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Crystalline lifts of two-dimensional mod p automorphic Galois representations

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We show that a sufficient condition for an irreducible automorphic Galois representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ of a totally real field F to have an automorphic crystalline lift is that for each place v of F above p the restriction $\det\rho|_{I_v}$ is a fixed power of the mod p cyclotomic character. Moreover, we show that the only obstruction to controlling the level and character of such automorphic lifts arises for badly dihedral representations.

1. Introduction

Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be a continuous irreducible odd representation of the absolute Galois group of the rationals. By Serre's conjecture, now a theorem

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of Khare-Wintenberger and Kisin, ρ is automorphic. It is moreover known that ρ arises from a modular form of level prime to p.

The analogue of the last statement for automorphic mod p representations of the absolute Galois group of a totally real number field is false in general. The purpose of this note is to give sufficient conditions for a mod p Hilbert modular form to have a lift of level prime to p, or equivalently for the associated Galois representation to have an automorphic lift which is crystalline at all primes over p. The result is motivated by a question of Dieulefait and Pacetti; see [5, Lemma 8.32] where a special case of Theorem 1.1 is used in the construction of "chains" of compatible systems of Galois representations.

We first recall the existence of an obvious obstruction. Let p be a prime number and F a totally real number field. Denote by Σ the set of embeddings of F in $\mathbf{R}$. For integers $k \geq 2$ and w having the same parity denote by $D_{k,w}$ the discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ having Blatter parameters (k, w). In particular $D_{k,w}$ has central character $t \mapsto t^{-w}$. Fix a tuple $\vec{k} = (k_\tau)_{\tau\in\Sigma} \in \mathbf{Z}^{\Sigma}_{\geq 2}$ and an integer w all sharing the same parity. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ which is holomorphic of weight $(\vec{k}, w)$, *i.e.*, such that $\pi_\tau \simeq D_{k_\tau,w}$ for all $\tau \in \Sigma$. Assume moreover that the level of π is coprime with p. If $\rho_\pi : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ denotes the p-adic Galois representation attached to π, we have $\det\rho_\pi|_{I_v} = \epsilon_v^{w-1}$ for all primes v of F dividing p, where I_v is the inertia subgroup of a decomposition group of G_F at v, and ϵ_v is the p-adic cylotomic character restricted to I_v (cf. [2, Corollary 2.11] and [4]).

We show that this condition on $\det\rho_\pi|_{I_v}$ is the only obstruction to the existence of a crystalline lift of an irreducible automorphic representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$. Moreover we control the conductor and central character of such a lift provided only that ρ is not badly dihedral (see Definition 3.3). More precisely, we prove:

Theorem 1.1. *Suppose that $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is automorphic, irreducible, and that for some integer k, we have $\det\rho|_{I_v} = \overline{\epsilon}_v^{k-1}$ for all $v|p$. Then there exists n_0 such that if $n \geq n_0$, there exists a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ such that*

- *if $v|p$, then π_v is unramified principal series;*
- *if $v|\infty$, then $\pi_v \cong D_{k+n\delta,k+n\delta}$ where $\delta = \mathrm{lcm}\{\, (p-1)/\gcd(p-1, e_v) \mid v|p \,\}$;*
- *$\overline{\rho}_\pi \cong \rho$.*

Suppose further that ρ is automorphic with prime-to-p conductor dividing $\mathfrak{n} \subset \mathcal{O}_F$, and that ψ is a finite order Hecke character of $\mathbf{A}_F^\times$ of conductor dividing $\mathfrak{n}$, totally of parity $w = k + n\delta$, and satisfying $\det\rho = \overline{\psi}\overline{\epsilon}^{w-1}$. Then if ρ is not badly dihedral, we can choose π as above with conductor dividing $\mathfrak{n}$ and central character $\psi^{-1}|\ |^{-w}$.

(Here e_v denotes the ramification degree of v over p.) We remark that we have ensured in the conclusion that the lift has parallel weight since it seems no harder to achieve and slightly simplifies the statement.

There are two main ingredients to the proof of Theorem 1.1. The first of these is Proposition 2.1 below, which is a statement purely about mod p representations of $\mathrm{GL}_2(\mathbf{F})$ where $\mathbf{F}$ is a finite field of characteristic p. The result can be deduced from the main result of [12], but we give instead a short self-contained proof that could be useful if one wishes to extract an explicit value of n_0 in the conclusion of Theorem 1.1.

We will then deduce Theorem 1.1 from standard arguments for producing congruences and liftings of cohomology classes (cf. Section 3.2). We must do some work however to show that the only obstruction to controlling the level and character is in the case of badly dihedral representations, even for $p = 2$ (cf. Sections 3.3 and 3.5). A related result is proved in [2, Lemma 4.11] using the Galois action on the cohomology of Shimura curves, but since we also wish to work with forms on definite quaternion algebras, we give a different argument in this paper by interpreting the obstruction in terms of the Hecke action. We remark that this obstruction is genuine, but that even in this case one can obtain slightly weaker results by modifying our arguments or by constructing CM lifts. We remark also that we could instead have attempted to deduce a version of Theorem 1.1 using level lowering or automorphy lifting theorems, as in [10] or [8], but this approach would have required additional hypotheses, such as adequacy of the image of ρ, and made the case of $p = 2$ even more problematic.

Finally we also give two more refined variants of the main result (Theorems 4.2 and 4.5) in the special case where the initial automorphic representation has weight $(2, \ldots, 2)$ and is unramified or special at all primes over p. This is again with a view to applications along the lines of those in [5].

2. Grothendieck ring relations

In this section we denote by $\mathbf{F}$ a fixed finite field of characteristic $p > 0$. Fix an embedding $\tau_0 : \mathbf{F} \to \overline{\mathbf{F}}_p$ and let $\tau_i = \tau_0 \circ \mathsf{Frob}^i$ where Frob is the (arithmetic) Frobenius automorphism of $\mathbf{F}$ and we sometimes view $i \in \mathbf{Z}/f\mathbf{Z}$ where $f = [\mathbf{F} : \mathbf{F}_p]$.

For $i = 0, 1, \ldots, f-1$ and $n \geq 0$, we let $\mathrm{Sym}^n_{[i]} = \overline{\mathbf{F}}_p \otimes_{\mathbf{F},\tau_i} \mathrm{Sym}^n \mathbf{F}^2$ where $\mathrm{Sym}^n \mathbf{F}^2$ denotes the nth symmetric power of the standard representation of $\mathrm{GL}_2(\mathbf{F})$ for $n \geq 0$; by convention we let $\mathrm{Sym}^{-1}_{[i]} \mathbf{F}^2 = 0$. For $\vec{n} = (n_0, n_1, \ldots, n_{f-1})$ with $n_0, \ldots, n_{f-1} \geq -1$, we let

$$S_{\vec{n}} = \otimes_{i=0}^{f-1} \mathrm{Sym}^{n_i}_{[i]}.$$

Recall that $S_{\vec{n}}$ is an irreducible representation of $\mathrm{GL}_2(\mathbf{F})$ if and only if $0 \leq n_i \leq p-1$ for all i, and that every irreducible representation of $\mathrm{GL}_2(\mathbf{F})$ over $\overline{\mathbf{F}}_p$ is of the form $\det^a \otimes S_{\vec{n}}$ for some $a \in \mathbf{Z}$ and $\vec{n}$ as above.

We let $G_0(\overline{\mathbf{F}}_p[\mathrm{GL}_2(\mathbf{F})])$ denote the Grothendieck group on finite-dimensional representations of $\mathrm{GL}_2(\mathbf{F})$ over $\overline{\mathbf{F}}_p$, which is thus isomorphic to the free abelian group generated by the classes $[\det^a \otimes S_{\vec{n}}]$ for $a = 0, \ldots, p^f - 2$ and $\vec{n} = (n_0, n_1, \ldots, n_{f-1})$ as above.

We use $\leq$ for the natural partial ordering on $G_0(\overline{\mathbf{F}}_p[\mathrm{GL}_2(\mathbf{F})])$; thus $R \leq R'$ whenever $R' - R$ is in the submonoid of $G_0(\overline{\mathbf{F}}_p[\mathrm{GL}_2(\mathbf{F})])$ consisting of classes of (actual) $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_2(\mathbf{F})$. Note that if σ and σ' are $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_2(\mathbf{F})$, then $[\sigma] \leq [\sigma']$ if and only if there is an embedding of the semisimplification of σ in that of σ'. In particular if σ is irreducible, then $[\sigma] \leq [\sigma']$ if and only if σ is a Jordan-Hölder factor of σ'.

Assume $f \geq 1$ is arbitrary. If $k < -1$ for each $i \in \mathbf{Z}/f\mathbf{Z}$ define (cf. [13]):

$$\left[\mathrm{Sym}^k_{[i]}\right] := -\left[\det^{p^i(k+1)} \otimes \mathrm{Sym}^{-k-2}_{[i]}\right].$$

In what follows we slightly abuse notation by allowing taking brackets of virtual representations in $G_0(\overline{\mathbf{F}}_p[\mathrm{GL}_2(\mathbf{F})])$.

Denote by $\mathbf{N}_{\mathbf{F}/\mathbf{F}_p}$ the field norm map for the extension $\mathbf{F}/\mathbf{F}_p$.

Proposition 2.1. *Let σ be an irreducible representation of $\mathrm{GL}_2(\mathbf{F})$ over $\overline{\mathbf{F}}_p$ with central character of the form $\mathbf{N}^{es}_{\mathbf{F}/\mathbf{F}_p}$ for some $e, s \geq 1$. Then $[\sigma] \leq \left[S^{\otimes e}_{(t,\ldots,t)}\right]$ for all sufficiently large $t \equiv s \bmod (p-1)/\gcd(p-1, e)$.*

The proof will be based on the following lemmas, which can be viewed as providing algebraic analogues of theta operators and Hasse invariants in order to shift weights of automorphic forms in characteristic p.

Let $n, m \geq 0$. The usual identification of graded $\mathbf{F}$-algebras $\operatorname{Sym}\mathbf{F}^2 \simeq \mathbf{F}[t_1, t_2]$ induces an action of $\mathrm{GL}_2(\mathbf{F})$ on $\mathbf{F}[t_1, t_2]$. When $f = 1$, multiplication by the Dickson invariant $t_1^p t_2 - t_1 t_2^p \in \mathbf{F}[t_1, t_2]^{\mathrm{SL}_2(\mathbf{F})}$ induces a $\mathrm{GL}_2(\mathbf{F})$-equivariant embedding $S_n \hookrightarrow \det^{-1} \otimes S_{n+p+1}$ ([1, Section 3]). Similarly, when $f > 1$ we obtain a $\mathrm{GL}_2(\mathbf{F})$-equivariant embedding $\mathrm{Sym}^m_{[0]} \otimes \mathrm{Sym}^n_{[1]} \hookrightarrow \det^{-p} \otimes \mathrm{Sym}^{m+p}_{[0]} \otimes \mathrm{Sym}^{n+1}_{[1]}$ induced by $t_1^p \otimes t_2 - t_2^p \otimes t_1$ ([11, Section 3.5.1]). We obtain in particular:

Lemma 2.2. *Suppose $f = 1$ and $n \geq 0$. Then $[S_n] \leq [\det^{-1} \otimes S_{n+p+1}]$.*

Lemma 2.3. *Suppose $f > 1$ and $m, n \geq 0$. Then*

$$\left[\mathrm{Sym}^m_{[0]} \otimes \mathrm{Sym}^n_{[1]}\right] \leq \left[\det{}^{-p} \otimes \mathrm{Sym}^{m+p}_{[0]} \otimes \mathrm{Sym}^{n+1}_{[1]}\right].$$

Lemma 2.4. *Suppose $f = 1$ and $n \geq 0$. Then $[S_n] \leq [S_{n+p-1}]$ unless $n = r(p+1)$ for some r, in which case we have $[S_n] \leq [S_{n+p-1} + \det^r]$.*

Proof. Serre's periodic relation:

$$[S_{n+p-1} - S_n] = [\det \otimes (S_{n-2} - S_{n-p-1})],$$

valid in $G_0(\overline{\mathbf{F}}_p[\mathrm{GL}_2(\mathbf{F})])$ for all $n \in \mathbf{Z}$ (cf. [13]) implies that positivity of $[S_{n+p-1} - S_n]$ depends only on $n \mod p+1$. If $1 \leq n < p+1$ the term $[S_{n-p-1}]$ is non-positive, so that $[S_n] \leq [S_{n+p-1}]$ when $n \not\equiv 0 \mod p+1$. When $n = r(p+1)$ we see by induction on r that $[S_{n+p-1} - S_n] = [\det^r \otimes (S_{p-1} - 1)]$. □

Lemma 2.5. *Suppose $f > 1$ and $np > m \geq 0$. Then*

$$\left[\mathrm{Sym}^m_{[0]} \otimes \mathrm{Sym}^n_{[1]}\right] \leq \left[\mathrm{Sym}^{m+p}_{[0]} \otimes \mathrm{Sym}^{n-1}_{[1]}\right].$$

Proof. We proceed by induction on n. The statement for $n = 1$ follows from the identity $\left[\mathrm{Sym}^{m+p}_{[0]}\right] = \left[\mathrm{Sym}^m_{[0]} \otimes \mathrm{Sym}^1_{[1]} - \det^p \otimes \mathrm{Sym}^{m-p}_{[0]}\right]$ (cf. last equation in [11, Theorem 2.7]), together with the fact that $\left[\mathrm{Sym}^{m-p}_{[0]}\right] \leq 0$ since $m < p$. Assuming now the statement for a fixed $n = n_0 > 0$ and letting

$0 \le m < (n_0+1)p$, we have, using again the above identity:

$$\begin{aligned}&\left[\mathrm{Sym}^{m+p}_{[0]} \otimes \mathrm{Sym}^{n_0}_{[1]}\right]\\ &= \left[\mathrm{Sym}^{m}_{[0]} \otimes \mathrm{Sym}^{1}_{[1]} \otimes \mathrm{Sym}^{n_0}_{[1]} - \det{}^p \otimes \mathrm{Sym}^{m-p}_{[0]} \otimes \mathrm{Sym}^{n_0}_{[1]}\right]\\ &= \left[\mathrm{Sym}^{m}_{[0]} \otimes \mathrm{Sym}^{n_0+1}_{[1]} + \det{}^p \otimes \left(\mathrm{Sym}^{m}_{[0]} \otimes \mathrm{Sym}^{n_0-1}_{[1]} - \mathrm{Sym}^{m-p}_{[0]} \otimes \mathrm{Sym}^{n_0}_{[1]}\right)\right],\end{aligned}$$

If $m < p$, then $\left[\mathrm{Sym}^{m-p}_{[0]}\right] \le 0$ in $G_0(\overline{\mathbf{F}}_p[\mathrm{GL}_2(\mathbf{F})])$, so the expression between rounded parenthesis is positive, implying the statement for n_0+1. If $m \ge p$, then $0 \le m-p < n_0 p$ so that $\left[\mathrm{Sym}^{m-p}_{[0]} \otimes \mathrm{Sym}^{n_0}_{[1]}\right] \le \left[\mathrm{Sym}^{m}_{[0]} \otimes \mathrm{Sym}^{n_0-1}_{[1]}\right]$. The result follows. □

We now proceed with the proof of Proposition 2.1. Suppose that $\sigma = \det^a \otimes S_{\vec{n}}$ with $a \ge 0$ and $\vec{n} = (n_0, n_1, \ldots, n_{f-1})$ with $0 \le n_i \le p-1$.

We first treat the case $e = f = 1$. By Lemma 2.2 and induction on n , we have $[\sigma] \le [S_{n+a(p+1)}]$. Let $t_0 = n + a(p+1)$, and note that $t_0 \equiv s \bmod p-1$. If $n = 0$, then we may replace a by $a+p-1$ so as to assume $t_0 \ge p^2-1$. We claim that $[\sigma] \le [S_t]$ for all $t \equiv s \bmod p-1$ with $t \ge t_0$. Indeed it suffices to prove that if $t \ge t_0$ and $[\sigma] \le [S_t]$, then $[\sigma] \le [S_{t+p-1}]$, and this is immediate from Lemma 2.4 except in the case $\sigma = \det^r$, $t = r(p+1)$. Note however that Lemma 2.2 implies that $(b+1)[\det^u] \le [S_{u(p+1)}]$ if $u \ge (p-1)b$; in particular $2[\det^r] \le [S_t]$ if $t = r(p+1) \ge p^2-1$, so in this case it again follows from Lemma 2.4 that $[\sigma] = [\det^r] \le [S_{t+p-1}]$.

Next we treat the case $e = 1$, $f > 1$. Note that Lemma 2.3 implies that

$$\left[\mathrm{Sym}^{m}_{[i]} \otimes \mathrm{Sym}^{n}_{[i+1]}\right] \le \left[\det{}^{-p^{i+1}} \otimes \mathrm{Sym}^{m+p}_{[i]} \otimes \mathrm{Sym}^{n+1}_{[i+1]}\right]$$

for any $m, n \ge 0$, $i \in \mathbf{Z}/f\mathbf{Z}$, and hence that

$$[S_{\vec{m}}] \le \left[\det{}^{-\sum_{i=0}^{f-1} b_i p^i} \otimes S_{\vec{m}'}\right] \tag{1}$$

for any $m_0, \ldots, m_{f-1}, b_0, \ldots, b_{f-1} \ge 0$, where $\vec{m} = (m_0, \ldots, m_{f-1})$ and $\vec{m}' = (m_0 + b_0 + pb_1, \ldots, m_{f-1} + b_{f-1} + pb_0)$. In particular, if $a = \sum_{i=0}^{f-1} b_i p^i$ with $b_0, \ldots, b_{f-1} \ge 0$, then $[\sigma] \le [S_{\vec{n}'}]$ where $\vec{n}' = (n'_0, \ldots, n'_{f-1})$ with $n'_i = n_i + b_i + pb_{i+1}$. Note that

$$\sum_{i=0}^{f-1} n'_i p^i \equiv 2a + \sum_{i=0}^{f-1} n_i p^i \equiv s\left(\sum_{i=0}^{f-1} p^i\right) \bmod p^f - 1,$$

from which it follows that $\sum_{i=0}^{f-1} n'_{i+j}p^i$ is divisible by $\sum_{i=0}^{f-1} p^i$ for $j = 0, \dots, f-1$.

Now consider the system of equations

$$n'_0 - x_0 + px_1 = n'_1 - x_1 + px_2 = \cdots = n'_{f-1} - x_{f-1} + px_0. \tag{2}$$

For any $x_0 \in \mathbf{Z}$, we obtain a solution with $x_0, \dots, x_{f-1} \in \mathbf{Z}$ by setting

$$x_j = x_{j-1} - n'_{j-1} + \left(\sum_{i=0}^{f-1} n'_{i+j}p^i\right) \Big/ \left(\sum_{i=0}^{f-1} p^i\right)$$

for $j = 1, \dots, f-1$. In particular we may choose a solution of (2) with $x_0, \dots, x_{f-1}$ non-negative integers.

We now wish to apply Lemma 2.5, or rather its twist by Frob^i, iteratively x_{i+1} times for $i = 0, \dots, f-1$ in order to conclude that $[\sigma] \le [S_{(t,\dots,t)}]$ where t is the common value of $n'_i - x_i + px_{i+1}$, but we must first ensure that the inequality $np > m$ in the hypothesis of the lemma is satisfied at each stage. To this end, note that we may replace $\vec{n}'$ by $\vec{n}'' = \vec{n}' + r(p^2-1, \dots, p^2-1)$ for any integer $r \ge 0$; indeed $[\sigma] \le [S_{\vec{n}'}] \le [S_{\vec{n}''}]$ by (1), and (2) still holds with each n'_i replaced by $n''_i = n'_i + r(p^2-1)$. Choosing r so that

$$r(p-1)(p^2-1) > n'_i - pn'_{i+1} + 2px_{i+1}$$

for each i, we find that $p(n''_{i+1} - x_{i+1}) > n''_i + px_{i+1}$. Now by Lemma 2.5 and induction on $\sum_{i=0}^{f-1} d_i$, we see that if $0 \le d_i \le x_i$ for $i = 0, \dots, f-1$, then $[S_{\vec{n}''}] \le [S_{\vec{n}'''}]$ where $n'''_i = n''_i - d_i + pd_{i+1}$. It follows that $[\sigma] \le [S_{(t,\dots,t)}]$ where t is the common value of $n''_i - x_i + px_{i+1}$. Similarly we find that if $t > 0$, then $[S_{(t,\dots,t)}] \le [S_{(t+p-1,\dots,t+p-1)}]$, which completes the proof in the case $e = 1$, $f > 1$.

Finally we treat the case $e > 1$. From the case $e = 1$, we have that $[\sigma] \le [S_{(u,\dots,u)}]$ for all sufficiently large $u \equiv es \bmod p-1$, hence $[\sigma] \le [S_{(et,\dots,et)}]$ for all sufficiently large $t \equiv s \bmod (p-1)/\gcd(e, p-1)$. From the natural surjection

$$\left(\mathrm{Sym}^t_{[i]}\right)^{\otimes e} \to \mathrm{Sym}^{et}_{[i]}$$

we see that $[S_{(et,\dots,et)}] \le \left[S^{\otimes e}_{(t,\dots,t)}\right]$, concluding the proof of Proposition 2.1.

Remark 2.6. When $f = 1$, multiplication by the Dickson invariant

$$(t_1^{p^2} t_2 - t_1 t_2^{p^2})/(t_1^p t_2 - t_1 t_2^p) \in (\mathrm{Sym}\,\mathbf{F}^2)^{\mathrm{GL}_2(\mathbf{F})}$$

induces a $\mathrm{GL}_2(\mathbf{F})$-equivariant injection $S_k \to S_{k+p(p-1)}$ for all $k \geq 0$. Notice that the change in weight produced by this operator does not allow us to prove the desired result.

3. Lifting to characteristic zero

The first part of Theorem 1.1 is proved in Section 3.2 below. In Sections 3.3 and 3.5 we refine the argument to control the level and character of the crystalline lifts we produce, thus proving the second part of Theorem 1.1. We begin by fixing some notation.

3.1. Notation

We normalize local and global class field theory so that geometric Frobenius elements correspond to uniformizers, and we adopt Hecke's normalizations of local-global compatibility when associating a Galois representation to an automorphic form. These are the normalizations adopted in [4] and [2].

Let p be a prime number. We fix an algebraic closure $\overline{\mathbf{Q}}$ (resp. $\overline{\mathbf{Q}}_p$) of the field $\mathbf{Q}$ of rational numbers (resp. of the field $\mathbf{Q}_p$ of p-adic numbers). We choose an embedding $\overline{\mathbf{Q}} \to \mathbf{C}$ and an isomorphism $\overline{\mathbf{Q}}_p \cong \mathbf{C}$, so that we also identify $\overline{\mathbf{Q}}$ with a subfield of $\overline{\mathbf{Q}}_p$. Denote by $\overline{\mathbf{F}}_p$ a fixed algebraic closure of the field $\mathbf{F}_p$ with p elements.

Let $F \subset \overline{\mathbf{Q}}$ be a totally real number field. Denote by $G_F = \mathrm{Gal}(\overline{\mathbf{Q}}/F)$ its absolute Galois group, and by $\epsilon : G_F \to \mathbf{Z}_p^\times$ the p-adic cyclotomic character. Let Σ be the set of embeddings of F in $\overline{\mathbf{Q}}$.

For each finite place x of F we denote by F_x the completion of F at x, and by $\mathcal{O}_{F_x}$ its ring of integers. We let $\mathbf{A}_F$ (resp. $\mathbf{A}_{F,f}$) denote the topological ring of adèles (resp. finite adèles) of F.

Let v be a place of F lying above p. Let G_v denote the decomposition group of G_F at v induced by $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p$, and I_v be its inertia subgroup. We let k_v be the residue field of $\mathcal{O}_{F_v}$, and we set $f_v := [k_v : \mathbf{F}_p]$. Denote by e_v the absolute inertial degree of v, and by $\overline{\Sigma}_v$ be the set of embeddings of k_v in $\overline{\mathbf{F}}_p$. We let ϵ_v denote the restriction of the p-adic cyclotomic character to G_v or to I_v. The reduction modulo p of ϵ_v is denoted by $\overline{\epsilon}_v$.

For any $\tau \in \overline{\Sigma}_v$ we denote by ω_τ the corresponding fundamental character of I_v, defined as the composition $I_v \longrightarrow \mathcal{O}_v^\times \longrightarrow k_v^\times \xrightarrow{\tau} \overline{\mathbf{F}}_p^{\,\times}$, where the first map is the restriction of the inverse of the reciprocity isomorphism of local class field theory. Recall that the restriction to I_v of the local mod p cyclotomic character $\bar{\epsilon}_v$ of G_v is given by $\prod_{\tau\in\overline{\Sigma}_v}\omega_\tau^{e_v}$.

For integers $k\geq 2$ and w having the same parity we denote by $D_{k,w}$ the discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ having Blatter parameters (k,w), and hence central character $t\mapsto t^{-w}$.

3.2. Existence of lifts

For each place $v|p$ of F, fix an embedding $\tau_{v,0}: k_v\to\overline{\mathbf{F}}_p$ and, as in Section 2, set $\tau_{v,i}=\tau_{v,0}\circ\mathsf{Frob}_v^i$ where Frob_v denotes the arithmetic Frobenius of k_v and $i\in\mathbf{Z}/f_v\mathbf{Z}$. For any $\vec{n}=(n_0,\ldots,n_{f_v-1})$ with $n_0,\ldots,n_{f_v-1}\geq -1$ let

$$S_{v,\vec{n}}=\bigotimes_{i=0}^{i=f_v-1}\left(\overline{\mathbf{F}}_p\otimes_{k_v,\tau_{v,i}}\mathrm{Sym}^{n_i}k_v^2\right),$$

viewed as an $\overline{\mathbf{F}}_p$-linear representation of $\mathrm{GL}_2(k_v)$.

Suppose now that $\rho: G_F\to\mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is the modular Galois representation in the statement of Theorem 1.1. Assume that σ is a Serre weight for ρ in the sense of [2]; in particular σ is an $\overline{\mathbf{F}}_p$-linear irreducible representation of $\mathrm{GL}_2(\mathcal{O}_F/(p))=\prod_{v|p}\mathrm{GL}_2(\mathcal{O}_F/(v^{e_v}))$, and therefore it can be written as $\sigma=\otimes_{v|p}\sigma_v$ where each σ_v is an irreducible representation of $\mathrm{GL}_2(k_v)$.

Denote by $\mathbf{N}_{k_v/\mathbf{F}_p}$ the norm map attached to the field extension $k_v/\mathbf{F}_p$. Slightly modifying the proof of [2, Corollary 2.11] by taking the map $N:(\mathcal{O}_F/(p))^\times\to\mathbf{F}_p^\times$ considered there to be $\prod_{v|p}\mathbf{N}_{k_v/\mathbf{F}_p}^{e_v}$, and by applying the generalization to the ramified settings of [2, Proposition 2.10], we deduce that if the central character of σ_v is given by $\prod_{\tau\in\overline{\Sigma}_v}\tau^{c_\tau-2e_v}$ for some integers c_τ, then

$$\det\rho|_{I_v}=\prod_{\tau\in\overline{\Sigma}_v}\omega_\tau^{c_\tau-e_v}.$$

For any prime v of F above p we have $\det\rho|_{I_v}=\bar{\epsilon}_v^{k-1}=\prod_{\tau\in\overline{\Sigma}_v}\omega_\tau^{e_v(k-1)}$ by assumption. We deduce therefore that the central character of σ_v is given by:

$$\mathbf{N}_{k_v/\mathbf{F}_p}^{e_v(k-2)}.$$

Proposition 2.1 implies that σ_v is a Jordan-Hölder factor of $S_{v,(t_v,\ldots,t_v)}^{\otimes e_v}$ for all sufficiently large $t_v\equiv k-2 \bmod (p-1)/\gcd(p-1,e_v)$. Define

$$\delta:=\mathrm{lcm}\{\,(p-1)/\gcd(p-1,e_v)\,|\,v|p\,\}.$$

We can thus find a non-negative integer n_0 such that for all $n \geq n_0$ we have $k-2+n\delta \geq 0$ and the weight σ is a Jordan-Hölder factor of the $\mathrm{GL}_2(\mathcal{O}_F/(p))$-representation $\otimes_{v|p} S_{v,(k-2+n\delta,\ldots,k-2+n\delta)}^{\otimes e_v}$. By (a generalization to the ramified settings of) [2, Proposition 2.5] we deduce that ρ arises from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ having level prime to p and such that $\pi_v \simeq D_{k+n\delta,k+n\delta}$ for all places $v|\infty$ of F. This proves the first part of Theorem 1.1.

3.3. The refinement of the argument in the case $[F:\mathbf{Q}]$ even

We now explain how to refine the argument above in order to control the level and character of π, and prove the second half of Theorem 1.1. The case of $[F:\mathbf{Q}]$ odd can be treated by modifying the argument above and using results in the proof of [2, Lemma 4.11] to show that obstructions arise only for badly dihedral representations. For the case of $[F:\mathbf{Q}]$ even, we will need to use forms on definite quaternion algebras and prove analogous results concerning the obstructions, which we proceed to do first.

3.3.1. Automorphic forms on definite quaternion algebras. Suppose that $[F:\mathbf{Q}]$ is even and let D be the totally definite quaternion algebra over F which splits at all finite places of F. Let $\mathcal{O}_D$ be a fixed maximal order in D, and choose isomorphisms of $\mathcal{O}_{F_x}$-algebras $\mathcal{O}_{D,x} \cong M_2(\mathcal{O}_{F_x})$ for each finite place x of F. Let $U = \prod_x U_x$ be an open compact subgroup of $D_f^\times := (D \otimes_F \mathbf{A}_{F,f})^\times$ such that $U_x \subset \mathcal{O}_{D,x}^\times$ for each finite place x. Let A denote the field $\overline{\mathbf{F}}_p$ or a topological $\mathbf{Z}_p$-algebra of finite type, and fix a continuous representation of $U\mathbf{A}_{F,f}^\times/F^\times$ on a finitely generated (topological) A-module V. Let

$$S_V(U) = \{\ f : D_f^\times \to V \mid f(\gamma g u) = u^{-1} f(g) \quad \text{for all } \gamma \in D^\times,\ g \in D_f^\times,\ u \in U\mathbf{A}_{F,f}^\times\ \}.$$

Write $D_f^\times = \coprod_{i\in I} D^\times t_i U \mathbf{A}_{F,f}^\times$ where I is a finite set, and let Γ_i denote the finite group $F^\times \backslash (U\mathbf{A}_{F,f}^\times \cap t_i^{-1} D^\times t_i)$, so that we have an isomorphism of A-modules:

$$S_V(U) \xrightarrow{\simeq} \oplus_{i\in I} V^{\Gamma_i} \tag{3}$$

induced by $f \mapsto \oplus_{i\in I} f(t_i)$.

Let S be a finite set of finite places of F containing the places dividing p and the places x such that U_x is not maximal. Let $U_S := \prod_{x\in S} U_x$

and suppose further that the action of U on V factors through the projection to U_S. For $x \notin S$ fix a choice of uniformizer ϖ_x of $\mathcal{O}_{F_x}$, and write $U_x \Pi_x U_x = \coprod_\alpha h_\alpha U_x$, where $\Pi_x = \left(\begin{smallmatrix} \varpi_x & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(F_x) \cong (D \otimes_F F_x)^\times$. We define the Hecke operator T_x acting on $f \in S_V(U)$ by

$$(T_x f)(g) := \sum_\alpha f(g h_\alpha)$$

for all $g \in D_f^\times$. The Hecke algebra $\mathbf{T}_A^S := A[T_x : x \notin S]$ acts on $S_V(U)$. With a slight abuse of notation, we will often not indicate the weight and level of automorphic forms on which $\mathbf{T}_A^S$ acts.

Let $\vec{k} = (k_\tau)_{\tau\in\Sigma} \in \mathbf{Z}_{\geq 2}^\Sigma$ and $w \in \mathbf{Z}$ such that $k_\tau \equiv w \bmod 2$ for all $\tau \in \Sigma$, and let ψ a finite order Hecke character of $\mathbf{A}_F^\times$, totally of parity w, so $\psi(x) = \psi_f(x_f)\prod_{\tau|\infty} \mathrm{sign}(x_\tau)^w$ for some character ψ_f of $\mathbf{A}_{F,f}^\times$. Let $E \subset \overline{\mathbf{Q}}_p \cong \mathbf{C}$ be a sufficiently large finite extension of $\mathbf{Q}_p$; in particular we assume E contains the values of ψ and the images of all the embeddings $F \to \overline{\mathbf{Q}}_p$. We suppose further that for each embedding $\tau : F \to E$, we have a splitting $D \otimes_{F,\tau} E \cong M_2(E)$ so that if v is the place of F induced by τ, then the projection of U to $(D \otimes_F F_v)^\times$ is contained in $\mathrm{GL}_2(\mathcal{O}_E)$. Thus for each $\tau \in \Sigma$, we obtain a map $U \to \mathrm{GL}_2(\mathcal{O}_E)$, and hence an action of U on $\det^{(w-k_\tau+2)/2} \otimes \mathrm{Sym}^{k_\tau-2}\mathcal{O}_E^2$. Suppose now that ψ_f is trivial on $U \cap \mathbf{A}_{F,f}^\times$, and let $V_{\vec{k},w,\psi}$ be the representation of $U\mathbf{A}_{F,f}^\times$ whose restriction to U is defined by

$$\otimes_{\tau\in\Sigma}(\det^{(w-k_\tau+2)/2} \otimes \mathrm{Sym}^{k_\tau-2}\mathcal{O}_E^2),$$

and whose restriction to $\mathbf{A}_{F,f}^\times$ is defined by the character

$$x \mapsto \mathbf{N}(x_p)^w |x|^w \psi_f(x).$$

We write $S_{\vec{k},w,\psi}(U)$ for $S_{V_{\vec{k},w,\psi}}(U)$, and we define $S^{\mathrm{triv}}_{\vec{k},w,\psi}(U) := \{0\}$, unless $\vec{k} = \vec{2}$, in which case we let $S^{\mathrm{triv}}_{\vec{k},w,\psi}(U)$ consist of those functions in $S_{\vec{k},w,\psi}(U)$ that factor through the reduced norm map $D_f^\times \cong \mathrm{GL}_2(\mathbf{A}_{F,f}) \xrightarrow{\det} \mathbf{A}_{F,f}^\times$. Setting $S_0 = S_{\vec{k},w,\psi}(U)/S^{\mathrm{triv}}_{\vec{k},w,\psi}(U)$, we have by the Jacquet-Langlands correspondence that $S_0 \otimes_{\mathcal{O}_E} \mathbf{C} \cong \oplus_\pi \pi_f^U$, the direct sum running over all holomorphic cuspidal automorphic representations $\pi = \pi_\infty \otimes \pi_f$ of $\mathrm{GL}_2(\mathbf{A}_F)$ such that $\pi_\tau \cong D_{k_\tau,w}$ for all $\tau \in \Sigma$, and π has central character $\psi^{-1}|\ |^{-w}$.

3.3.2. Conclusion of the argument. We fix an irreducible representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ as in Theorem 1.1. Assume that ρ arises from a holomorphic cuspidal automorphic form π' for $\mathrm{GL}_2(\mathbf{A}_F)$ of paritious weight

$(\vec{k},w) \in \mathbf{Z}^{\Sigma}_{\geq 2} \times \mathbf{Z}$, central character $\psi^{-1}|\ |^{-w}$, and level $U = U_p U^p$, where $U_p = \prod_{v|p} U_v$, $U^p = \prod_{x \nmid p} U_x$, and $U_x \subset \mathrm{GL}_2(\mathcal{O}_{F_x})$ for all x. Let S be a finite set of finite places of F containing the places of F above p and the places at which π' is ramified. Let $\mathfrak{m}_\rho$ denote the maximal ideal of the Hecke algebra $\mathbf{T}^S_{\mathcal{O}_E}$ attached to ρ; thus with our normalizations, $\mathfrak{m}_\rho$ is the kernel of the homomorphism $\mathbf{T}^S_{\mathcal{O}_E} \to \overline{\mathbf{F}}_p$ defined by

$$T_x \mapsto \det(\rho(\mathsf{Frob}_x))^{-1}\mathbf{N}(x)\mathrm{tr}(\rho(\mathsf{Frob}_x)) = \overline{\psi}(\varpi_x)^{-1}\mathbf{N}(x)^w\mathrm{tr}(\rho(\mathsf{Frob}_x))$$

for all $x \notin S$. By what was recalled in 3.3.1 we know that $S_{\vec{k},w,\psi}(U)_{\mathfrak{m}_\rho} \neq 0$.

In the next section we will prove the following:

Lemma 3.1. *If ρ is not badly dihedral in the sense of Definition 3.3 below, then the functor $\overline{V} \mapsto S_{\overline{V}}(U)_{\mathfrak{m}_\rho}$ from finite dimensional $\overline{\mathbf{F}}_p$-vector spaces endowed with a continuous action of $U_S\mathbf{A}^\times_{F,f}/F^\times$ to $\mathbf{T}^S_{\overline{\mathbf{F}}_p,\mathfrak{m}_\rho}$-modules is exact.*

Let $U_* := \mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p) \cdot U^p$, and notice there is a Hecke equivariant injection $S_{\vec{k},w,\psi}(U) \to S_{V'}(U_*)$ where $V' = \mathrm{Ind}^{U_*\mathbf{A}^\times_{F,f}}_{U\mathbf{A}^\times_{F,f}} V_{\vec{k},w,\psi}$. In particular, we also have that $S_{V'}(U_*)_{\mathfrak{m}_\rho} \neq 0$. Fix an embedding of the residue field of $\mathcal{O}_E$ into $\overline{\mathbf{F}}_p$. Using Lemma 3.1 we see that $S_\sigma(U_*)_{\mathfrak{m}_\rho} \neq 0$ for some Jordan-Hölder constituent σ of $V' \otimes_{\mathcal{O}_E} \overline{\mathbf{F}}_p$ for the action of $\mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbf{Z}} \mathbf{Z}_p)$. The assumption on $\rho|_{I_v}$ implies by Proposition 2.1 that σ is a constituent of the $\overline{\mathbf{F}}_p$-linear representation

$$\otimes_{v|p}(\det \otimes S^{\otimes e_v}_{v,(k'-2,\ldots,k'-2)})$$

for all sufficiently large $k' \equiv k \bmod \delta$, where δ as in the statement of Theorem 1.1. It follows that $\mathfrak{m}_\rho$ is in the support of $S_W(U_*)$, where W is the $\mathcal{O}_E$-linear representation $V_{\vec{k'},k',\psi}$ (this again uses Lemma 3.1). Now the result follows by applying the Jacquet-Langlands correspondence to $S_W(U_*) \otimes_{\mathcal{O}_E} \mathbf{C}$ to produce a holomorphic Hilbert modular form with desired weight, level, and central character.

3.4. Proof of Lemma 3.1

We keep the assumptions and notation from the previous section. Suppose that $0 \to V \to V_1 \to V_2 \to 0$ is an exact sequence of finite dimensional $\overline{\mathbf{F}}_p$-vector spaces endowed with a continuous action of $U\mathbf{A}^\times_{F,f}$ factoring through $U_S\mathbf{A}^\times_{F,f}$. By (3) we obtain the exact sequence:

$$(4) \qquad 0 \to S_V(U) \to S_{V_1}(U) \to S_{V_2}(U) \to \oplus_{i\in I} H^1(\Gamma_i, V),$$

where $\Gamma_i = F^\times\backslash(U\mathbf{A}_{F,f}^\times \cap t_i^{-1}D^\times t_i)$ and $D_f^\times = \coprod_{i\in I} D^\times t_i U\mathbf{A}_{F,f}^\times$. Notice the last term in (4) vanishes if $[F(\mu_p):F] > 2$, which occurs for example when $p > 3$ is unramified in $F/\mathbf{Q}$. We will show that, in general, it vanishes after localization at $\mathfrak{m}_\rho$ if ρ is not badly dihedral.

Note that if we choose another representative $t_i' = \delta t_i u$ for the double coset $D^\times t_i U\mathbf{A}_{F,f}^\times$ with $\delta \in D^\times$, $u \in U\mathbf{A}_{F,f}^\times$ and set $\Gamma_i' = F^\times\backslash(U\mathbf{A}_{F,f}^\times \cap (t_i')^{-1}D^\times t_i')$, then $\Gamma_i = u\Gamma_i' u^{-1}$ and we obtain canonical isomorphisms $H^j(\Gamma_i, V) \to H^j(\Gamma_i', V)$ induced by the isomorphisms

$$\begin{array}{ccccccc} \Gamma_i' & \to & \Gamma_i, & \qquad & V & \to & \mathrm{Res}_{\Gamma_i}^{\Gamma_i'} V \\ g & \mapsto & ugu^{-1}, & & v & \mapsto & uv. \end{array}$$

We let $X_U = D^\times\backslash D_f^\times/U\mathbf{A}_{F,f}^\times$ and write $H^j(X_U, V)$ for $\oplus_{i\in I} H^j(\Gamma_i, V)$; this is independent of the choice of the t_i up to canonical isomorphism[1], which is moreover compatible with the isomorphism $S_V(U) \cong H^0(X_U, V)$ in the evident sense. We may thus rewrite the exact sequence (4) as

$$0 \to H^0(X_U, V) \to H^0(X_U, V_1) \to H^0(X_U, V_2) \to H^1(X_U, V).$$

3.4.1. The Hecke action on $H^1(X_U, V)$. For $x \notin S$, the Hecke operator T_x acting on $S_V(U)$ can be defined as a composite $\mathrm{tr} \circ (\Pi_x)_* \circ \mathrm{res}$ where res (resp. tr) is a restriction (resp. trace) map to (resp. from) forms with respect to a smaller open compact subgroup, and $(\Pi_x)_*$ is induced by Π_x. More precisely, consider the natural projection $X_{U'} \to X_U$ where $U' = U \cap \Pi_x^{-1} U \Pi_x$, and for each double coset $D^\times t_i U\mathbf{A}_{F,f}^\times$ in X_U, let $\{t_{ij}\}$ be representatives of the preimage in $X_{U'}$ (so $D^\times t_i U\mathbf{A}_{F,f}^\times$ is the disjoint union over j of the $D^\times t_{ij} U'\mathbf{A}_{F,f}^\times$). Let Γ_{ij}' be the corresponding stabilizers, so $\Gamma_{ij}' = F^\times\backslash(U'\mathbf{A}_{F,f}^\times \cap t_{ij}^{-1}D^\times t_{ij})$. Writing $t_i = \delta t_{ij} u$ for some $\delta \in D^\times$, $u \in U'\mathbf{A}_{F,f}^\times$, we see that $v \mapsto uv$ defines a map $V \to V$ compatible with the inclusion $\Gamma_{ij}' \to \Gamma_i$ defined by conjugation by u, and this gives a map

$$H^1(\Gamma_i, V) \to \oplus_j H^1(\Gamma_{ij}', V).$$

Taking the direct sum over i of these maps, the resulting map $\mathrm{res} : H^1(X_U, V) \to H^1(X_{U'}, V)$ is independent of the choices of double coset representatives.

[1] Alternatively one can arrive at this notation by defining a Grothendieck topology on the groupoid fibered over X_U by the Γ_i and viewing V as a sheaf on the associated site.

Similarly we define $(\Pi_x)_*$ using the bijection $X_{U''} \to X_{U'}$ induced by right multiplication by Π_x, where $U'' = \Pi_x U' \Pi_x^{-1}$. Since

$$D_f^\times = \coprod_{i,j} D^\times t_{ij} U' \mathbf{A}_{F,f}^\times,$$

we have $D_f^\times = \coprod_{i,j} D^\times t_{ij} \Pi_x^{-1} U'' \mathbf{A}_{F,f}^\times$ and the corresponding stabilizer Γ''_{ij} equals $\Pi_x \Gamma'_{ij} \Pi_x^{-1}$. Since U_x acts trivially on V, the isomorphism between the groups Γ'_{ij} and Γ''_{ij} defined by conjugation by Π_x is compatible with their action on V, so it induces an isomorphism $H^1(\Gamma'_{ij}, V) \to H^1(\Gamma''_{ij}, V)$. Taking the direct sum of these isomorphisms gives a well-defined map $(\Pi_x)_* : H^1(X_{U'}, V) \to H^1(X_{U''}, V)$.

Finally tr is defined similarly to res but using $X_{U''} \to X_U$ and transfer maps on cohomology. It is then easy to see that the composite $\mathrm{tr} \circ (\Pi_x)_* \circ \mathrm{res}$ is compatible with (4) and the Hecke operators on the $S_V(U)$ since each of res, Π_x^* and tr is compatible with (4) in the obvious sense. Note that in fact $T_x = \mathrm{tr} \circ (\Pi_x)_* \circ \mathrm{res}$ on $H^0(X_U, V) = S_V(U)$.

More generally if $x_1, x_2, \ldots, x_m$ are distinct primes of F not in S, then we define a Hecke operator $T_{x_1 x_2 \cdots x_m}$ exactly as above, but replacing Π_x by the product of the Π_{x_i}, which we denote by $\Pi_{x_1 \cdots x_m}$.

Lemma 3.2. *We have $T_{x_1 x_2 \cdots x_m} = T_{x_1} T_{x_2} \cdots T_{x_m}$. In particular the operators T_{x_i} commute and $\mathbf{T}^S_{\overline{\mathbf{F}}_p}$ acts on $H^1(X_U, V)$.*

Proof. Consider the diagram:

$$\begin{array}{ccccccc}
H^1(X_U,V) & \to & H^1(X_{U_1'},V) & \to & H^1(X_{U_1''},V) & \to & H^1(X_U,V)\\
& \searrow & \downarrow & & \downarrow & & \downarrow\\
& & H^1(X_{U_1'\cap U_2'},V) & \to & H^1(X_{U_1''\cap U_2'},V) & \to & H^1(X_{U_2'},V)\\
& & & \searrow & \downarrow & & \downarrow \\
& & & & H^1(X_{U_1''\cap U_2''},V) & \to & H^1(X_{U_2''},V)\\
& & & & & \searrow & \downarrow\\
& & & & & & H^1(X_U,V),
\end{array}$$

where

- $U_1' = U \cap \Pi_{x_1}^{-1} U \Pi_{x_1}$, $U_1'' = \Pi_{x_1} U_1' \Pi_{x_1}^{-1}$, $U_2' = U \cap \Pi_{x_2\cdots x_m}^{-1} U \Pi_{x_2\cdots x_m}$ and $U_2'' = \Pi_{x_2\cdots x_m} U_2' \Pi_{x_2\cdots x_m}^{-1}$;
- the first row of vertical arrows and the first column of arrows (including the first diagonal) are defined by the evident restriction maps;
- the last row of arrows and the last column of horizontal arrows (including the last diagonal) are defined by the evident transfer maps;
- the middle column (resp. row) of horizontal (resp. vertical) arrows is of the form $(\Pi_{x_1})_*$ (resp. $(\Pi_{x_2\cdots x_m})_*$) and the middle diagonal arrow is $(\Pi_{x_1 x_2\cdots x_m})_*$.

Note that the top row comprises T_{x_1}, the last column $T_{x_2\cdots x_m}$ and the diagonal $T_{x_1 x_2\cdots x_m}$, so the lemma follows by induction from the commutativity of all the triangles and squares in the diagram.

We only sketch the proof of commutativity of the top right corner, the rest being immediate from the definitions of the maps. Moreover to prove commutativity of the top right corner reduces to checking it for the corresponding diagram associated to each summand of $H^1(X_{U_2'}, V)$. More precisely, given a double coset $D^\times t U_2' \mathbf{A}_{F,f}^\times$ in $X_{U_2'}$ mapping to $D^\times s U \mathbf{A}_{F,f}^\times$ in X_U, let $\Delta = F^\times \backslash (U_2' \mathbf{A}_{F,f}^\times \cap t^{-1} D^\times t)$ and $\Gamma = F^\times \backslash (U \mathbf{A}_{F,f}^\times \cap s^{-1} D^\times s)$. Writing

$$D^\times t U_2' \mathbf{A}_{F,f}^\times = \coprod_{j\in J} D^\times t_j (U_2' \cap U_1'') \mathbf{A}_{F,f}^\times$$
$$\text{and} \quad D^\times s U \mathbf{A}_{F,f}^\times = \coprod_{i\in I} D^\times s_i U_1'' \mathbf{A}_{F,f}^\times,$$

we must check the commutativity of the diagram

$$\begin{array}{ccc} \bigoplus_{i\in I} H^1(\Gamma_i, V) & \to & H^1(\Gamma, V) \\ \downarrow & & \downarrow \\ \bigoplus_{j\in J} H^1(\Delta_j, V) & \to & H^1(\Delta, V) \end{array}$$

where $\Delta_j = F^\times \backslash ((U_2' \cap U_1'') \mathbf{A}_{F,f}^\times \cap t_j^{-1} D^\times t_j)$, $\Gamma_i = F^\times \backslash (U_1'' \mathbf{A}_{F,f}^\times \cap s_i^{-1} D^\times s_i)$ and the maps are defined as follows:

- Writing $s = \alpha t w$ with $\alpha \in D^\times$, $w \in U \mathbf{A}_{F,f}^\times$, the right-hand arrow is the composite

$$H^1(\Gamma, V) \to H^1(\Delta, \mathrm{Res}_\Gamma^\Delta V) \xrightarrow{\sim} H^1(\Delta, V)$$

where the inclusion $\Delta \to \Gamma$ is defined by $g \mapsto w^{-1}gw$ and $\mathrm{Res}^{\Delta}_{\Gamma} V \to V$ is defined by $v \mapsto wv$.

- The left-hand arrow is similarly defined component-wise as the composite

$$H^1(\Gamma_i, V) \to \bigoplus_j H^1(\Delta_j, \mathrm{Res}^{\Delta_j}_{\Gamma_i} V) \stackrel{\sim}{\longrightarrow} \bigoplus_j H^1(\Delta_j, V),$$

where the direct sum is over j such that $s_i = \alpha_j t_j w_j$ for some $\alpha_j \in D^\times$, $w_j \in U_1'' \mathbf{A}^\times_{F,f}$.

- Writing $s = \beta_i s_i y_i$ with $\beta_i \in D^\times$, $y_i \in U\mathbf{A}^\times_{F,f}$ for each $i \in I$, the top arrow is defined component-wise as the composite

$$H^1(\Gamma_i, V) \stackrel{\sim}{\longrightarrow} H^1(\Gamma, \mathrm{Ind}^{\Gamma}_{\Gamma_i} V) \stackrel{\sim}{\longrightarrow} H^1(\Gamma, \mathrm{Ind}^{\Gamma}_{\Gamma_i} \mathrm{Res}^{\Gamma_i}_{\Gamma} V) \to H^1(\Gamma, V),$$

where the inclusion $\Gamma_i \to \Gamma$ is $g \mapsto y_i^{-1} g y_i$, the first isomorphism is that of Shapiro's Lemma, the second is induced by $V \stackrel{\sim}{\longrightarrow} \mathrm{Res}^{\Gamma_i}_{\Gamma} V$ defined by $v \mapsto y_i^{-1} v$, and the last map is given by the trace $\mathrm{Ind}^{\Gamma}_{\Gamma_i} \mathrm{Res}^{\Gamma_i}_{\Gamma} V \to V$.

- Writing $t = \gamma_j t_j z_j$ with $\gamma_j \in D^\times$, $z_j \in U_2' \mathbf{A}^\times_{F,f}$ for each j, the bottom arrow is defined similarly by the composite

$$H^1(\Delta_j, V) \stackrel{\sim}{\longrightarrow} H^1(\Delta, \mathrm{Ind}^{\Delta}_{\Delta_j} V) \stackrel{\sim}{\longrightarrow} H^1(\Delta, \mathrm{Ind}^{\Delta}_{\Delta_j} \mathrm{Res}^{\Delta_j}_{\Delta} V) \to H^1(\Delta, V).$$

Note that for each $j \in J$, the resulting diagram of inclusions

$$\begin{array}{ccc} \Delta_j & \to & \Delta \\ \downarrow & & \downarrow \\ \Gamma_i & \to & \Gamma \end{array}$$

commutes up to conjugation by the element $g_j = y_i^{-1} w_j^{-1} z_j w F^\times \in \Gamma$. Unravelling definitions and applying standard functorialities, one is reduced to checking commutativity of the following diagram of homomorphisms of Δ-modules

$$\begin{array}{ccccccc} \bigoplus_{i\in I} \mathrm{Res}^{\Delta}_{\Gamma} \mathrm{Ind}^{\Gamma}_{\Gamma_i} V & \to & & \bigoplus_{i \in I} \mathrm{Res}^{\Delta}_{\Gamma} \mathrm{Ind}^{\Gamma}_{\Gamma_i} \mathrm{Res}^{\Gamma_i}_{\Gamma} V & & \to & \mathrm{Res}^{\Delta}_{\Gamma} V \\ \downarrow & & & & & & \downarrow \\ \bigoplus_{j\in J} \mathrm{Ind}^{\Delta}_{\Delta_j} \mathrm{Res}^{\Delta_j}_{\Gamma_i} V & \to & \bigoplus_{j\in J} \mathrm{Ind}^{\Delta}_{\Delta_j} V & \to & \bigoplus_{j \in J} \mathrm{Ind}^{\Delta}_{\Delta_j} \mathrm{Res}^{\Delta_j}_{\Delta} V & \to & V, \end{array}$$

where the maps are defined as follows:

- The first downward arrow is defined by maps

$$\mathrm{Res}^{\Delta}_{\Gamma}\mathrm{Ind}^{\Gamma}_{\Gamma_i}V \to \mathrm{Ind}^{\Delta}_{\Delta_j}\mathrm{Res}^{\Delta_j}_{\Gamma_i}V$$

  sending $f : \Gamma \to V$ to the map $\Delta \to V$ defined by $g \mapsto f(g_j w^{-1} g w)$.

- The final horizontal map in each row is defined by the evident trace map.

- The remaining maps are induced by the evident ones of the form $v \mapsto xv$ where $x = w$, w_j, y_i^{-1} or z_j^{-1}.

Choosing coset representatives $u_a \in \mathrm{GL}_2(F_{x_1})$ so that $U = \coprod_{a\in A} u_a U_1''$, decomposing $A = \coprod_{i\in I} A_i$ where

$$A_i = \{\, a \in A \,|\, tu_a \in D^\times t_i U_1'' \mathbf{A}_{F,f}^\times \,\},$$

and writing $tu_a = \delta_a t_i r_a$ with $\delta_a \in D^\times$, $r_a \in U_1''\mathbf{A}_{F,f}^\times$ for each $a \in A_i$, we find that

$$\Gamma = \coprod_{a\in A_i} y_i^{-1}\Gamma_i y_i h_a$$

for each $i \in I$, where $h_a = y_i^{-1} r_a u_a^{-1}$. One can similarly choose coset representatives for each Δ_j in Δ, and the desired commutativity then follows from a direct calculation using the resulting description of the trace maps as sums over A. □

3.4.2. Badly dihedral representations.

Definition 3.3. We say that F' is a *p-bad quadratic extension* of F if F' is a quadratic totally imaginary extension of F of the form $F(\delta)$ for some δ such that $\delta^p \in F^\times$ and $\delta^p\mathcal{O}_F = I^p\mathcal{O}_F$ for some fractional ideal I of F. We say that an irreducible representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is *badly dihedral* [2] if ρ is induced from a character $G_{F'} \to \overline{\mathbf{F}}_p^{\times}$ for some p-bad quadratic extension F' of F.

Remark 3.4. Note that F has a p-bad quadratic extension if and only if F contains the maximal real subfield of $\mathbf{Q}(\zeta_p)$. If this is the case and p is

[2]There is a typo in the definition of *badly dihedral* in the discussion before Lemma 4.11 of [2]: $\delta^\ell \in K$ should be $\delta^\ell \in \mathcal{O}_K$. The definition here differs slightly from the one intended in [2] in the case $p = 2$ since we also wish to control the central character of the lift.

odd, then the only p-bad quadratic extension of F is $F(\zeta_p)$, but if $p = 2$, then there are still only finitely many such extensions, as follows for example from the fact that such an extension is necessarily unramified outside the primes dividing 2 and ∞.

Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be the modular Galois representation from Theorem 1.1. Assume that ρ arises from the definite quaternion algebra D split at all finite places of F, in level U and weight V (here V is a possibly reducible finite dimensional $\overline{\mathbf{F}}_p$-linear representation of $\mathrm{GL}_2(\mathcal{O}_F/(p))$). We keep the assumptions and notation from the previous section, so that in particular $\mathfrak{m}_\rho$ is the ideal of $\mathbf{T}^S_{\mathcal{O}_E}$ attached to ρ, where S is a finite set containing the primes of F dividing p and the primes at which ρ is ramified.

Let $F_1, F_2, \ldots, F_r$ denote the p-bad quadratic extensions of F (so $r \leq 1$ unless $p = 2$), and for each $i = 1, \ldots, r$, let J_i denote the ideal of $\mathbf{T}^S_{\mathcal{O}_E}$ generated by the elements T_x for those finite places x of F such that $x \notin S$ and x is inert in F_i.

Lemma 3.5. *If $J_1 J_2 \cdots J_r \subset \mathfrak{m}_\rho$ then ρ is badly dihedral.*

Proof. Since $\mathfrak{m}_\rho$ is prime, we may assume that $J_i \subset \mathfrak{m}_\rho$ for some i. We thus have that $\mathrm{tr}(\rho(\mathsf{Frob}_v)) = 0$ for all $v \notin S$ inert in F_i. By the Chebotarev Density Theorem, it follows that $\mathrm{tr}(\rho(g)) = 0$ for all $g \in G_F \setminus G_{F_i}$. Let L denote the projective splitting field of ρ, i.e., the fixed field of the kernel of the composite of ρ with the projection to $\mathrm{PGL}_2(\overline{\mathbf{F}}_p)$.

We claim that $F_i \subset L$. Indeed if not, then we may choose $g \in G_L \setminus G_{F_i}$ and observe that for any $h \in G_F$, we have that either $h \notin G_{F_i}$ so that $\mathrm{tr}(\rho(h)) = 0$, or $gh \notin G_{F_i}$ in which case $\mathrm{tr}(\rho(g)\rho(h)) = \mathrm{tr}(\rho(gh)) = 0$ also implies that $\mathrm{tr}(\rho(h)) = 0$ since $\rho(g)$ is a scalar. If $p > 2$, then taking h to be the identity immediately gives a contradiction; if $p = 2$, then we see that every element of $\mathrm{Gal}(L/F)$ has order dividing 2, which contradicts the irreducibility of ρ.

It follows that $H = \mathrm{Gal}(L/F_i)$ is subgroup of index 2 in $G = \mathrm{Gal}(L/F)$, and that every element of $G \setminus H$ has order 2. Moreover G is isomorphic to a finite subgroup of $\mathrm{PGL}_2(\overline{\mathbf{F}}_p)$ which is not contained in a Borel subgroup. By Dickson's classification of such subgroups, we see the only possibility is that G is isomorphic to a dihedral group and H is a cyclic subgroup of index 2. Therefore the projective image of $\rho(G_{F_i})$ is cyclic, from which it follows that $\rho|_{G_{F_i}}$ is reducible, and hence that ρ is induced from a character of G_{F_i}. □

Lemma 3.6. *There is a finite set of places S' such that if $x_\nu \notin S'$ and x_ν is inert in F_ν for $\nu = 1, \ldots, r$, then $T_{x_1} \cdots T_{x_r}$ annihilates $H^1(X_U, V)$.*

Proof. Write $D_f^\times = \coprod_i D^\times t_i U \mathbf{A}_{F,f}^\times$ and choose a representative $t_i^{-1}\gamma t_i$ with $\gamma \in D^\times$ for each conjugacy class of elements of order p in each of the groups

$$\Gamma_i = F^\times \backslash (U\mathbf{A}_{F,f}^\times \cap t_i^{-1} D^\times t_i).$$

Then $F[\gamma]$ is p-bad, so $F[\gamma] = F_\nu$ for some ν. Let S_γ be the finite set of places x of F inert in F_ν such that $\mathcal{O}_{F_x}[\gamma] \neq \mathcal{O}_{F_\nu,x}$, and let S' contain the union of the S_γ for all γ as above.

Now let $x_1, \ldots, x_r$ be as in the statement of the lemma and let $T = T_{x_1\cdots x_r}$, which by Lemma 3.2 coincides with $T_{x_1}\cdots T_{x_r}$. Let U', t_{ij} and Γ'_{ij} be as in the definition of the Hecke operator T on $\oplus_i H^1(\Gamma_i, V)$ (cf. 3.4.1). We claim that Γ'_{ij} has order prime to p. Indeed if $t_{ij}^{-1}\gamma' t_{ij}$ is a representative of an element of order p in Γ'_{ij}, then its image in Γ_i is of the form $t_i^{-1}\gamma t_i$ for some γ as above, so γ is conjugate in $\mathrm{GL}_2(F_x)$ to an element of $U'_x F_x^\times$, where $x = x_\nu$ for ν chosen so that $F[\gamma] = F_\nu$, and

$$U'_x = U_0(x) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_{F,x}) \,\middle|\, c \equiv 0 \bmod x \right\}.$$

Since $x \notin S_\gamma$, we see that $\det\gamma \in \mathcal{O}_{F,x}^\times$, so in fact γ is conjugate to an element of $U_0(x)$, hence its characteristic polynomial is reducible mod x. On the other hand since $x \notin S_\gamma$, we see also that γ generates the ring of integers of an unramified quadratic extension of F_x, so the characteristic polynomial of γ is irreducible mod x, giving a contradiction.

Since all the Γ'_{ij} have order prime to p, it follows that $\oplus_{i,j} H^1(\Gamma'_{ij}, V) = 0$, and therefore $T = 0$. □

Lemma 3.1 follows easily from Lemmas 3.5 and 3.6. Indeed it suffices to prove that if $\mathfrak{m}_\rho$ is in the support of $H^1(X_U, V)$, then ρ is badly dihedral. Note that we may enlarge S since if $\mathfrak{m}_\rho$ is in the support of $H^1(X_U, V)$, then so is $\mathfrak{m}'_\rho = \mathfrak{m}_\rho \cap \mathbf{T}^{S'}$ for any finite $S' \supset S$. In particular choosing S' as in Lemma 3.6 we see that the ideal $J_1 \cdots J_r$ of Lemma 3.5 is contained in the annihilator of $H^1(X_U, V)$, so if $H^1(X_U, V)_{\mathfrak{m}_\rho} \neq 0$, then ρ is badly dihedral.

3.5. The refinement of the argument in the case $[F : \mathbf{Q}]$ odd

Assume now that $[F : \mathbf{Q}] > 1$ is odd, and let ρ be as in the statement of Theorem 1.1.

Fix an infinite place τ_0 of F and let D denote the quaternion algebra over F whose ramification set equals $\Sigma - \{\tau_0\}$. Fix moreover isomorphisms $D_f \cong M_2(\mathbf{A}_{F,f})$ and $D \otimes_{F,\tau_0} \mathbf{R} \cong M_2(\mathbf{R})$. Let $U = \prod_x U_x$ be an open compact subgroup of $D_f^\times \cong \mathrm{GL}_2(\mathbf{A}_f)$. Set $\mathfrak{h}^\pm = \mathbf{P}^1(\mathbf{C}) - \mathbf{P}^1(\mathbf{R})$. We denote by X_U

the Shimura curve over F which is the canonical model for the complex analytic space

$$X_U(\mathbf{C}) = D^\times \backslash (D_f^\times \times \mathfrak{h}^\pm)/U\mathbf{A}_{F,f}^\times$$

as in [3] (we see $F \subset \mathbf{C}$ via τ_0). Notice that we quotient $D_f^\times$ also by the action of $\mathbf{A}_{F,f}^\times$ in order to keep track of central characters in what follows.

We have the decomposition into connected components:

$$X_U(\mathbf{C}) = \coprod_{i \in I} \Gamma_i \backslash \mathfrak{h},$$

where I is a finite set, and $\Gamma_i = F^\times \backslash \tilde{\Gamma}_i$ acts properly discontinuously on $\mathfrak{h}$. Here $\tilde{\Gamma}_i = U\mathbf{A}_{F,f}^\times \cap t_i^{-1} D_+^\times t_i$ where $D_f^\times = \coprod_{i \in I} D_+^\times t_i U \mathbf{A}_{F,f}^\times$ and $D_+^\times$ is the set of elements of D of totally positive reduced norm. The groups Γ_i are torsion free and act freely and properly on $\mathfrak{h}$ if U is small enough, but not in general. Each component $\Gamma_i \backslash \mathfrak{h}$ has a canonical model defined over a finite abelian extension of F by [3, 1.2].

Let S denote a finite set of finite places of F containing the places above p and the places at which ρ is ramified. Let $\vec{k} \in \mathbf{Z}_{\geq 2}^{\Sigma}$ and $w \in \mathbf{Z}$ such that $k_\tau \equiv w \mod 2$ for all $\tau \in \Sigma$ and let ψ be a finite order Hecke character of $\mathbf{A}_F^\times$ totally of parity w. The systems of Hecke eigenvalues for the action of $\mathbf{T}_{\mathbf{C}}^S$ on the space of holomorphic Hilbert modular forms of level U, weight $(\vec{k}, w)$, and central character $\psi^{-1}|\cdot|^{-w}$ coincide with those arising from the étale cohomology $H^1(X_{U,\overline{F}}, V_{\vec{k},w,\psi} \otimes_{\mathcal{O}_E} E)$, where $V_{\vec{k},w,\psi}$ is the $\mathcal{O}_E$-sheaf associated to the homonymous representation of $U_p\mathbf{A}_{F,f}^\times$ on a finite free $\mathcal{O}_E$-module (see 3.3.1 and [4]). Here E is a large enough finite extension of $\mathbf{Q}_p$, and we see $E \subset \overline{\mathbf{Q}}_p \cong \mathbf{C}$. These Hecke eigensystems also coincide with the Hecke eigensystems arising from $\oplus_{i \in I} H^1(\Gamma_i, V_{\vec{k},w,\psi} \otimes_{\mathcal{O}_E} E)$. (The action of the Hecke operators on the cohomology of the Γ_i's is defined as in Section 3.4.1.) We denote by $\mathfrak{m}_\rho$ the prime ideal of $\mathbf{T}_{\mathcal{O}_E}^S$ attached to ρ.

To prove the second half of Theorem 1.1 we use the same strategy adopted in the case of a definite quaternion algebra, modulo guaranteeing that an analogue of Lemma 3.1 holds. We now consider the natural action of $\mathbf{T}_{\mathcal{O}_E}^S$ on spaces $(\oplus_{i \in I} H^j(\Gamma_i, V))_{\mathfrak{m}_\rho}$ where V is a finite dimensional continuous $\overline{\mathbf{F}}_p$-linear representation of $U_p\mathbf{A}_{F,f}^\times/F^\times$; we wish to prove that if ρ is not badly dihedral, then the functor $V \mapsto (\oplus_{i \in I} H^1(\Gamma_i, V))_{\mathfrak{m}_\rho}$ is exact, so it is enough to prove that $(\oplus_{i \in I} H^j(\Gamma_i, V))_{\mathfrak{m}_\rho} = 0$ for $j = 0, 2$.

First note that by the strong approximation theorem, the reduced norm det induces a bijection $D_+^\times \backslash D_f^\times / U\mathbf{A}_{F,f}^\times \to I$ where

$$I = \mathbf{A}_F^\times / F_{\infty,>0}^\times F^\times \det(U)(\mathbf{A}_{F,f}^\times)^2.$$

It follows that in the definition of the Hecke operator T_x for places $x \notin S$, we may use the same index set I and representatives t_i when U is replaced by $U' = U \cap \Pi_x^{-1} U \Pi_x$ and $U'' = \Pi_x U' \Pi_x^{-1}$. We may thus write T_x as a direct sum of composite maps

$$H^j(\Gamma_i, V) \to H^j(\Gamma_i', V) \to H^j(\Gamma_{i'}'', V) \to H^j(\Gamma_{i'}, V)$$

where $i \mapsto i'$ is induced by multiplication by $(\varpi_x)_x$ on I. Let F_I denote the abelian extension of F corresponding to I by class field theory; we see that if x splits completely in F_I, then T_x acts componentwise on $\oplus_{i\in I} H^j(\Gamma_i, V)$. Moreover if $j = 0$, then for such x this action is simply multiplication by the index $[\Gamma_i : \Gamma_i'] = [\Gamma_i : \Gamma_i''] = \mathbf{N}(x) + 1$ on each component.

Suppose now that $\mathfrak{m}_\rho$ is in the support of $\oplus_{i\in I} H^0(\Gamma_i, V)$, and let F' denote the composite of $F_I(\zeta_p)$ with the splitting field of $\det(\rho)$. If x splits completely in F', then $T_x - \mathbf{N}(x) - 1 \in \mathfrak{m}_\rho$ for all such x, which implies that $\mathrm{tr}(\rho(\mathsf{Frob}_x)) = 2$. The Brauer-Nesbitt and Chebotarev Density Theorems then imply that $\rho|_{G_{F'}}$ has trivial semi-simplification; since F' is an abelian extension of F, this contradicts the irreducibility of ρ.

To treat the case of $j = 2$, we use Farrell cohomology groups $\widehat{H}^j(\Gamma, V)$ (defined in [7]) for finite index subgroups Γ of the groups Γ_i. Note that if $F = \mathbf{Q}$, then such Γ have virtual cohomological dimension one, so that $H^2(\Gamma, V) = \widehat{H}^2(\Gamma, V)$. If $F \neq \mathbf{Q}$, then Γ is a virtual duality group of dimension 2 with dualizing module isomorphic to $\mathbf{Z}$ (with trivial Γ action)[3], so that [7, Thm. 2] yields an exact sequence:

$$H_0(\Gamma, V) \to H^2(\Gamma, V) \to \widehat{H}^2(\Gamma, V) \to 0.$$

We will assume $F \neq \mathbf{Q}$ since the case $F = \mathbf{Q}$ is easier and can be treated by minor modifications to the arguments below.

For Γ' a finite index subgroup of Γ, we have restriction maps $H_j(\Gamma, V) \to H_j(\Gamma', V)$ and $\widehat{H}^j(\Gamma, V) \to \widehat{H}^j(\Gamma', V)$, as well as corestriction maps $H_j(\Gamma', V) \to H_j(\Gamma, V)$ and $\widehat{H}^j(\Gamma', V) \to \widehat{H}^j(\Gamma, V)$, allowing us to define Hecke operators T_x on $\oplus_{i\in I} H_j(\Gamma_i, V)$ and $\oplus_{i\in I} \widehat{H}^j(\Gamma_i, V)$ for $x \notin S$ exactly as on $\oplus_{i\in I} H^j(\Gamma_i, V)$. By the following lemma (and the fact that the isomorphisms $\Gamma_i' \cong \Gamma_i''$ are orientation-preserving), the homomorphisms

$$\bigoplus_{i\in I} H_0(\Gamma_i, V) \to \bigoplus_{i\in I} H^2(\Gamma_i, V) \to \bigoplus_{i\in I} \widehat{H}^2(\Gamma_i, V) \tag{5}$$

[3]Note that there is a canonical choice of orientation $H^2(\Gamma, \mathbf{Z}\Gamma) \cong \mathbf{Z}$ provided by the complex analytic structure on $X_U(\mathbf{C})$.

are compatible with the operators T_x.

Lemma 3.7. *Let* Γ *be a virtual duality group of dimension* n *with dualizing module* D*. Let* M *be a left* $\mathbf{Z}\Gamma$*-module and* Γ' *a finite index subgroup of* Γ*. Then the diagram:*

$$\begin{array}{ccccccc} \cdots \to H_{n-j}(\Gamma, D\otimes_{\mathbf{Z}} M) & \to & H^j(\Gamma, M) & \to & \widehat{H}^j(\Gamma, M) & \to & H_{n-j-1}(\Gamma, D\otimes_{\mathbf{Z}} M)\to\cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to H_{n-j}(\Gamma', D\otimes_{\mathbf{Z}} M) & \to & H^j(\Gamma', M) & \to & \widehat{H}^j(\Gamma', M) & \to & H_{n-j-1}(\Gamma', D\otimes_{\mathbf{Z}} M)\to\cdots \end{array}$$

commutes, where the rows are the exact sequences given by [7, Thm. 2] and the vertical arrows are the natural restriction maps. Similarly the diagram commutes with the downward arrows replaced by the upward corestriction maps.

Proof. Let $(P_\bullet, d_\bullet)$ be a projective resolution of finite type of the trivial (left[4]) Γ-module $\mathbf{Z}$, which we view also as a projective resolution of $\mathbf{Z}$ as a Γ'-module. We define $P_\bullet^* := \mathrm{Hom}_\Gamma(P_\bullet, \mathbf{Z}\Gamma)$ and $P_\bullet'^* := \mathrm{Hom}_{\Gamma'}(P_\bullet, \mathbf{Z}\Gamma')$ and denote by $(P_\bullet^*, d_\bullet^*)$ and $(P_\bullet'^*, d_\bullet'^*)$ the corresponding *cochain* complexes of right $\mathbf{Z}\Gamma$- and $\mathbf{Z}\Gamma'$-modules respectively. There is a natural map of cochain complexes of right $\mathbf{Z}\Gamma'$-modules $\rho_\bullet : P_\bullet^* \to P_\bullet'^*$ induced by the map $\mathbf{Z}\Gamma \to \mathbf{Z}\Gamma'$ given by $\sum_{\gamma\in\Gamma} n_\gamma\gamma \mapsto \sum_{\gamma\in\Gamma'} n_\gamma\gamma$. Note that $\rho_\bullet$ is an isomorphism, with inverse $\sigma_\bullet$ defined by $(\sigma_j(f))(x) = \sum_{\gamma\in\mathfrak{R}} \gamma^{-1} f(\gamma x)$ for $f \in P_j'^*$ and $x \in P_j$, where $\Gamma = \sqcup_{\gamma\in\mathfrak{R}}\Gamma'\gamma$.

Recall that the dualizing module D is defined as $H^n(\Gamma, \mathbf{Z}\Gamma)$, which we view as a right $\mathbf{Z}\Gamma$-module, and let $(Q_\bullet, e_\bullet)$ be a projective resolution of D as a right $\mathbf{Z}\Gamma$-module. Note that $D = H^n(P_\bullet^*) = \ker d_n^*/\mathrm{Im}\, d_{n-1}^*$, and then the natural inclusion $D \hookrightarrow \mathrm{coker}\, d_{n-1}^*$ can be extended to a map of *chain* complexes $f_\bullet : Q_\bullet \to P_{n-\bullet}^*$. Moreover if we let $D' = H^n(\Gamma', \mathbf{Z}\Gamma')$ denote the dualizing module of Γ', then $\rho_\bullet$ induces the canonical isomorphism $D \cong D'$ of $\mathbf{Z}\Gamma'$-modules, so that we may also view $(Q_\bullet, e_\bullet)$ as a projective resolution of D', and extend the natural inclusion $D' \hookrightarrow \mathrm{coker}\,(d_n')^*$ to a map of chain complexes $f_\bullet' : Q_\bullet \to (P_{n-\bullet}')^*$ where $f_\bullet' = \rho_\bullet \circ f_\bullet$.

We now let $X_\bullet$ denote the mapping cone of the chain map $f_\bullet$, so that $X_\bullet = Q_\bullet \oplus P_{n-\bullet-1}^*$, and similarly let $X_\bullet'$ be the mapping cone of $f_\bullet'$. Then $\mathrm{id} \oplus \rho_\bullet$ defines a chain map, giving a commutative diagram of morphisms

[4]Some of the modules we consider will be naturally right $\mathbf{Z}\Gamma$-modules; they can be regarded as left $\mathbf{Z}\Gamma$-modules via the involution $\gamma \mapsto \gamma^{-1}$ of Γ; and vice versa. Some of the chain complexes we consider will be sometimes regarded as cochain complexes, after relabelling; and vice versa.

cochain complexes of right $\mathbf{Z}\Gamma'$-modules:

(6)
$$\begin{array}{ccccccccc} 0 & \to & P^*_\bullet & \to & X_{n-\bullet-1} & \to & Q_{n-\bullet-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P'^*_\bullet & \to & X'_{n-\bullet-1} & \to & Q_{n-\bullet-1} & \to & 0 \end{array}$$

in which the rows are exact and the vertical maps are isomorphisms.

We now apply the functor $(\,\cdot\,)\otimes_\Gamma M$ to the first line of (6), and the functor $(\,\cdot\,)\otimes_{\Gamma'} M$ to the second. For a right Γ-module A and a left Γ-module B, we define the trace map $\mathrm{tr} : A\otimes_\Gamma B \to A\otimes_{\Gamma'} B$ by $\mathrm{tr}(a\otimes b) = \sum_{\gamma\in\mathfrak{R}} a\gamma^{-1}\otimes\gamma b$. We thus obtain a commutative diagram of complexes with exact rows:

(7)
$$\begin{array}{ccccccccc} 0 & \to & P^*_\bullet\otimes_\Gamma M & \to & X_{n-\bullet-1}\otimes_\Gamma M & \to & Q_{n-\bullet-1}\otimes_\Gamma M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P'^*_\bullet\otimes_{\Gamma'} M & \to & X'_{n-\bullet-1}\otimes_{\Gamma'} M & \to & Q_{n-\bullet-1}\otimes_{\Gamma'} M & \to & 0 \end{array}$$

where the left vertical arrow is given by $P^*_\bullet\otimes_\Gamma M \xrightarrow{\mathrm{tr}} P^*_\bullet\otimes_{\Gamma'} M \xrightarrow{\rho_\bullet\otimes\mathrm{id}_M} P'^*_\bullet\otimes_{\Gamma'} M$, the right vertical arrow is the trace map, and the middle vertical arrow is their direct sum.

Taking cohomology in (7) then yields the desired commutative diagram. Indeed the long exact sequences of [7, Thm. 2] are precisely those associated to the rows of (7), and it is straightforward to check that the vertical maps induce the corresponding restriction maps on homology and cohomology (for Farrell cohomology, this follows from the characterization of res following [7, Rem. 2]).

The proof of compatibility with corestriction is similar, so we omit the details. One just uses $\sigma_\bullet$ instead of $\rho_\bullet$ to obtain a diagram as in (6), but with upward arrows, and use the canonical projection $A\otimes_{\Gamma'} B\to A\otimes_\Gamma B$ to obtain the analogue of (7), again with upward arrows. □

We can now use (5) to prove that $\mathfrak{m}_\rho$ is not in the support of $\oplus_{i\in I}H^2(\Gamma_i,V)$. Indeed the same argument as for H^0 shows that if x splits completely in F_I, then $T_x = \mathbf{N}(x)+1$ on $\oplus_{i\in I}H_0(\Gamma_i,V)$, so that the irreducibility of ρ implies that $\mathfrak{m}_\rho$ is not in the support of the image of $\oplus_{i\in I}H_0(\Gamma_i,V)$. Note also that the surjectivity of

$$\oplus_{i\in I}H^2(\Gamma_i,V)\to\oplus_{i\in I}\widehat{H}^2(\Gamma_i,V)$$

implies that the operators T_x commute, hence $\mathbf{T}^S_{\mathcal{O}_E}$ acts, on $\oplus_{i\in I}\widehat{H}^2(\Gamma_i,V)$. Thus it suffices to prove that if ρ is not badly dihedral, then $\mathfrak{m}_\rho$ is not in the support of $\oplus_{i\in I}\widehat{H}^2(\Gamma_i,V)$.

Let S' be a finite set of finite places of F constructed as in the proof of Lemma 3.6. (Now D is an indefinite quaternion algebra, so the groups Γ_i are infinite, but each still has only finitely many conjugacy classes of elements of order p.) For each $\nu = 1, \ldots, r$ let $x_\nu \notin S' \cup S$ be a finite place of F inert in F_ν; let $T = T_{x_1 \cdots x_r} = T_{x_1} \cdots T_{x_r}$, and let Γ'_i be as in the definition of the Hecke operator T. (Note that we can use the same index set I and representatives t_i.) The same proof as in Lemma 3.6 shows that Γ'_i does not contain any element of order p. Therefore Γ'_i has a torsion-free subgroup of finite index prime to p, so $\widehat{H}^2(\Gamma'_i, V) = 0$. It follows that the operator T annihilates $\oplus_{i \in I} \widehat{H}^2(\Gamma_i, V)$, since it factors through $\oplus_{i \in I} \widehat{H}^2(\Gamma'_i, V)$. To prove that $(\oplus_{i \in I} \widehat{H}^2(\Gamma_i, V))_{\mathfrak{m}_\rho} = 0$, we may enlarge S so that $S \supset S'$. For each $\nu = 1, \ldots, r$ we denote by J_ν the ideal of $\mathbf{T}^S_{\mathcal{O}_E}$ generated by the Hecke operators T_x for those finite places x of F such that $x \notin S$ and x is inert in F_ν. Observe that the ideal $J_1 J_2 \cdots J_r$ annihilates $(\oplus_{i \in I} \widehat{H}^2(\Gamma_i, V))_{\mathfrak{m}_\rho}$. By Lemma 3.5 (which holds, *mut. mut.*, also in the current setting) we deduce that $(\oplus_{i \in I} \widehat{H}^2(\Gamma_i, V))_{\mathfrak{m}_\rho}$ vanishes, since ρ is not badly dihedral. This completes the proof that $(\oplus_{i \in I} H^2(\Gamma_i, V))_{\mathfrak{m}_\rho} = 0$, and hence the functor $V \mapsto (\oplus_{i \in I} H^1(\Gamma_i, V))_{\mathfrak{m}_\rho}$ is exact.

4. A variant in a special case

We now give a variant of the main result in a special case, with a view to producing forms satisfying the hypotheses of Assumption 8.15 in Section 8.3 of [5].

We must first introduce some notation. Recall that we have fixed an embedding $\overline{\mathbf{Q}} \to \mathbf{C}$ and an isomorphism $\overline{\mathbf{Q}}_p \cong \mathbf{C}$. Note that these choices induce a bijection between the set Σ of embeddings $\tau : F \to \mathbf{R}$ and the set of pairs (v, ϑ) where $v|p$ and ϑ is an embedding $F_v \to \overline{\mathbf{Q}}_p$. For each $v|p$ we let Σ_v denote the set of embeddings $\vartheta : F_v \to \overline{\mathbf{Q}}_p$, which we identify with a subset of Σ via this bijection.

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ which is holomorphic of weight $(\vec{k}, w)$, where as usual $\vec{k} \in \mathbf{Z}^\Sigma_{\geq 2}$ and $w \in \mathbf{Z}$ is such that $w \equiv k_\tau \bmod 2$ for all $\tau \in \Sigma$. Suppose further that for all $v|p$, the local factor π_v is either unramified principal series or an unramified twist of the Steinberg representation. Let $a_v(\pi)$ denote the eigenvalue of the Hecke operator $T_v = U_v \Pi_v U_v$ on the one-dimensional vector space $\pi_v^{U_v}$, where $\Pi_v = \left(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(F_v)$ and $U_v = \mathrm{GL}_2(\mathcal{O}_{F,v})$ or $U_0(v)$ according to whether π_v is unramified. Since $a_v(\pi)$ is algebraic, we may view it as an element of $\overline{\mathbf{Q}}$ via our choices of embeddings $\overline{\mathbf{Q}} \to \mathbf{C}$ and $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. We say that π is *ordinary* at v (with respect to our choices of embeddings) if

$$|a_v(\pi)|_p = p^{\sum_{\tau\in\Sigma_v}(k_\tau-2-w)/2e_v}.$$

Remark 4.1. In general we have the expression on the right as an upper bound on $|a_v(\pi)|_p$; this follows for example from [9, Thm. 4.11], but will also be clear from the proof of Theorem 4.2 below. Moreover if equality holds then Theorem 1 of [14] implies that the local Galois representation $\rho_\pi|_{G_{F_v}}$ is reducible. Note that if π_v is an unramified twist of the Steinberg representation, then π is ordinary at v if and only if $k_\tau = 2$ for all $\tau \in \Sigma_v$.

Theorem 4.2. *Suppose that $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is such that $\rho \cong \overline{\rho}_\pi$ for some cuspidal, holomorphic, automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ of weight $(\vec{k}, w) = (2, \dots, 2)$ such that for each $v|p$, π_v is either unramified principal series or an unramified twist of the Steinberg representation. For any finite set of primes T of F, there exist a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_F)$ and a character $\xi : G_F \to \overline{\mathbf{F}}_p^\times$ of order at most 2 such that*

- *if $\tau \in \Sigma$, then $\pi'_\tau \cong D_{k'_\tau,w'}$ with $k'_\tau \in \{2, w'\}$ where $w' = p+1$ if p is odd and $w' = 4$ if $p = 2$;*
- *if $v|p$, then π'_v is unramified principal series, and is ordinary if $k'_\tau = w'$ for some $\tau \in \Sigma_v$;*
- *the prime-to-p part of the conductor of ξ divides a prime $y \notin T$ which splits completely in F;*
- *$\overline{\rho}_{\pi'} \cong \xi \otimes \rho$.*

Suppose further that π has prime-to-p conductor dividing $\mathfrak{n} \subset \mathcal{O}_F$, and that ψ is a totally even finite order Hecke character of $\mathbf{A}_F^\times$ of conductor dividing $\mathfrak{n}$ satisfying $\det\rho = \overline{\psi\epsilon}$. Then if ρ is not badly dihedral, we can choose π' as above with conductor dividing $\mathfrak{n}y^2$, central character $\psi^{-1}|\ |^{-w'}$ and $\xi_y \otimes \pi'_y$ unramified principal series.

Remark 4.3. We will see from the proof that the conclusion can be made more precise as follows: For each $v|p$ such that π_v is ramified, we can ensure that π'_v is ordinary and the set of $\tau \in \Sigma_v$ such that $k'_\tau = w'$ maps bijectively to $\overline{\Sigma}_v$ under the natural projection.

Proof. Let R denote the set of primes $v|p$ such that π_v is ramified, and as usual let S be a sufficiently large finite set of primes containing all those dividing p and all those at which π is ramified. We suppose that E is a sufficiently large finite extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$ that contains the eigenvalue $a_x(\pi)$ of T_x on $\pi_x^{\mathrm{GL}_2(\mathcal{O}_{F,x})}$ for all $x \notin S$ and (necessarily also) the eigenvalue $a_v(\pi)$ of T_v on $\pi_v^{U_0(v)}$ for all $v \in R$.

Let $S' = S \setminus R$ and let $\mathbf{T} = \mathbf{T}^{S'}_{\mathcal{O}_E}$ denote the $\mathcal{O}_E$-algebra generated by the operators T_x for $x \notin S'$. Let $\mathfrak{m}$ denote the kernel of the homomorphism $\mathbf{T} \to \overline{\mathbf{F}}_p$ defined by sending T_x to the reduction of of $a_x(\pi \otimes |\det|)) = a_x(\pi)\mathbf{N}(x)^{-1}$ for $x \notin S'$. (For convenience in keeping track of ordinariness, we have replaced π by its twist by $|\det|$ to ensure that $T_v \notin \mathfrak{m}$ for $v \in R$.)

Let $U = U_1(\mathfrak{n}) \cap U_0(\prod_{v \in R} v)$ where $\mathfrak{n}$ is the prime-to-p conductor of π. If $[F : \mathbf{Q}]$ is even, then we let D be the definite quaternion algebra over F ramified at precisely the infinite places of F. By the Jacquet–Langlands correspondence, $\mathfrak{m}$ is in the support of $S_{V'}(U)$ where V' is the representation of $U\mathbf{A}^\times_{F,f}$ on $\overline{\mathbf{F}}_p$ on which U acts trivially and $\mathbf{A}^\times_{F,f}$ acts via $\overline{\psi}$. If $[F : \mathbf{Q}]$ is odd, then we let D be a quaternion algebra over F ramified at precisely all but one infinite place. In this case $\mathfrak{m}$ is in the support of $\oplus_{i \in I} H^1(\Gamma_i, V')$, where $D_f^\times = \coprod_{i \in I} D_+^\times t_i U \mathbf{A}^\times_{F,f}$ and $\Gamma_i = F^\times \backslash (U\mathbf{A}^\times_{F,f} \cap t_i^{-1} D_+^\times t_i)$ as before. Since the argument is the same in the case of either parity, we will ease notation by writing $S(U, V')$ for $\oplus_{i \in I} H^j(\Gamma_i, V')$ where $j = 0$ or 1 according to the parity of $[F : \mathbf{Q}]$ (so $S(U, V') = S_{V'}(U)$ if $j = 0$).

We will now show that $\mathfrak{m}$ is in the support of $S(U_1(\mathfrak{n}), V)$ where $V = V' \otimes (\otimes_{v \in R} S_{v,(p-1,\ldots,p-1)})$. We proceed by induction on $|R'|$ to show that $\mathfrak{m}$ is in the support of $S(U_{R'}, V_{R'})$ for $R' \subset R$, where $U_{R'} = U \prod_{v \in R'} \mathrm{GL}_2(\mathcal{O}_{F,v})$ and $V_{R'} = V' \otimes (\otimes_{v \in R'} S_{v,(p-1,\ldots,p-1)})$. Note that $U_R = U_1(\mathfrak{n})$, $U_\emptyset = U$, and we already know that $\mathfrak{m}$ is in the support of $S(U_\emptyset, V_\emptyset)$.

Suppose now that $v \in R \setminus R'$. The canonical isomorphisms

$$S(U_{R'}, V_{R'}) \cong S(U_{R' \cup \{v\}}, \mathrm{Ind}_{U_{R'}}^{U_{R' \cup \{v\}}}(V_{R'}))$$

and $\mathrm{Ind}_B^{\mathrm{GL}_2(k_v)} \overline{\mathbf{F}}_p \cong \overline{\mathbf{F}}_p \oplus S_{(p-1,\ldots,p-1)}$ give rise to an exact sequence

$$(8) \qquad 0 \to S(U_{R' \cup \{v\}}, V_{R'}) \xrightarrow{\alpha_v} S(U_{R'}, V_{R'}) \xrightarrow{\beta_v} S(U_{R' \cup \{v\}}, V_{R' \cup \{v\}}) \to 0$$

such that α_v and β_v are compatible with the operators T_x for $x \notin S' \cup \{v\}$, but not necessarily with the operator T_v. Note however that the matrix $w_v = \left(\begin{smallmatrix} 0 & 1 \\ \varpi_v & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(F_v)$ normalizes $U_{R'}$ so that $W_v = (w_v)_*$ defines an automorphism of $S(U_{R'}, V_{R'})$ which is compatible with T_x for $x \notin S' \cup \{v\}$ and satisfies $W_v^2 = \overline{\psi}(\varpi_v)^{-1}$. Moreover unravelling the definitions of the operator T_v one finds that $T_v W_v \alpha_v = 0$ and $\beta_v W_v T_v = T_v \beta W_v$. Therefore $\beta_v W_v$ is $\mathbf{T}$-linear, and $\mathfrak{m}$ is not in the support of its kernel since $T_v \notin \mathfrak{m}$. It follows that if $\mathfrak{m}$ is in the support of $S(U_{R'}, V_{R'})$ then it is also in the support of $S(U_{R'\cup\{v\}}, V_{R'\cup\{v\}})$.

To set the stage for lifting to characteristic zero, we distinguish between the cases $p = 2$ and $p > 2$.

If $p > 2$, then we let $\mathfrak{a}$ denote the ideal of $\mathcal{O}_F$ such that $p\mathcal{O}_F = \mathfrak{a}\prod_{v\in R} v$. Let y be any prime ideal of $\mathcal{O}_F$ such that $y \notin S$ and $[y] = [\mathfrak{a}]^{-1}$ in the ray class group of conductor $4\mathcal{O}_F$ (if non-trivial; otherwise we can let $y = \mathcal{O}_F$), and choose α such that $y\mathfrak{a} = \alpha\mathcal{O}_F$ and $\alpha \equiv 1 \bmod 4\mathcal{O}_F$. Finally let ξ be the character of G_F with splitting field $F(\sqrt{\alpha})$. Thus ξ is ramified precisely at y and at certain $v|p$, namely those such that either $v \in R$ and e_v is even, or $v \notin R$ and e_v is odd. Note also that y can be chosen to split completely in F.

If $p = 2$, then we let ξ be the trivial character, but we must make another modification instead of introducing a quadratic twist. For each $v \in R$ we have a $\mathrm{GL}_2(k_v)$-equivariant inclusion $S_{(1,\dots,1)} \to S_{(2,\dots,2)}$, and these induce a $\mathbf{T}$-equivariant map $S(U_1(\mathfrak{n}), V) \to S(U_1(\mathfrak{n}), V' \otimes (\otimes_{v\in R} S_{v,(2,\dots,2)}))$. One checks as usual that $\mathfrak{m}$ is not in the support of the kernel, so we can replace each $S_{v,(p-1,\dots,p-1)}$ by $S_{v,(2,\dots,2)}$ in the definition of V for $p = 2$.

Now let $\tilde{R} \subset \Sigma = \coprod_{v|p} \Sigma_v$ be a set of embeddings $F \to \overline{\mathbf{Q}}$ such that the $\tilde{R} \cap \Sigma_v = \emptyset$ if $v \notin R$ and the natural map $\tilde{R} \cap \Sigma_v \to \overline{\Sigma}_v$ is bijective if $v \in R$. Define $\vec{k}' \in \mathbf{Z}^{\Sigma}_{\geq 2}$ by setting $k'_\tau = 2$ if $\tau \notin \tilde{R}$ and $k'_\tau = w'$ if $\tau \in \tilde{R}$. (Recall that $w' = p + 1$ if $p > 2$ and $w' = 4$ if $p = 2$.) Consider the representation $V_{\vec{k}', w'-2, \psi}$ of $U_1(\mathfrak{n})\mathbf{A}^\times_{F,f}$; recall that this is the free $\mathcal{O}_E$-module defined by

$$V_{\vec{k}', w'-2, \psi} = \left(\bigotimes_{\tau\in\tilde{R}} \mathrm{Sym}^{w'-2}\mathcal{O}_E^2\right) \bigotimes \left(\bigotimes_{\tau\notin\tilde{R}} \det{}^{(w'-2)/2}\right)$$

as a representation of $\mathrm{GL}_2(\mathcal{O}_{F,p})$, with $x \in \mathbf{A}^\times_{F,f}$ acting via

$$\mathbf{N}(x_p)^{w'-2}|x|^{w'-2}\psi(x).$$

We let $V_\xi = V_{\vec{k}', w'-2, \psi}$ if $p = 2$ (or if $y = \mathcal{O}_F$); otherwise we let V_ξ denote the twist of $V_{\vec{k}', w'-2, \psi}$ by the Teichmuller lift of the character $\xi \circ \det$ of $\mathrm{GL}_2(\mathcal{O}_{F,y})$.

One finds that if $p > 2$, then the reduction of $\prod_{\tau\in\Sigma_v\setminus\tilde{R}} \det^{(p-1)/2}$ is $\det^{e_v(p^{f_v}-1)/2}$ if $v \in R$ and $\det^{(e_v-1)(p^{f_v}-1)/2}$ if $v \notin R$, from which it follows that $\overline{V}_\xi$ is isomorphic to the twist of V by the character $\xi \circ \det$ of $U_1(\mathfrak{n})\mathbf{A}^\times_{F,f}$. The isomorphisms $V \to \overline{V}_\xi$ of $\Gamma_i = F^\times\backslash(U_1(\mathfrak{n})\mathbf{A}^\times_{F,f} \cap t_i^{-1}D^\times t_i)$–modules defined by $v \mapsto \xi(\det(t_i))v$ induce an isomorphism $S(U_1(\mathfrak{n}), V) \cong S(U_1(\mathfrak{n}), \overline{V}_\xi)$

under which $\xi(\varpi_v)T_v$ corresponds to T_v for v not dividing yp and to T_v^0 for $v \in R$, where T_v^0 is compatible with an operator on $S(U, V_\xi)$ such that $T_v = T_v^0 \prod_{\tau \in \Sigma_v \setminus \tilde{R}} \tau(\varpi_v)^{(w'-2)/2}$. (Note that T_v^0 may depend on the choice of uniformizer ϖ_v.)

Now let $\mathbf{T}'$ denote the $\mathcal{O}_E$-algebra generated by the operators T_x for $x \notin S \cup \{y\}$ and T_v^0 for $v \in R$, and let $\mathfrak{m}'$ denote the kernel of the homomorphism $\mathbf{T}' \to \overline{\mathbf{F}}_p$ sending each T_x to the reduction of $\xi(\varpi_x)a_x(\pi)\mathbf{N}(x)^{-1}$ and each T_v^0 to the reduction of $\xi(\varpi_v)a_v(\pi)\mathbf{N}(v)^{-1}$. We then have that $\mathfrak{m}'$ is in the support of $S(U_1(\mathfrak{n}), \overline{V}_\xi)$. If ρ is not badly dihedral, it follows as in the proof of Theorem 1.1 that $\mathfrak{m}'$ is in the support of $S(U_1(\mathfrak{n}), V_\xi \otimes_{\mathcal{O}_E} E)$, and hence that there is an automorphic representation whose twist by $|\det|^{-1}$ is the required π'. If ρ is badly dihedral, then the proof goes through after replacing $U_1(\mathfrak{n})$ by a smaller open compact subgroup U so that the groups Γ_i are torsion-free. □

Remark 4.4. The necessity of the quadratic twist ξ in the conclusion follows from consideration of the local Galois representations $\overline{\rho}_{\pi'}|_{G_{F_v}}$ for $v|p$. Furthermore one can construct explicit examples showing that ξ may need to be ramified outside p; we are grateful to L. Dembélé for providing the following one. Over $F = \mathbf{Q}(\alpha)$ with $\alpha = \sqrt{10}$, there is a Hilbert modular form[5] of weight $(2, 2)$, level $(\alpha + 2)$ and trivial character with leading Hecke eigenvalues (ordered by norm):

v	$(2,\alpha)$	$(3,\alpha+2)$	$(3,\alpha+4)$	$(5,\alpha)$	$(13,\alpha+6)$	$(13,\alpha+7)$	$(31,\alpha+17)$	$(31,\alpha+14)$
a_v	-1	-1	3	1	-7	0	-3	4

The corresponding automorphic representation π then satisfies the hypotheses of the theorem for $p = 3$, but one can show that the character ξ in the conclusion must be ramified at $(3, \alpha + 4)$ but not $(3, \alpha + 2)$, from which it follows that ξ must also be ramified at a prime y not dividing 3; in fact one can let y be any non-principal prime of $\mathcal{O}_F$ not dividing 6.

We now give another variant which under the same hypotheses produces lifts of parallel weight without the quadratic twist, at the expense of ordinariness.

[5]Dembélé also offers the equation

$$y^2 = x^3 + (990144\alpha + 3127248)x - 545501952\alpha - 1726178688$$

for the associated elliptic curve.

Theorem 4.5. *Suppose that $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is such that $\rho \cong \overline{\rho}_\pi$ for some cuspidal, holomorphic, automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ of weight $(\vec{k}, w) = (2, \dots, 2)$ such that for each $v|p$, π_v is either unramified principal series or an unramified twist of the Steinberg representation. Let $k' = 2 + (p-1)n$ for any positive integer n. Then there exist a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_F)$, holomorphic of weight $(k', \dots, k')$ such that*

- *if $v|p$, then π'_v is unramified principal series;*
- *$\overline{\rho}_{\pi'} \cong \rho$.*

Suppose further that π has prime-to-p conductor dividing $\mathfrak{n} \subset \mathcal{O}_F$, and that ψ is a finite order Hecke character of $\mathbf{A}_F^\times$ of conductor dividing $\mathfrak{n}$, totally of parity k', and satisfying $\det\rho = \overline{\psi\epsilon}$. Then if ρ is not badly dihedral, we can choose π' as above with conductor dividing $\mathfrak{n}$ and central character $\psi^{-1}|\ |^{-k'}$.

Remark 4.6. Note that we may take $k' = p+1$ in the conclusion, but in the case $p = 2$, this precludes using the Teichmüller lift of $\overline{\epsilon}^{-1}\det\rho$ for ψ as this requires k' to be even.

Proof. By Lemmas 2.4 and 2.5, we see that $[S_{(p-1,\dots,p-1)}] \le [S^{\otimes e}_{(n(p-1),\dots,n(p-1))}]$ for all $n, e \ge 1$, so by the arguments of Section 3, it suffices to prove that $\mathfrak{m}_{\rho\otimes\overline{\epsilon}^{-1}} \subset \mathbf{T}^S_{\mathcal{O}_E}$ is in the support of $S(U_1(\mathfrak{n}), V_{\{v|p\}})$ (with notation as in the proof of Theorem 4.2, so in particular $V_{\{v|p\}}$ is the representation of $U_1(\mathfrak{n})\mathbf{A}^\times_{F,f}$ on which U_p acts as $\otimes_{v|p} S_{v,(p-1,\dots,p-1)}$, U_x acts trivially for x not dividing p, and $\mathbf{A}^\times_{F,f}$ acts via $\overline{\psi}$).

For $v|p$, define the representation L_v of $\mathrm{GL}_2(\mathcal{O}_{F,v})$ to be the cokernel of the natural inclusion $\mathcal{O}_E \to \mathrm{Ind}_{U_0(v)}^{\mathrm{GL}_2(\mathcal{O}_{F,v})}\mathcal{O}_E$. We let $L_{\{v|p\}}$ denote the representation of $U_1(\mathfrak{n})\mathbf{A}^\times_{F,f}$ on which U_p acts as $\bigotimes_{v|p} L_v$, U_x acts trivially for x not dividing p, and $\mathbf{A}^\times_{F,f}$ acts via ψ_π, where $\psi_\pi^{-1}|\ |^{-2}$ is the central character of π. Note that $\overline{\psi}_\pi = \overline{\psi}$, so that $L_{\{v|p\}} \otimes_{\mathcal{O}_E} \overline{\mathbf{F}}_p \cong V_{\{v|p\}}$; moreover the induced inclusion $S(U_1(\mathfrak{n}), L_{\{v|p\}}) \otimes_{\mathcal{O}_E} \overline{\mathbf{F}}_p \to S(U_1(\mathfrak{n}), V_{\{v|p\}})$ is compatible with the natural action of $\mathbf{T}^S_{\mathcal{O}_E}$. Therefore it suffices to prove that $\mathfrak{m}_{\rho\otimes\overline{\epsilon}^{-1}}$ is in the support of

$$S_0(U_1(\mathfrak{n}), L_{\{v|p\}}) = S(U_1(\mathfrak{n}), L_{\{v|p\}})/S^{\mathrm{triv}}(U_1(\mathfrak{n}), L_{\{v|p\}}),$$

which in turn follows from it being in the support of

$$S_0(U_1(\mathfrak{n}), L_{\{v|p\}}) \otimes_{\mathcal{O}_E} \mathbf{C} \cong \bigoplus_{\Pi} (\Pi_f \otimes_{\mathcal{O}_E} L_{\{v|p\}})^{U_1(\mathfrak{n})\mathbf{A}^\times_{F,f}},$$

where the sum is over all cuspidal automorphic representations $\Pi = \Pi_f \otimes \Pi_\infty$ of $\mathrm{GL}_2(\mathbf{A}_F)$ such that $\Pi_\tau \cong D_{2,0}$ for all $\tau \in \Sigma$. For $v|p$,

$$(\Pi_v \otimes_{\mathcal{O}_E} L_v)^{\mathrm{GL}_2(\mathcal{O}_{F,v})} \neq 0$$

if and only if Π_v has unramified central character and conductor dividing v, so that $(\Pi_f \otimes_{\mathcal{O}_E} L_{\{v|p\}})^{U_1(\mathfrak{n})\mathbf{A}_{F,f}^\times} \neq 0$ if and only if Π has central character ψ_π^{-1} and conductor dividing $\mathfrak{n}\prod_{v|p} v$. Furthermore note that $\mathfrak{m}_{\rho\otimes\bar{\epsilon}^{-1}}$ is in the support of such a summand if and only if $\overline{\rho}_\Pi \cong \rho \otimes \overline{\epsilon}^{-1}$. Finally our hypotheses ensure that $\Pi = \pi \otimes |\det|$ is exactly such an automorphic representation. □

Remark 4.7. Under the assumption that π_v is an unramified principal series for all $v|p$ and other technical conditions (p unramified in F, $[F : \mathbf{Q}]$ even or $F = \mathbf{Q}$), Theorem 4.5 is proved in Sections 2 and 4 of [6].

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