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Convexity for twisted conjugation

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Let G be a compact, simply connected Lie group. If $\mathcal{C}_1, \mathcal{C}_2$ are two G-conjugacy classes, then the set of elements in G that can be written as products $g = g_1 g_2$ of elements $g_i \in \mathcal{C}_i$ is invariant under conjugation, and its image under the quotient map $G \to G/\operatorname{Ad}(G) = \mathfrak{A}$ is a convex polytope. In this note, we will prove an analogous statement for *twisted conjugations* relative to group automorphisms. The result will be obtained as a special case of a convexity theorem for group-valued moment maps which are equivariant with respect to the twisted conjugation action.

1. Introduction

Let G be a compact connected Lie group, with maximal torus T, and let $\mathfrak{g}, \mathfrak{t}$ be their Lie algebras. Fix a positive Weyl chamber $\mathfrak{t}_+ \subseteq \mathfrak{t}$, and denote by $p\colon \mathfrak{g} \to \mathfrak{t}_+$ the quotient map, with fiber $p^{-1}(\xi) = \mathcal{O}_\xi$ the adjoint orbit of ξ. For any $r > 1$, the set

$$(1) \qquad \{(\xi_1, \ldots, \xi_r) \in \mathfrak{t}_+ \times \cdots \times \mathfrak{t}_+ \big|\ \exists \zeta_i \in \mathcal{O}_{\xi_i} \colon \zeta_1 + \cdots + \zeta_r = 0\}$$

is a convex polyhedral known as the *Horn cone*. Fixing $\xi_1, \ldots, \xi_{r-1}$, the Horn cone describes the set of adjoint orbits contained in the sum of adjoint orbits $\mathcal{O}_{\xi_1} + \cdots + \mathcal{O}_{\xi_{r-1}}$. For the case of $G = \mathrm{U}(N)$, the projection $p(\zeta)$ signifies the set of eigenvalues of a Hermitian matrix ζ, hence the Horn cone thus describes the possible eigenvalues of sums of Hermitian matrices with prescribed eigenvalues. The defining inequalities for the $\mathfrak{u}(n)$-Horn cone were obtained by Klyachko [14], who gave a description in terms of the Schubert calculus of the Grassmannian. This was extended to arbitrary compact groups by Berenstein-Sjamaar [7]. See Ressayre [20] and Vergne-Walter [23] for recent developments.

Suppose in addition that G is simply connected. Let $\mathfrak{A} \subseteq \mathfrak{t}_+$ be the Weyl alcove. Then $\mathfrak{A}$ labels the set of conjugacy classes in G, in the sense that there is a quotient map $q\colon G \to \mathfrak{A}$, with fiber $q^{-1}(\xi) = \mathcal{C}_\xi$ the conjugacy class of $\exp(\xi)$. As observed in Meinrenken-Woodward [17, Corollary 4.13], the set

$$\{(\xi_1,\ldots,\xi_r)\in\mathfrak{A}\times\cdots\times\mathfrak{A}\big|\ \exists g_i\in\mathcal{C}_{\xi_i}\colon g_1\cdots g_r=e\} \tag{2}$$

is a convex polytope. Put differently, this polytope describes the conjugacy classes arising in products of a collection of prescribed conjugacy classes. In the case of $G=\mathrm{SU}(n)$, it describes the possible eigenvalues of *products* of special unitary matrices with prescribed eigenvalues; these eigenvalue inequalities were determined, in terms of quantum Schubert calculus on flag manifolds, by Agnihotri-Woodward [1] and Belkale [5]. (See also Belkale-Kumar [6].) This was extended to general G by Teleman-Woodward [22].

In this note we will show that there are similar polytopes for *twisted* conjugations. Recall that the twisted conjugation action relative to a group automorphism $\kappa\in\mathrm{Aut}(G)$ is the action

$$\mathrm{Ad}^{(\kappa)}_g(a)=g\,a\,\kappa(g^{-1}). \tag{3}$$

As we will explain, it suffices to consider automorphisms κ defined by Dynkin diagram automorphisms. These automorphisms preserve $\mathfrak{t}$, with fixed point set $\mathfrak{t}^\kappa$, and there is a convex polytope (*alcove*) $\mathfrak{A}^{(\kappa)}\subseteq\mathfrak{t}^\kappa$ with a quotient map $q^{(\kappa)}\colon G\to\mathfrak{A}^{(\kappa)}$ whose fiber $(q^{(\kappa)})^{-1}(\xi)=\mathcal{C}^{(\kappa)}_\xi$ is the κ-twisted conjugacy class of $\exp(\xi)$.

Theorem 1.1. *Let $\kappa_1,\ldots,\kappa_r$ be diagram automorphisms with $\kappa_r\circ\cdots\circ\kappa_1=1$. Then the set*

$$\{(\xi_1,\ldots,\xi_r)\in\mathfrak{A}^{(\kappa_1)}\times\cdots\times\mathfrak{A}^{(\kappa_r)}\big|\ \exists g_i\in\mathcal{C}^{(\kappa_i)}_{\xi_i}\colon g_1\cdots g_r=e\} \tag{4}$$

is a convex polytope.

It would be interesting to obtain an explicit description of the defining inequalities of the polytopes (4). (In Section 5, we will work out the case of $G=\mathrm{SU}(3)$ and $r=3$ by direct computation.) Note that these polytopes (4) arise if one considers products of conjugacy classes of *disconnected* compact Lie groups K; indeed each conjugacy class of K is a finite union of twisted conjugacy classes of the identity component $G=K_0$.

We will obtain Theorem 1.1 as a special case of a convexity result for group-valued moment maps that are equivariant under *twisted conjugation.* Examples of such spaces are the twisted conjugacy classes, or components of moduli spaces of flat connections for disconnected groups on surfaces with boundary. We have (cf. Theorem 4.4):

Theorem 1.2. *Let (M,ω,Φ) be a compact, connected q-Hamiltonian G-space with a κ-twisted equivariant moment map $\Phi\colon M\to G$. Then the fibers of the moment map are connected, and the image*

$$\Delta(M) := q^{(\kappa)}(\Phi(M)) \subseteq \mathfrak{A}^{(\kappa)}$$

is a convex polytope.

In a very recent paper, Boalch and Yamakawa [10] independently considered twisted group-valued moment maps in the context of twisted wild character varieties, generalizing earlier results of Boalch [8, 9]. In particular, their work has a discussion of twisted moduli spaces, similar to Section 3.2. I also learned about a forthcoming article by Alex Takeda, using twisted group-valued moment maps in the setting of shifted symplectic geometry.

2. Twisted conjugation

We begin by reviewing some background material on twisted conjugation actions. References include Baird [4], Kac [13], Mohrdieck [18], Mohrdieck-Wendt [19], and Springer [21].

2.1. Twisted conjugation

Let $\mathrm{Aut}(G)$ be the group of automorphisms of a Lie group G, and let $\mathrm{Inn}(G)\cong G/Z(G)$ be the normal subgroup of inner automorphisms Ad_a, $a\in G$. The quotient group is denoted $\mathrm{Out}(G)=\mathrm{Aut}(G)/\mathrm{Inn}(G)$. For $\kappa\in\mathrm{Aut}(G)$, define the κ-twisted conjugation action as

$$\mathrm{Ad}_g^{(\kappa)}(h) = gh\kappa(g^{-1}).$$

Its orbits $\mathcal{C}\subseteq G$ are called the *κ-twisted conjugacy classes.* In terms of the semi-direct product $G\rtimes\mathrm{Aut}(G)$, the twisted conjugation action can be regarded as an ordinary conjugation,

$$(g,1)(h,\kappa)(g^{-1},1) = (gh\kappa(g^{-1}),\kappa).$$

For this reason, we will sometimes use the notation $G\kappa$ for the space G, regarded as a G-space under κ-twisted conjugation. For later reference, we note that if κ_1, κ_2 are two automorphisms, then

$$\mathrm{Ad}_g^{(\kappa_2\kappa_1)}(h_1h_2) = \mathrm{Ad}_g^{(\kappa_1)}(h_1)\,\mathrm{Ad}_{\kappa_1(g)}^{(\kappa_2)}(h_2) \tag{5}$$

for all $g, h_1, h_2 \in G$.

The differential of $\kappa \in \mathrm{Aut}(G)$ at the group unit is an automorphism of the Lie algebra $\mathfrak{g}$, still denoted by κ. The generating vector fields for the κ-twisted conjugation action are $\xi_G = \kappa(\xi)^L - \xi^R$ for $\xi \in \mathfrak{g}$. In terms of *right* trivialization of the tangent bundle, we have $\xi_G(h) = (\mathrm{Ad}_h \circ \kappa - I)\xi$. Hence, the Lie algebra of the stabilizer of $h \in G$ is

$$\mathfrak{g}_h = \ker(\mathrm{Ad}_h \circ \kappa - I), \tag{6}$$

while the tangent space to the twisted conjugacy class $\mathcal{C} = \mathrm{Ad}_G^{(\kappa)}(h)$ is

$$T_h\mathcal{C} = \mathrm{ran}(\mathrm{Ad}_h \circ \kappa - I), \tag{7}$$

in *right* trivialization $T_hG = \mathfrak{g}$.

Suppose $\kappa' = \mathrm{Ad}_a \circ \kappa$ for some $a \in G$. Then the corresponding twisted conjugations are related by right multiplication $r_a \colon G \to G$:

$$r_a \circ \mathrm{Ad}_g^{(\kappa')} = \mathrm{Ad}_g^{(\kappa)} \circ r_a.$$

That is, $g \mapsto ga^{-1}$ defines a G-map $G\kappa \to G\kappa'$. In particular, if $\mathcal{C}$ is a κ-twisted conjugacy class then $\mathcal{C}' = r_{a^{-1}}(\mathcal{C})$ is a κ'-twisted conjugacy class.

Example 2.1. Suppose $\kappa_1, \ldots, \kappa_r \in \mathrm{Aut}(G)$, and let $\mathcal{C}_i$ be κ_i-twisted conjugacy classes. Then the subset

$$\mathcal{C}_1 \cdots \mathcal{C}_r := \{h_1 \cdots h_r |\ h_i \in \mathcal{C}_i\} \subseteq G$$

is invariant under $\kappa := \kappa_r \cdots \kappa_1$-twisted conjugation. This follows by induction from (5). Let $\kappa_i' = \mathrm{Ad}_{a_i} \circ \kappa_i$ for some $a_i \in G$, and put $\mathcal{C}_i' = r_{a_i^{-1}}(\mathcal{C}_i)$ and $\kappa' = \kappa_r' \cdots \kappa_1'$. Then the problem of finding $h_i \in \mathcal{C}_i$ with product $h_1 \cdots h_r$ in a prescribed κ-twisted conjugacy class $\mathcal{C}$ is equivalent to a similar problem for the $\mathcal{C}_i'$.

To see this, let $u_1, \ldots, u_{r+1}$ be inductively defined as $u_{i+1} = a_i\kappa_i(u_i)$ with $u_1 = e$, and put $a = u_{r+1}$. Then $\kappa' = \mathrm{Ad}_a \circ \kappa$, hence $\mathcal{C}' = r_{a^{-1}}(\mathcal{C})$ is a κ'-twisted conjugacy class. A straightforward calculation shows that if

$h_i \in \mathcal{C}_i$ satisfy $h := h_1 \cdots h_r \in \mathcal{C}$, then the elements

$$h_i' = \mathrm{Ad}_{u_i}^{(\kappa_i)}(h_i)\; a_i^{-1} \in \mathcal{C}_i'$$

have product $h' = ha^{-1} \in \mathcal{C}'$.

2.2. Diagram automorphisms

Let G be a compact and simply connected Lie group, with maximal torus T and Weyl group $W = N_G(T)/T$. Fix a positive Weyl chamber $\mathfrak{t}_+ \subseteq \mathfrak{t}$, with corresponding alcove $\mathfrak{A} \subseteq \mathfrak{t}_+$. The walls of the Weyl chamber are defined by the simple roots $\alpha_1, \dots, \alpha_l \in \mathfrak{t}^*$. Let $\alpha_i^\vee \in \mathfrak{t}$ be the simple coroots, and let $e_i,\ f_i \in \mathfrak{g}^{\mathbb{C}}$ be the Chevalley generators, for $i = 1, \dots, l$.

Consider an automorphisms of the Dynkin diagram, given by a bijection $i \mapsto i'$ of its set of vertices preserving all Cartan integers: $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_{i'}, \alpha_{j'}^\vee \rangle$. Any diagram automorphism defines a unique Lie algebra automorphism $\kappa \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ such that $\kappa(e_i) = e_{i'},\ \kappa(f_i) = f_{i'}$. This automorphism preserves the real Lie algebra $\mathfrak{g} \subseteq \mathfrak{g}^{\mathbb{C}}$, and exponentiates to the Lie group G. We will refer to the resulting $\kappa \in \mathrm{Aut}(G)$ itself as a diagram automorphism. Every element of $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is represented by a unique diagram automorphism, and the resulting splitting $\mathrm{Out}(G) \hookrightarrow \mathrm{Aut}(G)$ identifies

$$\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes \mathrm{Out}(G).$$

That is, any automorphism of G can be written as $\kappa' = \mathrm{Ad}_a \circ \kappa$ with $a \in G$ and $\kappa \in \mathrm{Out}(G)$. To understand the orbit structure of κ-twisted conjugation actions, it hence suffices to consider the case that $\kappa \in \mathrm{Aut}(G)$ is a diagram automorphism. In particular, κ preserves T, with fixed point set $T^\kappa \subseteq G^\kappa$. Let $\mathfrak{t}^\kappa,\ \mathfrak{t}_\kappa$ be the kernel and range of $\kappa|_{\mathfrak{t}} - I \colon \mathfrak{t} \to \mathfrak{t}$. Then $\mathfrak{t}^\kappa$ is the Lie algebra of T^κ, and $\mathfrak{t}_\kappa = (\mathfrak{t}^\kappa)^\perp$ is the orthogonal space in $\mathfrak{t}$ (relative to a W-invariant metric). Put $T_\kappa = \exp(\mathfrak{t}_\kappa)$. Then $T = T^\kappa\, T_\kappa$, with finite intersection

$$T^\kappa \cap T_\kappa.$$

Let $W^\kappa \subseteq W$ the subgroup of elements w whose action on $\mathfrak{t}$ commutes with κ. For $a \in G$, denote by G_a the stabilizer under the κ-twisted adjoint action. For Propositions 2.2 and 2.3 below, see [19], [21], and references therein.

Proposition 2.2. *Let $\kappa \in \mathrm{Aut}(G)$ be a diagram automorphism. Then:*

a) *The group G^κ contains T^κ as a maximal torus, with Weyl group W^κ. The intersection $\mathfrak{t}_+^\kappa = \mathfrak{t}^\kappa \cap \mathfrak{t}_+$ is a positive Weyl chamber for G^κ.*

b) *Every κ-twisted conjugacy class $\mathcal{C} \subseteq G$ intersects the torus T^κ in an orbit of the finite group $(T^\kappa \cap T_\kappa) \rtimes W^\kappa$. Here $T^\kappa \cap T_\kappa$ acts by multiplication on T^κ.*

c) *For all $a \in T^\kappa$, the stabilizer group G_a under the twisted conjugation action contains T^κ as a maximal torus.*

Let $\Lambda = \exp_T^{-1}(e) \subseteq \mathfrak{t}$ be the integral lattice of T. Since G is simply connected, it coincides with the coroot lattice of (G, T). The fixed point set $\Lambda^\kappa \subseteq \mathfrak{t}^\kappa$ is the integral lattice of T^κ. It is contained in the lattice,

$$\Lambda^{(\kappa)} = \exp_{T^\kappa}^{-1}(T^\kappa \cap T_\kappa).$$

Proposition 2.3. *There is a unique closed convex polytope $\mathfrak{A}^{(\kappa)} \subseteq \mathfrak{t}_+^\kappa$, containing the origin, such that $G_{\exp\xi} = T^\kappa$ for elements $\xi \in \mathrm{int}(\mathfrak{A}^{(\kappa)})$, and such that the map*

$$\mathfrak{A}^{(\kappa)} \xrightarrow{\exp} G \longrightarrow G/\operatorname{Ad}_G^{(\kappa)}$$

is a bijection. Furthermore,

a) *The cone over $\mathfrak{A}^{(\kappa)}$ is $\mathfrak{t}_+^\kappa$.*

b) *For each open face $\sigma \subseteq \mathfrak{A}^{(\kappa)}$, the stabilizer group $G_\sigma := G_{\exp\xi}$ of elements $\xi \in \sigma$ does not depend on ξ, The stabilizer groups satisfy $G_\sigma \supseteq G_\tau$ for $\sigma \subseteq \overline{\tau}$.*

c) *The group $W_{\mathrm{aff}}^{(\kappa)} = \Lambda^{(\kappa)} \rtimes W^\kappa$ is an affine reflection group, generated by reflections across the facets of $\mathfrak{A}^{(\kappa)}$, and having $\mathfrak{A}^{(\kappa)}$ as a fundamental domain.*

Clearly, if $\kappa = 1$ then $\mathfrak{A}^{(\kappa)}$ is just the usual Weyl alcove, parametrizing the set of (untwisted) conjugacy classes in G. Note that in general, $\Lambda^{(\kappa)}$, $\mathfrak{A}^{(\kappa)}$ are different from the coroot lattice and alcove of $(\mathfrak{g}^\kappa, \mathfrak{t}^\kappa)$. The group $W_{\mathrm{aff}}^{(\kappa)}$ is the Weyl group of the twisted affine Lie algebra defined by κ, see Kac [13].

2.3. Slices

The conjugation action of G on itself has distinguished slices, labeled by the faces of the alcove. We will generalize this fact to twisted conjugation actions.

Lemma 2.4. *Let $\kappa \in \mathrm{Aut}(G)$, where G is compact. Let $\mathcal{C} \subseteq G$ be the κ-twisted conjugacy class of an element $a \in G^\kappa$. Then*

$$T_a G = T_a(G_a) \oplus T_a\mathcal{C}. \tag{8}$$

Proof. Pick an $\mathrm{Aut}(G)$-invariant inner product on $\mathfrak{g}$, defining a bi-invariant Riemannian metric on the Lie group G which is also invariant under κ. Since $\kappa(a) = a$, we have $a \in G_a$, and we obtain $T_a G_a = \mathfrak{g}_a$ in right trivialization. On the other hand, by (6) and (7) we have $T_a\mathcal{C} = \mathrm{ran}(\mathrm{Ad}_a \circ \kappa - I) = \mathfrak{g}_a^\perp$ in right trivialization. Since the two spaces are orthogonal, the Lemma follows. □

Using again that $\kappa(a) = a$, the twisted conjugation action of a on G restricts to the usual conjugation action on G_a. In particular, G_a is a $\mathrm{Ad}_a^{(\kappa)}$-invariant submanifold of G. The Lemma shows that any sufficiently small invariant open neighborhood of a in G_a is a slice for the twisted conjugation action.

If G is also simply connected, and κ is a diagram automorphism, there is a specific 'largest' slice, as follows. For any face $\sigma \subseteq \mathfrak{A}^{(\kappa)}$, let $\mathfrak{A}_\sigma^{(\kappa)}$ be the relatively open subset of $\mathfrak{A}^{(\kappa)}$ given as the union of faces $\tau \subseteq \mathfrak{A}^{(\kappa)}$ such that $\sigma \subseteq \overline{\tau}$. Put

$$U_\sigma = \mathrm{Ad}_{G_\sigma}^{(\kappa)} \exp(\mathfrak{A}_\sigma^{(\kappa)}), \tag{9}$$

a subset of $G_\sigma \subseteq G$.

Proposition 2.5. *The subset $U_\sigma \subseteq G_\sigma$ is open, and invariant under the twisted conjugation action of G_σ. The map*

$$G \times_{G_\sigma} U_\sigma \to G,\ \ [(g, a)] \mapsto \mathrm{Ad}_g^{(\kappa)}\, a \tag{10}$$

is an embedding as an open subset of G. That is, U_σ is a slice for the twisted conjugation action.

Proof. Pick $\zeta \in \sigma$, and put $c = \exp\zeta$ so that $G_c = G_\sigma$. For all $\xi \in \mathfrak{t}^\kappa$ and $g \in G_\sigma$,

$$\mathrm{Ad}_g^{(\kappa)} \exp(\xi) = \mathrm{Ad}_g^{(\kappa)}(\exp(\xi - \zeta)c) = \mathrm{Ad}_g(\exp(\xi - \zeta))\ c. \tag{11}$$

It follows that $U_\sigma = U'_\sigma\, c$ where

$$U'_\sigma = \mathrm{Ad}_{G_\sigma} \exp\left(\mathfrak{A}_\sigma^{(\kappa)} - \zeta\right).$$

Equation (11) also shows that for $\xi \in \mathfrak{A}_\sigma^{(\kappa)} \subseteq \mathfrak{t}^\kappa$, the stabilizer of $\exp(\xi)$ under the twisted conjugation action of G (which lies in G_σ, by definition of $\mathfrak{A}_\sigma^{(\kappa)}$) equals the stabilizer of $\exp(\xi - \zeta)$ under the usual conjugation action of G_σ. Consequently, $\mathfrak{A}_\sigma^{(\kappa)} - \zeta$ is a relatively open subset of an alcove of (G_σ, T^κ). It follows that U'_σ is open in G_σ, and hence U_σ is open in G_σ.

We next show that the map (10) is injective. Thus suppose $\mathrm{Ad}_g^{(\kappa)} a = \mathrm{Ad}_{g'}^{(\kappa)} a'$, where $a, a' \in U_\sigma$ and $g, g' \in G$. Since a, a' are in the same twisted conjugacy class, there is a unique $\xi \in \mathfrak{A}_\sigma$ and elements $h, h' \in G_\sigma$ such that

$$a = \mathrm{Ad}_h^{(\kappa)} \exp(\xi), \qquad a' = \mathrm{Ad}_{h'}^{(\kappa)} \exp(\xi).$$

We thus obtain

$$\mathrm{Ad}_{gh}^{(\kappa)} \exp(\xi) = \mathrm{Ad}_{g'h'}^{(\kappa)} \exp(\xi),$$

which implies $ghk = g'h'$ for some $k \in G_{\exp \xi} \subseteq G_\sigma$. Setting $u = h'k^{-1}h^{-1} \in G_\sigma$, we obtain $g' = gu^{-1}$, while $a' = \mathrm{Ad}_u^{(\kappa)} a$. That is, $[(g, a)] = [(g', a')]$.

To complete the proof, it suffices to show that (10) has surjective differential. By equivariance, it is enough to verify this at elements $[(e, a)]$ with $a \in \exp(\mathfrak{A}_\sigma^{(\kappa)}) \subseteq T^\kappa$. The range of the differential of (10) at such a point contains $T_a G_\sigma + T_a \mathcal{C}$. Since $G_a \subseteq G_\sigma$, hence $T_a G_a \subseteq T_a G_\sigma$, Lemma 2.4 shows that this is all of $T_a G$. □

3. q-Hamiltonian spaces

Let G be a Lie group, with an invariant inner product $\cdot$ on its Lie algebra $\mathfrak{g}$, and let $\eta \in \Omega^3(G)$ be the bi-invariant closed 3-form

$$\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] = \frac{1}{12}\theta^R \cdot [\theta^R, \theta^R]$$

where $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ are the left, right invariant Maurer-Cartan forms. Suppose $\kappa \in \mathrm{Aut}(G)$ is an automorphism. It will be convenient to denote the group G, viewed as a G-manifold under κ-twisted conjugation, by $G\kappa$.

3.1. $G\kappa$-valued moment maps

A *q-Hamiltonian G-space with $G\kappa$-valued moment map* is a G-manifold M, together with an G-invariant 2-form ω and a G-equivariant smooth map $\Phi \colon M \to G\kappa$. These are required to satisfy the following axioms:

a) $\mathrm{d}\omega = -\Phi^*\eta$,

b) $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\kappa(\xi)\cdot\theta^L + \xi\cdot\theta^R)$,

c) $\ker(\omega)\cap\ker(T\Phi) = 0$.

These axioms generalize the G-valued moment maps from [3]. In terms of equivariant de Rham forms, the first two properties may be combined into a single condition $\mathrm{d}_G\omega = -\Phi^*\eta_G^{(\kappa)}$, where

$$\eta_G^{(\kappa)}(\xi) = \eta - \tfrac{1}{2}(\kappa(\xi)\cdot\theta^L + \xi\cdot\theta^R).$$

is a closed equivariant 3-form on $G\kappa$.

Example 3.1 (Twisted conjugacy classes). κ-twisted conjugacy classes $\mathcal{C}\subseteq G$ are q-Hamiltonian G-spaces, with the $G\kappa$-valued moment map given as the inclusion. The 2-form is uniquely determined by the moment map condition (b), and is given by

$$\omega(\xi_{\mathcal{C}},\tau_{\mathcal{C}}) = \tfrac{1}{2}((\mathrm{Ad}_\phi\circ\kappa) - (\mathrm{Ad}_\phi\circ\kappa)^{-1})\xi\cdot\tau.$$

Note that the twisted conjugacy classes can be odd-dimensional. For example, in the case of $G = \mathrm{SU}(3)$ with κ given by complex conjugation, the generic stabilizer under twisted conjugation is a circle, and hence the generic twisted conjugacy classes are 7-dimensional.

Example 3.2 (Twisted moduli spaces). These are associated to any compact oriented surface with boundary, with marked points on the boundary components, with a prescribed homomorphism from the fundamental groupoid into $\mathrm{Aut}(G)$. This will be discussed in Section 3.2 below.

Example 3.3. Further examples are created by *fusion*: Suppose M_i for $i = 1,2$ are two q-Hamiltonian G-spaces with $G\kappa_i$-valued moment map. Then $M_1\times M_2$ with the new G-action $g.(m_1,m_2) = (g.m_1,\kappa_1(g).m_2)$ and the 2-form

$$\omega = \omega_1 + \omega_2 + \tfrac{1}{2}\Phi_1^*\theta^L\cdot\Phi_2^*\theta^R$$

becomes a q-Hamiltonian G-space with $G\,\kappa_2\kappa_1$-valued moment map $\Phi_1\Phi_2$. Properties (a) and (b) may be verified directly; for property (c) it is best to use the Dirac-geometric approach as in Remark 3.5.

For example, if $\mathcal{C}$ is a κ-twisted conjugacy class in G, and M is a q-Hamiltonian G-space with (non-twisted) G-valued moment map, then the fusion product $M\times\mathcal{C}$ is a q-Hamiltonian G-space with $G\kappa$-valued moment map. Also, if $\mathcal{C}_i\subseteq G$ are κ_i-twisted conjugacy classes, for $i = 1,\ldots,r$, then their fusion product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ is a q-Hamiltonian space with $G\kappa$-valued moment map, where $\kappa = \kappa_r \cdots \kappa_1$. See example 2.1.

Remark 3.4. Let $L^{(\kappa)}G$ be the twisted loop group, consisting of paths $g\colon \mathbb{R} \to G$ with the property that $g(t+1) = \kappa(g(t))$ for all t. There is a notion of Hamiltonian $L^{(\kappa)}G$-space generalizing that of a Hamiltonian LG-space [17], and by the same proof as for $\kappa = 1$ [3] one sees that there is a 1-1 correspondence between Hamiltonian $L^{(\kappa)}G$-spaces with proper moment maps and κ-twisted q-Hamiltonian G-spaces.

Remark 3.5. The definition of $G\kappa$-valued moment maps has a Dirac-geometric interpretation, similar to [11] and [2]. Using the notation from [2], let $\mathbb{A} = \mathbb{T}G_\eta$ be the standard Courant algebroid over G, with the Courant bracket twisted by the closed 3-form η. It has a canonical trivialization $\mathbb{A} = G \times (\overline{\mathfrak{g}} \oplus \mathfrak{g})$, where $\overline{\mathfrak{g}}$ stands for $\mathfrak{g}$ with the opposite metric. Any Lagrangian Lie subalgebra $\mathfrak{s} \subseteq \overline{\mathfrak{g}} \oplus \mathfrak{g}$ defines a Dirac structure $E_\mathfrak{s} = G \times \mathfrak{s} \subseteq \mathbb{A}$. Taking $\mathfrak{s}$ to be the diagonal, one obtains the Cartan-Dirac structure $E_\Delta = E$. Taking $\mathfrak{s} = \{(\xi, \kappa(\xi)|\ \xi \in \mathfrak{g}\}$ for $\kappa \in \mathrm{Aut}(G)$, one obtains a Dirac structure $E_\mathfrak{s} = E^{(\kappa)}$ generalizing the Cartan-Dirac structure. As a Lie algebroid, it is the action Lie algebroid for the κ-twisted conjugation action. For a q-Hamiltonian space (M, ω, Φ) with $G\kappa$-valued moment map, the pair (Φ, ω) defines a full morphism of Manin pairs, $(\mathbb{T}M, TM) \dashrightarrow (\mathbb{A}, E^{(\kappa)})$. Conversely, such a morphism defines a $\mathfrak{g}$-action on M for which the underlying map $\Phi\colon M \to G$ is equivariant, and it also defines an invariant 2-form on M satisfying the axioms above. The Dirac-geometric approach explains many of the properties of $G\kappa$-valued moment maps; for example the fusion construction finds a conceptual explanation in terms of a Dirac morphism

$$(\mathrm{Mult}_G,\ \varsigma)\colon (\mathbb{A}, E^{(\kappa_1)}) \times (\mathbb{A}, E^{(\kappa_2)}) \to (\mathbb{A}, E^{(\kappa_2\kappa_1)})$$

with the 2-form $\varsigma = \frac{1}{2}\,\mathrm{pr}_1^*\,\theta^L \cdot \mathrm{pr}_2^*\,\theta^R$. See [2, Section 4.4].

3.2. Twisted moduli spaces

Let $\Sigma = \Sigma_h^r$ be a compact, connected, oriented surface of genus h with $r > 0$ boundary components, and let $\mathcal{V} = \{x_1, \ldots, x_r\}$ be a collection of base points on the boundary components, $x_i \in (\partial\Sigma)_i \cong S^1$. Let

$$\pi_1(\Sigma, \mathcal{V}) \rightrightarrows \mathcal{V}$$

denote the fundamental groupoid, consisting of homotopy classes λ of paths for which both the initial point $\mathsf{s}(\lambda)$ and the end point $\mathsf{t}(\lambda)$ are in $\mathcal{V}$. Suppose we are given a groupoid homomorphism ('twist')

$$\kappa \in \mathrm{Hom}\big(\pi_1(\Sigma,\mathcal{V}),\mathrm{Aut}(G)\big).$$

Such a κ may be obtained by assigning elements of $\mathrm{Aut}(G)$ to a system of free generators of the fundamental groupoid, and extending by the homomorphism property. Let

$$M = \mathrm{Hom}_\kappa\big(\pi_1(\Sigma,\mathcal{V}),G\big) \tag{12}$$

be the space of κ-twisted homomorphisms, consisting of maps $\lambda \mapsto \phi_\lambda$ such that

$$\phi_{\lambda_1\circ\lambda_2} = \phi_{\lambda_1}\ \kappa_{\lambda_1}(\phi_{\lambda_2})$$

whenever $\mathsf{s}(\lambda_1) = \mathsf{t}(\lambda_2)$. (The space M may be regarded as a certain moduli space of flat connections.) [1] The group $\mathrm{Map}(\mathcal{V},G) = G\times\cdots\times G$ act on the space (12) as

$$(g.\phi)_\lambda = g_{\mathsf{t}(\lambda)}\ \phi_\lambda\ \kappa_\lambda(g^{-1}_{\mathsf{s}(\lambda)}).$$

Let $\kappa_1,\ldots,\kappa_r \in \mathrm{Aut}(G)$ be the values of κ on the oriented boundary loops $\lambda_1,\ldots,\lambda_r$. Then M is a q-Hamiltonian G^r-space, with a $G\kappa_1\times\cdots\times G\kappa_r$-valued moment map Φ given by evaluation on boundary loops. We won't describe the 2-form here, since for the case that κ takes values in diagram automorphisms it may be regarded as a component of the moduli space of flat $G\rtimes\mathrm{Out}(G)$-bundles – see Section 3.4 below.

Remark 3.6. This construction also gives new examples of *non-twisted* q-Hamiltonian spaces. For example, take $\Sigma = \Sigma_1^1$ be the surface of genus 1 with one boundary component. Its fundamental group(oid) has free generators α,β, with the boundary loop given as $\alpha\beta\alpha^{-1}\beta^{-1}$. Attach an automorphism $\sigma\in\mathrm{Aut}(G)$ to β, and 1 to α, and extend to a homomorphism κ as above. Then the corresponding M is $G\times G$, with elements (a,b) corresponding to holonomies along α,β. The group G acts on a by conjugation and on b by κ-twisted conjugation. The boundary holonomy is a G-valued moment map

$$(a,b)\mapsto ab\kappa(a^{-1})\kappa(b^{-1}),$$

a twisted group commutator.

---

[1] Alternatively, Hom_κ is the lift of κ to the space $\widetilde{M} = \mathrm{Hom}\big(\pi_1(\Sigma,\mathcal{V}),G\rtimes\mathrm{Aut}(G)\big)$.

Remark 3.7. Let K be a disconnected group with identity component $G = K_0$. The space $\mathrm{Hom}(\pi_1(\Sigma, \mathcal{V}), K)$ is a moduli space of flat K-bundles over Σ, with framings at the base points. This space is disconnected, in general. The conjugation action of K on its identity component defines a group homomorphism $K \to \mathrm{Aut}(G)$. Hence, any element $x \in \mathrm{Hom}(\pi_1(\Sigma, \mathcal{V}), K)$ determines an element $\kappa \in \mathrm{Hom}(\pi_1(\Sigma, \mathcal{V}), \mathrm{Aut}(G))$, and the connected component containing x is identified with a connected component of $\mathrm{Hom}_\kappa(\pi_1(\Sigma, \mathcal{V}), G)$.

The example giving rise to the convex polytope in Theorem 1.1 is obtained from the r-holed sphere $\Sigma = \Sigma_0^r$. Here $\pi_1(\Sigma, \mathcal{V})$ is freely generated by $\lambda_1, \ldots, \lambda_{r-1}$, represented by oriented boundary loops based at $x_1, \ldots, x_{r-1}$, together with $\mu_1, \ldots, \mu_{r-1}$ represented by non-intersecting paths connecting these points to x_r. The element λ_r represented by the remaining boundary loop satisfies

$$\prod_{i=1}^{r-1} \mu_i \lambda_i \mu_i^{-1} = \lambda_r^{-1}. \tag{13}$$

Given $\kappa \in \mathrm{Hom}(\pi_1(\Sigma, \mathcal{V}), \mathrm{Aut}(G))$, we denote by κ_i the images of the λ_i's, and by σ_i the images of the μ_i's. Then

$$\prod_{i=1}^{r-1} \sigma_i \kappa_i \sigma_i^{-1} = \kappa_r^{-1}. \tag{14}$$

We find $\mathrm{Hom}_\kappa(\pi_1(\Sigma, \mathcal{V}), G) = G^{2r-2}$, consisting of tuples $(d_1, \ldots, d_{r-1}, a_1, \ldots, a_{r-1})$, where d_i are holonomies attached to the λ_i, and a_i are attached to the μ_i. The holonomy d_r around the r-th boundary loop is determined from

$$\prod_{i=1}^{r-1} (a_i, \sigma_i)(d_i, \kappa_i)(a_i, \sigma_i)^{-1} = (d_r, \kappa_r)^{-1}. \tag{15}$$

Lemma 3.8. *Let $\kappa_1, \ldots, \kappa_r$ be holonomies attached to the boundaries of Σ_0^r, with $\kappa_r \kappa_{r-1} \cdots \kappa_1 = 1$. Then there is an extension to a homomorphism*

$$\kappa \in \mathrm{Hom}(\pi_1(\Sigma, \mathcal{V}), \mathrm{Aut}(G)),$$

in such a way that the moment map image of $M = \mathrm{Hom}_\kappa(\pi_1(\Sigma, \mathcal{V}), G)$ consists of all $(d_1, \ldots, d_r) \in G^r$ for which there exists $(g_1, \ldots, g_r)$ with $g_i \in \mathrm{Ad}_G^{(\kappa_i)}(d_i)$ and $\prod_{i=1}^r g_i = e$.

Proof. Using the notation above, put $\sigma_1 = 1$, $\sigma_2 = \kappa_1^{-1}$, $\ldots$, $\sigma_{r-1} = \kappa_1^{-1} \cdots \kappa_{r-2}^{-1}$. Equation (14) becomes the condition $\kappa_r \kappa_{r-1} \cdots \kappa_1 = 1$. Introducing

$$a_1' = a_1,\ a_2' = \kappa_1(a_2),\ a_3' = \kappa_2(\kappa_1(a_3)),\ \ldots$$

the equation for the holonomies becomes

$$\prod_{i=1}^{r} a_i' d_i \kappa_i((a_i')^{-1}) = e$$

where we put $a_r' = e$. That is $\prod g_i = e$ where

$$g_i = a_i' d_i \kappa_i((a_i')^{-1}) \in \mathrm{Ad}_G^{(\kappa_i)}(d_i).$$

The moment map for M is the map taking $(d_1, \ldots, d_{r-1}, a_1, \ldots, a_{r-1})$ to $(d_1, \ldots, d_r)$, with d_r determined from the condition $\prod_i g_i = e$. □

3.3. Basic properties of $G\kappa$-valued moment maps

The following statement extends a well-known property of moment maps in symplectic geometry.

Proposition 3.9. *Let (M, ω, Φ) be a q-Hamiltonian G-space with $G\kappa$-valued moment map. For all $m \in M$ we have*

$$\ker(T_m\Phi)^\omega = T_m(G \cdot m), \quad \mathrm{ran}(\Phi^*\theta^R)_m = \mathfrak{g}_m^\perp.$$

(For any subspace $V \subseteq T_mM$, the notation V^ω means the set of all $v \in T_mM$ such that $\omega(v, w) = 0$ for all $w \in V$.)

Proof. In terms of $A = \mathrm{Ad}_{\Phi(m)} \circ \kappa$, the moment map condition gives

$$\iota(\xi_M)\omega_m = -\tfrac{1}{2}((A + I)\xi) \cdot (\Phi^*\theta^R)_m. \tag{16}$$

In particular, for $\xi \in \mathfrak{g}_m$, we get that

$$\tfrac{1}{2}((A + I)\xi) \cdot (\Phi^*\theta^R)_m = 0.$$

But $\mathfrak{g}_m \subseteq \mathfrak{g}_{\Phi(m)} = \ker(A - I)$, so A acts as the identity on $\mathfrak{g}_m$. Hence we obtain $\xi \cdot (\Phi^*\theta^R)_m = 0$, proving $\mathrm{ran}(\Phi^*\theta^R)_m \subseteq \mathfrak{g}_m^\perp$. On the other hand, it is immediate from the moment map condition that $\ker(T_m\Phi)^\omega \supseteq T_m(G\cdot m)$. Equality of both inclusions follows by a dimension count:

$$\begin{aligned}\dim(G\cdot m) &\le \dim(\ker(T_m\Phi)^\omega)\\ &= \dim T_mM - \dim(\ker(T_m\Phi))\\ &= \dim(\mathrm{ran}(\Phi^*\theta^R)_m)\\ &\le \dim \mathfrak{g}_m^\perp = \dim(G\cdot m).\end{aligned}$$

Here we used $\ker(\omega)\cap\ker(T\Phi)=0$ for the first equality sign. □

Proposition 3.10. *The map* $\mathfrak{g}\to T_mM$ *given by the infinitesimal action restricts to an isomorphism,*

$$\ker(\mathrm{Ad}_{\Phi(m)}\circ\kappa + I) \xrightarrow{\cong} \ker(\omega_m).$$

Proof. Here, the Dirac-geometric viewpoint from Remark 3.5 is convenient. Let $\mathbb{T}G_\eta$ be as in that remark. The subspace

$$E_1 = \{T\Phi(v)+\alpha \in \mathbb{T}G_\eta |\ v\in T_mM,\ \alpha\in T^*_{\Phi(m)}G,\ \Phi^*\alpha = \iota(v)\omega_m\}$$

is the 'forward image' of $T_mM \subseteq \mathbb{T}M = TM\oplus T^*M$ under the linear Dirac morphism $(T_m\Phi,\omega_m)$; in particular it satisfies $E_1 = E_1^\perp$. The axioms show that E_1 contains the space

$$E = \left\{\xi_G + \frac{1}{2}\theta^R\cdot(A+I)\xi \ \Big|\ \xi\in\mathfrak{g}\right\}$$

(everything evaluated at $\Phi(m)$). Here ξ_G are the generating vector fields for the κ-twisted conjugation,

$$\xi_G = \kappa(\xi)^L - \xi^R = ((A-I)\xi)^R.$$

But it is easily checked that $E=E^\perp$, which together with $E\subseteq E_1$ implies $E_1 = E$. In particular, taking $\alpha=0$ in the definition of E_1 we see that

$$(T_m\Phi)(\ker\omega_m) = \big\{\xi_G(\Phi(m))\,\big|\,(A+I)\xi = 0\big\}.$$

Since $\ker(\omega_m)\cap\ker(T_m\Phi)=0$, the map $T_m\Phi$ is injective on $\ker(\omega_m)$. Consequently, $\ker(\omega_m) = \{\xi_M(m)\mid (A+I)\xi=0\}$. □

3.4. Changing κ by inner automorphisms

Let (M, ω, Φ) be a q-Hamiltonian G-space with $G\kappa$-valued moment map. Suppose

$$\kappa' = \mathrm{Ad}_a \circ \kappa.$$

Then the manifold M with the same G-action and 2-form, but with a shifted moment map $\Phi' = r_{a^{-1}} \circ \Phi$, is a q-Hamiltonian G-space with $G\kappa'$-valued moment map. For this reason, if G is compact and simply connected, it usually suffices to consider the case of diagram automorphism $\kappa \in \mathrm{Out}(G)$. But for $\kappa \in \mathrm{Out}(G)$, the q-Hamiltonian G-spaces with $G\kappa$-valued moment map are simply q-Hamiltonian spaces with moment maps valued in the disconnected group $G \rtimes \mathrm{Out}(G)$, whose image is contained in the component $G \times \{\kappa\}$. (The only wrinkle is that we only consider the action of the identity component G of this group, but this doesn't affect the theory from [3].) In this sense, the examples considered above are not new, at least for G compact and simply connected. For instance, in the fusion procedure 3.3, first apply the automorphism κ_1 to the second space, thus obtaining (M_2, ω_2, Φ_2') with the new G-action $m \mapsto \kappa_1(g).m$, and a $G\kappa'$-valued moment map $\Phi_2' = \kappa_1^{-1} \circ \Phi_2$, where $\kappa' = \kappa_1^{-1}\kappa_2\kappa_1$. Since

$$(\Phi_1, \kappa_1)(\kappa_1^{-1} \circ \Phi_2, \kappa_2) = (\Phi_1\Phi_2, \kappa_2 \circ \kappa_1),$$

we recognize the fusion product 3.3 as a standard fusion product [3] for q-Hamiltonian G-spaces with $G \rtimes \mathrm{Out}(G)$-valued moment maps.

4. Convexity properties

We now turn to the convexity properties of $G\kappa$-valued moment maps. The arguments are mostly straightforward adaptations of those in [17] and [15]. Throughout, we will assume that G is compact and simply connected, and that $\kappa \in \mathrm{Aut}(G)$ is a diagram automorphism. We denote by $\mathfrak{A}^{(\kappa)} \subseteq \mathfrak{t}^\kappa$ the alcove, and by

$$q^{(\kappa)} \colon G \to \mathfrak{A}^{(\kappa)}$$

the quotient map, with fibers $(q^{(\kappa)})^{-1}(\xi)$ the κ-twisted conjugacy classes of $\exp(\xi)$. Recall from 2.3 the definition of the slices U_σ. Let κ_σ denote the restriction of κ to G_σ.

Proposition 4.1 (Cross-section theorem). *Let (M, ω, Φ) be a connected q-Hamiltonian G-space with $G\kappa$-valued moment map. For any face $\sigma \subseteq \mathfrak{A}^{(\kappa)}$,* *the pre-image $Y_\sigma = \Phi^{-1}(U_\sigma)$ is a q-Hamiltonian $G_\sigma\kappa_\sigma$-space, with the pull-back of ω as the 2-form and the restriction of Φ as the moment map.*

The proof is parallel to the result for non-twisted q-Hamiltonian spaces, see [3], which in turn is a version of the cross-section theorem for Hamiltonian spaces, due to Guillemin-Sternberg [12] and Marle [16].

Recall that for any connected G-manifold M, the principal stratum M_{prin} is the set of all points whose stabilizer is subconjugate to any other stabilizer. It is connected, and open and dense in M.

Proposition 4.2. *Let (M, ω, Φ) be a connected q-Hamiltonian G-space with $G\kappa$-valued moment map. Then:*

a) *The stabilizer G_m of any point $m \in M_{\mathrm{prin}}$ is an ideal in $G_{\Phi(m)}$.*

b) *All points in $M_{\mathrm{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)}))$ have the same stabilizer H.*

c) *The image $q^{(\kappa)}(\Phi(M_{\mathrm{prin}}))$ is a connected, relatively open subset of*

$$(x + \mathfrak{h}^\perp) \cap \mathfrak{A}^{(\kappa)},$$

where $\mathfrak{h}$ is the Lie algebra of H, and x is any point of $q^{(\kappa)}(\Phi(M_{\mathrm{prin}}))$.

Proof. The parallel statements for ordinary Hamiltonian G-spaces are proved in [15, Section 3.3]. In particular, if N is a connected Hamiltonian G-space, with moment map $\Psi\colon N \to \mathfrak{g}^*$, then for each $n \in N_{\mathrm{prin}}$, the stabilizer G_n is an ideal in $G_{\Phi(n)}$, and the stabilizer $H = G_n$ of points in $N_{\mathrm{prin}} \cap \Psi^{-1}(\mathfrak{t}^*_+)$ is independent of n. We will use cross-sections Y_σ to reduce to the Hamiltonian case. As noted in the proof of Proposition 2.5, the automorphism $\kappa_\sigma = \kappa|_{G_\sigma}$ is inner, and is given by $\mathrm{Ad}_{a^{-1}}$ for any choice of $a \in \exp(\sigma)$. Hence, Y_σ becomes a q-Hamiltonian G_σ-space with (untwisted) G_σ-valued moment map $r_{a^{-1}} \circ \Phi_\sigma$. Furthermore, this then becomes an ordinary Hamiltonian G_σ-space with a moment map

$$\Phi_{0,\sigma}\colon Y_\sigma \to \mathfrak{g}_\sigma \cong \mathfrak{g}_\sigma^*, \quad m \mapsto \log(\Phi_\sigma(m)a^{-1}).$$

We conclude that for all $m \in (Y_\sigma)_{\mathrm{prin}} = Y_\sigma \cap M_{\mathrm{prin}}$, the stabilizer G_m is an ideal in the stabilizer of $\Phi_{0,\sigma}(m)$ under the adjoint action. The latter coincides with stabilizer of $\Phi(m) = \exp(\Phi_{0,\sigma}(m))a$ under twisted conjugation. Hence G_m is an ideal in $G_{\Phi(m)}$. Since the flow-outs of all the Y_σ's under twisted conjugation cover M, this proves (a).

The map $M_{\text{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)})) \to M_{\text{prin}}/G$ is surjective, and has connected fibers $G_{\Phi(m)}.m = G_{\Phi(m)}/G_m$. Since the target of this map is connected, it follows that $M_{\text{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)}))$ is connected. Consider the decomposition of each Y_σ into its connected components Y_σ^i. Passing to the corresponding Hamiltonian G_σ-space as above, and using the general results for connected Hamiltonian spaces, we see that all points of $Y_\sigma^i \cap M_{\text{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)}))$ have the same stabilizer. Since the union of these sets, over all σ, i, covers $M_{\text{prin}} \cap \Phi^{-1}(\exp(\mathfrak{A}^{(\kappa)}))$, it follows that all points of this intersection have the same stabilizer, proving (b).

Each $q^{(\kappa)}(\Phi(M_{\text{prin}}) \cap Y_\sigma^i)$ is a connected, relatively open subset of $(x + \mathfrak{h}^\perp) \cap \mathfrak{A}_\sigma^{(\kappa)}$, for any choice of $x \in q^{(\kappa)}(\Phi(M_{\text{prin}}) \cap Y_\sigma^i)$. (Once again, this follows from the corresponding statement for Hamiltonian spaces, see [15, Section 3.3].) This implies (c). $\square$

Theorem 4.3 (Principal cross-section). *Let (M, ω, Φ) be a connected q-Hamiltonian G-space with $G\kappa$-valued moment map. Then there exists a unique open face σ of $\mathfrak{A}^{(\kappa)}$ such that*

$$q^{(\kappa)}(\Phi(M)) \subseteq \overline{q^{(\kappa)}(\Phi(M)) \cap \sigma}.$$

(Equality holds if M is compact.) Alternatively, σ is characterized as the smallest face such that the corresponding cross-section Y_σ satisfies $\Phi(Y_\sigma) \subseteq$ $\exp(\sigma)$*. This* principal cross-section *Y_σ is a connected q-Hamiltonian T^κ-space, with the restriction of Φ as the moment map, and*

$$M = \overline{G \cdot Y_\sigma}.$$

Proof. Using the notation from the previous proposition, let σ be the lowest dimensional face of $\mathfrak{A}^{(\kappa)}$ whose closure contains $(x + \mathfrak{h}^\perp) \cap \mathfrak{A}^{(\kappa)}$. Since $q^{(\kappa)}(\Phi(M_{\text{prin}}))$ is a relatively open subset of $(x + \mathfrak{h}^\perp) \cap \mathfrak{A}^{(\kappa)}$, its intersection with σ is non-empty. It follows that $q^{(\kappa)}(\Phi(M)) \cap \sigma = q^{(\kappa)}(\Phi(Y_\sigma))$. That is, $\Phi(Y_\sigma) \subseteq \exp(\sigma) \subseteq T^\kappa$, so that Y_σ may be regarded as a q-Hamiltonian T^κ-space, for the restriction of the moment map.

By construction, $G \cdot Y_\sigma = \Phi^{-1}((q^{(\kappa)})^{-1}(\sigma))$. The difference

$$M_{\text{prin}} - ((G \cdot Y_\sigma) \cap M_{\text{prin}}) = M_{\text{prin}} - \big(\Phi^{-1}((q^{(\kappa)})^{-1}(\sigma)) \cap M_{\text{prin}}\big) \tag{17}$$

is the union over all $\Phi^{-1}((q^{(\kappa)})^{-1}(\tau)) \cap M_{\text{prin}}$ where τ ranges over proper faces of $\overline{\sigma}$. But those are submanifolds of codimension at least 3, hence removing them will not disconnect M_{prin}. Thus $(G \cdot Y_\sigma) \cap M_{\text{prin}}$ is connected, which implies that $G \cdot Y_\sigma = G \times_{G_\sigma} Y_\sigma$ is connected, and therefore Y_σ is connected. □

Note that since the principal cross-section Y_σ is a q-Hamiltonian T^κ-space, it is in particular symplectic.

Theorem 4.4. *Let* (M, ω, Φ) *be a compact, connected q-Hamiltonian G-space with* $G\kappa$*-valued moment map. Then the fibers of the moment map* Φ *are connected, and the image*

$$\Delta(M) := q^{(\kappa)}(\Phi(M)) \subseteq \mathfrak{A}^{(\kappa)}$$

is a convex polytope.

Proof. The principal cross-section $Y = Y_\sigma$ is a connected q-Hamiltonian T^κ-space, with the restriction $\Phi_Y = \Phi|_Y$ as its moment map. We can regard Y as an ordinary Hamiltonian T^κ-space, with a moment map $\Phi_{Y,0} = q^{(\kappa)} \circ \Phi_Y$ that is proper as a map to $\sigma \subseteq \mathfrak{t}^\kappa$.

Since σ is convex, [15, Theorem 4.3] shows that $\Phi_{Y,0}$ has connected fibers, and its image is a convex set of the form

$$q^{(\kappa)}(\Phi(Y)) = \Phi_{Y,0}(Y) = P \cap \sigma,$$

where P is some convex polytope in $\overline{\sigma}$. But then $q^{(\kappa)}(\Phi(M)) = \overline{q^{(\kappa)}(\Phi(Y))} = P$. Finally, if $x \in q^{(\kappa)}(\Phi(M))$, then the same argument as in [15] shows that for any open ball B around x, the pre-image $\Phi^{-1}((q^{(\kappa)})^{-1}(B))$ is connected. By a continuity argument [15, Lemma 5.1] this implies that $\Phi^{-1}(x)$ is connected. □

We obtain Theorem 1.1 as a special case:

Proof of Theorem 1.1. Consider again the twisted moduli space for the r-holed sphere Σ_0^r, corresponding to $\kappa_i \in \mathrm{Out}(G)$ with $\kappa_r \kappa_{r-1} \cdots \kappa_1 = 1$, as in Lemma 3.8. We had found that the moment map image consists of all $(d_1, \ldots, d_r)$ for which there exist elements $g_i \in G$ in the κ_i-twisted conjugacy class of d_i, such that $g_1 \cdots g_r = e$. Hence, by Theorem 4.4 the set (4) is a convex polytope. □

5. An example

We will illustrate Theorem 1.1 in a simple setting, were the resulting polytope can be computed by hand. Let $G = A_2 \cong \mathrm{SU}(3)$, with its standard maximal torus T consisting of diagonal matrices, and its usual choice of positive roots. We denote by α, β the simple roots, and let $\gamma = \alpha + \beta$ be their sum. The fundamental alcove $\mathfrak{A} \subseteq \mathfrak{t}$ is defined by the inequalities $\langle \alpha, \xi \rangle \geq 0$, $\langle \beta, \xi \rangle \geq 0$, $\langle \gamma, \xi \rangle \leq 1$. Let $\kappa \in \mathrm{Aut}(G)$ be the nontrivial diagram automorphism of G given by $\kappa(\alpha) = \beta$ and $\kappa(\beta) = \alpha$.

The Lie algebra $\mathfrak{t}^\kappa$ consists of all ξ such that $\langle \alpha, \xi \rangle = \langle \beta, \xi \rangle$; it is thus the line spanned by the coroot $\gamma^\vee$. The alcove $\mathfrak{A}^{(\kappa)}$ is 'half' of the intersection $\mathfrak{A} \cap \mathfrak{t}^\kappa$, i.e. it consists of elements of $\mathfrak{t}^\kappa$ with $\langle \gamma, \xi \rangle \in [0, \frac{1}{2}]$. We thus label the κ-twisted conjugacy classes by a parameter $s \in [0, \frac{1}{2}]$, where $\mathcal{C}_s^{(\kappa)}$ contains $\exp(\xi_s)$ for a unique $\xi_s \in \mathfrak{A}^\kappa$ with $\langle \gamma, \xi_s \rangle = s$.

Consider the setting of Theorem 1.1, with $r = 3$. Unless all $\kappa_i = 1$, two of the automorphisms $\kappa_1, \kappa_2, \kappa_3$ have to be κ, and the third is the identity. We may assume $\kappa_1 = \kappa_2 = \kappa$ and $\kappa_3 = 1$. Hence,

$$\mathfrak{A}^{(\kappa_1)} \times \mathfrak{A}^{(\kappa_2)} \times \mathfrak{A}^{(\kappa_3)} = \big[0, \frac{1}{2}\big] \times \big[0, \frac{1}{2}\big] \times \mathfrak{A}.$$

Proposition 5.1. *For $G = A_2 \cong \mathrm{SU}(3)$ with its non-trivial diagram automorphism κ, the polytope of all $(s_1, s_2, \xi) \in \big[0, \frac{1}{2}\big] \times \big[0, \frac{1}{2}\big] \times \mathfrak{A}$ such that there exists $(g_1, g_2, g_3) \in \mathcal{C}_{s_1}^{(\kappa)} \times \mathcal{C}_{s_2}^{(\kappa)} \times \mathcal{C}_\xi$ with $g_1 g_2 g_3 = e$, is given by the inequalities $0 \leq s_i \leq \frac{1}{2}$ together with*

$$\begin{aligned} &|s_1 - s_2| \leq \langle \alpha + \beta, \xi \rangle \leq 1 \qquad |s_1 - s_2| \leq 1 - \langle \alpha, \xi \rangle \leq 1, \\ &|s_1 - s_2| \leq 1 - \langle \beta, \xi \rangle \leq 1. \end{aligned}$$

Proof. The problem of computing this polytope is equivalent to computing the moment polytope of the fusion product $\mathcal{C}_{s_1}^{(\kappa)} \times \mathcal{C}_{s_2}^{(\kappa)}$ for any s_1, s_2. This fusion product is an untwisted q-Hamiltonian G-space, with action

$$h \cdot (g_1, g_2) = \big(h\, g_1\, \kappa(h)^{-1},\ \kappa(h) g_2\, h^{-1}\big)$$

and moment map $(g_1, g_2) \mapsto g_1 g_2$; its moment polytope is a 2-dimensional convex polytope inside $\mathfrak{A}$. Observe that the set of $g_1 g_2$ with $g_i \in \mathcal{C}_{s_i}^{(\kappa)}$ is invariant under left-translation by central elements $c \in Z(G) \cong \mathbb{Z}_3$. This follows from

$$\mathrm{Ad}^\kappa_{c^{-1}}(g) = c^{-1} g \kappa(c) = c^{-1} g c^2 = c g.$$

Left multiplication of the center on G induces an action on the set of conjugacy classes, and the resulting action of $\mathbb{Z}_3$ on the alcove $\mathfrak{A}$ is by 'rotation'.

Hence, the moment polytope is invariant under 'rotations' of the alcove. If $s_1 = s_2 = 0$, this implies that the moment polytope must be all of $\mathfrak{A}$, since it contains the origin. If at least one of s_1, s_2 is non-zero, the moment polytope does *not* contain the origin. Using standard results from symplectic geometry, applied to the symplectic cross-section, it is cut out from the alcove by affine half-spaces orthogonal to 1-dimensional stabilizer groups. But the generic stabilizer for the twisted conjugation action of G on itself is T^κ, and all other 1-dimensional stabilizers are W-conjugate to T^κ. (The fixed point set of T is trivial.) Together with the rotational symmetry, it follows that the moment polytope is cut out from the alcove by inequalities of the form $r \leq \langle \gamma, \xi \rangle, \quad r \leq 1 - \langle \alpha, \xi \rangle, \quad r \leq 1 - \langle \beta, \xi \rangle$, for some $0 < r < \frac{1}{2}$. To find r, it suffices to determine the fixed point set of T^κ on the product of twisted conjugacy classes, and takes its image under the multiplication map. Since the action of T^κ is just ordinary conjugation, and since T^κ contains regular elements, the fixed point set for each factor is

$$\mathcal{C}_{s_i}^{(\kappa)} \cap T = \exp(\xi_{s_i} + \mathfrak{t}_\kappa) \cup \exp(-\xi_{s_i} + \mathfrak{t}_\kappa),$$

and the image under multiplication is $\exp(\xi_{\pm s_1 \pm s_2} + \mathfrak{t}^\kappa) \subseteq T$. We conclude $r = |s_1 - s_2|$. □

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