Relative K-stability and extremal Sasaki metrics

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We define K-stability of a polarized Sasakian manifold relative to a maximal torus of automorphisms. The existence of a Sasaki-extremal metric in the polarization is shown to imply that the polarization is K-semistable. Computing this invariant for the deformation to the normal cone gives an extention of the Lichnerowicz obstruction, due to Gauntlett, Martelli, Sparks, and Yau, to an obstruction of Sasaki-extremal metrics. We use this to give a list of examples of Sasakian manifolds whose Sasaki cone contains no extremal representatives. These give the first examples of Sasaki cones of dimension greater than one that contain no extremal Sasaki metrics whatsoever. In the process we compute the unreduced Sasaki cone for an arbitrary smooth link of a weighted homogeneous polynomial.

1. Introduction

Sasaki-extremal metrics were introduced in [BGS08] and provide a more general notion of a canonical metric than constant scalar curvature, which is obstructed by the Futaki invaraiant. On a Sasakian manifold it is also natural to deform the Reeb vector field by Hamiltonian holomorphic vector fields, giving the notion of the Sasaki cone $\mathfrak{t}^+$ of a Sasakian manifold, a notion analogous to the Kähler cone in Kähler geometry. It is natural to ask which Reeb vector fields $\xi \in \mathfrak{t}^+$ admit Sasaki-extremal representatives. This subset, called the *extremal set*, $\mathfrak{e} \subseteq \mathfrak{t}^+$, was shown to be open by the above authors. But little is known about this set because there are no known obstructions, besides the obstructions of Gauntlett, Martelli, Sparks, and Yau [GMSY07] which obstruct the existence of a Sasaki-Einstein metric in

the positive definite case and thus a Sasaki-extremal metric if the Futaki invariant vanishes.

It is natural to conjecture that the existence of a Sasaki-extremal metric is equivalent to some form of GIT stability of the affine cone (Y, ξ), where $Y = C(M) \cup \{o\}$ is the Kähler cone on M with the vertex added, polarized by the Reeb vector field ξ. Interesting work has been done in this direction [CS18, CS15] using K-stability in particular. Collins and Székelyhidi [CS18] extended K-stability to irregular Sasakian manifolds using the Hilbert series of (Y, ξ), and they proved that the existence of a constant scalar curvature metric implies K-semistability.

We define a relative version of the K-stability of Collins and Székelyhidi to get an obstruction to Sasaki-extremal metrics. This is K-stability relative to a maximal torus $T \subset \mathfrak{Aut}(Y, \xi)$. We prove that the existence of a Sasaki-extremal structure compatible with a polarized cone (Y, ξ) implies that (Y, ξ) is K-semistable relative to a maximal torus T.

Considering the Rees algebra degenerations which, as shown in [CS18], give the Lichnerowicz obstructions of Gauntlett, Martelli, Sparks, and Yau, we give a generalized Lichnerowicz obstruction to the existence of a Sasaki-extremal metric depending on a T-homogenous $f \in H^0(Y, \mathcal{O}_Y)$. We apply our construction to a large class of weighted homogeneous polynomials by giving a sufficient condition for the obstruction of extremal Sasaki metrics in the entire Sasaki cone. This is Theorem 13 below. We also present a list of Sasaki manifolds in Table 1 which has families of Sasaki cones of dimension greater than one with no extremal Sasaki metrics. In particular, our results can be rephrased in terms of the moduli space of positive Sasaki classes with vanishing first Chern class, viz.

Theorem 1. *Let M be one of the smooth manifolds listed in the first five entries or the last entry of Table 1. Then the moduli space $\mathfrak{M}^c_{+,0}(M)$ of positive Sasaki classes with $c_1(\mathcal{D}) = 0$ has a countably infinite number of components of dimension greater than one and which contain no extremal Sasaki metrics, i.e. $\mathfrak{e} = \emptyset$. Moreover, these different components correspond to isomorphic transverse holomorphic structures.*

2. Background

In this section we present a brief discussion of Sasakian geometry and refer to [BG08] for details. A Riemannian manifold (M, g) is *Sasakian* if the metric cone $(C(M), \bar{g})$, $C(M) = M \times \mathbb{R}^+, \bar{g} = dr^2 + r^2 g$, is Kähler. A Sasakian structure is a special case of a *contact metric structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ which

is a quadruple where $\mathcal{D} = \ker \eta$ is a contact structure, ξ is the Reeb vector field of the contact form η, Φ is an endomorphism field that annihilates ξ and such that $(\mathcal{D}, \Phi|_{\mathcal{D}})$ is a strictly pseudoconvex almost CR structure, and g is a compatible Riemannian metric, i.e. $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X, Y. Then the structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ is *Sasakian* if $\mathcal{L}_\xi \Phi = 0$ and the almost CR structure $(\mathcal{D}, \Phi|_{\mathcal{D}})$ is integrable. A contact structure $\mathcal{D}$ is said to be of *Sasaki type* if there exists such a quadruple (ξ, η, Φ, g) which is Sasakian with $\mathcal{D} = \ker \eta$. Note that all Sasaki manifolds are oriented and here we shall only consider compact Sasaki manifolds.

Since the Reeb vector field ξ is nowhere vanishing it defines a 1-dimensional foliation $\mathcal{F}_\xi$ known as the *characteristic foliation* which is a *Riemannian foliation* when (ξ, η, Φ, g) is Sasakian. In fact, in the Sasakian case, the transverse geometry is Kählerian. The *basic first Chern class* $c_1(\mathcal{F}_\xi)$ is an important invariant of a Sasakian structure. A Sasakian structure (ξ, η, Φ, g) is *of positive type* or simply *positive* if $c_1(\mathcal{F}_\xi)$ can be represented by a positive definite basic $(1,1)$ form. We denote the *space of Sasakian structures* with Reeb vector field ξ and transverse holomorphic structure $\overline{J}$ by $\mathcal{S}(\xi, \overline{J})$. Note that $c_1(\mathcal{F}_\xi)$ and hence positivity depends only on the space $\mathcal{S}(\xi, \overline{J})$ and not on a particular element $(\xi, \eta, \Phi, g) \in \mathcal{S}(\xi, \overline{J})$. So we can think of $c_1(\mathcal{F}_\xi)$ as representing the *Sasaki class* of $\mathcal{S} = (\xi, \eta, \Phi, g)$. By the transverse version [EKA90] of Yau's theorem a positive Sasakian structure has a Sasaki metric of positive Ricci curvature. Recall that in the basic cohomology exact sequence

$$0 \longrightarrow H_B^0(\mathcal{F}_\xi) \overset{\delta}{\longrightarrow} H_B^2(\mathcal{F}_\xi) \overset{\iota_*}{\longrightarrow} H^2(M, \mathbb{R}) \longrightarrow \cdots$$

we have $\iota_* c_1(\mathcal{F}_\xi) = c_1(\mathcal{D})$. Thus, if $c_1(\mathcal{D}) = 0$ there exists $a \in \mathbb{R}$ such that $c_1(\mathcal{F}_\xi) = a[d\eta]_B$. Following [BMvK16] we denote by $\mathfrak{M}_{+,0}^c(M)$ the *moduli space of positive Sasaki classes on M with $c_1(\mathcal{D}) = 0$*. Clearly, in the positive case we have $a > 0$.

If the type of a Sasakian structure on M is positive or indefinite it is possible to deform the characteristic foliation through Sasakian structures by deforming the Reeb vector field. This gives rise to the (reduced) *Sasaki cone* [BGS08] of M. However, here, as in [CS18], it is more convenient to work on the Kähler cone $C(M)$ or better yet on the Kähler cone with the vertex added $Y = C(M) \cup \{o\}$ which uniquely has the structure of a normal affine variety. We will see that the Reeb vector field ξ gives a notion of a polarization on Y. Choose a torus $T \subset \mathfrak{Aut}(Y, \xi)$, which we assume to be maximal, so that $\xi \in \mathfrak{t} = \text{Lie}(T)$. We say that a Kähler cone metric on Y is compatible with $\zeta \in \mathfrak{t}$ if ζ is its Reeb vector field. Of course the link M then

has a Sasakian structure with Reeb vector field ζ. We define the *Sasaki cone* to be the subset $\mathfrak{t}^+ \subset \mathfrak{t}$

$$(1) \qquad \mathfrak{t}^+ = \{\zeta \in \mathfrak{t} \mid \exists \text{ a Kähler cone metric compatible with } (Y, \zeta)\}$$

Given a polarized affine scheme (Y, ξ) with a torus action $T \subset \mathfrak{Aut}(Y)$ where we assume $\xi \in \mathfrak{t} = \text{Lie}(T)$, the ring of global functions $\mathcal{H} = H^0(Y, \mathcal{O}_Y)$ has a weight space decomposition

$$\mathcal{H} = \bigoplus_{\alpha \in \mathcal{W}} \mathcal{H}_\alpha$$

where $\mathcal{W} \subset \mathfrak{t}^*$ is the set of weights.

If (Y, ξ) has a single singular point it turns out that $\xi \in \mathfrak{t}$ is a Reeb vector field for some Sasakian structure with affine cone Y if and only if for all $\alpha \in \mathcal{W}, \alpha \neq 0$ we have $\alpha(\xi) > 0$. Following [CS18] we give a second definition of the *Sasaki cone* (where it is called the *Reeb cone*) in $\mathfrak{t}$

$$(2) \qquad \mathfrak{t}^+ = \{\xi \in \mathfrak{t} \mid \forall \, \alpha \in \mathcal{W}, \alpha \neq 0 \text{ we have } \alpha(\xi) > 0\} \subset \mathfrak{t}.$$

Proposition 2. *The Sasaki cones as defined in (1) and in (2) are identical. Thus $\mathfrak{t}^+ \subset \mathfrak{t}$ is an open rational polyhedral cone.*

Proof. As described in [vC11] by the second author, or in [CS18], by choosing enough homogenous elements $f_i \in \mathcal{H}$ one can define a T-equivariant embedding $\Psi = (f_1, \ldots, f_N) : Y \to \mathbb{C}^N$, where T acts diagonally on $\mathbb{C}^N$ with weight α_j acting on z_j. If $\zeta \in \mathfrak{t}^+$ according to definition (2) then the polarized affine cone (Y, ζ) has a Sasakian structure induced by the Kähler cone structure on $\mathbb{C}^N$ with Reeb vector field $\sum_{j=1}^N \alpha_i(\zeta)(x_j \partial_{y_j} - y_j \partial_{x_j})$, where $z_j = x_j + iy_j$. The Sasakian metric on S^{2N-1} is the weighted Sasakian structure with weights $(w_1, \ldots, w_N) = (\alpha_1(\zeta), \ldots, \alpha_N(\zeta))$, as defined in [BG08].

The other direction, that if $\zeta \in \mathfrak{t}$ satisfies definition (1) then it satisfies (2), is proved in [CS18]. $\qquad\square$

Generally, the type of a Sasakian structure can change as one moves in the Sasaki cone; however, this does not happen if $c_1(\mathcal{D}) = 0$.

Lemma 3. *Let $(M, \mathcal{D})$ be a contact manifold of Sasaki type with $c_1(\mathcal{D}) = 0$, and let $\mathcal{S} = (\xi, \eta, \Phi, g)$ be a Sasakian structure satisfying $\eta = \ker \mathcal{D}$. Suppose further that the Sasaki cone $\mathfrak{t}^+$ has dimension greater than one. Then all Sasakian structures whose Reeb vector field ξ' is in $\mathfrak{t}^+$ are positive.*

Proof. Let $\mathcal{S}' = (\xi', \eta', \Phi', g')$ be any Sasakian structure in $\mathfrak{t}^+$. Then as stated above there exists $a \in \mathbb{R}$ such that $c_1(\mathcal{F}_{\xi'}) = a[d\eta']_B$. But since the dimension of $\mathfrak{t}^+$ is greater than one, $a > 0$ by Proposition 8.2.15 of [BG08]. So $\mathcal{S}'$ is positive. $\square$

3. Relative K-stability

Relative K-stability and its relation with extremal Kähler metrics were first studied by Székelyhidi [Szé06, Szé07]. Here following Collins and Székelyhidi [CS18] we adopt these ideas to the Sasaki setting. They defined the *T-invariant index character* as a Hilbert series for $\xi \in \mathfrak{t}^+$, viz.

$$(3) \qquad F(\xi, t) := \sum_{\alpha \in \mathcal{W}} e^{-t\alpha(\xi)} \dim \mathcal{H}_\alpha.$$

It is proved in [CS18] that $F(\xi, t)$ converges and has a meromorphic extension with expression

$$(4) \qquad F(\xi, t) = \frac{a_0(\xi) n!}{t^{n+1}} + \frac{a_1(\xi)(n-1)!}{t^n} + O(t^{1-n}).$$

Given $\zeta \in \mathfrak{t}$ we define the weight characters:

$$C_\zeta(\xi, t) = \sum_{\alpha \in \mathcal{W}} e^{-t\alpha(\xi)} \alpha(\zeta) \dim \mathcal{H}_\alpha$$

$$C_{\zeta^2}(\xi, t) = \sum_{\alpha \in \mathcal{W}} e^{-t\alpha(\xi)} \big(\alpha(\zeta)\big)^2 \dim \mathcal{H}_\alpha.$$

Again, these have meromorphic extensions

$$(5) \qquad C_\zeta(\xi, t) = \frac{b_0(\xi)(n+1)!}{t^{n+2}} + \frac{b_1(\xi) n!}{t^{n+1}} + O(t^n),$$

$$(6) \qquad C_{\zeta^2}(\xi, t) = \frac{c_0(\xi)(n+2)!}{t^{n+3}} + O(t^{-n-2}).$$

The coeficients a_0, a_1, b_0, b_1, c_0 depend smoothly on $\xi \in \mathfrak{t}^+$ and

$$(7) \qquad \begin{aligned} b_0 &= \frac{-1}{n+1} D_\zeta a_0 \\ b_1 &= \frac{-1}{n} D_\zeta a_1 \\ c_0 &= \frac{1}{(n+1)(n+2)} D_\zeta^2 a_0. \end{aligned}$$

We also have a norm defined on $\mathfrak{t}$ defined by

$$\tag{8} \|\zeta\|_\xi^2 = c_0(\xi) - \frac{b_0(\xi)^2}{a_0(\xi)}.$$

A *T-equivariant test configuration* is a flat family of affine schemes

$$Y = Y_1 \subset \Upsilon \xrightarrow{\varpi} \mathbb{C},$$

such that ϖ is $\mathbb{C}^*$-equivariant and T acts on the fibers. The $\mathbb{C}^*$ action induces an action on the central fiber Y_0. Let $\zeta \in \mathrm{Lie}(\mathbb{C}^*)$ be the generator of this action. The *Donaldson-Futaki invariant* of the test configuration is

$$\mathrm{Fut}(Y_0, \xi, \zeta) = \frac{a_1(\xi)}{a_0(\xi)} b_0(\xi) - b_1(\xi).$$

In the following we assume that $T \subset \mathfrak{Aut}(Y)$ is a maximal torus with $\xi \in \mathfrak{t} = \mathrm{Lie}(T)$. Define $\chi \in \mathfrak{t}$ to be the element dual to $\mathrm{Fut}(Y_0, \xi, \cdot) : \mathfrak{t} \to \mathbb{R}$ with respect to (8). Notice that both the bilinear form $\langle \cdot, \cdot \rangle_\xi$ defined by (8) and $\mathrm{Fut}(Y_0, \xi, \cdot)$ vanish on $\mathbb{R}\xi$, so χ is well defined modulo ξ. Up to a constant χ is just the transversely extremal vector field, as defined in [FM95]. We define the *Donaldson-Futaki invariant relative to T* of a test configuration

$$Fut_\chi(Y_0, \xi, \zeta) = Fut(Y_0, \xi, \zeta) - \langle \zeta, \chi \rangle.$$

Definition 4. *A polarized affine variety (Y, ξ) with a unique singular point is* K-semistable relative to T *if for every T-equivariant test configuration*

$$\mathrm{Fut}_\chi(Y_0, \xi, \zeta) \geq 0.$$

Recall that for any $(\eta, \xi, \Phi, g) \in \mathcal{S}(\xi, \bar{J})$ the *Calabi functional* is defined by

$$\tag{9} \mathrm{Cal}_{\xi, \bar{J}}(g) = \left(\int_M (S_g - \bar{S}_g)^2 \, d\mu_g \right)^{1/2},$$

where S_g is the scalar curvature and $\bar{S}_g$ is the average scalar curvature. A Sasaki metric g is *extremal* if it is a critical point of $\mathrm{Cal}_{\xi, \bar{J}}^2$. This amounts to the fact that the $(1, 0)$ component of the gradient of the scalar curvature S_g is a transversely holomorphic vector field. Equivalently, the transverse Kähler structure is extremal. Note that when g is extremal, this transversely holomorphic vector field is, up to a constant factor, χ defined above.

An interesting property of the Calabi functional follows from the Donaldson lower bound [Don05] extended to Sasakian manifolds in [CS18].

Theorem 5.

$$\inf_{(\eta,\xi,\Phi,g)\in\mathcal{S}(\xi,\overline{J})} \mathrm{Cal}_{\xi,\overline{J}}(g) \geq c(n)\|\chi\|, \tag{10}$$

where $c(n)$ the constant, depending only on n, follows from the different scaling of the algebraic and analytic Futaki invariant. The lower bound $\mathrm{Cal}_{\xi,\overline{J}}(g) = c(n)\|\chi\|$ is achieved if and only if g is Sasaki-extremal.

Proof. This is an application of the lower bound [CS18, Thm. 5]

$$Cal_{\xi,\overline{J}}(g) \geq -c(n)\frac{\mathrm{Fut}_\chi(Y_0,\xi,\zeta)}{\|\zeta\|}, \tag{11}$$

for any $(\eta,\xi,\Phi,g) \in \mathcal{S}(\xi,\overline{J})$ and test configuration $\mathcal{X}$ with central fiber Y_0 and generator of $\mathbb{C}^*$ action ζ. Consider a product test configuration with $-\chi$ acting fiber-wise, in particular $\zeta = -\chi$. The fact that χ is not necessarily rational is not a problem, because one can use an approximation argument.

For the second statement, if the lower bound is achieved at (η,ξ,Φ,g), then it is obviously a critical point of $\mathrm{Cal}^2_{\xi,\overline{J}}$ and thus Sasaki-extremal. Conversely, if (η,ξ,Φ,g) is Sasaki-extremal, then we can by conjugating by an element of $\mathfrak{Aut}(Y,\xi)$ assume that $T \subset \mathfrak{Aut}(\eta,\xi,\Phi,g)$ is a maximal torus. Then $S_g - \overline{S}_g$ is the normalized holomorphy potential of χ and $\mathrm{Cal}_{\xi,\overline{J}}(g)^2 = c(n)^2\|\chi\|^2$ be the equality, up to a constant, of the norm on $\mathfrak{t}$ defined in [FM95] and the algebraic norm. $\qquad\square$

Remark 6. This inequality follows easily from the arguments of [FM95] for T-invariant metrics for a maximal torus $T \subset \mathfrak{Aut}(Y,\xi)$. But this extends the lower bound to all metrics in $\mathcal{S}(\xi,\overline{J})$.

The following is the Sasakian version of a theorem in [Szé06]:

Theorem 7. *Let $T \subset \mathfrak{Aut}(Y,\xi)$ be a maximal torus, and let $\mathcal{X}$ be a test configuration compatible with T so that $\mathrm{Fut}_\chi(Y_0,\xi,\zeta) < 0$. Then*

$$\mathrm{Cal}_{\xi,\overline{J}}(g)^2 \geq c(n)^2 \left(\frac{\mathrm{Fut}_\chi(Y_0,\xi,\zeta)^2}{\|\zeta\|^2} + \|\chi\|^2 \right), \tag{12}$$

for any $(\eta,\xi,\Phi,g) \in \mathcal{S}(\xi,\overline{J})$.

In particular, for any T-invariant $(\eta, \xi, \Phi, g) \in \mathcal{S}(\xi, \overline{J})$

$$\left(\int_M (S_g - \overline{S}_g - h_g^\chi)^2 \, d\mu_g \right)^{1/2} \geq -c(n) \frac{\mathrm{Fut}_\chi(Y_0, \xi, \zeta)}{\|\zeta\|},$$

where h_g^χ is the normalized holomorphy potential for χ with respect to g.

Proof. Choose a constant λ so that $\overline{\zeta} = \zeta - \lambda\chi$ satisfies $\langle \overline{\zeta}, \chi \rangle = 0$. Then $\mathrm{Fut}(Y_0, \xi, \overline{\zeta}) = \mathrm{Fut}_\chi(Y_0, \xi, \zeta) < 0$. Choose $\mu > 0$ so that $\mathrm{Fut}(Y_0, \xi, \mu\overline{\zeta}) = -\|\mu\overline{\zeta}\|^2$. If we define $\gamma = \mu\overline{\zeta} - \chi$, then

$$\mathrm{Fut}(Y_0, \xi, \gamma) = -\|\mu\overline{\zeta}\|^2 - \|\chi\|^2 = -\|\gamma\|^2,$$

and

$$\frac{\mathrm{Fut}(Y_0, \xi, \gamma)}{\|\gamma\|^2} = \frac{\mathrm{Fut}_\chi(Y_0, \xi, \zeta)}{\|\overline{\zeta}\|^2} + \|\chi\|^2.$$

The inequality then follows from $\|\overline{\zeta}\| \leq \|\zeta\|$ and (11). $\qquad\square$

The following easily follows from (12).

Corollary 8. *If we have equality in (10), in particular if there exists a Sasaki-extremal structure in $\mathcal{S}(\xi, \overline{J})$, then (Y, ξ) is K-semistable relative to any maximal torus T.*

4. Modified Lichnerowicz obstruction

We will give a modified version of the Lichnerowicz obstruction of [GMSY07], which will be useful in obstructing Sasaki-extremal metrics in examples constructed from hypersurface singularities.

A necessary condition that (M, η, ξ, Φ, g), with polarized cone (Y, ξ), admits a Sasaki-Einstein metric is that $c_1(Y) = 0$ (equivalently $c_1(\mathcal{D}) = 0$), and $o \in Y$ is a $\mathbb{Q}$-Gorenstein singularity. In fact, a stronger condition must hold. One must have

$$(13) \qquad\qquad c_1(\mathcal{F}_\xi) = (n+1)[d\eta]_B.$$

One can easily show, assuming M is simply connected, that (13) is equivalent to the existence of a non-vanishing $\Omega \in \Gamma(\Lambda^{n+1,0}C(M))$ with

$$(14) \qquad\qquad \mathcal{L}_\xi \Omega = \sqrt{-1}(n+1)\Omega.$$

(cf. [FOW09]). If M is not simply connected, then Ω can be defined as multivalued. Following [MSY08] we define

$$(15) \qquad \Sigma = \{\xi \in \mathfrak{t}^+ \mid \mathcal{L}_\xi\Omega = \sqrt{-1}(n+1)\Omega\}.$$

We recall the Lichnerowicz obstruction of Gauntlett, Martelli, Sparks, and Yau.

Theorem 9 ([GMSY07]). *Suppose g is a holomorphic function on Y of charge $0 < \lambda < 1$, i.e.*

$$\mathcal{L}_\xi g = \sqrt{-1}\lambda g,$$

then (Y, ξ) admits no Ricci-flat Kähler cone metric with Reeb vector field ξ.

Remark 1. *The Lichnerowicz obstruction was also refined by Obata to include $\lambda = 1$ in which case (Y, ξ) must be the flat structure $(\mathbb{C}^{n+1}, \xi_0)$. See Section 11.3.3 of [BG08] and references therein.*

Suppose that f is a weighted homogeneous holomorphic function with weight $\alpha \in \mathfrak{t}^*$. Let $Y \cong Y_1 \subset \Upsilon \to \mathbb{C}$ be the deformation to the normal cone of $V = \{f = 0\} \subset Y$.

Theorem 10. *The following expression holds:*

$$(16) \qquad \mathrm{Fut}_\chi(Y_0, \xi, \zeta) = \frac{a_0(\xi)}{2}\left(1 - \frac{1}{\alpha(\xi)}\right) - \frac{1}{(n+2)(n+1)^2}\frac{D_\chi a_0(\xi)}{\alpha(\xi)}$$
$$- \frac{1}{(n+2)(n+1)}\frac{a_0(\xi)\alpha(\chi)}{\alpha(\xi)^2}.$$

Notice that the righthand side is independent of the choice of χ modulo ξ. If we choose $\chi \in \mathfrak{t}$ tangent to Σ, then this can be written as

$$(17) \qquad \mathrm{Fut}_\chi(Y_0, \xi, \zeta) = \frac{a_0(\xi)}{2}\left(1 - \frac{1}{\alpha(\xi)}\right) - \frac{2}{(n+2)(n+1)^2}\frac{\|\chi\|^2}{\alpha(\xi)}$$
$$- \frac{1}{(n+2)(n+1)}\frac{a_0(\xi)\alpha(\chi)}{\alpha(\xi)^2}.$$

Proof. It was shown in [CS18] that the Hilbert series of the central fiber Y_0 of Υ is

$$F(\xi + s\zeta) = \frac{a_0(\xi)\alpha(\xi)n!}{(\alpha(\xi) + s)t^{n+1}} + \frac{\alpha(\xi)(a_1(\xi) + \frac{sn}{2}a_0(\xi))(n-1)!}{(\alpha(\xi) + s)t^n} + O(t^{1-n}).$$

We denote the coefficients of the series as

$$\hat{a}_0(\xi + s\zeta) = \frac{a_0(\xi)\alpha(\xi)}{\alpha(\xi) + s}, \qquad \hat{a}_1(\xi + s\zeta) = \frac{\alpha(\xi)\big(a_1(\xi) + \frac{sn}{2}a_0(\xi)\big)}{\alpha(\xi) + s}.$$

Then (7) gives

$$\hat{b}_0(\xi) = \frac{1}{n+1}\frac{a_0(\xi)}{\alpha(\xi)}, \qquad \hat{b}_1(\xi) = \frac{1}{n}\left(\frac{a_1(\xi)}{\alpha(\xi)} - \frac{n}{2}a_0(\xi)\right).$$

We compute

$$\begin{aligned}
\mathrm{Fut}(Y_0, \xi, \zeta) &= \frac{1}{n+1}\frac{a_1(\xi)}{\alpha(\xi)} + \frac{1}{n}\left(\frac{n}{2}a_0(\xi) - \frac{a_1(\xi)}{\alpha(\xi)}\right) \\
&= \frac{a_0(\xi)}{2}\left(1 - \frac{2}{n(n+1)}\frac{a_1(\xi)}{a_0(\xi)\alpha(\xi)}\right).
\end{aligned}$$

But the Calabi-Yau condition (13) implies for $\xi \in \Sigma$ that $a_1(\xi) = \frac{n(n+1)}{2}a_0(\xi)$ [CS18], so we get

$$(18) \qquad\qquad \mathrm{Fut}(Y_0, \xi, \zeta) = \frac{a_0(\xi)}{2}\left(1 - \frac{1}{\alpha(\xi)}\right),$$

which was originally obtained in [CS18].

Similarly, using (7), we compute

$$\begin{aligned}
(19) \qquad \langle \zeta, \chi \rangle &= \frac{1}{(n+1)(n+2)}D_\zeta D_\chi \hat{a}_0 - \frac{1}{(n+1)^2}\frac{1}{a_0(\xi)}D_\zeta \hat{a}_0 D_\chi a_0 \\
&= \frac{-1}{(n+1)(n+2)}D_\chi\left(\frac{a_0(\xi)}{\alpha(\xi)}\right) + \frac{1}{(n+1)^2}\frac{1}{\alpha(\xi)}D_\chi a_0(\xi) \\
&= \frac{1}{(n+1)^2(n+2)}\frac{D_\chi a_0(\xi)}{\alpha(\xi)} + \frac{1}{(n+1)(n+2)}\frac{a_0(\xi)\alpha(\chi)}{\alpha(\xi)^2},
\end{aligned}$$

and (17) follows from (18) and (19).

To prove (17) first observe that (16) is independent of the choice of χ in $\mathrm{Lie}(T)/\mathbb{R}\xi$. This follows easily from $D_\xi a_0 = -(n+1)a_0(\xi)$. It is well known that if χ is tangent to Σ then $\frac{1}{2}D_\chi a_0(\xi) = \mathrm{Fut}(Y, \xi, \chi)$. Thus by the definition of χ we have $\frac{1}{2}D_\chi a_0(\xi) = \|\chi\|^2$, and (17) follows. $\qquad\square$

Since $\xi \in \Sigma$ Equation (16) implies

Corollary 11. *If f is a homogeneous holomorphic function with weight $\alpha \in \mathfrak{t}^*$ satisfying the Lichnerowicz condition in Theorem 9 and $\alpha|_{T\Sigma} = 0$, then the entire Sasaki cone is obstructed from admitting extremal Sasaki metrics.*

5. Applications to links of weighted homogeneous polynomials

Let f be a weighted homogeneous polynomial in $\mathbb{C}^{n+1}$ of degree d and weight vector $\mathbf{w} = (w_0, \ldots, w_n)$ with only an isolated singularity at the origin. It is well known [BG08] that the *link* of f defined by $L_f = \{f = 0\} \cap S^{2n+1}$ is an $(n-2)$-connected smooth manifold with a natural Sasakian structure $\mathcal{S}_{\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$, that we call the *standard* Sasakian structure. Furthermore, according to [BGK05] the Sasaki automorphism group will be finite if $2w_i < d$ for all but one $i = 0, \ldots, n$. So we assume to the contrary that $2w_i \geq d$ for at least two of the $i = 1, \ldots, n$. In fact, we shall assume that f has the form

$$(20) \qquad f(z_0, \ldots, z_n) = f'(z_0, \ldots, z_k) + z_{k+1}^2 + \cdots + z_n^2$$

with $n - k \geq 2$ and all weights w_i with $i = 0, \ldots, k$ satisfy $2w_i < d'$, the degree of f'. In this case the connected component of the Sasaki automorphism group is $U(1) \times SO(n-k)$. For convenience we order the weights

$$w_0 \leq w_1 \leq \cdots \leq w_k.$$

We shall also assume that there is no linear factor as this implies that the link is a standard sphere. First we describe the Sasaki cone.

Proposition 12. *Let f be a weighted homogeneous polynomial of the form of Equation (20). The Sasaki cone $\mathfrak{t}^+$ of the link L_f is given by*

$$(21) \qquad \mathfrak{t}^+ = \left\{ b_0 \xi_{\mathbf{w}} + \sum_{j=1}^{r} b_j \zeta_j \in \mathfrak{t} \;\middle|\; b_0 > 0, \; -\frac{db_0}{2} < b_j < \frac{db_0}{2} \right\}$$

where $\xi_{\mathbf{w}}$ is the standard Reeb field on L_f.

Proof. First note that the dimension of the Sasaki cone $\mathfrak{t}^+$ is $r + 1$ where $r = \lfloor \frac{n-k}{2} \rfloor$. We define variables $u_j = z_{k+j} + iz_{k+j+1}$ and $v_j = z_{k+j} - iz_{k+j+1}$

for $j = 1, \ldots, r$ in which case f takes the form

$$f(z_0, \ldots, z_n) = \begin{cases} f'(z_0, \ldots, z_k) + \sum_{j=1}^r u_j v_j, & \text{if } n - k \text{ is even}; \\ f'(z_0, \ldots, z_k) + \sum_{j=1}^r u_j v_j + z_n^2. & \text{if } n - k \text{ is odd}. \end{cases}$$

Now the Sasaki cone $\mathfrak{t}^+$ is spanned by elements of the form

$$(22) \qquad \xi = b_0 \xi_{\mathbf{w}} + \sum_{j=1}^r b_j \zeta_j$$

where ζ_j has weight $(1, -1)$ with respect to (u_j, v_j) and 0 elsewhere. Now it follows from Theorem 3 of [CS18] that the Sasaki cone is determined by the domain of the smooth function $a_0(\xi)$ in the Laurent expansion of the index character $F(\xi, t)$ of Equation (4). To find this we compute the Hilbert series for the ring $(R = \mathbb{C}[z_0, \ldots, z_n])/I$ where I is the ideal generated by f, that is, $I = Rf$. See Proposition 4.3 of [CS18]. The Hilbert series for R is

$$\frac{1}{(1 - e^{-w_0(\xi))t})} \cdots \frac{1}{(1 - e^{-w_n(\xi))t})}$$

and that of I is $1 - e^{-W(\xi)t}$ where $W(\xi)$ is the weight of the polynomial f, that is, $W(\xi) = d$. This gives the Hilbert series of R/I as

$$(23) \quad F(\xi, t) = \frac{1 - e^{-db_0 t}}{\prod_{i=0}^k (1 - e^{-w_i b_0 t}) \prod_{j=1}^r (1 - e^{-(\frac{d}{2} b_0 + b_j)t})(1 - e^{-(\frac{d}{2} b_0 - b_j)t})}$$

if $n - k$ is even, and
$$(24)$$
$$F(\xi, t) = \frac{1 - e^{-db_0 t}}{\prod_{i=0}^k (1 - e^{-w_i b_0 t}) \prod_{j=1}^r (1 - e^{-(\frac{d}{2} b_0 + b_j)t})(1 - e^{-(\frac{d}{2} b_0 - b_j)t})(1 - e^{-\frac{d}{2} b_0 t})}$$

if $n - k$ is odd. Then using Equation (4) with $n \mapsto n - 1$ we have for $n - k$ even

$$(25) \qquad a_0(\xi)(n - 1)! = \frac{d}{w_0 \cdots w_k b_0^{k-1}(\frac{d^2 b_0^2}{4} - b_1^2) \cdots (\frac{d^2 b_0^2}{4} - b_r^2)}.$$

So $a_0(\xi)$ is well defined and positive precisely for the range indicated. This finishes the proof when $n - k$ is even. There are similar expressions when $n - k$ is odd. $\qquad\square$

Theorem 13. *Let f be a weighted homogeneous polynomial of the form of Equation (20) with no linear factors. Suppose further that the inequality holds*

$$\sum_{i=0}^{k} w_i - w_0 n + \frac{d}{2}(n - k - 2) \geq 0.$$

Then there are no extremal Sasaki metrics in the entire Sasaki cone of the link L_f.

Proof. We need to check that the conditions of Corollary 11 are satisfied. To check the Lichnerowicz condition Theorem 9 we consider the homogeneous function $g = z_0$. The Reeb vector field of the link L_f is $\xi_{\mathbf{w}}$ which has Fano index $\mathfrak{I} = |\mathbf{w}| - d$. So the Reeb vector field which satisfies Equation (14) is

$$\xi = \frac{n}{|\mathbf{w}| - d}\xi_{\mathbf{w}}.$$

Thus, the charge of g is $\frac{nw_0}{|\mathbf{w}|-d}$. There are two cases to consider: (1) the degree d' of f' is even in which case $d = d'$, and (2) the degree d' of f' is odd and $d = 2d'$. In both cases $w_{k+1} = \cdots = w_n = \frac{d}{2}$ and the Fano index $\mathfrak{I}$ is

$$(26) \qquad \mathfrak{I} = |\mathbf{w}| - d = \sum_{i=0}^{k} w_i + \frac{d}{2}(n - k) - d = \sum_{i=0}^{k} w_i + \frac{d}{2}(n - k - 2).$$

Thus, the charge $\frac{nw_0}{|\mathbf{w}|-d}$ of g will be less than one if and only if the strict inequality holds. Moreover, since there are no linear factors in f, Remark 1 also implies the result when equality holds.

Next, we check the condition $\alpha|_{T\Sigma} = 0$. Note first that $\xi \in \Sigma$ if and only if we have

$$b_0 = \frac{n}{\sum_{i=0}^{k} w_i + \frac{d}{2}(n - k - 2)} < \frac{1}{w_0}$$

so the range on Σ becomes

$$-\frac{1}{4}\frac{dn}{\sum_{i=0}^{k} w_i + \frac{d}{2}(n - k - 2)} < b_j < \frac{1}{4}\frac{dn}{\sum_{i=0}^{k} w_i + \frac{d}{2}(n - k - 2)}.$$

Now from (22) the tangent space to Σ at ξ is spanned by the set $\{\zeta_j\}_{j=1}^{r}$, so if α is the weight of f then $\alpha(\zeta_j) = 0$ since (u_j, v_j) has weight $(1, -1)$ with respect to ζ_j. $\qquad \square$

6. Explicit examples

In this section our results are presented in the form of tables. The first table presents examples of families of Sasakian structures whose Sasaki cone has dimension greater than one, and contains no extremal Sasaki metric at all. We shall only present details for a representative case. In the first four examples of Table 1 the oriented diffeomorphism type is stated explicitly where Σ_1^{4n+1} is a generator of the group bP_{4n+2} of homotopy spheres which bound a parallelizable manifold of dimension $4n + 2$. Note that $\Sigma_k^{4n+1} = \Sigma_1^{4n+1}$ if $(2k + 1) \equiv \pm 3 \mod 8$, and is the standard sphere when $(2k + 1) \equiv \pm 1 \mod 8$. It is known that $bP_{4n+2} \approx \mathbb{Z}_2$ for $4n + 2 \neq 2^j - 2$ for some j and equals the identity if $n = 1, 3, 7, 15$, so in these cases $\Sigma_1^{4n+1} = \{id\}$. (It is still an open question for the remaining cases.) Thus, in these four cases there are a countably infinite number of families of Sasakian structures with $n + 1$ dimensional Sasaki cones having no extremal Sasaki metrics. Moreover, these belong to a countably infinite number of inequivalent underlying contact structures [Ust99, KvK16, Gut17, BMvK16, Ueb16].

The oriented diffeomorphism type can also be determined for the fifth example. These are homotopy spheres of dimension $4n - 1$ which bound a parallelizable manifold of dimension $4n$. A well known result of Kervaire and Milnor [KM63] says that the group bP_{4n} is cyclic of order

$$|bP_{4n}| = 2^{2n-2}(2^{2n-1} - 1) \text{ numerator}\left(\frac{4B_n}{n}\right),$$

where B_n is the n-th Bernoulli number. Thus, for example $|bP_8| = 28, |bP_{12}| = 992, |bP_{16}| = 8128$. Thus, there are a countably infinite number of families of Sasakian structures with n dimensional Sasaki cones having no extremal Sasaki metrics. Moreover, these also belong to a countably infinite number of underlying contact structures (actually almost contact structures in this case) since $\Sigma_{k'}^{4n-1} \equiv \Sigma_k^{4n-1} \mod |bP_{4n}|$, cf. Section 9.5.3 of [BG08]. Here Σ_1^{4n-1} is called the *Milnor generator*.

The sixth example consists of a family of rational homology spheres of dimension $4n - 1$ with $H_{2n} \approx \mathbb{Z}_3$ described by Durfee [Dur77]. There are two families of oriented diffeomorphism types that are equivalent as non-oriented manifolds. We begin with the links K_2 and K_4 where K_k is the link of the polynomial $z_0^k + z_1^3 + z_2^2 + \cdots + z_{2n}^2$. Then K_{6l+2} is diffeomorphic to $K_2\#(-1)^{\frac{n}{2}}l\Sigma_1^{4n-1}$ and K_{6l+4} is diffeomorphic to $K_4\#(-1)^{\frac{n}{2}}l\Sigma_1^{4n-1}$. Note that the Milnor generator $\Sigma_1^{4n-1} \approx K_5$. Thus, in this case there are also a

countably infinite number of families of Sasakian structures with n dimensional Sasaki cones having no extremal Sasaki metrics. We conjecture that these also belong to inequivalent underlying contact structures, although this has not been proven yet to the best of the authors' knowledge.

In example seven we have Sasaki manifolds that are homeomorphic to $2k(S^{2n+1} \times S^{2n+2})$. The oriented diffeomorphism type is not known explicitly and not all possible diffeomorphism types occur. In [BG06] a formula is given for the number of diffeomorphism types $D_n(k)$ obtained by our method, and tables are given for dimension 7 and 11. Nevertheless, since there is a periodicity modulo a subgroup of bP_{4n+4}, we do have countably infinite families of Sasakian structures with an $n + 1$ dimensional Sasaki cones having no extremal Sasaki metrics. Whether the underlying contact structures are inequivalent is also not known at this time.

Finally we consider the connected sums $\#m(S^2 \times S^3)$ which are all the compact smooth Sasaki 5-manifolds with a 2-torus of Sasaki automorphisms. Here $m = 0$ means S^5. These are represented by links of the polynomial $z_0^p + z_1^q + z_2^2 + z_3^2$ with $m = \gcd(p, q) - 1$. They are perhaps the most interesting case owing to the recent work of Collins and Székelyhidi [CS15] which shows that the standard Sasakian structure on the link admits a Sasaki-Einstein metric if and only if $2p > q$ and $2q > p$. Thus, when these inequalities are violated, namely when $p \geq 2q$ or $q \geq 2p$, there are for each $m = 0, 1, \cdots$ a countably infinite number of families of Sasakian structures with 2-dimensional Sasaki cones having no extremal Sasaki metrics. Moreover, they belong to a countably infinite number of inequivalent underlying contact structures [BMvK16, Ueb16].

Although in most cases of Table 1 the members of a given family belong to inequivalent contact structures, they do have isomorphic transverse holomorphic structures. This can be seen by showing that the S^1 quotients of the link for each member are isomorphic as algebraic varieties. We refer to the proof of Proposition 4.1 of [BGN03] for the details in the case of the homotopy spheres. This completes the proof of Theorem 1. □

Question 14. *In the case that Sasaki-Einstein metrics exist, that is when $2p > q$ and $2q > p$, is the entire Sasaki cone exhausted by extremal Sasaki metrics?*

We give the simple details of applying Theorem 13 for a sample case. Consider the unit tangent sphere bundle T of S^{2n+1} represented by the link $L(4l + 2, 2, \ldots, 2)$ given by the polynomial

$$f = z_0^{4l+2} + z_1^2 + \cdots + z_{2n+1}^2.$$

Diffeo-(homeo)-morphism Type	f	dim $\mathfrak{t}^+$
$S^{2n} \times S^{2n+1}$	$z_0^{8l} + z_1^2 + \cdots + z_{2n+1}^2,\ n, l \geq 1$	$n+1$
$S^{2n} \times S^{2n+1} \# \Sigma_1^{4n+1}$	$z_0^{8l+4} + z_1^2 + \cdots + z_{2n+1}^2 = 0,\ n \geq 1, l \geq 0$	$n+1$
Unit tangent bundle of S^{2n+1}	$z_0^{4l+2} + z_1^2 + \cdots + z_{2n+1}^2,\ n > 1, l \geq 1$	$n+1$
Homotopy sphere Σ_k^{4n+1}	$z_0^{2k+1} + z_1^2 + \cdots + z_{2n+1}^2,\ n > 1, k \geq 1$	$n+1$
Homotopy sphere Σ_k^{4n-1}	$z_0^{6k-1} + z_1^3 + z_2^2 + \cdots + z_{2n}^2,\ n \geq 2, k \geq 1$	n
Rat. homology sphere $H_{2n} \approx \mathbb{Z}_3$	$z_0^k + z_1^3 + \cdots + z_{2n}^2,\ n, k > 1$	n
$2k(S^{2n+1} \times S^{2n+2}),\ D_{n+1}(k)$	$z_0^{2(2k+1)} + z_1^{2k+1} + z_2^2 + \cdots + z_{2n+2}^2,\ n, k \geq 1$	$n+1$
$\#m(S^2 \times S^3), m = \gcd(p,,q) - 1$	$z_0^p + z_1^q + z_2^2 + z_3^2,\quad p \geq 2q \text{ or } q \geq 2p$	2

Table 1: Manifolds having Sasaki cones with no extremal metrics.

We mention that there are no exotic structures in this case since $T \# \Sigma^{4n+1}$ is diffeomorphic to T. Applying Theorem 13 we take $n \mapsto 2n+1$, $k = 0, d = 4l + 2$ and $w_0 = 1$, so our inequality becomes

$$0 < w_0 - w_0(2n+1) + (2l+1)(2n-1) = 2l(2n-1) - 1.$$

Of course there are many other examples to which we can apply Theorem 13. For example, the rational homology spheres M_k^{4n-1} of dimension $4n - 1$ with $H_{2n-1} \approx \mathbb{Z}_k$ given by Brieskorn-Pham link of the polynomial $z_0^k + z_1^2 + \cdots + z_{2n}^2$, with $k > 2, n > 1$ has a Sasaki cone of dimension $n + 1$ and one can easily check that the inequality of Theorem 13 holds in this case. So for each $k > 2$ M_k^{4n-1} has an $n+1$-dimensional Sasaki cone having no extremal Sasaki metrics. We mention that the case $k = 2$ M_2^{4n-1} is a Stiefel manifold which admits a Sasaki-Einstein metric. Many more examples of rational homology spheres can be obtained by noting that as in Corollary 9.5.3 of [BG08] if the link of f' in the polynomial (20) is a rational homology sphere and $n - k$ is even, then the link of f is also a rational homology sphere.

Finally we give a table of ADE n-folds which admit no extremal Sasaki metrics in the entire Sasaki cone or equivalently no Ricci-flat Kähler metric on the corresponding Calabi-Yau cone. Notice that there is some overlap with Table 1. These are of particular interest since they have been used in Physics in conformal field theory [GVW00, GMSY07].

Acknowledgements

	f	$\mathbf{w}$	$\dim \mathfrak{t}^+$
A_{k-1}	$z_0^k + z_1^2 + \cdots + z_n^2,\ k \geq 3$	$(2, k, \cdots, k)$	$1 + \lfloor \frac{n}{2} \rfloor$
D_{k+1}	$z_0^k + z_0 z_1^2 + z_2^2 + \cdots + z_n^2,\ k \geq 2$	$(2, k-1, k, \cdots, k)$	$\lceil \frac{n}{2} \rceil$
E_6	$z_0^4 + z_1^3 + z_2^2 + \cdots + z_n^2$	$(4, 3, 6, \cdots, 6)$	$\lceil \frac{n}{2} \rceil$
E_7	$z_0^3 + z_0 z_1^3 + z_2^2 + \cdots + z_n^2$	$(6, 4, 9, \cdots, 9)$	$\lceil \frac{n}{2} \rceil$
E_8	$z_0^5 + z_1^3 + z_2^2 + \cdots + z_n^2$	$(6, 10, 15, \cdots, 15)$	$\lceil \frac{n}{2} \rceil$

Table 2: ADE n-folds with $n \geq 4$ whose Sasaki cones have no extremal metrics.

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