The Jacquet–Langlands correspondence, Eisenstein congruences, and integral L-values in weight 2

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We use the Jacquet–Langlands correspondence to generalize well-known congruence results of Mazur on Fourier coefficients and L-values of elliptic modular forms for prime level in weight 2 both to nonsquare level and to Hilbert modular forms.

A celebrated result of Mazur says that, for N a prime and p a prime dividing (the numerator of) $\frac{N-1}{12}$, there exists a cusp form $f \in S_2(N) = S_2(\Gamma_0(N))$ congruent to the Eisenstein series $E_{2,N}$ of weight 2 and level N mod p [Maz77, II(9.6)]. Further, if p is odd, one has a congruence for the algebraic part of the central L-value $L(1, f_K) = L(1, f)L(1, f \otimes \eta_K)$, where η_K is the quadratic character associated to a quadratic field $K/\mathbb{Q}$ [Maz79]. For instance, if $N = 11$ and K is not split at 11, there is one cusp form $f \in S_2(N)$ and one gets that $L^{\mathrm{alg}}(1, f_K) \not\equiv 0 \bmod 5$ if and only if $5 \nmid h_K$. (If K splits at 11, the root number is -1 so $L(1, f_K) = 0$.) A form of Mazur's L-value result was reproved by Gross [Gro87] for $K/\mathbb{Q}$ imaginary quadratic using quaternion algebras and the height pairing, whereas Mazur used modular symbols. Ramakrishnan pointed out to me that one can also deduce this from his average L-value formula with Michel [MR12]. (Note Gross's argument also involves an averaging type procedure, so these two arguments are not entirely different in spirit.)

In this article, we use the Jacquet–Langlands correspondence and an explicit L-value formula to extend these results of Mazur both to more general levels and to parallel weight 2 Hilbert modular forms over a totally real field F with K/F a quadratic CM extension. For simplicity, we only state our results precisely for $F = \mathbb{Q}$ in this introduction.

We first discuss the Hecke eigenvalue congruence result and a nonvanishing L-value result, and will state the more precise result on L-value congruences below in Theorem B.

Theorem A. *Let N be a nonsquare, and write $N = N_1 N_2$ where $(N_1, N_2) = 1$ and N_1 has an odd number of prime factors, all of which occur to odd exponents. Let p be a prime dividing (the numerator of) $\frac{1}{12}\varphi(N_1)N_2 \prod_{q|N_2}(1 + q^{-1})$ and $\mathfrak{p}$ a prime of $\overline{\mathbb{Q}}$ above p. Then there exists $f \in S_2(N)$ which is an eigenform for all T_n with $(n, N) = 1$ and is congruent to the Eisenstein series $E_{2,N}$ mod $\mathfrak{p}$ away from N.*

Suppose moreover $N = N_1$ with N_1 squarefree and that $p|\frac{\varphi(N_1)}{24}$. Then we can take f to be a newform such that $f \equiv E_{2,N}$ mod $\mathfrak{p}$ (as q-expansions). If $K/\mathbb{Q}$ is an imaginary quadratic field not split at any prime dividing N such that $p \nmid h_K$, then there exists such an f with $L(1, f_K) \neq 0$.

Here $E_{2,N}$ is the Eisenstein series of weight 2 and level N defined by $E_{2,N}(z) = \sum_{d|N} \mu(d) d E_2(dz)$, where μ is the Möbius function and E_2 is the weight 2 Eisenstein series for $\mathrm{SL}(2, \mathbb{Z})$ normalized so that the Fourier coefficient of q is 1.

The statement about congruence away from N means that f and $E_{2,N}$ have the same Hecke eigenvalues mod $\mathfrak{p}$ for T_ℓ for any prime $\ell \nmid N$. If $N = N_1$ is squarefree, then $E_{2,N}$ has constant term $\frac{\varphi(N)}{24}$, and the only reason we need to assume $p|\frac{\varphi(N)}{24}$ rather than $p|\frac{\varphi(N)}{12}$ in the second part is so the constant term of $E_{2,N}$ will be 0 mod p. E.g., if $N = 73$ there is a rational newform $f \in S_2(73)$ all of whose Hecke eigenvalues match with those of $E_{2,73}$ mod 2, but the constant term of $E_{2,73}$ is 1 mod 2 so they are not congruent as q-expansions.

Note $E_{2,N} = E_{2,N'}$ where $N' = \prod_{p|N} p$ is the "powerfree part" of N so $E_{2,N}$ really has level N', and $E_{2,N}$ is an eigenfunction of the Hecke operators T_ℓ on $M_2(N)$ for all primes ℓ having eigenvalue $\ell + 1$ when $\ell \nmid N$ and eigenvalue 1 when $\ell|N$. One might prefer to replace $E_{2,N}$ with a form whose T_ℓ eigenvalue is 0 when $\ell^2|N$ and look for a congruence mod ℓ as well. However, since we do not know the exact level of f in general, this does not make a difference for the statement of the theorem. Still, we give non-squarefree examples below where one can replace $E_{2,N}$ with a different Eisenstein series to get a congruence of all Fourier coefficients.

The Hecke eigenvalue congruence in the $N = N_1$ squarefree case of Theorem A was also proved by Ribet (unpublished, announced in 2010) under the additional assumptions $p \nmid N$ and $p \geq 5$. In fact, Ribet also obtained a converse (there are no other such $p \nmid N$ with $p \geq 5$) and more generally addressed finding newforms congruent to $E_{2,N}$ away from N with specified p_i-th Fourier coefficient $a_{p_i} \in \{\pm 1\}$ for each $p_i|N$. This was studied further by Yoo in [Yooa], which also explains Ribet's work in detail. (When $N = N_1$ is squarefree, our Theorem A only concerns finding newforms with

$a_{p_i} = 1$ for each $p_i | N$.) Their arguments involve studying Eisenstein ideals using Jacobian calculations and then appealing to the Jacquet–Langlands correspondence. See also [Yoob] for related results when N is not squarefree.

In contrast, our proof follows from using a (generalized) Eichler mass formula to construct a quaternionic cusp form ϕ "of level N" on the definite quaternion algebra B ramified at each prime dividing N_1 such that $\phi \equiv 1 \mod \mathfrak{p}$. (Here ϕ is just a function on a certain finite set of ideal classes with the cuspidality criterion amounting to a linear relation among the values of ϕ.) In addition to our method being more straightforward, we have no trouble dealing with N not squarefree and arbitrary p, or extending this to Hilbert modular forms as in Theorem 2.1 (though in the Hilbert modular case we make a simplyfing assumption that our totally real field F satisfies $h_F = h_F^+$ — see Remark 2.2). We do not get a converse or consider $a_{p_i} = -1$, though see Remark 2.4 regarding the converse.

Here are some simple consequences of Theorem A.

Corollary 1. *If $p > 2$, then there exists $f \in S_2(p^3)$ such that f and $E_{2,p^3} = E_{2,p}$ are congruent $\mod \mathfrak{p}$ away from p, for $\mathfrak{p}$ a prime above p in $\overline{\mathbb{Q}}$.*

For example, when $p = 3$, there is a unique normalized (new)form $f \in S_2(27)$, which has Fourier expansion

$$\sum a_n q^n = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19}$$
$$- 5q^{25} + 2q^{28} - 4q^{31} + 11q^{37} + 8q^{43} + \cdots$$

and it satisfies $a_\ell \equiv \ell + 1 \mod 3$ for all primes $\ell \neq 3$. In fact we have $f(z) \equiv E_{2,3}(z) - E_{2,3}(3z) \mod 3$. (Note the constant term of the right hand side is 0.)

We remark that the analogous statement of the corollary is not true in level p^2 — e.g., for $N = N_1 = 25$, then prime $p = 5$ divides the numerator of $\frac{1}{12}\phi(N_1)$, but $S_2(25) = \{0\}$.

In the case of $p = 2$, the theorem says that there exists $f \in S_2(32)$ which is congruent to $E_{2,2} \mod 2$ away from 2. Here $S_2(32)$ is 1 dimensional, and spanned by the form

$$\sum a_n q^n = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - q^{25}$$
$$- 10q^{29} - 2q^{37} + 10q^{41} + 6q^{45} - 7q^{49} + \cdots .$$

In fact, $f(z) \equiv E_{2,2}(z) - E_{2,2}(2z) \mod 2$.

Corollary 2. *Let p, q be distinct primes with $p > 3$. Then there exists $f \in S_2(qp^2)$ such that f and $E_{2,q}$ are congruent mod $\mathfrak{p}$ away from pq, for $\mathfrak{p}$ a prime of $\overline{\mathbb{Q}}$ above p.*

For instance, $S_2(50) = S_2(50)^{\mathrm{new}}$ has dimension 2, and one eigenform $f \in S_2(50)$ has Fourier expansion

$$q + q^2 - q^3 + q^4 - q^6 - 2q^7 + q^8 - 2q^9$$
$$- 3q^{11} - q^{12} + 4q^{13} - 2q^{14} + q^{16} + 3q^{17} + \cdots$$

In this case it happens that f and $E_{2,50}$ are congruent mod 5 everywhere away from 5, and $f(z) \equiv E_{2,10}(z) - E_{2,10}(5z) \bmod 5$.

The L-value result at the end of Theorem A is a direct generalization of [Gro87, Cor 11.8], and follows from our proof of Theorem 2.1 and Waldspurger's formula [Wal85] — see Proposition 3.1. The $p = 2$ case (not treated by Mazur) has consequences in the spirit of Goldfeld's conjecture:

Example 3. *Let C be the unique-up-to-isogeny elliptic curve over $\mathbb{Q}$ of conductor $N = 17$. Then among negative prime discriminants $-d$, the central value $L(1, C_{-d})$ for the quadratic twist C_{-d} is nonzero exactly 50% of the time.*

In fact, we can say precisely when $L(1, C_{-d}) \neq 0$ for $-d < 0$ a prime discriminant. We know C is associated to the unique normalized form $f \in S_2(17)$. We take $p = 2$ in our theorem and conclude $L(1, f_K) \neq 0$ when 17 is inert in $K = \mathbb{Q}(\sqrt{-d})$ and $2 \nmid h_K$. However, for d prime, Gauss' genus theory implies $2 \nmid h_K$. Thus, if 17 is inert in K, $L(1, f_K) = L(1, C)L(1, C_{-d}) \neq 0$. On the other hand, if 17 is split in $\mathbb{Q}(\sqrt{-d})$, then $L(1, C_{-d}) = 0$ because the root number is -1. In particular, at least 50% of these prime quadratic twists C_{-d} have finitely many rational points.

Now we discuss congruences of L-values. Let $K/\mathbb{Q}$ be a quadratic extension and χ an ideal class character of K. Let $L(s, f, \chi)$ denote the twist $L(s, f_K \otimes \chi)$ of the base change f_K of f to K (a degree 2 L-function over K), or equivalently $L(s, f \times \theta_\chi)$ the degree 4 Rankin–Selberg L-function over $\mathbb{Q}$ of f times the GL(2) theta series θ_χ attached to χ. When $\chi = 1$ is trivial, this L-function is just $L(s, f, 1) = L(s, f_K) = L(s, f)L(s, f \otimes \eta_K)$.

Given a congruence of modular forms, say as in Theorem A, we expect a congruence of algebraic parts of the special L-values $L(1, f, \chi)$. In the case where one of the modular forms is an Eisenstein series, this was studied

by Vatsal and Heumann [Vat99], [HV14] (after Mazur [Maz79] and Stevens [Ste82]) using modular symbols.

The Jacquet–Langlands correspondence, together with an explicit L-value formula of earlier joint work with Whitehouse [MW09], also yields a congruence of central L-values when one starts with a congruence of *quaternionic* modular forms. We just consider the case where one of these quaternionic modular forms is constant (i.e., corresponds to $E_{2,N}$) but the same approach can also be used when one starts with two quaternionic cusp forms which are congruent mod $\mathfrak{p}$.

Theorem B. *Suppose N is squarefree product of an odd number of primes and $f \in S_2(N)$ corresponds via Jacquet–Langlands to a quaternionic cusp form ϕ of level N such that $\phi \equiv 1 \bmod \mathfrak{p}^r$ for a prime $\mathfrak{p}$ of $\overline{\mathbb{Q}}$ above p. Then we may normalize $L^{\mathrm{alg}}(1, f, \chi)$ so that (i) it is an algebraic integer, and (ii) there exist an algebraic integer c_ϕ invertible mod $\mathfrak{p}$ such that, for any imaginary quadratic field K inert at each prime dividing N and any ideal class character χ of K, we have*

$$L^{\mathrm{alg}}(1, f, \chi) \equiv \begin{cases} |c_\phi|^2 h_K^2 \bmod \mathfrak{p}^r & \chi = 1, \\ 0 \bmod \mathfrak{p} & \chi \neq 1. \end{cases}$$

Given f, the normalization of $L^{\mathrm{alg}}(1, f, \chi)$ depends on K and ϕ in a simple way, and does not depend on χ — see (3.5). The integer c_ϕ can also be determined in a simple way from ϕ.

Note this L-value congruence can be viewed as a congruence between $L^{\mathrm{alg}}(1, f, \chi)$ and $L^{\mathrm{alg}}(1, E_{2,N}, \chi)$, though we have not written it in this form. For instance, if χ is trivial, then $L(1, E_{2,N}, \chi)$ is essentially $L(0, \eta_K)L(1, \eta_K)$, which is essentially h_K^2.

Vatsal and Heumann [Vat99], [HV14] show that if f is congruent to $E_{2,N}$ mod $\mathfrak{p}^r$, $p > 2$ and $p \nmid N$, then there is a congruence of special values between algebraic parts of $L(1, f, \chi)$ and $L(1, E_{2,N}, \chi)$. In fact they treat higher weights, other Eisenstein series and non-central special values (when $k \geq 4$). However their normalization of algebraic parts of special values makes use of canonical periods, which are only defined up to $\mathfrak{p}$-adic units, whereas our normalization determines $L^{\mathrm{alg}}(1, f, \chi)$ uniquely. On the other hand, we do not require any restriction on p, and we can easily treat Hilbert modular forms also.

Our method is similar in spirit to Gross's approach [Gro87], and also an approach by Quattrini [Qua11] using half-integral weight modular forms. (These works also make use of definite quaternion algebras.) In fact we get

a period congruence, which is stronger than the L-value congruence as the period also accounts for the sign of the "square root" of $L^{\mathrm{alg}}(1, f_K)$.

The main deficiency in our result is that we require a congruence of *quaternionic* modular forms, which is a priori stronger than a congruence of elliptic (or Hilbert) modular forms. However, we expect that these two notions of congruence are equivalent (see Remark 2.4 and (3.3)). This issue is also present in Quattrini's approach, who used work of Emerton [Eme02] to show these notions are equivalent when $r = 1$ and N is prime [Qua11, Thm 3.6]. Hence our Theorem B covers the case addressed in [Maz79]. We can also show that, for $r = 1$, under the hypotheses of Theorem A, if there is a unique cuspidal newform $f \equiv E_{2,N} \bmod \mathfrak{p}$, then it corresponds to a quaternionic form $\phi \equiv 1 \bmod \mathfrak{p}$. The difficulty in general is separating quaternionic eigenforms with congruent eigenvalues.

Now we outline the contents and briefly remark on other related literature.

In Section 1, we explain some preliminaries on weight 2 quaternionic modular forms for totally definite quaternion algebras B over totally real number fields F. These will be functions on the finite set of ideal classes $\mathrm{Cl}(\mathcal{O})$, for some order $\mathcal{O}$ of B.

In Section 2, we use the Eichler mass formula to show the existence of a quaternionic cusp form ϕ congruent to the constant function 1 modulo suitable primes. Then we apply the Jacquet–Langlands correspondence to get a Hilbert cusp form f corresponding to ϕ whose Hecke eigenvalues are congruent to those of a Hilbert Eisenstein series associated to the constant function on $\mathrm{Cl}(\mathcal{O})$. This (Theorem 2.1) is the first main result, which specializes to the Hecke eigenvalue congruence statements in Theorem A.

We note recent work of Berger, Klosin and Kramer [BKK14] gives an algebraic approach to counting congruences of Hecke eigenvalues in more general settings, which yields a refinement of Mazur's congruence of Fourier coefficients result for $S_2(N)$ with N is prime. In the case of Hilbert modular forms, some results are already known about finding primes of congruence between two *cusp* forms, e.g., the work of Ghate [Gha02] generalizing a result of Hida [Hid81] for elliptic cusp forms, but we are not aware of results along the lines of Theorem 2.1 guaranteeing the existence of Hilbert cusp forms congruent to Eisenstein series.

Then in Section 3, we use Waldspurger's formula [Wal85] relating central L-values to periods on quaternion algebras. This immediately gives the nonvanishing L-value statement in Theorem A. To get the precise congruence in Theorem B, we use a more explicit version of Waldspurger's formula

from [MW09]. That L-value formula applies to arbitrary quadratic extensions K/F (F totally real or not), but we need to restrict to K/F CM here in order for the relevant period to lie on the definite quaternion algebra B. Again, we are not aware of such L-value congruences in the Hilbert modular case.

On the other hand, there are some existence results of a different nature about even weight cusp forms with $L^{\mathrm{alg}}(1, f, \chi) \not\equiv 0 \bmod \mathfrak{p}$ using average value formulas, e.g., the aforementioned work of Michel and Ramakrishnan [MR12] for elliptic cusp forms and joint work of File and Pitale with the present author [FMP17] for Hilbert cusp forms (though excluding parallel weight 2 for simplicity). These results are of just the form that for a given $\mathfrak{p}$ and any suitably large level $\mathfrak{N}$, there exists some $f \in S_k(\mathfrak{N})$ such that $L^{\mathrm{alg}}(1, f, \chi) \not\equiv 0 \bmod \mathfrak{p}$, but the bound on the level *depends on K*. I am not aware of any other results on the vanishing of $L^{\mathrm{alg}}(1, f, \chi) \bmod \mathfrak{p}$ for χ nontrivial when the sign of the functional equation is $+1$.

Acknowledgements

1. Quaternionic modular forms

Throughout, we fix the following notation. Let F be a totally real number field of degree d, and $\mathfrak{o}_F$ its ring of integers. Let $\mathfrak{N} = \mathfrak{N}_1 \mathfrak{N}_2$ be a nonzero integral ideal in $\mathfrak{o}_F$ such that $\mathfrak{N}_1 = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ and $\mathfrak{N}_2 = \mathfrak{q}_1^{f_1} \cdots \mathfrak{q}_s^{f_s}$ where the $\mathfrak{p}_i$'s and $\mathfrak{q}_j$'s are distinct prime ideals of $\mathfrak{o}_F$, $e_i, f_j \in \mathbb{N}$, each e_i is odd, and $r \equiv d \bmod 2$. Note that $\mathfrak{N}$ can be any nonzero ideal when d is even, and $\mathfrak{N}$ can be any ideal which is not a square when d is odd. The letter ℓ will denote a finite prime of F.

Denote by B the unique (up to isomorphism) totally definite quaternion algebra over F ramified at each $\mathfrak{p}_i$ and no other finite prime. Let $\mathcal{O}$ be an order of level $\mathfrak{N}$ in B such that $\mathcal{O}_{\mathfrak{q}_i}$ is conjugate to $R_0(\mathfrak{q}_i^{f_i})$, the subring

of $M_2(\mathfrak{o}_{F,\mathfrak{q}_i})$ with lower left entry 0 mod $\mathfrak{q}_i^{f_i}$, for $1 \leq i \leq s$. Note this is an Eichler order if $\mathfrak{N}_1$ is squarefree — otherwise it is the intersection of an Eichler order of level $\prod \mathfrak{p}_i \cdot \mathfrak{N}_2$ with an order, which is unique up to isomorphism, of level $\mathfrak{N}_1$.

We want to work with a certain space of automorphic forms on $B^\times$. Adelically, we will be looking at functions on

$$(1.1) \qquad \mathrm{Cl}(\mathcal{O}) = B^\times \backslash \hat{B}^\times / \hat{\mathcal{O}}^\times = B^\times \backslash B^\times(\mathbb{A}_F)/(\hat{\mathcal{O}}^\times \times B_\infty^\times).$$

Here $\mathrm{Cl}(\mathcal{O})$ can naturally be viewed as the (finite) set of invertible (locally principal) right ideal classes of $\mathcal{O}$. We denote the class number $\#\mathrm{Cl}(\mathcal{O})$ by $h(\mathcal{O})$. Let $\mathcal{I}_1, \ldots, \mathcal{I}_h$ be a set of representatives for the right ideal classes of $\mathcal{O}$, and let $x_1, \ldots, x_h$ be a corresponding set of representatives for the double coset classes $B^\times \backslash \hat{B}^\times / \hat{\mathcal{O}}^\times$.

We consider the space of (weight 2, or weight 0, depending on convention) quaternionic modular forms on $\mathcal{O}$ defined by

$$M(\mathcal{O}) = \{\phi : \mathrm{Cl}(\mathcal{O}) \to \mathbb{C}\},$$

a complex vector space of dimension $h(\mathcal{O})$. To work adelically, we will typically view $\phi \in M(\mathcal{O})$ as a function on $\hat{B}^\times$ which is left invariant by $B^\times$ and right invariant by $\hat{\mathcal{O}}^\times$. For simplicity, we will work with the subspace of forms with trivial central character:

$$(1.2) \qquad \mathcal{M} = M(\mathcal{O}, 1) = \{\phi \in M(\mathcal{O}) : \phi(z) = \phi(1) \text{ for } z \in \hat{F}^\times\}.$$

Note that $M(\mathcal{O}, 1) = M(\mathcal{O})$ if F has class number 1, because in this case $\hat{F}^\times = F^\times \hat{\mathfrak{o}}_F^\times \subset B^\times \hat{\mathcal{O}}^\times$. In general $\mathcal{M}$ is a complex vector space of dimension $h(\mathcal{O}) - (m - 1)$ where m is size the image of the map $\mathrm{Cl}(\mathfrak{o}_F) \to \mathrm{Cl}(\mathcal{O})$ induced from inclusion $\mathbb{A}_F^\times \to B^\times(\mathbb{A}_F)$, so we always have $h(\mathcal{O}) - h_F + 1 \leq \dim \mathcal{M} \leq h(\mathcal{O})$. When h_F is odd, composing with the reduced norm shows the map $\mathrm{Cl}(\mathfrak{o}_F) \to \mathrm{Cl}(\mathcal{O})$ is injective so $\dim \mathcal{M} = h(\mathcal{O}) - h_F + 1$.

We make $\mathcal{M}$ into an inner product space by defining

$$(\phi, \phi') = \int_{\mathbb{A}_F^\times B^\times \backslash B^\times(\mathbb{A}_F)} \phi(g)\overline{\phi'(g)}\, dg = (4\pi^2)^d \int_{\hat{F}^\times B^\times \backslash \hat{B}^\times} \phi(g)\overline{\phi'(g)}\, dg,$$

where dg denotes the Haar measure on the relevant quotient induced by the product of local Tamagawa measures on $B_v^\times$ and $F_v^\times$ and the counting measure on $B^\times$. The inner product converges by compactness.

We can write

$$\hat{F}^\times B^\times x_i \hat{\mathcal{O}}^\times = \bigsqcup_z \hat{B}^\times z x_i \hat{\mathcal{O}}^\times,$$

where $z \in \hat{F}^\times / (\hat{F}^\times \cap B^\times x_i \hat{\mathcal{O}}^\times x_i^{-1})$. The latter set is finite of size at most h_F since $F^\times \hat{\mathfrak{o}}_F^\times \subset \hat{F}^\times \cap B^\times x_i \hat{\mathcal{O}}^\times x_i^{-1}$. Denote its cardinality by c_i.

Since ϕ and ϕ' are right invariant by $\hat{\mathcal{O}}^\times$, we may write

$$(\phi, \phi') = \sum_i \omega_i^{-1} \phi(\mathcal{I}_i) \overline{\phi'(\mathcal{I}_i)}, \qquad \phi, \phi' \in M(\mathcal{O}),$$

where $\phi(\mathcal{I}_i) = \phi(x_i)$ and

$$\omega_i = (4\pi^2)^{-d} \mathrm{vol}(\hat{F}^\times B^\times \backslash \hat{F}^\times B^\times x_i \hat{\mathcal{O}}^\times)^{-1} c_i.$$

Note

$$\mathrm{vol}(\hat{F}^\times B^\times \backslash \hat{F}^\times B^\times x_i \hat{\mathcal{O}}^\times) = \mathrm{vol}(\hat{F}^\times B^\times \backslash \hat{F}^\times B^\times x_i \hat{\mathcal{O}}^\times x_i^{-1})$$
$$= \mathrm{vol}(\hat{\mathfrak{o}}_F^\times B^\times \backslash B^\times \hat{x}_i \mathcal{O}^\times x_i^{-1}) \frac{c_i}{h_F}.$$

Since $\mathcal{O}_\ell(\mathcal{I}_i) = x_i \hat{\mathcal{O}}^\times x_i^{-1} \cap B^\times$, we have

$$(1.3) \qquad \qquad \omega_i = \frac{[\mathcal{O}_\ell(\mathcal{I}_i)^\times : \mathfrak{o}_F^\times]}{\mathrm{vol}(\hat{\mathcal{O}}^\times / \hat{\mathfrak{o}}_F^\times)} \frac{h_F}{(4\pi^2)^d}.$$

One can obtain the following Tamagawa volume computations from [Vig80].

First, $\mathrm{vol}(\mathfrak{o}_{F_v}^\times) = \Delta_{F_v}^{-\frac{1}{2}}$ for $v < \infty$.

For finite $v \nmid \mathfrak{N}$, we have

$$\mathrm{vol}(\mathcal{O}_v^\times / \mathfrak{o}_{F_v}^\times) = (1 - q_v^{-2}) \mathrm{vol}(\mathfrak{o}_{F_v}^\times)^3 = L(2, 1_{F_v})^{-1} \mathrm{vol}(\mathfrak{o}_{F_v}^\times)^3.$$

For $v = \mathfrak{p}_i | \mathfrak{N}_1$, we have

$$\mathrm{vol}(\mathcal{O}_v^\times / \mathfrak{o}_{F_v}^\times) = (1 - q_v^{-1})^{-1} q_v^{-e_i} L(2, 1_{F_v})^{-1} \mathrm{vol}(\mathfrak{o}_{F_v}^\times)^3.$$

For $v = \mathfrak{q}_j | \mathfrak{N}_2$, we have that

$$[\mathrm{GL}_2(\mathfrak{o}_{F, \mathfrak{q}_j}) : R_0(\mathfrak{q}_j^{f_j})^\times] = q_v^{f_j} (1 + q_v^{-1}),$$

and therefore

$$\mathrm{vol}(\mathcal{O}_v^\times / \mathfrak{o}_{F_v}^\times) = q_v^{-f_j} (1 + q_v^{-1})^{-1} L(2, 1_{F_v})^{-1} \mathrm{vol}(\mathfrak{o}_{F_v}^\times)^3.$$

Putting together the local measures gives

$$(1.4) \qquad \mathrm{vol}(\hat{\mathcal{O}}^\times / \hat{\mathfrak{o}}_F^\times) = \frac{\Delta_F^{-3/2}}{N(\mathfrak{N}) \zeta_F(2)} \prod_{v | \mathfrak{N}_1} \frac{1}{1 - q_v^{-1}} \prod_{v | \mathfrak{N}_2} \frac{1}{1 + q_v^{-1}}.$$

Note that computing (ϕ, ϕ) for the constant function $\phi_0 = 1$ gives

$$\sum \omega_i^{-1} = (\phi_0, \phi_0) = (2\pi)^{2d} \mathrm{vol}(\hat{F}^\times B^\times \backslash \hat{B}^\times) = \mathrm{vol}(B^\times \mathbb{A}_F^\times \backslash B^\times(\mathbb{A}_F)) = 2.$$

If we put $w_i = [\mathcal{O}_\ell(\mathcal{I}_i)^\times : \mathfrak{o}_F^\times] = \frac{(2\pi)^{2d} \mathrm{vol}(\hat{\mathcal{O}}^\times / \hat{\mathfrak{o}}_F^\times)}{h_F} \omega_i$ and define the normalized paring

$$
\begin{aligned}
(1.5) \qquad [\phi, \phi'] &= \frac{h_F}{(2\pi)^{2d} \mathrm{vol}(\hat{\mathcal{O}}^\times / \hat{\mathfrak{o}}_F^\times)} (\phi, \phi') \\
&= \sum w_i^{-1} \phi(\mathcal{I}_i) \overline{\phi'(\mathcal{I}_i)}, \qquad \phi, \phi' \in M(\mathcal{O}),
\end{aligned}
$$

then we recover a generalized form of the usual Eichler mass formula,

$$
\begin{aligned}
(1.6) \quad m(\mathcal{O}) &= \sum w_i^{-1} = [\phi_0, \phi_0] \\
&= 2^{1-2d} \pi^{-2d} |\Delta_F|^{3/2} h_F \zeta_F(2) N(\mathfrak{N}) \prod_{v | \mathfrak{N}_1} (1 - q_v^{-1}) \prod_{v | \mathfrak{N}_2} (1 + q_v^{-1}) \\
&= 2^{1-d} h_F |\zeta_F(-1)| N(\mathfrak{N}) \prod_{v | \mathfrak{N}_1} (1 - q_v^{-1}) \prod_{v | \mathfrak{N}_2} (1 + q_v^{-1}).
\end{aligned}
$$

Here $N(\mathfrak{N})$ is the level (reduced norm) of $\mathfrak{N}$. The rational number $m(\mathcal{O})$ is called the mass of $\mathcal{O}$. When $F = \mathbb{Q}$, $\mathfrak{N} = (N_1)$ and $\mathfrak{N}_2 = (N_2)$, this simplifies to

$$(1.7) \qquad m(\mathcal{O}) = \sum w_i^{-1} = \frac{\varphi(N_1)}{12} N_2 \prod_{p | N_2} (1 + p^{-1}).$$

Now we want to define the Eisenstein and cuspidal subspaces of $\mathcal{M}$. The Eisenstein space will be generated by the one-dimensional representations of $\hat{B}^\times$, which all factor through the reduced norm map $N : \hat{B}^\times \to \hat{F}^+$. The reduced norm induces a surjective map $N : \mathrm{Cl}(\mathcal{O}) \to \mathrm{Cl}^+(\mathfrak{o}_F)$ to the narrow class group of F. Define the Eisenstein subspace $\mathcal{E}$ of $\mathcal{M}$ to be the subspace of all $\phi \in \mathcal{M}$ which factor through N, and the cuspidal space $\mathcal{S}$ to be the orthogonal complement of $\mathcal{E}$ in $\mathcal{M}$. We can describe $\mathcal{S}$ in terms of our

normalized pairing (1.5) by

$$\mathcal{S} = \{\phi \in \mathcal{M} : [\phi, \psi \circ N] = 0 \text{ for all characters } \psi : \text{Cl}^+(\mathfrak{o}_F)/\text{Cl}(\mathfrak{o}_F) \to \mathbb{C}\}.$$

One has in the usual way Hecke operators T_ℓ for each prime $\ell \nmid \mathfrak{N}$, which commute with each other and are self-adjoint with respect to the inner product. One can also describe the action on $\mathcal{M}$ (or $M(\mathcal{O})$) in terms of (generalized) Brandt matrices. Hence $\mathcal{M}$, and also $\mathcal{S}$, has a basis consisting of eigenforms for each such T_ℓ. Via the Jacquet–Langlands correspondence, each eigenform $\phi \in \mathcal{S}$ transfers to an eigenform $f \in S_2(\mathfrak{N})$, the space of parallel weight 2 Hilbert modular forms of level $\mathfrak{N}$, with the same Hecke eigenvalues away from $\mathfrak{N}$.

Suppose now $\ell | \mathfrak{N}_1$ but $\ell^2 \nmid \mathfrak{N}_1$. Then one can define a Hecke operator by $(T_\ell \phi)(x) = \phi(x \varpi_{B_\ell})$ where ϖ_{B_ℓ} is a uniformizer for B_ℓ. This corresponds to the double coset $\mathcal{O}_\ell \varpi_{B_\ell} \mathcal{O}_\ell = \varpi_{B_\ell} \mathcal{O}_\ell$. Now one has bases for $\mathcal{M}$ and $\mathcal{S}$ which are eigenforms for all T_ℓ with $\ell \nmid \mathfrak{N}_2$ and $\ell^2 \nmid \mathfrak{N}_1$, and the Jacquet–Langlands lift of such an eigenform ϕ will be a modular form $f \in S_2(\mathfrak{N})$ which, at all such ℓ, is new and is an eigenform for T_ℓ with the same eigenvalue as ϕ.

We refer the reader to [Hid06, Chap 2] or [DV13] for details.

2. Congruence with Eisenstein series

We keep the notation of the previous section.

For $f \in M_2(\mathfrak{N})$ (resp. $\phi \in \mathcal{M}$) which is an eigenfunction of the Hecke operator T_ℓ, let $\lambda_\ell(f)$ (resp. $\lambda_\ell(\phi)$) denote the corresponding eigenvalue.

Let E be the normalized Eisenstein series on $M_2(\mathfrak{N})$ whose Hecke eigenvalues are $\lambda_\ell(E) = N(\ell) + 1$ for $\ell \nmid \mathfrak{N}$ and $\lambda_\ell(E) = 1$ for $\ell | \mathfrak{N}$. When $F = \mathbb{Q}$, this is the $E_{2,N}$ explicitly constructed in the introduction. Recall $\phi_0 = 1 \in \mathcal{E}$. Then ϕ_0 is an eigenform for the Hecke operators T_ℓ because the Brandt matrices have constant row sums, and in fact $\lambda_\ell(E) = \lambda_\ell(\phi_0)$ when $\ell \nmid \mathfrak{N}_2$ and $\ell^2 \nmid \mathfrak{N}_1$.

The field of rationality of an eigenform $\phi \in \mathcal{M}$ is the field $\mathbb{Q}(\phi)$ generated by its Hecke eigenvalues. We may normalize ϕ so that all values of ϕ are integers in $\mathbb{Q}(\phi)$, in which case we say ϕ is integral. Let $\mathbb{Q}(\mathfrak{N})$ be the compositum of the fields of rationality $\mathbb{Q}(\phi)$ of all eigenforms in $\phi \in \mathcal{M}$. Note this is a subfield of the compositum of all $\mathbb{Q}(f)$ where f ranges over eigenforms in $S_2(\mathfrak{N})$.

If ϕ, ϕ' are integral and $\mathfrak{p}$ is an ideal of a suitable rationality field, we write $\phi \equiv \phi' \bmod \mathfrak{p}$ if $\phi(\mathcal{I}) \equiv \phi'(\mathcal{I}) \bmod \mathfrak{p}$ for all $\mathcal{I} \in \text{Cl}(\mathcal{O})$. If ϕ, ϕ' are eigenforms, then $\phi(\mathcal{I}) \equiv \phi'(\mathcal{I}) \bmod \mathfrak{p}$ implies their Hecke eigenvalues are

also congruent mod $\mathfrak{p}$ because the Hecke operators act by integral Brandt matrices. We expect the converse to generally be true, though do not know how to show it. See Remark 2.4 and (3.3) below.

Denote by h_F^+ the narrow class number of F.

Theorem 2.1. *Assume $h_F^+ = h_F$. Let* num *be the numerator of $m(\mathcal{O})$. Suppose $p|$num and $\mathfrak{p}$ is a prime above p in $\mathbb{Q}(\mathfrak{N})$. Then there exists an eigenform $f \in S_2(\mathfrak{N})$ such that $\lambda_\ell(f) \equiv \lambda_\ell(E)$ mod $\mathfrak{p}$ for all ℓ with $\ell \nmid \mathfrak{N}_2$ and $\ell^2 \nmid \mathfrak{N}_1$. If $\mathfrak{N} = \mathfrak{N}_1$ is squarefree, we may take f to be a newform.*

Proof. First we show there exists an integral $\phi \in \mathcal{S}$ such that $\phi \equiv 1$ mod p. Consider $\phi \in M(\mathcal{O})$ integral with $\phi \equiv 1$ mod p so, for each $\mathcal{I}_i \in \mathrm{Cl}(\mathcal{O})$, $\phi(\mathcal{I}_i) = 1 + p a_i$ for some $a_i \in \mathbb{Z}$. Order our ideals so that $\mathcal{I}_1, \ldots, \mathcal{I}_m$ represent each of the ideal classes in the image of the map $\mathrm{Cl}(\mathfrak{o}_F) \to \mathrm{Cl}(\mathcal{O})$. Recall $m \leq h_F$. Then, by (1.2), $\phi \in \mathcal{M}$ just means: (i) $a_1 = \cdots = a_m$. Moreover, $\phi \in \mathcal{S}$ if in addition $[\phi, \phi_0] = 0$, i.e., $\phi \in \mathcal{S}$ if and only if (i) holds and we have (ii) $m(\mathcal{O}) = -p \sum_{i=1}^h w_i^{-1} a_i$.

We claim there exist $a_i \in \mathbb{Z}$ such that these conditions hold. Let $w = \prod_{i=1}^h w_i$ and $w_i^* = \frac{w}{w_i}$. We can rewrite (ii) as $\sum w_i^* = -p \sum w_i^* a_i$. To also account for (i), put $w_1' = w_1^* + \cdots + w_m^*$, $b_1 = a_1$ and $w_i' = w_{i+m-1}^*$, $b_i = a_{i+m-1}$ for $2 \leq i \leq k := h - m + 1$. Note $\mathcal{M} \neq \mathcal{E}$ means $h > m$, i.e., $k \geq 2$. Then (i) and (ii) are achievable if and only if $\sum_{i=1}^k w_i' = -p \sum_{i=1}^k b_i w_i'$ is solvable for $b_i \in \mathbb{Z}$. For this, it suffices to show $\gcd(w_1', \ldots, w_k')$ divides $p^{-1} \sum_{i=1}^k w_i'$, which is obvious at primes away from p. Say $p^j \parallel \gcd(w_1', \ldots, w_k')$. Then $p|$num implies $wp | \sum w_i' = wm(\mathcal{O})$, and therefore $p^{j+1} | \sum w_i'$. This proves the claim and gives us our desired ϕ. Note this argument also shows that $p|$num implies $\mathcal{M} \neq \mathcal{E}$.

Now let Φ be the set of integral $\phi \in \mathcal{S}$ which are congruent to a nonzero multiple of ϕ_0 mod $\mathfrak{p}$. Fix a basis of eigenforms $\phi_1, \ldots, \phi_s$ of $\mathcal{S}$. Let r be minimal such that, after a possible reordering of the ϕ_i's, there exists $\phi \in \Phi$ with $\phi = c_1 \phi_1 + \cdots + c_r \phi_r$, $c_i \in \mathbb{Q}(\mathfrak{N})$. Say $\phi \equiv c \phi_0$ mod $\mathfrak{p}$. Then, for ℓ such that $\ell \nmid \mathfrak{N}_2$ and $\ell^2 \nmid \mathfrak{N}_1$,

$$[T_\ell - \lambda_\ell(\phi_j)]\phi \equiv (\lambda_\ell(\phi_0) - \lambda_\ell(\phi_j)) c \phi_0 \text{ mod } \mathfrak{p},$$

and thus $[T_\ell - \lambda_\ell(\phi_j)]\phi \in \Phi$ unless $\lambda_\ell(\phi_0) \equiv \lambda_\ell(\phi_j)$ mod $\mathfrak{p}$. (Note $[T_\ell - \lambda_\ell(\phi_j)]\phi \in \Phi$ is also integral because $T_\ell \phi$ is, so it makes sense to consider this mod $\mathfrak{p}$.) However, $[T_\ell - \lambda_\ell(\phi_j)]\phi$ is a linear combination of $\phi_1, \ldots, \phi_{j-1}$, $\phi_{j+1}, \ldots, \phi_r$, which would contradict the minimality of r if $[T_\ell - \lambda_\ell(\phi_j)] \in \Phi$. Hence $\lambda_\ell(\phi_0) \equiv \lambda_\ell(\phi_j)$ mod $\mathfrak{p}$ for all ℓ as above and all $1 \leq j \leq r$.

Since f has the same Hecke eigenvalues as ϕ for T_ℓ with ℓ as above, this yields the theorem. $\qquad\square$

Remark 2.2. (a) The reason we assume $h_F^+ = h_F$ is to guarantee the existence of $\phi \in \mathcal{S}$ such that $\phi \equiv 1 \bmod p$. If $h_F^+ \neq h_F$, one gets an additional linear constraint $[\phi, \psi \circ N] = 0$ on the a_i's in the proof for each nontrivial character ψ of $\mathrm{Cl}^+(\mathfrak{o}_F)/\mathrm{Cl}(\mathfrak{o}_F)$. Then, at least a priori, one needs to place some conditions on the w_i's, $m(\mathcal{O})$ and p to guarantee the existence of $a_i \in \mathbb{Z}$ solving this system of $\mathbb{Q}$-linear equations.

(b) When $h_F^+ > 1$, there are other Eisenstein series in $M(\mathcal{O})$, and one might ask for congruences of these Eisenstein series as well. However, such Eisenstein series which are also eigenforms can be obtained by twisting ϕ_0 by narrow ideal class characters of F, and analogous congruences are just obtained by twisting both E and f.

We observe that there are often multiple choices for B for a given $\mathfrak{N}$. For instance, if $F = \mathbb{Q}$ and $N = 11 \cdot 13$, we get $\mathrm{num} = 5 \cdot 7$ if we take $N_1 = 11$, $N_2 = 13$ and $\mathrm{num} = 12$ if we take $N_1 = 13$, $N_2 = 11$. Thus one gets congruences of cusp forms in level $11 \cdot 13$ with Eisenstein series modulo the primes $p = 2, 3, 5, 7$.

Some examples with $F = \mathbb{Q}$ were given in the introduction. Here is a simple example with $F \neq \mathbb{Q}$.

Example 2.3. Let $F = \mathbb{Q}(\sqrt{5})$, which has narrow class number 1. Suppose $\mathfrak{N} = \mathfrak{N}_2$ is one of the two prime ideals with $N(\mathfrak{N}) = 31$. Then $\zeta_F(-1) = \frac{1}{30}$ and $m(\mathcal{O}) = \frac{32}{60} = \frac{8}{15}$, so we take $p = 2$. Here $S_2(\mathfrak{N})$ is one dimensional and has rationality field $\mathbb{Q}$. Then for nonzero $f \in S_2(\mathfrak{N})$ our theorem says that $\lambda_\ell(f) \equiv N(\ell) + 1 \bmod 2$ for $\ell \neq \mathfrak{N}$. It is also true that $\lambda_{\mathfrak{N}}(f) \equiv 1 \bmod 2$. Indeed, the first few nonzero Hecke eigenvalues $\lambda_\ell(f)$ satisfy the values listed in the following table:

$N(\ell)$	4	5	9	11	11	19	19	29	29	31	31	41	41
$\lambda_\ell(f)$	-3	-2	2	4	-4	-4	4	-2	-2	-1	8	-6	-6

Remark 2.4. Suppose $\mathfrak{N} = \mathfrak{N}_1$ is squarefree. If $f \in S_2(\mathfrak{N})$ such that $f \equiv E \bmod \mathfrak{p}$, we expect (cf. (3.3) below) that f corresponds to a quaternionic form $\phi \in \mathcal{S}$ such that, after normalization, $\phi \equiv 1 \bmod \mathfrak{p}$. Let p be the rational prime below $\mathfrak{p}$. For such a cusp form ϕ to exist, we need $w[\phi, \phi_0] \equiv w[\phi_0, \phi_0] \equiv 0 \bmod \mathfrak{p}$ where as before $w = \prod w_i$. Thus if $(p, w) = 1$, a congruence $f \equiv E \bmod \mathfrak{p}$ should only exist if $p | \mathrm{num}$ as in the theorem. If $F = \mathbb{Q}$,

then no prime > 3 divides w, and this agrees with the converse obtained by Ribet.

We also note that when $p|\text{num}$, if there is a unique newform $f \in S_2(\mathfrak{N})$ such that $f \equiv E \bmod \mathfrak{p}$, the above proof implies f corresponds to a quaternionic form $\phi \in \mathcal{S}$ with $\phi \equiv 1 \bmod \mathfrak{p}$, as we must have $r = 1$ in the final paragraph. When $r = 1$, $F = \mathbb{Q}$ and $N = \mathfrak{N}$ is prime, this is also true by [Qua11], [Eme02].

3. Quadratic twist L-values

We keep our previous notation, but now assume $\mathcal{O}$ is a maximal order of B, i.e., $\mathfrak{N} = \mathfrak{N}_1$ and $\mathfrak{N}$ is squarefree. For consistency and clarity, we will denote complete L-functions by $L^*(s, -)$ and incomplete (i.e., the finite part of) L-functions just by $L(s, -)$. However, we follow the usual convention that automorphic L-functions are normalized so that $s = 1/2$ is the central point, whereas L-functions for $f \in S_2(\mathfrak{N})$ are normalized classically with $s = 1$ the central point.

Let K/F be a CM quadratic extension such that each $\mathfrak{p}|\mathfrak{N}$ is inert in K. This means K embeds in B and, since we are now assuming $\mathcal{O}$ is maximal, we may fix an embedding of K into B such that $\mathfrak{o}_K \subset \mathcal{O}$. Hence, to each $t \in \text{Cl}(\mathfrak{o}_K) = K^\times \backslash \hat{K}^\times / \hat{\mathfrak{o}}_K^\times$ we can associate the quaternionic ideal class $x(t) \in \text{Cl}(\mathcal{O})$ via $x(t) = B^\times t\hat{\mathcal{O}}^\times$. Fix a character χ of $\text{Cl}(\mathfrak{o}_K)$. Put the product of local Tamagawa measures on $\mathbb{A}_F^\times$ and $\mathbb{A}_K^\times$, and the counting measure on $K^\times$. For $\phi \in \mathcal{M}$, define the period

$$P_\chi(\phi) = \int_{K^\times \mathbb{A}_F^\times \backslash \mathbb{A}_K^\times} \phi(t)\overline{\chi(t)}\, dt = \frac{\text{vol}(K^\times \mathbb{A}_F^\times \backslash \mathbb{A}_K^\times)}{h_K} \sum_{t \in \text{Cl}(\mathfrak{o}_K)} \phi(x(t))\overline{\chi(t)},$$

where dt is the quotient measure. Let η_K denote the quadratic idele class character of F associated to K/F. Then

$$\frac{\text{vol}(K^\times \mathbb{A}_F^\times \backslash \mathbb{A}_K^\times)}{h_K} = \frac{2L^*(1, \eta_K)}{h_K} = \frac{2^{d+1}\sqrt{\Delta_F}}{w_K h_F Q_{K/F} \sqrt{|\Delta_K|}},$$

where $Q_{K/F} = [\mathfrak{o}_K^\times : \mathfrak{o}_F^\times \mu(K)] = 2^{d-1}\frac{R_F}{R_K}$, the R_* denoting a regulator. We remark $Q_{K/F}$ is 1 or 2. Consider the normalized period,

$$(3.1) \qquad P_\chi^0(\phi) = \sum_{t \in \text{Cl}(\mathfrak{o}_K)} \phi(x(t))\overline{\chi(t)} = \frac{w_K h_F Q_{K/F}\sqrt{|\Delta_K|}}{2^{d+1}\sqrt{\Delta_F}} P_\chi(\phi).$$

Let $\phi \in \mathcal{S}$ be an eigenform. Then ϕ generates an irreducible cuspidal automorphic representation π of $B^\times(\mathbb{A}_F)$ with trivial central character. The period P_χ extends to a linear functional on π via the same defining integral. Suppose $P_\chi(\phi) \neq 0$ so P_χ is a nonzero functional. Then for each place v of F, π_v has a nonzero local $K_v^\times$-invariant linear functional ℓ_v. It is well known that the functional ℓ_v is unique up to scaling, and we normalize the ℓ_v's so that $P_\chi = \prod \ell_v$ (i.e., this factorization holds factorizable forms in π).

For each finite place v, Gross and Prasad [GP91] defined a test vector $\phi_v \in \pi_v$ such that $\ell_v(\phi_v) \neq 0$. In our setting, the Gross–Prasad test vector ϕ_v is the (unique up to scaling) vector in π_v fixed by $\mathcal{O}_v^\times$. (This is not true if $\mathcal{O}$ is not maximal.) This implies that $\phi = \prod \phi_v$ as a function on $B^\times(\mathbb{A}_F)$ where ϕ_v is a suitably normalized (to guarantee convergence) Gross–Prasad test vector for $v < \infty$ and $\phi_v = 1$ for $v|\infty$.

The central-value formula in [MW09], refining Waldspurger's formula [Wal85], specializes here to

$$(3.2) \qquad \frac{|P_\chi(\phi)|^2}{(\phi,\phi)} = \frac{\zeta_F(2)}{2\pi^{2d}} \sqrt{\frac{\Delta_F}{|\Delta_K|}} \prod_{v|\mathfrak{N}} (1 - q_v^{-1}) \frac{L^*(1/2, \pi_K \otimes \chi)}{L^*(1, \pi, Ad)}.$$

(Note [MW09] uses a slightly different choice of measure on $B^\times(\mathbb{A})$ there — a formulation with the present choice of measure is given in [FMP17].) Observe the left hand side of this formula is invariant under scaling. The L-functions for π are the same as the L-functions for its Jacquet–Langlands transfer to $\mathrm{GL}(2)$.

We will use this formula to deduce a precise congruence result on L-values. We remark that Zhang (e.g., [Zha04]), generalizing the work of Gross [Gro87], also obtained an explicit L-value formula in this setting. It is formulated somewhat differently than (3.2), but presumably Zhang's formula can be used in a similar manner. Alternatively one could use half-integral weight forms and the formula of Baruch–Mao [BM07] (generalizing an earlier formula of Waldspurger), similar to the approach in [Qua11] for $F = \mathbb{Q}$.

Unfortunately, we are not able to prove that a Hecke eigenvalue congruence implies a congruence of L-values as [Vat99], [HV14] do over $\mathbb{Q}$. Our method requires starting with a congruence of quaternionic modular forms. In fact we generally expect the statement

$$(3.3) \qquad \lambda_\ell(\phi) \equiv \lambda_\ell(\phi_0) \bmod \mathfrak{p} \text{ for all } \ell \implies c\phi \equiv \phi_0 \bmod \mathfrak{p} \text{ for suitable } c$$

to hold, but we do not know how to prove it. As remark earlier, the converse direction is true. Recall from Remark 2.4 that at least (3.3) is true if there is a

unique (up to scaling) eigenform $\phi \in \mathcal{S}$ satisfying the above Hecke eigenvalue congruence, i.e. a unique cuspidal newform f congruent to E). This is also true if $F = \mathbb{Q}$ and $N = \mathfrak{N}$ is prime ([Qua11, Thm 3.6], relying on [Eme02]).

Our main evidence for believing (3.3) holds more generally is that it seems necessary to obtain a congruence of L-values for newforms f congruent to E as in [HV14]. This is because a congruence of L-values would say that (up to constants) we have $|P_K^0(\phi)|^2 \equiv |P_K^0(\phi_0)|^2 \bmod \mathfrak{p}$ for infinitely many K. On the other hand, the map $\mathrm{Cl}(\mathfrak{o}_K) \to \mathrm{Cl}(\mathcal{O})$ appears to behave essentially randomly, so an infinitude of such congruences would seem to happen with probability 0 if $c\phi \not\equiv \phi_0$. We remark that the expectation (3.3) relies on the specific action of the Hecke algebra on $\mathcal{M}$ — e.g., if one were to replace the action of the Brandt matrices on $\mathcal{M}$ by diagonal matrices the analogous statement fails — so (3.3) may be a rather deep arithmetic statement (indeed, the analysis in [Eme02] is not trivial).

Before deducing our precise congruence result of L-values under a congruence of quaternionic modular forms we observe that we can at least get some statement about nonvanishing of L-values among forms congruent to Eisenstein series. For this we can also allow K/F to be ramified at primes dividing the level.

Proposition 3.1. *We continue the hypotheses and notation of Theorem 2.1, with the further assumption that $\mathfrak{N} = \mathfrak{N}_1$ is squarefree. Let K/F be a CM extension not split at each prime dividing $\mathfrak{N}$. If $p \nmid h_K$, then there exists a newform $f \in S_2(\mathfrak{N})$ such that $\lambda_\ell(f) \equiv \lambda_\ell(E) \bmod \mathfrak{p}$ for all ℓ and $L(1, f_K) \neq 0$.*

When $F = \mathbb{Q}$, $p > 2$, and $p \nmid N = \mathfrak{N}$, one can deduce stronger nonvanishing results from Theorem 2.1 using [HV14].

Proof. By the proof of Theorem 2.1, there exists an integral $\phi \in \mathcal{S}$ such that $\phi \equiv 1 \bmod \mathfrak{p}$ and $\phi = \sum \phi_i$ where the ϕ_i are eigenforms with T_ℓ-eigenvalues congruent to $\lambda_\ell(E) \bmod \mathfrak{p}$. Now if $p \nmid h_K$, then

$$P^0(\phi) \equiv \sum_{t \in \mathrm{Cl}(\mathfrak{o}_K)} 1 \equiv h_K \bmod \mathfrak{p}$$

implies $P^0(\phi) \neq 0$. Since $P^0(\phi) = \sum P^0(\phi_i)$ at least one $P^0(\phi_i)$ is nonzero. Hence, by Waldspurger's formula [Wal85] (or the more refined (3.2)), $L(1, f_{i,K}) \neq 0$ where $f_i \in S_2(\mathfrak{N})$ is the newform corresponding to ϕ_i via Jacquet–Langlands. (While we have only stated (3.2) if each prime dividing $\mathfrak{N}$ is inert in K, [Wal85] and [MW09] allow these primes to be ramified in K, with a suitable change in local factors.) □

From now on we will suppose that we have a newform $f \in S_2(\mathfrak{N})$ corresponds to an integral eigenform $\phi \in \mathcal{S}$ such that $\phi \equiv 1 \bmod \mathfrak{p}^r$ for some $r \geq 1$. As remarked above, this holds for $r = 1$ under the conditions of Theorem 2.1 if there is a unique cuspidal newform f congruent to $E \bmod \mathfrak{p}$ or if $F = \mathbb{Q}$ and $N = \mathfrak{N}$ is prime.

Consider the Petersson norm on $S_2(\mathfrak{N})$ normalized so that

$$(f, f) = 2^{1-2d} h_F \Delta_F^2 N(\mathfrak{N}) L^*(1, \pi, Ad),$$

which corresponds to the definition of (f, f) in [Hid91] (cf. [GG12, Thm 5.16]). Here π is the automorphic representation associated to f. Then (3.2) becomes

$$(3.4) \qquad \frac{|P_\chi(\phi)|^2}{(\phi, \phi)} = \frac{\zeta_F(2) \Delta_F^{5/2} h_F N(\mathfrak{N})}{(2\pi^2)^{2d} \sqrt{|\Delta_K|}} \prod_{v|\mathfrak{N}} (1 - q_v^{-1}) \frac{L(1, f, \chi)}{(f, f)}.$$

We normalize ϕ so that the gcd of all values of ϕ is minimal. This specifies ϕ only up to a root of unity in $\mathbb{Q}(\phi)$, but specifies both $|P_\chi(\phi)|^2$ and (ϕ, ϕ) uniquely. Now it need not be that $\phi \equiv 1 \bmod \mathfrak{p}$, but we will have $\phi \equiv c_\phi \bmod \mathfrak{p}$ for some integer c_ϕ of $\mathbb{Q}(\phi)$ which is nonzero mod $\mathfrak{p}$. With this normalization, put

$$(3.5) \qquad L^{\text{alg}}(1, f, \chi) = |P_\chi^0(\phi)|^2 = \frac{h_F^2 w_K^2 Q_{K/F}^2 \sqrt{|\Delta_K|}}{4(2\pi)^{2d}} [\phi, \phi] \frac{L(1, f, \chi)}{(f, f)},$$

which is an algebraic integer since $P_\chi^0(\phi)$ is.

Note that our algebraic special value $L^{\text{alg}}(1, f, \chi)$ is normalized differently from what is typically found in the literature. For instance, when $F = \mathbb{Q}$, Shimura [Shi76] essentially considered the values

$$A(1, f, \chi) = -\frac{g(\eta_K)}{(2\pi)^2} \frac{L(1, f, \chi)}{\langle f, f \rangle},$$

where $g(\cdot)$ denotes the Gauss sum and $\langle f, f \rangle = \frac{1}{2}(f, f)$ is the usual Petersson norm. In our case, with $F = \mathbb{Q}$, we see

$$(3.6) \qquad L^{\text{alg}}(1, f, \chi) = -\frac{w_K^2}{8} \frac{\sqrt{|\Delta_K|}}{g(\eta_K)} [\phi, \phi] A(1, f, \chi),$$

which, for fixed f, just depends on K in a simple way (and not at all on χ). Note the factor $[\phi, \phi] = \sum w_i^{-1} |\phi(\mathcal{I}_i)|^2$ is also algebraic.

Theorem 3.2. *Suppose $\mathfrak{N} = \mathfrak{N}_1$ is squarefree and $f \in S_2(\mathfrak{N})$ is a newform corresponding to a quaternionic form $\phi \in \mathcal{S}$ such that $\phi \equiv 1 \bmod \mathfrak{p}^r$. Then there exists an algebraic integer $c_\phi \in \mathbb{Q}(\phi)$ which is nonzero mod $\mathfrak{p}$ such that, for any quadratic extension K/F which is inert at each prime dividing $\mathfrak{N}$ and any ideal class character χ of K,*

$$L^{\mathrm{alg}}(1, f, \chi) \equiv \delta_{\chi,1} |c_\phi|^2 h_K^2 \bmod \mathfrak{p}^r,$$

where $\delta_{\chi,1}$ is 1 if $\chi = 1$ and 0 otherwise. In particular, if $\mathfrak{p}^r \nmid h_K^2$, then $L(1, f_K) \neq 0$.

We do not require $\mathfrak{p} | \mathrm{num}$ as in Theorem 2.1. One could also allow K to be ramified at primes dividing $\mathfrak{N}_1$, but then one needs to exclude the L-factors at such primes (cf. [MW09]).

Moreover, the number c_ϕ can be read off directly from ϕ (and can be chosen independent of $\mathfrak{p}$), which can be determined from Brandt matrices. The calculations of c_ϕ and $[\phi, \phi]$ (and thus the normalization of $L^{\mathrm{alg}}(1, f, \chi)$) may be simpler than the calculation of canonical periods arising in, e.g., [HV14].

Proof. With ϕ normalized as above, we get $P_\chi^0(\phi) \equiv c_\phi P_\chi^0(\phi_0) \bmod \mathfrak{p}^r$, and $P_\chi^0(\phi_0)$ is either h_K or 0 according to whether χ is trivial or not. Now apply (3.5). $\qquad\square$

Finally, we briefly illustrate how one can use L-values to recover information about the map $\mathrm{Cl}(\mathfrak{o}_K) \to \mathrm{Cl}(\mathcal{O})$, and use this to give an example where $L(1/2, f_K)^{\mathrm{alg}} \not\equiv 0 \bmod p$ and $L(1/2, f, \chi) \neq 0$ but $L(1/2, f, \chi) \equiv 0 \bmod p$ for a nontrivial ideal class character χ of K.

Example 3.3. *Suppose $F = \mathbb{Q}$, $N = 11$ and $p = 5$. Then the quaternion algebra B ramified at 11 and ∞ has class number 2. Write $\mathrm{Cl}(\mathcal{O}) = \{x_1, x_2\}$. Then, up to reordering, $w_1 = 3$ and $w_2 = 2$. Here $\dim \mathcal{S} = 1$, and we can define $\phi \in \mathcal{S}$ by $\phi(x_1) = 3$ and $\phi(x_2) = -2$. This is normalized as we specified above, and $[\phi, \phi] = 5$ and we can take $c_\phi = 3$.*

Let $K = \mathbb{Q}(\sqrt{-23})$, which has class number 3, and $\mathcal{O}$ be a maximal order of B containing $\mathfrak{o}_K$. Our theorem says that $L^{\mathrm{alg}}(1, f_K) \equiv -h_K^2 \equiv 1 \bmod 5$, where $f \in S_2(11)$ is the unique normalized cusp form. Indeed, one can compute $L(1/2, f_K)^{\mathrm{alg}} = 1$. On the other hand, the normalized period for $\chi = 1$ is $P_1^0(\phi) = \sum_{t \in \mathrm{Cl}(\mathfrak{o}_K)} \phi(x(t)) = 5a - 6$, where $0 \leq a \leq 3$ is the number of classes of $\mathfrak{o}_K$ mapping to x_1. Since $|P_1^0(\phi)|^2 = 1$, we deduce $a = 1$. Hence $x(t_1) = x_1$ and $x(t_2) = x(t_3) = x_2$ for some ordering t_1, t_2, t_3 of the ideal

classes of $\mathfrak{o}_K$, i.e., the map $\mathrm{Cl}(\mathfrak{o}_K) \to \mathrm{Cl}(\mathcal{O})$ is surjective with fibers of size 1 and 2 over x_1 and x_2, respectively.

Now let χ be one of the nontrivial idele class characters. Then the $\zeta_i := \chi(t_i)$'s are the distinct 3rd roots of unity in some order, and $P_\chi^0(\phi) = 3\zeta_1 - 2\zeta_2 - 2\zeta_3 = 5\zeta_1$. Hence we see $L^{\mathrm{alg}}(1, f, \chi)$ is nonzero, but 0 mod 5.

We remark that with f as in the previous example, one can use the values of ϕ to conclude $L^{\mathrm{alg}}(1, f_K) \in \{(5a - 2h_K)^2 : 0 \le a \le h_K\}$ for any imaginary quadratic K unramified at 11.

Remark 3.4. *One can similarly use this method to prove that congruences of quaternionic cusp forms yield congruences of L-values. That is, if ϕ_1 and ϕ_2 are integral quaternionic eigenforms such that $\phi_1 \equiv \phi_2 \bmod \mathfrak{p}^r$, then with a suitable definition of algebraic L-values, one will have $L^{\mathrm{alg}}(1, f_1, \chi) \equiv L^{\mathrm{alg}}(1, f_2, \chi) \bmod \mathfrak{p}^r$. See [DK17] for an approach using half-integral weight forms to this situation when $F = \mathbb{Q}$.*

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