*

$$K_1^{\text{hp}}(\text{mod } \Lambda) \cong D_1(\text{mod } \Lambda).$$

Proof. We have already seen how the differential modules admitting projective flags are all homotopically projective. Conversely, by Lemma 3.5 each differential module K allows a quasi-isomorphism $\mathfrak{p}K \rightarrow K$ where $\mathfrak{p}K$ admits a projective flag, and by Lemma 3.8 this will be invertible already in $K_1(\text{mod } \Lambda)$ whenever K is homotopically projective. The last claim is clear, as density of the restricted localization functor is Lemma 3.5 while full faithfulness is Lemma 3.8. \square

Remark. With no restriction on $\text{gldim } \Lambda$, it is clear how to devise homotopically projective resolutions of differential Λ -modules whose underlying modules are not necessarily finitely generated. Indeed, using the idea of the proof of Lemma 3.5, the resolutions will appear as colimits of possibly infinite systems

$$P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \dots$$

of differential modules in which each map is split mono over Λ and each quotient P_{i+1}/P_i has vanishing differential and underlying projective module. Plainly, as in the filtration (3.7), each P_i is homotopically projective, and so it follows from

$$\text{Hom}_{K_1(\Lambda)}(\text{colim } P_i, N) \cong \lim \text{Hom}_{K_1(\Lambda)}(P_i, N)$$

that also the colimit itself satisfies the required vanishing condition. Hence, in analogy to the unbounded resolutions of differential complexes of [16], there is a triangle equivalence $D_1(\text{Mod } \Lambda) \cong K_1^{\text{hp}}(\text{Mod } \Lambda)$.

Let us denote by $K_1(\text{proj } \Lambda)$ the triangulated subcategory of $K_1(\text{mod } \Lambda)$ consisting of *relatively projectives*, i.e. differential modules whose underlying modules are projective. In a sense, the most convenient scenario imaginable,

and also the most striking analogy to the context of differential complexes and their resolutions, is that the relatively projectives are all homotopically projective. A priori, this would be just as surprising as it would be beneficial. For instance, let G be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

$$\quad \quad \quad \longleftarrow \gamma$$

and consider the algebra $kG/(\beta\alpha)$. Then, denoting by P_2 the indecomposable projective corresponding to vertex 2, it might seem unreasonable to expect that the relatively projective differential module $(P_2, \alpha\gamma\beta)$ admits a projective flag, even in the 1-periodic homotopy category. We nevertheless have the following.

Proposition 3.10. *The categories $K_1^{\text{hp}}(\text{mod } \Lambda)$ and $K_1(\text{proj } \Lambda)$ coincide.*

Proof. Each homotopically projective in $K_1(\text{mod } \Lambda)$ is isomorphic to a differential module admitting a projective flag, and so is relatively projective. Conversely, let (P_0, ε) be relatively projective. Observe that $\text{Im}(\varepsilon)$ admits a projective resolution

$$P = 0 \rightarrow P_l \xrightarrow{\partial_l} P_{l-1} \rightarrow \dots \rightarrow P_1 \xrightarrow{\partial_1} P_0 \rightarrow 0,$$

obtained by splicing a finite resolution of $\text{Ker}(\varepsilon)$ with its inclusion into P_0 . Let $(\Delta P, \varepsilon_{\Delta P})$ be the compression of P . The cone

$$C_{-1_{\Delta P}} = \left(\Delta P \oplus \Delta P, \begin{pmatrix} -\varepsilon_{\Delta P} & 0 \\ -1_{\Delta P} & \varepsilon_{\Delta P} \end{pmatrix} \right)$$

vanishes in the homotopy category, and it therefore suffices to show that

$$(Q, \varepsilon_Q) = (P_0, \varepsilon) \oplus C_{-1_{\Delta P}} = \left(\Delta P \oplus P_0 \oplus \Delta P, \begin{pmatrix} -\varepsilon_{\Delta P} & 0 & 0 \\ 0 & \varepsilon & 0 \\ -1_{\Delta P} & 0 & \varepsilon_{\Delta P} \end{pmatrix} \right)$$

is isomorphic to a differential module admitting a projective flag. To this end,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & P_l & \xrightarrow{\partial_l} & P_{l-1} & \longrightarrow & \dots & \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & \text{Im}(\varepsilon_Q) & \longrightarrow & 0 \\ & & \downarrow 0 & & \downarrow 0 & & & & \downarrow 0 & & \downarrow \varepsilon & & \downarrow 0 & & \\ 0 & \longrightarrow & P_l & \xrightarrow{\partial_l} & P_{l-1} & \longrightarrow & \dots & \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & \text{Im}(\varepsilon_Q) & \longrightarrow & 0 \end{array}$$

is a pivotal diagram. Indeed, its commutativity reveals that the indicated chain map $P \rightarrow P$ is a lift of the zero endomorphism on $\text{Im}(\varepsilon_Q)$, and must therefore be null-homotopic. This means there are $s_i: P_i \rightarrow P_{i+1}$ for $0 \leq i \leq l - 1$ such that $\partial_1 s_0 = \varepsilon$, $\partial_{i+1} s_i + s_{i-1} \partial_i = 0$ for $1 \leq i \leq l - 1$ and $s_{l-1} \partial_l = 0$. Denoting by $e: \Delta P \rightarrow P_0$ and $m: P_0 \rightarrow \Delta P$ the canonical split epi- and monomorphism, respectively, and letting $h_i = (-1)^i s_i$, it is clear that the latter assemble to $h: \Delta P \rightarrow \Delta P$ such that

$$(3.11) \quad m\varepsilon e = \varepsilon_{\Delta P} h - h\varepsilon_{\Delta P}.$$

Miraculously, it turns out that (Q, ε_Q) is isomorphic to

$$(Q', \varepsilon_{Q'}) = \left(\Delta P \oplus P_0 \oplus \Delta P, \begin{pmatrix} -\varepsilon_{\Delta P} & 0 & 0 \\ -\varepsilon e & 0 & 0 \\ -(1_{\Delta P} + h) & -m & \varepsilon_{\Delta P} \end{pmatrix} \right),$$

whose differential is clearly of the required lower triangular form. Indeed,

$$f = \begin{pmatrix} 1_{\Delta P} + h & m & 0 \\ e & 0 & -\varepsilon_{\Delta P} \\ 0 & 0 & 1_{\Delta P} \end{pmatrix}$$

gives an isomorphism $(Q', \varepsilon_{Q'}) \rightarrow (Q, \varepsilon_Q)$. The reader so inclined is invited to check that $\varepsilon_Q f = f \varepsilon_{Q'}$ using (3.11) together with the obvious equalities

$$\varepsilon_{\Delta P} m = 0, \quad \varepsilon e \varepsilon_{\Delta P} = 0 = e h \quad \text{and} \quad e m = 1_{P_0}.$$

□

Example. Let us revisit the algebra $\mathbb{k}G/(\beta\alpha)$ with the relatively projective differential module $(P_2, \varepsilon = \alpha\gamma\beta)$, discussed just prior to Proposition 3.10. The image of ε is the simple module S_2 , so the relevant diagram is

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{\alpha} & P_2 & \xrightarrow{\varepsilon} & S_2 & \longrightarrow & 0 \\ & & \downarrow 0 & & \downarrow \varepsilon & & \downarrow 0 & & \\ 0 & \longrightarrow & P_1 & \xrightarrow{\alpha} & P_2 & \xrightarrow{\varepsilon} & S_2 & \longrightarrow & 0. \end{array}$$

In the notation of the proof of Proposition 3.10, (P_2, ε) is hence a summand, with null-homotopic complement, of the homotopically projective differential

module

$$(Q', \varepsilon_{Q'}) = \left(P_1 \oplus P_2 \oplus P_2 \oplus P_1 \oplus P_2, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & 0 \\ -1_{P_1} & -\gamma\beta & 0 & 0 & 0 \\ 0 & -1_{P_2} & -1_{P_2} & \alpha & 0 \end{pmatrix} \right).$$

We moreover know that $\varepsilon_{Q'}$ is obtained by conjugation of the differential of

$$(Q, \varepsilon_Q) = \left(P_1 \oplus P_2 \oplus P_2 \oplus P_1 \oplus P_2, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 \\ -1_{P_1} & 0 & 0 & 0 & 0 \\ 0 & -1_{P_2} & 0 & \alpha & 0 \end{pmatrix} \right),$$

and the reader may verify this by checking that $f^{-1}\varepsilon_Q f = \varepsilon_{Q'}$ for

$$f = \begin{pmatrix} 1_{P_1} & \gamma\beta & 0 & 0 & 0 \\ 0 & 1_{P_2} & 1_{P_2} & 0 & 0 \\ 0 & 1_{P_2} & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 1_{P_1} & 0 \\ 0 & 0 & 0 & 0 & 1_{P_2} \end{pmatrix} \quad \text{with} \quad f^{-1} = \begin{pmatrix} 1_{P_1} & 0 & -\gamma\beta & 0 & 0 \\ 0 & 0 & 1_{P_2} & \alpha & 0 \\ 0 & 1_{P_2} & -1_{P_2} & -\alpha & 0 \\ 0 & 0 & 0 & 1_{P_1} & 0 \\ 0 & 0 & 0 & 0 & 1_{P_2} \end{pmatrix}.$$

Remark. The results of the current subsection clearly extend beyond the case $n = 1$. In particular combining counterparts of Proposition 3.9 and Proposition 3.10 yields

$$D_n(\text{mod } \Lambda) \cong K_n(\text{proj } \Lambda).$$

An independent discussion of the case $n = 2$ can be found in [12, Section 9.1].

Observation 3.12. We do not have a triangle equivalence of the above form when $\text{gldim } \Lambda$ is infinite. In particular the restriction

$$K_n(\text{proj } \Lambda) \rightarrow D_n(\text{mod } \Lambda)$$

of the localization functor is no longer faithful in this case. For an easy example take $\Lambda = \mathbb{k}[t]/(t^2)$. Then the identity on the acyclic differential module (Λ, t) cannot be null-homotopic, that is, (Λ, t) is non-zero in $K_1(\text{proj } \Lambda)$ but vanishes in $D_1(\text{mod } \Lambda)$.

3.4. An embedding of the orbit category

If $X, Y \in \mathbf{D}^b(\text{mod } \Lambda)$ then, since Λ is of finite global dimension, we may replace X by a projective resolution to get

$$\prod_{i \in \mathbb{Z}} \text{Hom}_{\mathbf{D}^b(\Lambda)}(X, \Sigma^i Y) \cong \prod_{i \in \mathbb{Z}} \text{Hom}_{\mathbf{K}^b(\Lambda)}(X, \Sigma^i Y).$$

In this case ΔX is homotopically projective by Proposition 3.10 and hence

$$\text{Hom}_{\mathbf{D}_1(\Lambda)}(\Delta X, \Delta Y) \cong \text{Hom}_{\mathbf{K}_1(\Lambda)}(\Delta X, \Delta Y)$$

by Lemma 3.8. Further, it is straightforward to write down an isomorphism

$$\prod_{i \in \mathbb{Z}} \text{Hom}_{\mathbf{K}^b(\Lambda)}(X, \Sigma^i Y) \rightarrow \text{Hom}_{\mathbf{K}_1(\Lambda)}(\Delta X, \Delta Y),$$

as an element of the left hand side is just a sequence $(f_i)_i$ of, up to homotopy, sequences of morphisms $f_i = (f_i^j)_j$ with $f_i^j: X^j \rightarrow Y^{j+i}$. It is clear that these isomorphisms comprise a proof of the following.

Lemma 3.13. *Compression of complexes induces a fully faithful embedding*

$$\Delta: \mathbf{D}^b(\text{mod } \Lambda)/\Sigma \rightarrow \mathbf{D}_1(\text{mod } \Lambda).$$

In other words, the orbit category is equivalent to the full subcategory of gradable objects in $\mathbf{D}_1(\text{mod } \Lambda)$.

Remark. There is a fully faithful embedding, abusively denoted by

$$\Delta_n: \mathbf{D}^b(\text{mod } \Lambda)/\Sigma^n \rightarrow \mathbf{D}_n(\text{mod } \Lambda),$$

for each positive integer n , given by taking a complex

$$0 \rightarrow X^0 \rightarrow X^1 \rightarrow \dots \rightarrow X^{l-1} \rightarrow X^l \rightarrow 0$$

to the n -periodic

$$\dots \rightarrow \prod_{i \equiv 1(n)} X^i \rightarrow \prod_{i \equiv 2(n)} X^i \rightarrow \dots \rightarrow \prod_{i \equiv 0(n)} X^i \rightarrow \prod_{i \equiv 1(n)} X^i \rightarrow \dots$$

whose differentials, with respect to the obvious decompositions, are matrices of ‘gradable’ shape. The embedding of Lemma 3.13 is clearly the case $n = 1$.

4. Main result on the triangulated hull

The aim of the current section is to show that the embedding Δ of Lemma 3.13, and more generally the Δ_n in the remark following it, is exactly the embedding of the orbit category into its triangulated hull (Theorem 4.3). For the sake of brevity, we stick to our scheme of providing details only for the case $n = 1$.

From here on, \mathcal{B} denotes the DG category $\mathcal{B}_\Sigma = \text{per } \Lambda/\Sigma$ from Subsection 2.2 and \mathcal{B}_0 is its full DG subcategory on the single object Λ viewed as a stalk complex. For $X, Y \in \mathcal{B}$, note that each degree of the ‘mapping complex’ $\mathcal{B}(X, Y)$ is simply $\text{Hom}_\Lambda(\Delta X, \Delta Y)$ and that the differential $\mathcal{B}(X, Y)^i \rightarrow \mathcal{B}(X, Y)^{i+1}$ is given by

$$f \mapsto \varepsilon_{\Delta Y} f - (-1)^i f \varepsilon_{\Delta X}.$$

Indeed,

$$\text{per } \Lambda(X, \Sigma^j Y)^i = \prod_{l \in \mathbb{Z}} \text{Hom}_\Lambda(X^l, Y^{l+j+i}),$$

so letting j run through the integers we obtain

$$\mathcal{B}(X, Y)^i = \prod_{j \in \mathbb{Z}} \text{per } \Lambda(X, \Sigma^j Y)^i = \text{Hom}_\Lambda(\Delta X, \Delta Y)$$

for each $i \in \mathbb{Z}$. It is straightforward to check that the differential acts as claimed.

Lemma 4.1. *There is an equivalence of DG categories*

$$\text{dg-mod } \mathcal{B} \cong \text{dg-mod } \mathcal{B}_0.$$

Proof. By Lemma 2.3 and Lemma 2.2, the inclusion of \mathcal{B}_0 in \mathcal{B} yields a fully faithful DG induction functor

$$\mathcal{J}: \mathcal{B}_0^{\text{pre-tr}} \rightarrow \mathcal{B}^{\text{pre-tr}}.$$

By Lemma 2.1 it suffices to show that \mathcal{J} is dense. In our specific setup, i.e. where the objects of the DG category \mathcal{B} are complexes, all shifts and certain cones exist already in $Z^0(\mathcal{B})$. Indeed, the notion of shift is the obvious one,

and it is clear how to define the cone of each morphism

$$f = (f_i)_i \in Z^0(\mathcal{B})(X, Y) = \coprod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{C}(\Lambda)}(X, \Sigma^i Y)$$

with the property that f_i is non-zero for at most one i . This is in stark contrast to the general setup where we must pass to the module category before such constructions become available. Denoting by $\text{thick}^0(\mathcal{B}_0)$ the full subcategory of \mathcal{B} obtained as the closure of \mathcal{B}_0 under summands, shifts and cones of maps from $Z^0(\mathcal{B}_0)$ of the form of the above f , the key observation is that

$$\text{thick}^0(\mathcal{B}_0) = \mathcal{B}.$$

This is easily verified. Indeed, the complex

$$0 \rightarrow P^0 \xrightarrow{\partial^0} P^1 \rightarrow 0$$

is the cone of $\partial^0 \in Z^0(\mathcal{B})(P^0, P^1)$ and therefore belongs to $\text{thick}^0(\mathcal{B}_0)$. We hence inductively obtain each perfect complex

$$0 \rightarrow P^0 \xrightarrow{\partial^0} P^1 \rightarrow \dots \rightarrow P^{l-1} \xrightarrow{\partial^{l-1}} P^l \rightarrow 0$$

as the cone of the chain map

$$\begin{array}{ccccccc} 0 & \longrightarrow & P^0 & \xrightarrow{-\partial^0} & P^1 & \longrightarrow & \dots & \longrightarrow & P^{l-2} & \xrightarrow{-\partial^{l-2}} & P^{l-1} & \longrightarrow & 0 \\ & & & & & & & & & & & \downarrow \partial^{l-1} & & \\ & & & & & & & & & & & 0 & \longrightarrow & P^l & \longrightarrow & 0 \end{array}$$

between objects in this subcategory. Moreover, taking shifts and cones commutes with the Yoneda embedding in the sense that there are canonical isomorphisms

$$(\Sigma X)^\wedge \cong \Sigma(X^\wedge) \text{ and } (C_f)^\wedge \cong C_{f^\wedge}$$

in $\mathcal{C}(\mathcal{B})$ for each $X \in \mathcal{B}$ and $f \in Z^0(\mathcal{B})(X, Y)$ of the above form. Combining these observations with the commutativity of

$$\begin{array}{ccc} \mathcal{B}_0 & \hookrightarrow & \mathcal{B} \\ \downarrow \iota & & \downarrow \iota \\ \mathcal{B}_0^{\text{pre-tr}} & \xrightarrow{j} & \mathcal{B}^{\text{pre-tr}} \end{array}$$

it is straightforward to verify that $\mathcal{J}(\mathcal{B}_0^{\text{pre-tr}})$ is all of $\mathcal{B}^{\text{pre-tr}}$. □

The point of view that a category is merely a ring with several objects is justified also in the differential graded context. Hence we may identify the DG category \mathcal{B}_0 with the DG algebra $\Gamma = \mathcal{B}(\Lambda, \Lambda)$, which in combination with Lemma 4.1 reads

$$\text{dg-mod } \mathcal{B} \cong \text{dg-mod } \Gamma.$$

Γ is clearly the formal DG algebra $\Lambda[t, t^{-1}]$ with $|t| = 1$, and we proceed by observing that DG Γ -modules are nothing but modules over the algebra of dual numbers.

Lemma 4.2. *There is an equivalence of exact categories*

$$\mathbf{C}(\Gamma) \cong \mathbf{C}_1(\text{Mod } \Lambda).$$

Proof. A graded ring Π is said to be *strongly graded* if $\Pi_i \Pi_j = \Pi_{i+j}$ for all $i, j \in \mathbb{Z}$. By a classical theorem of Dade [8], Π is strongly graded if and only if the functor

$$- \otimes_{\Pi_0} \Pi: \text{Mod } \Pi_0 \rightarrow \text{Gr } \Pi$$

is an equivalence. In this case a quasi-inverse takes a graded Π -module M to the Π_0 -module M_0 , and we also have $\text{mod } \Pi_0 \cong \text{gr } \Pi$. Recall from Lemma 2.4 that there is an equivalence $\mathbf{C}(\Gamma) \cong \text{Gr } \Gamma(\varepsilon)$ where, since Γ is formal,

$$\Gamma(\varepsilon) = \Gamma\langle \varepsilon \rangle / (\varepsilon^2, \varepsilon\gamma - (-1)^{|\gamma|} \gamma\varepsilon)$$

is the graded algebra with $|\varepsilon| = 1$. It is easy to check that $\Gamma(\varepsilon)$ is strongly graded, and hence the proof is completed by the straightforward calculation

$$\Gamma(\varepsilon)_0 \cong \Lambda[t]/(t^2). \quad \square$$

Upon passage to the level of derived categories, the two previous lemmas reveal a triangle equivalence

$$\mathbf{D}(\mathcal{B}) \cong \mathbf{D}_1(\text{Mod } \Lambda),$$

which sets us up for proving Theorem 4.3. First, under the DG equivalence of Lemma 4.1, $\text{per } \mathcal{B}$ corresponds to $\text{per } \Gamma$. Moreover the equivalence of Lemma 4.2 identifies the free Γ -module of rank one with the differential module $(\Lambda, 0)$, making the exact category $\text{per } \mathcal{B}$ equivalent to the full subcategory of $\mathbf{C}_1(\text{mod } \Lambda)$ whose class of objects is the iterated extensions of objects

of the form $(P, 0)$ with $P \in \text{proj } \Lambda$. We have seen, using the filtration (3.7), that this is precisely the class of differential modules admitting projective flags, and hence passing to the derived level it follows that the triangulated hull $D_\Sigma(\text{mod } \Lambda)$ is all of $D_1(\text{mod } \Lambda)$. On the other hand, each perfect complex $X \in H^0(\mathcal{B}) \cong D^b(\text{mod } \Lambda)/\Sigma$ sits in $\text{dg-mod } \mathcal{B}$ as the associated representable functor and corresponds by Lemma 4.1 to the DG Γ -module $\mathcal{B}(\Lambda, X)$. Passing through the equivalence of Lemma 4.2 amounts simply to taking degree zero, and so X is further identified with

$$\mathcal{B}(\Lambda, X)^0 = \prod_{i \in \mathbb{Z}} \text{per } \Lambda(\Lambda, \Sigma^i X)^0 = \prod_{i \in \mathbb{Z}} \text{Hom}_\Lambda(\Lambda, X^i) \cong \Delta X,$$

equipped with the differential $\varepsilon_{\Delta X}$.

Theorem 4.3. *Compression of complexes, considered as a functor*

$$\Delta: D^b(\text{mod } \Lambda)/\Sigma \rightarrow D_1(\text{mod } \Lambda),$$

is precisely the embedding of the orbit category into its triangulated hull.

Remark. More generally, for each positive integer n the embedding

$$\Delta_n: D^b(\text{mod } \Lambda)/\Sigma^n \rightarrow D_n(\text{mod } \Lambda)$$

described in the remark following Lemma 3.13, is precisely the embedding of the orbit category into its triangulated hull. This is of course dense, and hence an equivalence, if and only if the compression $\Delta_n: D^b(\text{mod } \Lambda) \rightarrow D_n(\text{mod } \Lambda)$ is dense.

5. Applications

Our next goal is to determine when the orbit category $D^b(\text{mod } \Lambda)/\Sigma^n$ actually coincides with its triangulated hull. The following homogeneity property shows that for such purposes we need only consider the case $n = 1$.

Theorem (See [21, Theorem 1]). *If the orbit category $D^b(\text{mod } \Lambda)/\Sigma^n$ is triangulated for one choice of n , then it is triangulated for each n .*

5.1. Iterated tilted algebras

Our first application is Proposition 5.3 which recovers a weak version of [17, Theorem 1]. An analogous result in the context of root categories (i.e.

for $n = 2$) can be found in [20, Corollary 7.1]. Recall that if \mathbf{H} is a hereditary abelian category, then in $\mathbf{D}^b(\mathbf{H})$ there is an isomorphism

$$X \cong \prod_{i \in \mathbb{Z}} \Sigma^{-i} H^i(X)$$

for each X . We start by observing that the analogous property holds in $\mathbf{D}_1(\mathbf{H})$.

Lemma 5.1. *If \mathbf{H} is hereditary abelian, then in $\mathbf{D}_1(\mathbf{H})$ there is an isomorphism*

$$(M, \varepsilon_M) \cong (H(M), 0)$$

for each (M, ε_M) .

Proof. The idea is of course to think of global dimension i not as Ext^{i+1} vanishing, but rather as Ext^i being right exact. The assumption on \mathbf{H} hence tells us that the epimorphism $M \xrightarrow{\varepsilon_M} \text{Im}(\varepsilon_M)$ induces an exact sequence

$$\text{Ext}_{\mathbf{H}}^1(H(M), M) \rightarrow \text{Ext}_{\mathbf{H}}^1(H(M), \text{Im}(\varepsilon_M)) \rightarrow 0.$$

In particular there is a commutative diagram

$$(5.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{m} & E & \xrightarrow{p} & H(M) \longrightarrow 0 \\ & & \downarrow \varepsilon_M & & \downarrow s & & \downarrow 1_{H(M)} \\ 0 & \longrightarrow & \text{Im}(\varepsilon_M) & \longrightarrow & \text{Ker}(\varepsilon_M) & \longrightarrow & H(M) \longrightarrow 0 \end{array}$$

in \mathbf{H} with exact rows. Notice that the chain map

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{m} & E & \longrightarrow & 0 \\ & & & & \downarrow p & & \\ 0 & \longrightarrow & 0 & \longrightarrow & H(M) & \longrightarrow & 0 \end{array}$$

is a quasi-isomorphism, and so after compressing we get

$$(H(M), 0) \cong \left(M \oplus E, \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \right)$$

in $D_1(\mathbf{H})$. The kernel and image of the latter differential is E and $\text{Im}(m)$, respectively, viewed as subobjects of $M \oplus E$, and the map induced in homology by

$$f: \left(M \oplus E, \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \right) \xrightarrow{(1_M \ s)} (M, \varepsilon_M)$$

therefore appears as the following cokernel.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im}(m) & \longrightarrow & E & \longrightarrow & H(M \oplus E) \longrightarrow 0 \\ & & \downarrow s & & \downarrow s & & \downarrow Hf \\ 0 & \longrightarrow & \text{Im}(\varepsilon_M) & \longrightarrow & \text{Ker}(\varepsilon_M) & \longrightarrow & H(M) \longrightarrow 0 \end{array}$$

This, however, is nothing but diagram (5.2), which means that Hf is invertible. □

Proposition 5.3. *The orbit category $D^b(\text{mod } \Lambda)/\Sigma^n$ is triangulated whenever Λ is an iterated tilted algebra.*

Proof. It suffices to show that the compression $\Delta: D^b(\text{mod } \Lambda) \rightarrow D_1(\text{mod } \Lambda)$ is dense. This is easy, since when Λ is iterated tilted there is a hereditary algebra Λ' and a standard equivalence $D^b(\text{mod } \Lambda') \rightarrow D^b(\text{mod } \Lambda)$. From (3.2) and Lemma 3.3 this fits in a commutative diagram

$$\begin{array}{ccc} D^b(\text{mod } \Lambda') & \longrightarrow & D^b(\text{mod } \Lambda) \\ \downarrow \Delta & & \downarrow \Delta \\ D_1(\text{mod } \Lambda') & \longrightarrow & D_1(\text{mod } \Lambda) \end{array}$$

where also the bottom row is an equivalence. By Lemma 5.1 it is clear that the left hand compression functor is dense, forcing density of the right hand one. □

5.2. Non-gradable objects

We now turn to the perhaps more intriguing problem of finding algebras Λ for which $D^b(\text{mod } \Lambda)/\Sigma^n$ does *not* coincide with its triangulated hull. Akin to [2, Theorem 7.1], it turns out that the existence of an oriented cycle in the ordinary quiver of Λ is sufficient (Theorem 5.5). A fundamental tool for proving this is the following.

Proposition 5.4. *If there exists an indecomposable periodic complex of finitely generated projective Λ -modules, then the orbit category $D^b(\text{mod } \Lambda)/\Sigma^n$ is strictly smaller than its triangulated hull.*

Proof. It suffices, using the resolutions of differential modules from Subsection 3.3, to show that the compression functor

$$\Delta: K^b(\text{proj } \Lambda) \rightarrow K_1(\text{proj } \Lambda)$$

is not dense, i.e. to produce an object in $K_1(\text{proj } \Lambda)$ that is not gradable. So let

$$Y = \dots \rightarrow Y^l \xrightarrow{\partial^l} Y^1 \xrightarrow{\partial^1} Y^2 \rightarrow \dots \rightarrow Y^{l-1} \xrightarrow{\partial^{l-1}} Y^l \xrightarrow{\partial^l} Y^1 \rightarrow \dots$$

be an indecomposable complex of finitely generated projectives. Then Y is indecomposable also after we factor out homotopy, and the coproduct

$$\coprod_{i=1}^l \Sigma^i Y$$

is 1-periodic, that is a differential module. Moreover, as such it cannot be gradable. Indeed, assume it belongs to the essential image of Δ . This means there is some indecomposable $X \in K^b(\text{proj } \Lambda)$ such that

$$\coprod_{i=1}^l \Sigma^i Y \cong \coprod_{i \in \mathbb{Z}} \Sigma^i X$$

in $K_1(\text{proj } \Lambda)$. Since $\text{End}_{K^b(\Lambda)}(X)$ is local this implies that X must be a summand of at least one of the left hand summands. This is a contradiction, as Y cannot have a summand in $K^b(\text{proj } \Lambda)$. □

Theorem 5.5. *If Λ is non-triangular then $D^b(\text{mod } \Lambda)/\Sigma^n$ is strictly smaller than its triangulated hull.*

Proof. When Λ is non-triangular, its ordinary quiver contains an oriented cycle

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \longrightarrow \dots \longrightarrow (l-1) \xrightarrow{\alpha_{l-1}} l. \\ \xleftarrow{\alpha_l} 1$$

The following algorithm produces an indecomposable periodic complex of finitely generated projective Λ -modules, which is sufficient by Proposition 5.4. Start with the doubly infinite periodic sequence

$$\cdots \rightarrow P_l \xrightarrow{\alpha_l} P_1 \xrightarrow{\alpha_1} P_2 \xrightarrow{\alpha_2} P_3 \rightarrow \cdots \rightarrow P_{l-1} \xrightarrow{\alpha_{l-1}} P_l \xrightarrow{\alpha_l} P_1 \rightarrow \cdots .$$

If this is a complex, the algorithm terminates. If not, insert the maximal non-zero composition

$$a_1: P_{i_1} = P_1 \xrightarrow{\alpha_1} P_2 \rightarrow \cdots \rightarrow P_{i-1} \xrightarrow{\alpha_{i-1}} P_i = P_{i_2}$$

with the property $\alpha_i a_1 = 0$ into the previous sequence to obtain

$$\cdots \rightarrow P_l \xrightarrow{\alpha_l} P_{i_1} \xrightarrow{a_1} P_{i_2} \xrightarrow{\alpha_{i_2}} P_{i_2+1} \rightarrow \cdots \rightarrow P_{l-1} \xrightarrow{\alpha_{l-1}} P_l \xrightarrow{\alpha_l} P_1 \rightarrow \cdots .$$

If this is a complex, the algorithm terminates. If not, insert the maximal non-zero composition

$$a_2: P_{i_2} \xrightarrow{\alpha_{i_2}} P_{i_2+1} \rightarrow \cdots \rightarrow P_{j-1} \xrightarrow{\alpha_{j-1}} P_j = P_{i_3}$$

with the property $\alpha_j a_2 = 0$ into the previous sequence to obtain

$$\cdots \rightarrow P_{i_1} \xrightarrow{a_1} P_{i_2} \xrightarrow{a_2} P_{i_3} \xrightarrow{\alpha_{i_3}} P_{i_3+1} \rightarrow \cdots \rightarrow P_{l-1} \xrightarrow{\alpha_{l-1}} P_l \xrightarrow{\alpha_l} P_{i_1} \rightarrow \cdots .$$

After finitely many steps this gives a sequence

$$\cdots \rightarrow P_{i_1} \xrightarrow{a_1} P_{i_2} \rightarrow \cdots \rightarrow P_{i_{s-1}} \xrightarrow{a_{s-1}} P_{i_s} \xrightarrow{\alpha_k} P_{k+1} \rightarrow \cdots \rightarrow P_l \xrightarrow{\alpha_l} P_{i_1} \rightarrow \cdots$$

in which $\alpha_k a_{s-1} = 0$ and $a_s = \alpha_l \cdots \alpha_k \neq 0$ (possibly $a_s = \alpha_l = \alpha_k$). If $a_1 a_s = 0$, then

$$\cdots \rightarrow P_{i_1} \xrightarrow{a_1} P_{i_2} \rightarrow \cdots \rightarrow P_{i_{s-1}} \xrightarrow{a_{s-1}} P_{i_s} \xrightarrow{a_s} P_{i_1} \xrightarrow{a_1} P_{i_2} \rightarrow \cdots$$

is a complex and the algorithm terminates. If $a_1 a_s \neq 0$, then the algorithm terminates with the complex

$$\cdots \rightarrow P_{i_s} \xrightarrow{a_1 a_s} P_{i_2} \rightarrow \cdots \rightarrow P_{i_{s-1}} \xrightarrow{a_{s-1}} P_{i_s} \xrightarrow{a_1 a_s} P_{i_2} \rightarrow \cdots .$$

This is a complex which is clearly periodic and indecomposable as such. Indeed, it has an indecomposable module in each degree and non-vanishing differentials. □

In light of [17, Theorem 1], and in analogy to [2, Conjecture 1.2], the highest hope one could have for the search for more algebras whose orbit categories do not coincide with their triangulated hulls, is the following.

Conjecture. For each positive integer n , the embedding

$$\Delta_n: \mathbf{D}^b(\text{mod } \Lambda)/\Sigma^n \rightarrow \mathbf{D}_n(\text{mod } \Lambda)$$

is dense if and only if Λ is piecewise hereditary.

Recall that the *strong global dimension* of Λ , a notion first proposed by Ringel, is the supremum of the widths of the indecomposable objects in $\mathbf{K}^b(\text{proj } \Lambda)$. By the work of Happel and Zacharia in [14], Λ is piecewise hereditary if and only if it has finite strong global dimension. One approach to the conjecture might be to look for a more explicit argument showing that arbitrarily wide indecomposable objects exist in $\mathbf{K}^b(\text{proj } \Lambda)$ whenever Λ is not piecewise hereditary. Indeed, if one can show that one of these indecomposable complexes is even periodic, then the conjecture will be settled by Proposition 5.4.

In a sense this has already been done in the commutative setting. That is, in [6] Buchweitz and Flenner classify the commutative noetherian rings of finite strong global dimension. As finiteness of the strong global dimension is a local property, the characterization is furnished by the construction of an ‘iterated Koszul complex’. Explicitly, if (R, \mathbf{m}) is local and admits a regular sequence of length $l \geq 2$ then the associated Koszul complex

$$K = 0 \rightarrow K^0 \rightarrow K^1 \rightarrow \cdots \rightarrow K^{l-1} \rightarrow K^l \rightarrow 0$$

can be ‘spliced’ with $\Sigma^l K$ by taking the cone of the chain map $K \rightarrow \Sigma^l K$ whose only non-zero component is an isomorphism $K^0 \rightarrow K^l$. Factoring out an acyclic subcomplex isomorphic to $0 \rightarrow K^0 \rightarrow K^l \rightarrow 0$ yields an indecomposable complex, and it is clear how to repeat the procedure in both directions to obtain periodicity.

Secretly, a modification of the iterated Koszul complex appeared already in the algorithm in the proof of Theorem 5.5. Let us explain how this works, as a rewriting of the process in homological terms will indicate that a characterization of piecewise hereditary algebras by minimal projective resolutions of simples may prove helpful in settling our conjecture. Also, the reader might find the rephrased algorithm to be more easily applied in certain examples. So assume that the algebra Λ admits a simple module S with $\text{Ext}_\Lambda^l(S, S) \neq 0$. Then a chain map $f: P \rightarrow \Sigma^l P$ which is not null-homotopic

must exist, where P is a minimal projective resolution of S . Note that this is precisely what happens with the simple R -module R/\mathfrak{m} above, whose minimal projective resolution is precisely the Koszul complex K . Factoring out the only acyclic subcomplex of the cone C_f results in some complex C , and a decomposition $C = C_1 \oplus C_2$ will induce a decomposition in homology, which is simply $\Sigma S \oplus \Sigma^l S$. Hence we may assume that $H(C_1) = \Sigma S$ and $H(C_2) = \Sigma^l S$, which implies $C_1 = \Sigma P$ and $C_2 = \Sigma^l P$ since P is indecomposable. This is a contradiction, since a splitting of the canonical exact sequence

$$0 \rightarrow \Sigma^l P \rightarrow C_f \rightarrow \Sigma P \rightarrow 0$$

of complexes implies the vanishing of f in the homotopy category. Hence C is indecomposable, and iterating results in periodicity. More generally, a cycle of non-zero extensions between simples allows a similarly flavored combination of minimal projective resolutions, yielding an indecomposable periodic complex of projectives.

Example. Let G be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4. \\ \underbrace{\hspace{10em}}_{\delta}$$

If $\Lambda = \mathbb{k}G/(\beta\alpha, \delta\gamma)$, then $\text{Ext}_\Lambda^2(S_1, S_3) \neq 0 \neq \text{Ext}_\Lambda^2(S_3, S_1)$, corresponding to

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_3 & \xrightarrow{\gamma} & P_4 & \xrightarrow{\delta} & P_1 \longrightarrow 0 \\ & & & & \downarrow 1_{P_3} & & \\ 0 & \longrightarrow & P_1 & \xrightarrow{\alpha} & P_2 & \xrightarrow{\beta} & P_3 \longrightarrow 0 \\ & & \downarrow 1_{P_1} & & & & \\ 0 & \longrightarrow & P_3 & \xrightarrow{\gamma} & P_4 & \xrightarrow{\delta} & P_1 \longrightarrow 0. \end{array}$$

Taking cones and factoring out acyclic subcomplexes yields

$$0 \rightarrow P_3 \xrightarrow{\gamma} P_4 \xrightarrow{\alpha\delta} P_2 \xrightarrow{\gamma\beta} P_4 \xrightarrow{\delta} P_1 \rightarrow 0,$$

revealing the 2-periodic

$$\dots \rightarrow P_2 \xrightarrow{\gamma\beta} P_4 \xrightarrow{\alpha\delta} P_2 \xrightarrow{\gamma\beta} P_4 \rightarrow \dots$$

In like manner, over the algebra $\mathbb{k}G/(\gamma\beta\alpha)$ there are chain maps

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & P_1 & \xrightarrow{\beta\alpha} & P_3 & \xrightarrow{\gamma} & P_4 & \longrightarrow & 0 \\
 & & & & & & \downarrow & & 1_{P_1} & & \\
 & & 0 & \longrightarrow & P_4 & \xrightarrow{\delta} & P_1 & \longrightarrow & 0 & & \\
 & & & & & & \downarrow & & 1_{P_4} & & \\
 0 & \longrightarrow & P_1 & \xrightarrow{\beta\alpha} & P_3 & \xrightarrow{\gamma} & P_4 & \longrightarrow & 0. & &
 \end{array}$$

between minimal projective resolutions of simples. Taking cones and factoring out acyclic subcomplexes yields

$$0 \rightarrow P_1 \xrightarrow{\beta\alpha} P_3 \xrightarrow{\beta\alpha\delta\gamma} P_3 \xrightarrow{\gamma} P_4 \rightarrow 0,$$

hence the 1-periodic

$$\dots \rightarrow P_3 \xrightarrow{\beta\alpha\delta\gamma} P_3 \xrightarrow{\beta\alpha\delta\gamma} P_3 \rightarrow \dots .$$

5.3. Functors induced on orbit categories

Given a category \mathbb{T} with an automorphism F , one might expect that F induces the identity functor on \mathbb{T}/F . This famously does happen in the cluster category $D^b(\text{mod } \Lambda)/S \circ \Sigma^{-2}$, and hence the mantra ‘the cluster category is 2-Calabi–Yau’. Interestingly enough however, we have on our hands a situation that does not conform to this intuition.

Proposition 5.6. *The identity functor and Σ do not in general coincide on the orbit category $D^b(\text{mod } \Lambda)/\Sigma$.*

Let us elaborate under the necessary assumption $\text{char } \mathbb{k} \neq 2$. First of all, note that the condition $\Sigma \cong \text{id}$ on a triangulated category is a highly restrictive one. In fact, as was pointed out to the author by Steffen Oppermann, it essentially implies semisimplicity. Hence we should not expect to find such an isomorphism of functors on $D_1(\text{mod } \Lambda) \cong K_1(\text{proj } \Lambda)$. And indeed, recalling that the shift functor changes the signs of differentials, it is easy to produce an algebra Λ and an object in $K_1(\text{proj } \Lambda)$ that does not admit an isomorphism to its shift, never mind one that commutes with induced maps. Of course, one might still suspect that Σ coincides with the identity functor on the orbit category $D^b(\text{mod } \Lambda)/\Sigma$ itself, i.e. on the subcategory of gradables in $K_1(\text{proj } \Lambda)$. And admittedly, each gradable does allow an isomorphism to its shift (even worse, there are often many). However, objectwise isomorphisms $\Sigma P \cong P$ cannot constitute a natural isomorphism $\Sigma \cong \text{id}$ of functors on $D^b(\text{mod } \Lambda)/\Sigma$ in general. For example, consider the path algebra of

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

$$\quad \quad \quad \underbrace{\hspace{1.5cm}}_{\gamma}$$

modulo the ideal generated by $\beta\alpha$, and the gradable differential modules

$$Q_1 = (P_1, 0) \text{ and } Q_2 = \left(P_2 \oplus P_3, \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \right).$$

If $\Sigma \cong \text{id}$ as functors on the orbit category, then the morphism

$$f: Q_1 \xrightarrow{\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}} Q_2$$

fits in a commutative square

$$(5.7) \quad \begin{array}{ccc} Q_1 & \xrightarrow{f} & Q_2 \\ \downarrow & & \downarrow \\ \Sigma Q_1 & \xrightarrow{\Sigma f = f} & \Sigma Q_2 \end{array}$$

whose vertical arrows are isomorphisms. As this takes place in a triangulated category, an isomorphism of cones $C_f \rightarrow C_{\Sigma f}$ is induced, i.e.

$$\left(P_1 \oplus P_2 \oplus P_3, \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{pmatrix} \right) \cong \left(P_1 \oplus P_2 \oplus P_3, \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & -\beta & 0 \end{pmatrix} \right).$$

However, a straightforward calculation shows that the latter isomorphism does not exist. One can also rebut the existence of (5.7) with no mention of the axioms of triangulated categories, using only the facts that an isomorphism $Q_1 \rightarrow \Sigma Q_1$ is multiplication by λe_1 with $\lambda \in \mathbb{k}^*$ and that each isomorphism $Q_2 \rightarrow \Sigma Q_2$ is some

$$\begin{pmatrix} \lambda' e_2 & 0 \\ \lambda'' \beta & -\lambda' e_3 \end{pmatrix}$$

with $\lambda' \in \mathbb{k}^*$ and $\lambda'' \in \mathbb{k}$.

Remark. For even n and regardless of $\text{char } \mathbb{k}$, the identity and the n 'th power of the shift are precisely the same functor on $C_n(\text{mod } \Lambda)$. In particular, it follows that there is always an isomorphism $\Sigma^n \cong \text{id}$ on the orbit

category $D^b(\text{mod } \Lambda)/\Sigma^n$ in these cases. Contrarily, as long as n is odd and $\text{char } \mathbb{k} \neq 2$ one can easily mimic the above example to produce occurrences of $D^b(\text{mod } \Lambda)/\Sigma^n$ that suffer from the imperfection $\Sigma^n \not\cong \text{id}$. In light of this, one could (very naively) worry that the n -Calabi–Yau property of the n -cluster category $D^b(\text{mod } \Lambda)/\mathbb{S} \circ \Sigma^{-n}$, introduced in [17], might in fact fail unless n is even. In [19], however, Keller shows that in this orbit category the signs add up to ensure $\mathbb{S} \cong \Sigma^n$ after all.

This behavior can be traced back to the formal DG algebra $\Gamma = \Lambda[t, t^{-1}]$ with $|t| = 1$, employed in Section 4. Recall that if M is a right DG module over a DG algebra \mathcal{A} , then the right DG \mathcal{A} -module ΣM is the shifted complex with \mathcal{A} -action

$$m * a = ma.$$

On the other hand, if N is a left DG \mathcal{A} -module, then the shifted complex ΣN is a left DG \mathcal{A} -module with action given by

$$a * n = (-1)^{|a|}an.$$

Lemma 5.8. *Γ and $\Sigma\Gamma$ are isomorphic as right and as left DG Γ -modules, but not as DG Γ -bimodules. In particular $\Sigma \cong - \otimes_{\Gamma} \Sigma\Gamma$ is not isomorphic to the identity functor on the category of right DG Γ -modules.*

Proof. Let $f: \Gamma \rightarrow \Sigma\Gamma$ be a homomorphism of right DG Γ -modules, i.e. $f = (f_i)$ where $f_i \in \text{End}_{\Lambda}(\Lambda)$ for each integer i such that

$$f_{|\gamma\gamma'|}(\gamma\gamma') = f_{|\gamma|}(\gamma) * \gamma' = f_{|\gamma|}(\gamma)\gamma'$$

for homogeneous $\gamma, \gamma' \in \Gamma$. In particular picking $\gamma' = 1_{\Lambda} \in \Gamma_1$ yields $f_{|\gamma|+1} = f_{|\gamma|}$, that is f must be given by the same homomorphism in each degree. Similarly, if $g = (g_i): \Gamma \rightarrow \Sigma\Gamma$ is a homomorphism of left DG Γ -modules, then

$$g_{|\gamma'\gamma|}(\gamma'\gamma) = \gamma' * g_{|\gamma|}(\gamma) = (-1)^{|\gamma'|}\gamma'g_{|\gamma|}(\gamma),$$

and picking $\gamma' = 1_{\Lambda} \in \Gamma_1$ yields $g_{|\gamma|+1} = -g_{|\gamma|}$. Hence $\Gamma \cong \Sigma\Gamma$ on each side, but the required signs reveal that no isomorphism of DG Γ -bimodules can exist. □

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