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Howe finiteness conjecture for covering groups

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In this paper, we prove Howe finiteness conjecture in the context of finite central extensions of connected, reductive p-adic groups.

1. Introduction

Let $\mathbb{G}$ be a connected reductive group over a p-adic field F of characteristic 0. Let $G = \mathbb{G}(F)$, and $\tilde{G}$ be a finite central extension of G by μ_n, i.e.

$$1 \to \mu_n \to \tilde{G} \xrightarrow{p} G \to 1.$$

We say a representation $\tilde{\pi}$ of $\tilde{G}$ is *genuine (resp. anti-genuine)* if

$$\tilde{\pi}(\epsilon\tilde{x}) = \epsilon\tilde{\pi}(\tilde{x}) \qquad (resp.\ \epsilon^{-1}\tilde{\pi}(\tilde{x}))$$

for all $\epsilon \in \mu_n,\ \tilde{x} \in \tilde{G}$, similarly for test functions. Throughout this paper, the preimage of a subset H of G in $\tilde{G}$ will be denoted by $\tilde{H}$, i.e. $\tilde{H} = p^{-1}(H)$.

Let $\mathcal{W}$ be a compact subset of G, $\mathcal{W}^G$ the G-conjugate invariant subset $\{gxg^{-1} :\ g \in G,\ x \in \mathcal{W}\}$. We say a closed invariant subset $\tilde{\Omega}$ of $\tilde{G}$ is *compact modulo conjugation* if $p(\tilde{\Omega}) \subset \mathcal{W}^G$ for some compact $\mathcal{W}$.

The purpose of this article is to extend Howe finiteness conjecture to finite central covering groups. Let $\mathscr{S}(\tilde{\Omega})$ be the set of invariant distributions on $\tilde{G}$ supported on $\tilde{\Omega}$. If K is a compact-open subgroup of G which splits in $\tilde{G}$, let $\mathscr{H}_{K,--}$ be the K-bi-invariant Hecke algebra of compactly supported anti-genuine functions on $\tilde{G}$.

Theorem (Howe finiteness conjecture). *Assume K is a compact-open subgroup of G which splits in $\tilde{G}$, and $\tilde{\Omega}$ is invariant, compact modulo conjugation. Then the space of distributions in $\mathscr{S}(\tilde{\Omega})$ restricting to $\mathscr{H}_{K,--}$ is finite-dimensional.*

Key words and phrases: Covering groups, Howe finiteness conjecture, p-adic groups.

As a corollary of Harish-Chandra's and Kazhdan's density theorems (see [Li12, Theorem 5.8.10]), we have an equivalent version of Howe finiteness conjecture concerning orbital integrals (see [Clo85, §5]) that is what we will prove in this article:

Proposition. *Assume* $T \subset G$ *is a Cartan subgroup,* ω_T *a compact subset of* T*. For* $\tilde{t} \in \tilde{T}_{reg}^{Bon} := \{\tilde{t} \in \tilde{T}_{reg} : Ad(\tilde{t})|_{\tilde{T}} = id\}$*, let* $O(\cdot, \tilde{t})$ *be the linear form on* $\mathscr{H}_{K,--}$ *defined by the orbital integral at* $\tilde{t}$*. Then these orbital integrals* $O(\cdot, \tilde{t})$*,* $\tilde{t} \in \tilde{\omega}_T \cap \tilde{T}_{reg}^{Bon}$*, span a finite-dimensional space in the dual of* $\mathscr{H}_{K,--}$*.*

Note that Howe finiteness conjecture for connected reductive groups has been proved first by L.Clozel, later by D.Barbasch and A.Moy using Bruhat-Tits theory. As the reduction step in [BM00] seems not so easy to be generalized, we would follow L.Clozel's argument (see [Clo89]). Notice that the main ingredients in [Clo89] are as follows:

- L.Clozel's "compact trace" formula.
- Finiteness of elliptic representations with K fixed vectors.

Thus the aim of this article is to generalize the above ingredients to covering groups.

2. An integration formula *à* la Clozel

Recall that $\mathbb{G}$ is a connected reductive group over a p-adic field F of characteristic 0, $G = \mathbb{G}(F)$, and $\tilde{G}$ is a finite central extension of G by μ_n. Let A_0 be a maximal split torus in G, P_0 a minimal parabolic subgroup containing A_0, Φ the set of roots of G relative to A_0, Δ the set of simple roots associated to P_0, we define

$$A_0^+ := \{x \in A_0 : |\alpha(x)| \leq 1,\ \forall \alpha \in \Delta\}.$$

Recall Deligne's construction in [Del76, Cas77]. Assume $g \in G$ is regular semi-simple, let $T = Z_G(g)^0$. Note that T is isogenous to $T_s \times T_a$ with T_s its maximal split subtorus and T_a its maximal anisotropic subtorus. Namely, there is a positive integer n such that for each $t \in T$ we have $t^n = sa$ with $s \in T_s$ and $a \in T_a$. Applying the decomposition to g, one gets an associated $s_g \in T_s$. On the other hand, there exists $y \in G$ such that $ys_gy^{-1} \in A_0^+$. Let $\Omega = \{\alpha \in \Delta : |\alpha(ys_gy^{-1})| = 1\}$, and define $P_g = M_gN_g$ to be the parabolic subgroup $y^{-1}P_\Omega y$, here $P_\Omega \supset P_0$ is the unique parabolic subgroup associated to $\Omega \subset \Delta$. We may then say g is M_g-compact. If $M_g = G$, we say g is compact for short, and denote by G_c the set of compact regular semi-simple elements, $\tilde{G}_c = p^{-1}(G_c)$.

If $f \in C_c^\infty(\tilde{G})$ and P is a parabolic subgroup of G, let $\bar{f}^{(P)}$ denote the constant term of f along P:

$$\bar{f}^{(P)}(\tilde{m}) = \delta_P(m)^{1/2} \int_N \bar{f}(\tilde{m}n)dn,$$

where

$$\bar{f}(\tilde{g}) = \int_{\tilde{K}_0} f(\tilde{k}\tilde{g}\tilde{k}^{-1})d\tilde{k}, \quad \delta_p(m) = |det(Ad(m)|_{\mathfrak{n}})|,$$

where K_0 is a good maximal compact subgroup of G in the sense of Bruhat-Tits, and N lifts to $\tilde{G}$ in a unique way in the sense of [Li14, Proposition 2.2.1].

If $P = MN$ is a parabolic subgroup of G, let A_M be the split component of the center of M, $\Delta(N, A_M)$ the set of reduced simple roots of N with respect to A_M, $X(M) = Hom(M,\ \mathbb{G}_m)$, and $\mathfrak{a}_M = Hom(X(M),\ \mathbb{R})$. Let

$$H_{\tilde{M}}:\ \tilde{M} \xrightarrow{p} M \xrightarrow{H_M} \mathfrak{a}_M$$

be the composed Harish-Chandra map such that

$$e^{\langle \chi,\ H_{\tilde{M}}(\tilde{m})\rangle} = |\chi(m)|$$

for all $\tilde{m} \in \tilde{M},\ \chi \in X(M)$. We will denote by $\tilde{M}^+(N)$ (*resp.* ${}^+\tilde{M}(N)$) the set of $\tilde{m} \in \tilde{M}$ such that $\alpha(H_{\tilde{M}}(\tilde{m})) < 0$ (*resp.* $\omega_\alpha(H_{\tilde{M}}(\tilde{m})) < 0$) for all $\alpha \in \Delta(N, A_M)$, here $\{\omega_\alpha\}_\alpha$ stands for the dual basis of $\mathfrak{a}_M^* := Hom(\mathfrak{a}_M, \mathbb{R})$ with respect to the coroot basis $\{\alpha^\vee\}_\alpha$ of $\mathfrak{a}_M$ (cf. [Art78, P.916]). Set χ_N to be the characteristic function of $\tilde{M}^+(N)$, and $\hat{\chi}_N$ the characteristic function of ${}^+\tilde{M}(N)$ as in [Clo90, P.259].

If $\tilde{\pi}$ is an admissible representation of finite length of $\tilde{G}$, we denote by $\tilde{\pi}_N$ the normalized Jacquet module of $\tilde{\pi}$.

Fix a minimal parabolic subgroup P_0 of G, and the Levi decomposition $P_0 = M_0N_0$, let $\mathscr{P}$ be the set of standard parabolic subgroups of G with respect to $P_0 = M_0N_0$. If $P = MN \in \mathscr{P}$, $\tilde{\pi}$ is a genuine admissible representation of finite length of $\tilde{M}$, and $f \in C_{c,--}^\infty(\tilde{M})$, we set

$$\langle trace\ \tilde{\pi},\ f\rangle_c = \int_{\tilde{M}_c} \chi_{\tilde{\pi}}(\tilde{m})f(\tilde{m})d\tilde{m}.$$

Here $\chi_{\tilde{\pi}}(\tilde{m})$ denotes the Harish-Chandra character of $\tilde{\pi}$ [Li12]; we choose Haar measure $d\tilde{g}$ on $\tilde{G}$ compatible with dg on G, i.e. $mes(\tilde{K_0}) = mes(K_0) = 1$. If we want to specify $\tilde{M}$ in this formula, we will write $\langle trace\ \tilde{\pi},\ f\rangle_{\tilde{M},c}$.

We will write G_{reg} for the set of regular semisimple elements of G. If X is any subset of G, we write X_{reg} for $X \cap G_{reg}$.

Proposition 1 (Clozel's integration formula). *Assume $\tilde{\pi}$ is a genuine admissible representation of finite length of $\tilde{G}$. Then, if $f \in C^{\infty}_{c,--}(\tilde{G})$:*

$$trace\ \tilde{\pi}(f) = \sum_{\substack{P \in \mathscr{P} \\ P=MN}} \left\langle trace\ \tilde{\pi}_N,\ \chi_N \bar{f}^{(P)} \right\rangle_{\tilde{M},c}.$$

Proof. We give the proof in Appendix A.2 which is an easy modification of Clozel's argument in [Clo90]. Note that one may also follow Clozel's argument in [Clo89, Proposition 1] once we have the following three ingredients which in its own right is important and will be used later on:

- Weyl's integration formula for $\tilde{G}$:

  $$\langle trace\ \tilde{\pi},\ f\rangle = \sum_{\substack{T \subset G \\ T\ mod\ G}} \int_{\substack{\tilde{t} \in \tilde{T}^{Bon}_{reg} \\ t\ mod\ G}} \theta_{\tilde{\pi}}(\tilde{t}) O(f,\tilde{t}) d\tilde{t}$$

  where $\theta_{\tilde{\pi}}(\tilde{t}) = \Delta_G(t)\chi_{\tilde{\pi}}(\tilde{t}), \quad \Delta_G(t) = |det(1 - Ad(t)|_{Lie\ G/Lie\ G_t})|^{1/2}$, and

  $$O(f,\tilde{t}) = \Delta_G(t) \int_{\tilde{G}_t \backslash \tilde{G}} f(\tilde{g}^{-1}\tilde{t}\tilde{g}) d\tilde{g}$$

  provided $\tilde{t} \in \tilde{T}^{Bon}_{reg}$, i.e. the character $\tilde{T} \xrightarrow{[\tilde{t},\cdot]} \mathbb{C}^{\times}$ is trivial.

  Note that this follows from the decomposition:

  $$\tilde{G}_{reg} = \bigcup_{\tilde{T}_{reg}} \bigcup_{\tilde{t} \in \tilde{T}_{reg}} \bigcup_{g \in T \backslash G} \tilde{g}^{-1}\tilde{t}\tilde{g}.$$

- Descent formula for orbital integral (see [Clo85, Lemma 1], [Li14, Proposition 3.2.2]):

  $$\text{For } \tilde{t} \in \tilde{T}_{reg} \subset \tilde{M}, \quad O^{\tilde{G}}(f,\tilde{t}) = O^{\tilde{M}}(\bar{f}^{(P)},\tilde{t}).$$

  Note that this follows from Langlands decomposition $G = MNK_0$, and the relation

  $$|det(1 - Ad(t)|_{LieN})| = \delta_N^{1/2}(t) \frac{\Delta_G(t)}{\Delta_M(t)}.$$

- Casselman's character formula [Cas77]:

$$\text{For } \tilde{t} \in \tilde{G}_{reg} \text{ with } P_t = P = MN, \quad \theta_{\tilde{\pi}}(\tilde{t}) = \theta_{\tilde{\pi}_N}(\tilde{t}).$$

The proof is an easy modification of Casselman's argument in [Cas77]. For the convenience of the reader, we would give the proof in Appendix A.1. □

Corollary 2 (Clozel, Waldspurger). *[Clo90, P.259]*

$$\langle trace\ \tilde{\pi}, \phi\rangle_c = \sum_{P\in\mathscr{P}} (-1)^{a_P - a_G} \left\langle trace\ \tilde{\pi}_N, \hat{\chi}_N \bar{\phi}^{(P)} \right\rangle_{\tilde{M}}.$$

Proof. For the convenience of readers, we should write down the short proof. We first recall Clozel's integration formula for covering groups:

$$trace\ \tilde{\pi}(\phi) = \sum_{\substack{P\in\mathscr{P} \\ P=MN}} \left\langle trace\ \tilde{\pi}_N,\ \chi_N \bar{\phi}^{(P)} \right\rangle_{\tilde{M}_c}.$$

By descent formula, we also have:

$$\left\langle trace\ \tilde{\pi}_N,\ \hat{\chi}_N \bar{\phi}^{(P)} \right\rangle_{\tilde{M}} = \sum_{\substack{P'\subset P \\ P'=M'N'}} \left\langle trace\ \tilde{\pi}_{N'},\ \hat{\chi}_N \chi_{N'}^{\tilde{M}} \bar{\phi}^{(P')} \right\rangle_{\tilde{M}'_c},$$

where $\chi_{N'}^{\tilde{M}}$ is the analogue of χ_N for the subgroup $N' \cap M$ of $\tilde{M}$. Let $\epsilon_P = (-1)^{a_P - a_G}$, the right-hand side of the identity in the corollary can be rewritten as:

$$\sum_{P'} \sum_{\substack{P: \\ P'\subset P\subset G}} \epsilon_P \left\langle trace\ \tilde{\pi}_{N'},\ \hat{\chi}_N \chi_{N'}^{\tilde{M}} \bar{\phi}^{(P')} \right\rangle_{\tilde{M}_c}.$$

Notice that the inner summation term $\sum_P \epsilon_P \hat{\chi}_N \chi_{N'}^{\tilde{M}}$ equals to 0 if $P' \neq G$, 1 otherwise, by a basic Arthur's formula [Art78, Lemma 6.1], whence the identity in the corollary holds. □

3. *K*-finiteness of elliptic representations

Recall Harish-Chandra's character theory in [Li12, Theorem 4.3.2], if $\tilde{\pi}$ is an irreducible tempered representation of $\tilde{G}$, $\chi_{\tilde{\pi}}$ would be a locally integrable function on $\tilde{G}$, smooth on $\tilde{G}_{reg}$. Let G_{rel} be the set of regular elliptic elements of G, i.e. the set of regular elements contained in some elliptic torus. We say $\tilde{\pi}$ is elliptic if $p(supp(\chi_{\tilde{\pi}})) \cap G_{rel} \neq \emptyset$. Let $\Pi_{2,-}(\tilde{G})$ and $\Pi_{ell,-}(\tilde{G})$ be the set of genuine irreducible discrete series and tempered elliptic representations of $\tilde{G}$ respectively.

Recall Harish-Chandra's map $H_G:\ G \to \mathfrak{a}_G$, let ${}^0G$ be the kernel of H_G, and Λ_G its image.

Lemma 3 (K-Finiteness). *Assume K is a compact-open subgroup of G which splits in $\tilde{G}$. Then there is a finite set of representation of finite length of ${}^0\tilde{G}$ such that, if $\tilde{\pi} \in \Pi_{ell,-}(\tilde{G})$ has a vector fixed by K, $\tilde{\pi}|_{{}^0\tilde{G}}$ belongs to that set.*

Proof. We follow the two steps argument as in [Clo89]:

- For $\tilde{\pi} \in \Pi_{2,-}(\tilde{G})_\chi$ which have the central character χ, the lemma holds by Harish-Chandra's Plancherel theorem which is stated as follows [Wal03, Li12]:
  Let $\mathcal{C}(\tilde{G})_\chi$ be the space of Schwartz-Harish-Chandra functions on $\tilde{G}$ with central character χ, ${}^0\mathcal{C}(\tilde{G})_\chi$(resp. ${}^0\mathcal{C}(\tilde{G}//K)_\chi$) the subspace of (resp. K-bi-invariant) cusp forms i.e. $f^{(P)} = 0$ for every proper parabolic subgroup P of G. Then $\overline{{}^0\mathcal{C}(\tilde{G})_\chi} = L^2_{disc}(\tilde{G})_\chi$ i.e. the discrete part of $L^2(\tilde{G})_\chi$. Moreover, dim ${}^0\mathcal{C}(\tilde{G}//K)_\chi < \infty$.
- By Knapp-Zuckerman classification theorem of tempered representations (see [Wal03, Proposition III.4.1]), it suffices to prove the following Lemma 4. □

Let $P = MN$ be a parabolic subgroup of G, we have A_M the split component of the center of M, and $X = Hom(A_M,\ \mathbb{G}_m)$. Fix a uniformizer ϖ of $p-adic$ field F, let $C_M = Hom_{\mathbb{Z}}(X,\ \varpi^{\mathbb{Z}})$, $C_{\tilde{M}} = \tilde{C}_M \cap Z(\tilde{M})$. Consider C_M as a subgroup of M, we then know that ${}^0MC_M$ $(resp.\ {}^0\tilde{M}C_{\tilde{M}})$ is a subgroup of finite index in M $(resp.\ \tilde{M})$.

Lemma 4. *There exists a finite set $X_{\tilde{M}}$ of unitary characters of $C_{\tilde{M}}$ such that, for any genuine discrete series $\tilde{\delta}$ of $\tilde{M}$, if $\tilde{\pi} = ind_{\tilde{M}N}^{\tilde{G}}(\tilde{\delta} \otimes 1)$ contains an elliptic submodule, then the central character $\omega_{\tilde{\delta}}$ belongs to $X_{\tilde{M}}$.*

Proof. The argument is the same as in [Clo85, Lemma 4] once the following property of elliptic representations holds for $\tilde{G}$:

Keypoint. *Let $W_{\tilde{\delta}}$ be the stabilizer of $\tilde{\delta}$ in $W(G, A_M)$. If $W_{\tilde{\delta}} \subset W(M_1, A_M)$ for a parabolic subgroup $P_1 = M_1N_1 \supset P$, then all irreducible sub-*

representations of $\tilde{\pi}$ *are induced from an irreducible sub-representations of* $\tilde{\pi}_1 = ind_{\tilde{M}(N\cap M_1)}^{\tilde{M}_1}(\tilde{\delta}\otimes 1)$.

This is equivalent to saying

$$dim\ Hom_{\tilde{G}}(\tilde{\pi},\ \tilde{\pi}) = dim\ Hom_{\tilde{M}_1}(\tilde{\pi}_1,\ \tilde{\pi}_1).$$

As Frobenius reciprocity says that

$$dim\ Hom_{\tilde{G}}(\tilde{\pi},\ \tilde{\pi}) = dim\ Hom_{\tilde{M}}(\tilde{\pi}_N,\ \tilde{\delta}),$$
$$dim\ Hom_{\tilde{M}_1}(\tilde{\pi}_1,\ \tilde{\pi}_1) = dim\ Hom_{\tilde{M}}((\tilde{\pi}_1)_{N\cap M_1},\ \tilde{\delta}).$$

Note that the unitary central character part of any Jacquet module is tempered (see [Wal03, Lemme III.3.2]), and discrete series is projective in the category of tempered representations (see [Wal03, Corollaire III.7.2]), then it suffices to show that the $\tilde{\delta}-isotypic$ part $\tilde{\pi}_{N,\tilde{\delta}}$ of $\tilde{\pi}_N$ is isomorphic to $(\tilde{\pi}_1)_{N\cap M_1,\tilde{\delta}}$. This follows from the transitivity and exactness properties of Jacquet functor, i.e.

$$\tilde{\pi}_N \simeq (\tilde{\pi}_{N_1})_{N\cap M_1} \twoheadrightarrow (\tilde{\pi}_1)_{N\cap M_1},$$

and the geometric lemma:

$$(\tilde{\pi}_{N,\tilde{\delta}})_{s.s} = \bigoplus_{w\in W_{\tilde{\delta}}^G} w.\tilde{\delta}, \quad ((\tilde{\pi}_1)_{N\cap M_1,\tilde{\delta}})_{s.s} = \bigoplus_{w\in W_{\tilde{\delta}}^{M_1}} w.\tilde{\delta}.$$

For the convenience of readers, we continue the proof of the lemma as follows which is almost a copy of Clozel's proof in [Clo85, Lemma 4].

Suppose that $\tilde{\pi} = ind_{\tilde{M}N}^{\tilde{G}}(\tilde{\delta}\otimes 1)$ contains an elliptic submodule, then there is no $P_1 \neq G$ satisfying the condition $W_{\tilde{\delta}} \subset W(M_1, A_M)$ by the above property of elliptic representations. Therefore for all $M_1 \subsetneq G$ which contains M, there exists $w \in W_{\tilde{\delta}}$ such that $w \notin W(M_1, A_M)$, We want to show that those central characters $\chi = \chi_{\tilde{\delta}}$ belong to a finite set determined by $W_{\tilde{\delta}}$.

Recall that W acts on C_M and $C_{\tilde{M}}$. As $w\tilde{\delta} = \tilde{\delta}$ for $w \in W_{\tilde{\delta}}$, we have $w\chi = \chi$. The map $H : M \to \mathfrak{a}_M$ realizes an isomorphism of C_M with a lattice $L_M \subset \mathfrak{a}_M$. So the group of characters of C_M identifies with $\mathfrak{a}^*_{M,\mathbb{C}}/L^*_M$, where

$$L^*_M = \{\lambda \in \mathfrak{a}^*_{M,\mathbb{C}} : \lambda(L_M) \subset 2\pi i\mathbb{Z}\}.$$

Assume that the cardinality of the set

$$\text{(A)} \qquad \{\chi \in Hom_{genuine}(C_{\tilde{M}}, \mathbb{C}^\times) : w\chi = \chi,\ \forall w \in W_{\tilde{\delta}}\}$$

is infinite. We fix one of such characters χ_0, then for χ in (A), one knows that $\chi\chi_0^{-1}$ is a character on $p(C_{\tilde{M}})$ which contains C_M^n, thus the cardinality of the set

$$\{\chi \in Hom(C_M, \mathbb{C}^\times) : w\chi = \chi,\ \forall w \in W_{\tilde{\delta}}\}$$

is also infinite. Therefore there are infinitely many fixed points of the action of $W_{\tilde{\delta}}$ on the complex torus

$$Hom(C_M, \mathbb{C}^\times) \simeq \mathfrak{a}^*_{M,\mathbb{C}}/L^*_M.$$

On the other hand, such characters on which the action of $W_{\tilde{\delta}}$ is fixed form a subgroup of the complex torus. Consider the action on the Lie algebra, we see that there exists a non-zero vector $v \in \mathfrak{a}^*_{M,\mathbb{C}}$ fixed by $W_{\tilde{\delta}}$.

As X is the character lattice of A_M, we have $\mathfrak{a}^*_{M,\mathbb{C}} = X \otimes_{\mathbb{Z}} \mathbb{C}$ and W acts on X. So $W_{\tilde{\delta}}$ fixes a non-trivial sub-lattice of X, this gives rise to a non-trivial sub-torus $\mathbf{H}$ of A_M which is fixed by $W_{\tilde{\delta}}$.

Recall that a standard torus of G is a split torus of the form $A_{M'}$ for a Levi subgroup M' of G. Let $H = \mathbf{H}(F)$, and A_1 the smallest standard torus of G containing H. It is clear that $A_1 \subset A_M$. By a lemma of Casselman [Cas77, Lemma 1.1], the centralizer of H in G is equal to that of A_1.

For $w \in W_{\tilde{\delta}}$, one knows that w centralizes H, then w centralizes A_1. Let M_1 be the centralizer of A_1 in G. Note that M_1 is a Levi subgroup of G containing M. If $\omega \in G$ is a representative of w, one would have $\omega \in M_1$. Thus $W_{\tilde{\delta}}$ is contained in $W_1 = W(M_1, A_M)$.

By the above **Keypoint**, this contradicts the hypothesis that $\tilde{\pi} = ind(\tilde{\delta} \otimes 1)$ contains an elliptic submodule. Therefore if $\tilde{\pi}$ contains an elliptic submodule, the set of points fixed by $W_{\tilde{\delta}}$ in $Hom_{genuine}(C_{\tilde{M}}, \mathbb{C}^\times)$ is finite, and the central character $\chi_{\tilde{\delta}}$ belongs to this finite set. According to the finiteness of the possible $W_{\tilde{\delta}}$, we see that the set of such $\chi_{\tilde{\delta}}$ is finite. □

4. Proof of the covering Howe finiteness conjecture

We now prove the covering Howe finiteness conjecture following the methods in [Clo89].

We fix a compact-open subgroup $K \subset G$ which splits in $\tilde{G}$. Let $\mathscr{H}_{K,--}$ be the K-bi-invariant Hecke algebra of compactly supported anti-genuine functions on $\tilde{G}$.

Lemma 5. *There exists a finite set $X({}^0\tilde{G}, K)$ of unitary representations of finite length of ${}^0\tilde{G}$ such that, if $\tilde{\pi} \in \Pi_{ell,-}(\tilde{G})$ such that the linear form*

$f \mapsto \langle trace\ \tilde{\pi}, f\rangle_c$ does not vanish identically on $\mathscr{H}_{K,--}$, then $\tilde{\pi}|_{{}^0\tilde{G}}$ belongs to $X({}^0\tilde{G}, K)$.

Proof. By Corollary 2, it is easy to see that

$$\langle trace\ \tilde{\pi}, f\rangle_c \neq 0$$

implies

$$\left\langle trace\ \tilde{\pi}_N, \hat{\chi}_N \bar{f}^{(P)} \right\rangle_{\tilde{M}} \neq 0$$

for some M. Note that we have the following good properties for $\bar{f}^{(P)}$ and $\tilde{\pi}_N$.

- For $f \in \mathscr{H}_{K,--}$, $\bar{f}^{(P)} \in \mathscr{H}_{K_M,--}$ for some splitting compact-open subgroup K_M of M determined by K [Clo85, Lemma 2].
- If $\tilde{\pi}_N$ has K_M fixed vector, then $\tilde{\pi}$ has K'-invariant vector for some compact-open splitting $K' \subset \tilde{G}$. This follows from the well-known fact:
  Assume K_M is compact-open in M splitting in $\tilde{M}$, and $\tilde{\sigma}$, a genuine irreducible representation of $\tilde{M}$, has non-zero K_M-fixed vectors. Then there exists a compact-open subgroup K' of G which splits in $\tilde{G}$ such that all subquotients of $\tilde{\pi} = ind_{\tilde{M}N}^{\tilde{G}}(\tilde{\sigma} \otimes 1)$ have non-zero K'-fixed vectors. This is a corollary of Jacquet's lemma [Cas95, BJ13].

Thus the lemma follows from Lemma 3. □

Recall the Harish-Chandra map $H_G : G \to \Lambda_G$, let ω be a finite subset of Λ_G. We will denote by $G_c(\omega)$ the inverse image of ω in G_c, and define

$$\langle trace\ \tilde{\pi},\ f\rangle_{c,\omega} = \int_{\tilde{G}_c(\omega)} \chi_{\tilde{\pi}}(\tilde{g}) f(\tilde{g}).$$

Lemma 6. *Assume $\omega \subset \Lambda_G$ is finite. Then the functionals*

$$f \mapsto \langle trace\ \tilde{\pi},\ f\rangle_{c,\omega}$$

span a finite-dimensional subspace of the dual of $\mathscr{H}_{K,--}$ as $\tilde{\pi}$ ranges over $\Pi_{ell,-}(\tilde{G})$.

Proof. The lemma follows by the same argument as in [Clo89, Lemma 6] once Lemma 5 holds. For the convenience of the reader, we should write down the details. We may assume $\omega = \{x\}$. Note that the characteristic function 1_x of $\tilde{G}(x)$ is bi-invariant by any compact-open K. By Lemma 5,

$$\langle trace\ \tilde{\pi},\ f\rangle_{c,x} = \langle trace\ \tilde{\pi},\ 1_x f\rangle_c$$

vanishes unless $\tilde{\pi}|_{{}^0\tilde{G}}$ belongs to a finite set. Moreover, if χ is an unramified character of $\tilde{G}$ which is trivial on μ_n:

$$\langle trace\ (\tilde{\pi}\otimes\chi),\ f\rangle_{c,x} = \chi(x)\,\langle trace\ \tilde{\pi},\ f\rangle_{c,x}\,.$$

Since the genuine elliptic representations in Lemma 5 form a finite set modulo twisting by unramified characters, the lemma is clear. □

Proposition. *Assume $T \subset G$ is a Cartan subgroup, ω_T a compact subset of T. For $\tilde{t} \in \tilde{T}_{reg}^{Bon}$, let $O(\cdot, \tilde{t})$ be the linear form on $\mathscr{H}_{K,--}$ defined by the orbital integral at $\tilde{t}$. Then the orbital integrals $O(\cdot,\tilde{t})$ ($\tilde{t} \in \tilde{\omega}_T \cap \tilde{T}_{reg}^{Bon}$) span a finite-dimensional space in the dual of $\mathscr{H}_{K,--}$.*

Proof. The Proposition follows by the same argument as in [Clo89, Proposition 2]. For the convenience of the reader, we recall the details here.

Assume an invariant, compact modulo conjugation set $\Omega \supset \tilde{\omega}_T \cap \tilde{T}_{reg}^{Bon}$, enlarging Ω if necessary, we may assume that it is open and closed, and contains $G_{c,\omega}$, where ω is the image of Ω under the map H_G.

- Reduction steps:
  - (i). By induction and Descent formula of orbital integrals: for non-elliptic $\tilde{t} \in \tilde{\Omega} \cap \tilde{G}_{reg}^{Bon}$, the linear forms $f \mapsto O(f,\tilde{t})$ span a finite subspace of the dual of $\mathscr{H}_{K,--}$.
  - (ii). By Lemma 6, there is a subspace of finite codimension $\mathscr{H}' \subset \mathscr{H}_{K,--}$ such that, the linear forms in (i) vanish and

    $$\langle trace\ \tilde{\pi},\ f\rangle_{c,\omega} = 0$$

    for all $f \in \mathscr{H}'$, and $\tilde{\pi} \in \Pi_{ell,-}(\tilde{G})$.
- Vanishing argument:
  In what follows, we would like to prove that all orbital integrals of $f \in \mathscr{H}'$ vanish on $\tilde{\Omega} \cap \tilde{G}_{reg}^{Bon}$. Let $g = \chi_{\tilde{\Omega}} f$, where $\chi_{\tilde{\Omega}}$ is the characteristic function of $\tilde{\Omega}$. Thus it deduces to show the vanishing of all regular semi-simple orbital integrals of g. This results from Kazhdan's density theorem [Li12, Theorem 5.8.10] as follows.

(iii). For genuine irreducible, tempered, non-elliptic representation $\tilde{\pi}$ of $\tilde{G}$, the Weyl integration formula and (i) show that

$$\langle trace\ \tilde{\pi},\ g\rangle = 0$$

(iv). For elliptic $\tilde{\pi}$,

$$\langle trace\ \tilde{\pi},\ g\rangle = \left\langle trace\ \tilde{\pi},\ \chi_{\tilde{\Omega}} f\right\rangle \overset{(ii)}{=} \langle trace\ \tilde{\pi},\ f\rangle_{c,\omega} \overset{(ii)}{=} 0. \qquad \square$$

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Appendix A.1: Casselman's character formula

We generalize the formula to covering groups following the argument in [Cas77]. For $\tilde{t} \in \tilde{G}_{reg}$, let $P_{\tilde{t}} = \tilde{P}_t,\ M_{\tilde{t}} = \tilde{M}_t$.

Lemma (Deligne). *There exits a decreasing sequence $\{K_i\}$ of splitting compact open subgroups in $\tilde{G}$ which form a basis of neighborhoods of the identity and such that, where $N_i = N_{\tilde{t}} \cap K_i,\ M_i = M_{\tilde{t}} \cap K_i,$ and $\bar{N}_i = \bar{N}_{\tilde{t}} \cap K_i$ with $\bar{N}_{\tilde{t}}$ the unique opposite nilpotent subgroup of $N_{\tilde{t}}$:*

(a) $K_i = \bar{N}_i M_i N_i$;

(b) $\tilde{t} N_i \tilde{t}^{-1} \subset N_i,\ \tilde{t} M_i \tilde{t}^{-1} = M_i,\ \tilde{t}^{-1} \bar{N}_i \tilde{t} \subset \bar{N}_i$;

(c) If U_1 and U_2 are any two compact open subgroups of N, then there exists $n \geq 0$ such that $\tilde{t}^n U_1 \tilde{t}^{-n} \subset U_2$, and similarly for $\bar{N}$ and $\tilde{t}^{-1}$;

(d) In the anti-genuine K_i-bi-invariant Hecke algebra $\mathscr{H}_{K_i,--}$ of $\tilde{G}$, for $n \geq 0$:

$$(\mu_{K_i \tilde{t} K_i})^n = \mu_{K_i \tilde{t}^n K_i},$$

where $\mu_{K_i \tilde{t} K_i}$ stands for the anti-genuine characteristic function of $K_i \tilde{t} K_i$ which is well-defined resulting from the Lemma (a), (b) and the splitting of $N_{\tilde{t}}$.

Proof. Note that $\tilde{t}$ is $M_{\tilde{t}}$-compact. This follows from [BJ13, Lemma 2.7, Proposition 2.11]. $\square$

For any compact subgroup $H \subset \tilde{G}$, let $\mathscr{P}_H$ be the operator

$$mes(H)^{-1}\int_H \tilde{\pi}(h)dh.$$

Recall that $(\tilde{\pi}, V)$ is a genuine admissible representation of finite length of $\tilde{G}$.

Lemma (Jacquet's lemma). *(a) The natural map from V^{K_i} to $V_{N_{\tilde{t}}}^{M_i}$ is surjective;*

For each K_i, there exists a subspace $V_{\tilde{t}}^{K_i} \subset V^{K_i}$ such that

(b) The projection from $V_{\tilde{t}}^{K_i}$ to $V_{N_{\tilde{t}}}^{M_i}$ is a linear isomorphism;

(c) For each $n \geq 0$, $V_{\tilde{t}}^{K_i}$ is stable with respect to $\tilde{\pi}(\mu_{K_i\tilde{t}^nK_i})$;

(d) There exists n such that $\tilde{\pi}(\mu_{K_i\tilde{t}^nK_i})V^{K_i} \subset V_{\tilde{t}}^{K_i}$.

Proof. Recall that for any compact subgroup $U \subset N_{\tilde{t}}$, the space $V(U)$ consists of all $v \in V$ such that

$$\int_U \tilde{\pi}(u)vdu = 0$$

and $V(N_{\tilde{t}})$ is the union of all the $V(U)$. Choose a fixed compact open subgroup $U \subset N_{\tilde{t}}$ such that $V(N_{\tilde{t}}) \cap V^{K_i} \subset V(U)$ and $N_i \subset U$.

Lemma. *If $\tilde{t}^nU\tilde{t}^{-n} \subset N_i$ and $v \in V(N_{\tilde{t}}) \cap V^{K_i}$, then $\tilde{\pi}(\mu_{K_i\tilde{t}^nK_i})v = 0$.*

Proof. This follows from the fact that $\mathscr{P}_{N_i}(\tilde{\pi}(\tilde{t}^n)v) = 0$. □

Choose n to be large enough so that $\tilde{t}^nU\tilde{t}^{-n} \subset N_i$, and define $V_{\tilde{t}}^{K_i}$ to be $\tilde{\pi}(\mu_{K_i\tilde{t}^nK_i})V^{K_i}$.

Proof of (a). This follows from an easy modification of the proof in [Cas95, Theorem 3.3.4].

Proof of (b). Surjectivity part: For $u \in V_{N_{\tilde{t}}}^{M_i}$, we have $\tilde{\pi}_{N_{\tilde{t}}}(\tilde{t}^{-n})u \in V_{N_{\tilde{t}}}^{M_i}$ as $\tilde{t}$ normalizes M_i. By Lemma (a), there exists $v \in V^{K_i}$ such that its image in $V_{N_{\tilde{t}}}$ is $\tilde{\pi}_{N_{\tilde{t}}}(\tilde{t}^{-n})u$, whence $\mathscr{P}_{K_i}(\tilde{\pi}(\tilde{t}^n)v)$ has image u.

Injectivity part: It suffices to show the following statement

let $v \in V_{\tilde{t}}^{K_i}$, if the projection of v in $V_{N_{\tilde{t}}}^{M_i}$ is 0, then $v = 0$.

Note that by definition $v = \tilde{\pi}(\mu_{K_i\tilde{t}^nK_i})v_0$, for some $v_0 \in V^{K_i}$. As the projection of v in $V_{N_{\tilde{t}}}$ is 0, so $v \in V(N_{\tilde{t}})$, then by the choice of U, $v \in$

$V(U)$. Notice also that v, up to a constant, is equal to $\mathscr{P}_{K_i}(\tilde{\pi}(\tilde{t}^n)v_0) = \mathscr{P}_{N_i}(\tilde{\pi}(\tilde{t}^n)v_0)$. Thus

$$0 = \int_U \tilde{\pi}(u)vdu \overset{N_i \subseteq U}{=} \tilde{\pi}(\tilde{t}^n) \int_{\tilde{t}^{-n}U\tilde{t}^n} \tilde{\pi}(u)v_0 du,$$

so $v_0 \in V(N_{\tilde{t}})$ as $\tilde{t}^{-n}U\tilde{t}^n \subset N_{\tilde{t}}$, then $v = 0$ by the above Lemma.

Proof of (c). By Lemma (b), all the spaces $\tilde{\pi}(\mu_{K_i\tilde{t}^nK_i})V^{K_i}$ have the same dimension for large n. Thus part (c) follows from Deligne (d).

Proof of (d). Statement (d) follows immediate from the definition. □

Theorem (Cassselman's character formula).

$$\textit{For } \tilde{t} \in \tilde{G}_{reg} \textit{ with } P_t = P = MN,\ \theta_{\tilde{\pi}}(\tilde{t}) = \theta_{\tilde{\pi}_N}(\tilde{t}).$$

Proof. Recall that $\mu_{K_i\tilde{t}K_i}$ is the anti-genuine characteristic function of $K_i\tilde{t}K_i$. It suffices to prove that

$$Tr(mes(K_i\tilde{t}K_i)^{-1}\pi(\mu_{K_i\tilde{t}K_i})) = Tr(mes(M_i)^{-1}\delta_P^{1/2}(t)\pi_N(\tilde{t}M_i)).$$

According to the following fact:

If X and Y are two endomorphisms of a finite-dimensional space such that $Tr(X^n) = Tr(Y^n)$ for $n \gg 0$, then $Tr(X^n) = Tr(Y^n)$ for all $n \geq 1$.

It reduces to prove that

$$Tr(mes(K_i\tilde{t}^nK_i)^{-1}\pi(\mu_{K_i\tilde{t}^nK_i})) = Tr(mes(M_i)^{-1}\delta_P^{1/2}(t^n)\pi_N(\tilde{t}^nM_i))$$

for $n \gg 0$. But this follows from Jacquet's lemma which is stated as above. □

Appendix A.2: Clozel's integration formula

One may modify the argument in [Clo89]. But here we adapt Rogawski's argument in [Clo90]. For $P = MN \subset G$, we denote by $\mathfrak{n}$ the Lie algebra of N, and $\bar{\mathfrak{n}}$ the Lie algebra of the opposite $\bar{N}$. Let

$$M_c^+ = \{m \in M_c \cap G_{reg} : \chi_N(m) = 1\} = \{m \in M^+ \cap G_{reg} : P_m = P\},$$
$$G(M_c^+) = \{gmg^{-1} : g \in G, m \in M_c^+\}.$$

Proposition (Rogawski formula). *If f is an integrable funcition on G,*

$$\int_{G(M_c^+)} f(g)dg = \int_{KM_c^+N} f(knmn^{-1}k^{-1})\frac{\Delta_G(m)^2}{\Delta_M(m)^2}dkdmdn.$$

Proof. This has been proved in [Clo90, Proposition 2.2]. □

Proposition (Clozel's integration formula). *Assume $\tilde{\pi}$ is a genuine admissible representation of finite length of $\tilde{G}$. Then, if $f \in C^{\infty}_{c,--}(\tilde{G})$:*

$$trace\ \tilde{\pi}(f) = \sum_{\substack{P\in\mathscr{P}\\P=MN}} \left\langle trace\ \tilde{\pi}_N,\ \chi_N \bar{f}^{(P)} \right\rangle_{\tilde{M},c}$$

Proof. Note that $\tilde{G}_{reg} = \bigsqcup_{P\in\mathscr{P}} \tilde{G}(M_c^+)$. It is then enough to show:

$$(\star) \qquad \int_{\tilde{G}(M_c^+)} f(\tilde{g}) trace\ \tilde{\pi}(\tilde{g}) d\tilde{g} = \left\langle trace\ \tilde{\pi}_N,\ \chi_N \bar{f}^{(P)} \right\rangle_{\tilde{M},c}.$$

The right-hand side of $(\star)$ is equal to

$$\int_{M_c^+} \int_{KN} f(k^{-1}mnk)\delta_P(m)^{1/2} trace\ \tilde{\pi}_N(m) dkdndm.$$

By Casselman's character formula,

$$\delta_P^{1/2}(m) trace\ \tilde{\pi}_N(\tilde{m}) = trace\ \tilde{\pi}(\tilde{m}).$$

It reduces to show that the Jacobian $D_{\bar{\mathfrak{n}}}(m)$ of the map $n \mapsto \tilde{m}^{-1}n^{-1}\tilde{m}n$ is equal to

$$\frac{\Delta_G(m)^2}{\Delta_M(m)^2}.$$

Note that $D_{\bar{\mathfrak{n}}}(m) = |det(1 - Ad(m)|_{\bar{\mathfrak{n}}})|$, and

$$\frac{\Delta_G(m)^2}{\Delta_M(m)^2} = D_{\mathfrak{n}}(m) D_{\bar{\mathfrak{n}}}(m) = D_{\bar{\mathfrak{n}}}(m)$$

as $m \in M_c^+$ whence $|m^\alpha| < 1$ for α a root in N, we see that this equals

$$\int_{M_c^+} \int_{KN} f(k^{-1}n^{-1}mnk) \frac{\Delta_G(m)^2}{\Delta_M(m)^2} trace\ \tilde{\pi}(m) dkdndm$$

therefore it equals the left-hand side of $(\star)$ by the above Rogawski formula. □

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