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Epimorphic subgroups of algebraic groups

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In this note, we show that the epimorphic subgroups of an algebraic group are exactly the pull-backs of the epimorphic subgroups of its affinization. We also obtain epimorphicity criteria for subgroups of affine algebraic groups, which generalize a result of Bien and Borel. Moreover, we extend the affinization theorem for algebraic groups to homogeneous spaces.

1. Introduction and statement of the results

The algebraic groups considered in this note are the group schemes of finite type over a field k. They form the objects of a category, with morphisms being the homomorphisms of k-group schemes. One of the most basic questions one may ask about this category is to describe *monomorphisms* and *epimorphisms*. Recall that a morphism $f : G \to H$ is a monomorphism if it satisfies the left cancellation property: for any algebraic group G' and for any morphisms $f_1, f_2 : G' \to G$ such that $f \circ f_1 = f \circ f_2$, we have $f_1 = f_2$. Likewise, f is an epimorphism if it satisfies the right cancellation property.

The answer to this question is very easy and well-known for monomorphisms: these are exactly the homomorphisms with trivial (scheme-theoretic) kernel, or equivalently the closed immersions of algebraic groups. Also, $f : G \to H$ is an epimorphism if and only if so is the inclusion of its scheme-theoretic image. This reduces the description of epimorphisms to that of the *epimorphic subgroups* of an algebraic group G, i.e., of those algebraic subgroups H such that every morphism $G \to G'$ is uniquely determined by its pull-back to H. The purpose of this note is to characterize such subgroups.

Examples of epimorphic subgroups include the parabolic subgroups of a smooth connected affine algebraic group G, i.e., the algebraic subgroups $H \subset G$ such that the homogeneous space G/H is proper, or equivalently projective. Indeed, for any morphisms $f_1, f_2 : G \to G'$ which coincide on H, the map $G \to G'$, $x \mapsto f_1(x)\, f_2(x^{-1})$ factors through a map $\varphi : G/H \to G'$. But every such morphism is constant: to see this, we may assume k algebraically closed; then G/H is covered by rational curves (images of morphisms $\mathbb{P}^1 \to G/H$), while every morphism $\mathbb{P}^1 \to G'$ is constant.

In the category of smooth connected affine algebraic groups over an algebraically closed field, the epimorphic subgroups have been studied by Bien and Borel in [2, 3]; see also [10, §23] for a more detailed exposition, and [4, §4] for further developments. In particular, [2, Thm. 1] presents several epimorphicity criteria in that setting. Our first result extends most of these criteria to affine algebraic groups over an arbitrary field. To state it, let us define *affine epimorphic subgroups* of an affine algebraic group G as those algebraic subgroups $H \subset G$ that are epimorphic in the category of affine algebraic groups. (Clearly, epimorphic implies affine epimorphic. In fact, the converse holds, as we will show in Corollary 3).

Theorem 1. *The following conditions are equivalent for an algebraic subgroup H of an affine algebraic group G:*

(i) *H is affine epimorphic in G.*

(ii) $\mathcal{O}(G/H) = k$.

(iii) *For any finite-dimensional G-module V, we have the equality of fixed point subschemes $V^H = V^G$.*

(iv) *For any finite-dimensional G-module V, if $V = V_1 \oplus V_2$, where V_1, V_2 are H-submodules, then V_1, V_2 are G-submodules.*

This result is proved in Section 2 by adapting the argument of [2, Thm. 1] (see also [14, Thm. 13]). When G is smooth and connected, condition (ii) is equivalent to the k-vector space $\mathcal{O}(G/H)$ being finite-dimensional. But this fails for non-connected groups (just take H to be the trivial subgroup of a non-trivial finite group G) and for non-smooth groups as well (take G, H as above with G infinitesimal).

For the category of finite-dimensional Lie algebras over a field of characteristic 0, the equivalence of conditions (i), (iii) and (iv) has been obtained by Bergman in an unpublished manuscript which was the starting point of [2]; see [1, Cor. 3.2], and [15] for recent developments based on the (related but not identical) notion of wide subalgebra of a semi-simple Lie algebra.

Our second result yields an epimorphicity criterion in the category of all algebraic groups. To formulate it, recall the affinization theorem (see [7, §III.3.8]): every algebraic group G has a smallest normal algebraic subgroup N such that the quotient G/N is affine. Moreover, N is smooth, connected, and contained in the center of the neutral component G^0. Also, N is anti-affine (i.e., $\mathcal{O}(N) = k$) and N is the largest algebraic subgroup of G satisfying this property; we denote N by G_{ant}. The quotient morphism $G \to G/G_{\mathrm{ant}}$

is the affinization morphism, i.e., the canonical map

$$\varphi_G : G \longrightarrow \operatorname{Spec} \mathcal{O}(G).$$

We may now state:

Theorem 2. *The following conditions are equivalent for an algebraic subgroup H of an algebraic group G:*

(i) *H is epimorphic in G.*

(ii) *$H \supset G_{\mathrm{ant}}$ and $\mathcal{O}(G/H) = k$.*

This result is proved in Section 4, after gathering auxiliary results in Section 3.

Note that the formations of $\mathcal{O}(G/H)$ and G_{ant} commute with base change by field extensions of k. In view of Theorem 2, this yields the first assertion of the following:

Corollary 3. *Let G be an algebraic group, and H an algebraic subgroup.*

(i) *H is epimorphic in G if and only if the base change $H_{k'}$ is epimorphic in $G_{k'}$ for some field extension k' of k.*

(ii) *When G is affine, H is epimorphic in G if and only if it is affine epimorphic.*

The second assertion follows readily from Theorems 1 and 2. As a consequence, *the epimorphic subgroups of an algebraic group G are exactly the pull-backs of the epimorphic subgroups of its affinization.*

Theorem 2 and Corollary 3 reduce the description of the epimorphic subgroups of an algebraic group G over k, to the case where G is affine and k is algebraically closed. When G is smooth, our next result yields further reductions:

Theorem 4. *The following conditions are equivalent for an algebraic subgroup H of a smooth algebraic group G over an algebraically closed field k:*

(i) *H is epimorphic in G.*

(ii) *The reduced subgroup H_{red} is epimorphic in G.*

(iii) *The reduced neutral component H^0_{red} is epimorphic in G^0 and the natural map $H_{\mathrm{red}}/H^0_{\mathrm{red}} \to G/G^0$ is surjective.*

This result is proved in Section 5.

In view of the above results, the class of homogeneous spaces $X = G/H$ such that $\mathcal{O}(X) = k$ deserves further consideration. These anti-affine homogeneous spaces feature in an extension of the affinization theorem for algebraic groups (see [7, III.3.8]), which is our final result. To state it, recall that a quasi-compact scheme Z is said to be quasi-affine if the affinization map $\varphi_Z : Z \to \operatorname{Spec} \mathcal{O}(Z)$ is an open immersion (see [9, II.5.1.2] for further characterizations).

Theorem 5. *Let G be an algebraic group, and H an algebraic subgroup.*

(i) *There exists a smallest algebraic subgroup L of G containing H such that G/L is quasi-affine. Moreover, $\mathcal{O}(G/L) = \mathcal{O}(G/H)$ and the affinization map*

$$\varphi_{G/H} : G/H \longrightarrow \operatorname{Spec} \mathcal{O}(G/H) =: X$$

is the composition

$$G/H \xrightarrow{u} G/L \xrightarrow{\varphi_{G/L}} \operatorname{Spec} \mathcal{O}(G/L) = X,$$

where u denotes the canonical morphism.

(ii) *The formation of L commutes with base change by arbitrary field extensions.*

(iii) *L is the largest subgroup of G containing H such that L/H is anti-affine.*

(iv) *L/H is geometrically irreducible.*

(v) *If G is affine, then L is the largest subgroup of G containing H as an epimorphic subgroup.*

(vi) *If G and H are smooth, then so is L.*

This result is proved in Section 6. In the setting of smooth affine algebraic groups over an algebraically closed field, it gives back a statement of Bien and Borel (see [2, Prop. 1]), proved by Grosshans in [10, §2, §23]; our proof, based on a descent argument, is somewhat more direct.

Also, Theorem 5 gives back most of the affinization theorem for an arbitrary algebraic group G. More specifically, taking for H the trivial subgroup and using the fact that every quasi-affine algebraic group is affine (see e.g. [8, VIB.11.11]), we obtain that G has a smallest algebraic subgroup L such that

G/L is affine, and L is the largest anti-affine subgroup of G; moreover, L is connected. But the smoothness property of anti-affine algebraic groups does not extend to homogeneous spaces, as shown by Example 11 at the end of Section 6.

Returning to the description of all the epimorphic subgroups H of a smooth algebraic group G over a field k, we may assume (by Theorem 2, Corollary 3 and Theorem 4) G to be affine and connected, H smooth and connected, and k algebraically closed; this is precisely the setting of [2, 3]. Even so, the structure of epimorphic subgroups is only partially understood; a geometric criterion of epimorphicity is obtained in [16] when G is semi-simple and k has characteristic 0.

The classification of epimorphic subgroups of non-smooth algebraic groups presents further open problems; see Example 11 again for a construction of such subgroups $H \subset G$, for which the quotient G/H is non-smooth as well. Examples with a smooth quotient may be obtained as follows: over any algebraically closed field k of prime characteristic, there exist rational homogeneous projective varieties X such that the automorphism group scheme Aut_X is non-smooth (see [5, Prop. 4.3.4]). Let then G denote the neutral component of Aut_X, and H the stabilizer of a k-rational point $x \in X$. Then G is affine, non-smooth, and $X \cong G/H$; as a consequence, H is non-smooth as well. Also, H is epimorphic in G, by Theorem 2 or a direct argument as for parabolic subgroups. Thus, the description of epimorphic subgroups of possibly non-smooth algebraic groups entails that of automorphism group schemes of rational homogeneous projective varieties, which seems to be completely unexplored.

Notation and conventions. We use the books [7] and [8] as general references, and the expository text [6] for some further results.

Throughout this note, we consider schemes over a fixed field k. By an *algebraic group*, we mean a group scheme G of finite type over k; we denote by $e = e_G \in G(k)$ the neutral element, and by G^0 the neutral component of G. The group law of G will be denoted multiplicatively: $(x, y) \mapsto xy$.

By a *subgroup* of G, we mean a k-subgroup scheme H; then H is closed in G. *Morphisms* of algebraic groups are understood to be homomorphisms of k-group schemes.

Given a subgroup $H \subset G$ and a normal subgroup $N \triangleleft G$, we denote by $N \rtimes H$ the corresponding semi-direct product, and by $N \cdot H$ the scheme-theoretic image of the morphism

$$N \rtimes H \longrightarrow G, \quad (x, y) \longmapsto xy.$$

Then $N \cdot H$ is a subgroup of G, and the natural map $H/N \cap H \to G/N$ is a closed immersion with image $N \cdot H/N$ (see e.g. [8, VIIA.5.3.3]).

2. Proof of Theorem 1

(i) $\Rightarrow$ (ii): The action of G on itself by right multiplication yields a G-module structure on the algebra $\mathcal{O}(G)$ (see [7, Ex. II.2.1.2]). Moreover, for any subgroup $K \subset G$ acting on $\mathcal{O}(G)$ by right multiplication, the natural map $\mathcal{O}(G/K) \to \mathcal{O}(G)^K$ is an isomorphism, as follows e.g. from [8, Cor. VIA.3.3.3]. In particular, $\mathcal{O}(G/H) \cong \mathcal{O}(G)^H$ and $\mathcal{O}(G)^G = k$. Thus, it suffices to show that every $f \in \mathcal{O}(G)^H$ is fixed by G.

Consider the action of G on itself by left multiplication; this yields another G-module structure on $\mathcal{O}(G)$, and $\mathcal{O}(G)^H$ is a G-submodule. By [7, II.2.3.1], we may choose a finite-dimensional G-submodule $V \subset \mathcal{O}(G)^H$ that contains f. View V as a vector group (the spectrum of the symmetric algebra of the dual vector space) equipped with a compatible G-action, and consider the semi-direct product $G' := V \rtimes G$. Then G' is an affine algebraic group; moreover, the maps

$$f_1 : G \longrightarrow G', \quad g \longmapsto (0, g), \qquad f_2 : G \longrightarrow G', \quad g \longmapsto (g \cdot f - f, g)$$

are two morphisms which coincide on H. Thus, $f_1 = f_2$, that is, f is fixed by G.

(ii) $\Rightarrow$ (iii): Recall from [11, I.3.2] that V^H is the subscheme of V associated with a linear subspace. So it suffices to show that every $v \in V^H(k)$ is fixed by G. Let $f \in \mathcal{O}(V)$. Then the assignment $g \mapsto f(g \cdot v)$ defines $f_v \in \mathcal{O}(G)^H \cong \mathcal{O}(G/H) = k$. Thus, we have $f_v(g) = f_v(e)$, that is, $f(g \cdot v) = f(v)$ identically on G. Since this holds for all $f \in \mathcal{O}(V)$, it follows that $g \cdot v = v$ identically on G.

(iii) $\Rightarrow$ (iv): Let $\pi : V \to V_1$ denote the projection with kernel V_2. Consider the action of G on $\mathrm{End}(V)$ by conjugation; then $\mathrm{End}(V)$ is a finite-dimensional G-module, and $\pi \in \mathrm{End}(V)^H$. As a consequence, $\pi \in \mathrm{End}(V)^G$. It follows that V_1 (the image of π) is normalized by G. Likewise, V_2 is normalized by G.

(iv) $\Rightarrow$ (i): Let G' be an affine algebraic group, and

$$f_1, f_2 : G \longrightarrow G'$$

two morphisms which coincide on H. We may view G' as a subgroup of $\mathrm{GL}(V)$ for some finite-dimensional vector space V (see [7, II.2.3.3]). This yields two linear representations

$$\rho_1, \rho_2 : G \longrightarrow \mathrm{GL}(V)$$

which coincide on H. Consider the morphism

$$\rho_1 \oplus \rho_2 : G \longrightarrow \mathrm{GL}(V \oplus V).$$

Then we have with an obvious notation:

$$V \oplus V = (V \oplus 0) \oplus \mathrm{diag}(V),$$

where $V \oplus 0$ is normalized by G, and $\mathrm{diag}(V)$ is normalized by H (as $\rho_1|_H = \rho_2|_H$). So $\mathrm{diag}(V)$ is normalized by G, that is, $\rho_1 = \rho_2$. Thus, $f_1 = f_2$.

3. Some auxiliary results

Throughout this section, G denotes an algebraic group, and $H \subset G$ a subgroup. We begin with a series of easy observations.

Lemma 6. *Assume that H is epimorphic in G.*

(i) *If $K \subset G$ is a subgroup containing H, then K is epimorphic in G.*

(ii) *If $N \lhd G$ is a normal subgroup, then $H/N \cap H$ is epimorphic in G/N.*

Proof. The assertion (i) is obvious, and implies that $N \cdot H$ is epimorphic in G. As a direct consequence, $N \cdot H/N$ is epimorphic in G/N; this yields the assertion (ii). □

Lemma 7. *Let H be an epimorphic subgroup of G.*

(i) *If G is finite, then $G = H$.*

(ii) *For an arbitrary G, we have $G = G^0 \cdot H$.*

Proof. (i) Since G is affine and $H \subset G$ is affine epimorphic, we have $\mathcal{O}(G/H) = k$ by Theorem 1. As the scheme G/H is finite and contains a k-rational point x, it follows that this scheme consists of the point x, hence $H = G$.

(ii) By Lemma 6 (ii), $G^0 \cdot H/G^0$ is epimorphic in G/G^0. Thus, we may replace G with G/G^0, and hence assume that G is finite and étale. Then $H = G$ by (i). □

Remark 8. (i) For finite étale groups, Lemma 7 (i) also follows by adapting the proof of the surjectivity of epimorphisms of abstract groups, given in [13].

(ii) Lemmas 6 and 7 also hold in the category of affine algebraic groups, with the same proofs.

Lemma 9. *Let $N \lhd G$ be a normal subgroup.*

If $H \supset G_{\mathrm{ant}}$, then $H/N \cap H \supset (G/N)_{\mathrm{ant}}$.

Conversely, if $H/N \cap H \supset (G/N)_{\mathrm{ant}}$ and N is affine, then $H \supset G_{\mathrm{ant}}$.

Proof. By [6, Lem. 3.3.6], the natural map $G_{\mathrm{ant}}/N \cap G_{\mathrm{ant}} \to (G/N)_{\mathrm{ant}}$ is an isomorphism. This yields the first assertion.

Conversely, assume that $H/N \cap H \supset (G/N)_{\mathrm{ant}}$; equivalently, we have $(H/N \cap H)_{\mathrm{ant}} = (G/N)_{\mathrm{ant}}$. Using [6, Lem. 3.3.6] again, it follows that $G_{\mathrm{ant}} \subset N \cdot H_{\mathrm{ant}}$. Thus, it suffices to show that $(N \cdot H)_{\mathrm{ant}} = H_{\mathrm{ant}}$. Using once more [6, Lem. 3.3.6], it suffices in turn to check that $(N \rtimes H)_{\mathrm{ant}} = H_{\mathrm{ant}}$. Since N is affine and $N \rtimes H \cong N \times H$ as schemes, the affinization morphism

$$\varphi_{N \rtimes H} : N \rtimes H \longrightarrow \operatorname{Spec} \mathcal{O}(N \rtimes H)$$

is identified with

$$\mathrm{id} \times \varphi_H : N \times H \longrightarrow N \times \operatorname{Spec} \mathcal{O}(H).$$

Taking fibers at e yields the desired equality. □

Next, we obtain a result of independent interest, which generalizes (and builds on) Lemma 7 (i):

Lemma 10. *If G is proper and H is epimorphic in G, then $H = G$.*

Proof. The largest anti-affine subgroup G_{ant} is smooth, connected and proper, that is, an abelian variety. Moreover, the quotient group G/G_{ant} is proper and affine, hence finite. Thus, using Lemma 6 (ii) and Lemma 7 (i), it suffices to show that H contains G_{ant}.

We now reduce to the case where *G and H are smooth*. For this, we may assume that k has prime characteristic p. Denote by G_n (resp. H_n) the kernel of the nth relative Frobenius morphism of G (resp. H). Then G_n and H_n are infinitesimal; also, G/G_n and H/H_n are smooth for $n \gg 0$ (see [8, VIIA.8.3]). Using Lemma 6 (ii) again together with Lemma 9, we see that it suffices to show that $H/H_n = H/G_n \cap H$ contains $(G/G_n)_{\mathrm{ant}}$. This yields the desired reduction.

Under this smoothness assumption, $G^0 = G_{\mathrm{ant}}$ is an abelian variety. Also, we have $G = G^0 \cdot H$ by Lemma 7 (ii). Thus, $G^0 \cap H$ is centralized by G^0 and normalized by H, and hence is a normal subgroup of G. Using Lemma 6 (ii) again, we may replace G, resp. H with $G/G^0 \cap H$, resp. $H/G^0 \cap H$, and hence assume in addition that $G^0 \cap H$ *is trivial.*

Under these assumptions, we may identify G with $G^0 \rtimes H$. Consider the diagonal action of H on $G^0 \times G^0$ and form the semi-direct product $G' := (G^0 \times G^0) \rtimes H$. Then the maps

$$f_1 : G \longrightarrow G', \quad (x, y) \longmapsto (x, e, y),$$
$$f_2 : G \longrightarrow G', \quad (x, y) \longmapsto (x, x, y),$$

are two morphisms which coincide on H. Thus, $f_1 = f_2$. But then G^0 must be trivial. □

4. Proof of Theorem 2

(i) ⇒ (ii): By [6, Thm. 2], G has a smallest normal subgroup N such that G/N is proper; moreover, N is affine. If H is epimorphic in G, then the quotient group $H/H \cap N$ is epimorphic in G/N by Lemma 6 (ii). Using Lemma 10, it follows that $H/H \cap N = G/N$. So $H \supset G_{\mathrm{ant}}$ by Lemma 9. Thus, $\bar{H} := H/G_{\mathrm{ant}}$ is epimorphic in $\bar{G} := G/G_{\mathrm{ant}}$ by Lemma 6 (ii) again. In view of Theorem 1, this yields $\mathcal{O}(\bar{G}/\bar{H}) = k$. As

$$\mathcal{O}(\bar{G}/\bar{H}) \cong \mathcal{O}(\bar{G})^{\bar{H}} = \mathcal{O}(G/G_{\mathrm{ant}})^H \cong \mathcal{O}(G/H),$$

we obtain $\mathcal{O}(G/H) = k$.

(ii) ⇒ (i): Let again $\bar{G} := G/G_{\mathrm{ant}}$ and $\bar{H} := H/G_{\mathrm{ant}}$. Then $\mathcal{O}(\bar{G}/\bar{H}) = k$ by the above argument. Using Theorem 1, it follows that $\bar{H}$ is affine epimorphic in $\bar{G}$. Together with Lemma 7 (ii) and Remark 8 (ii), this yields $\bar{G} = \bar{G}^0 \cdot \bar{H}$, and hence $\mathcal{O}(\bar{G}/\bar{H}) \cong \mathcal{O}(\bar{G}^0/\bar{G}^0 \cap \bar{H})$. By Theorem 1 again, it follows that $\bar{G}^0 \cap \bar{H}$ is affine epimorphic in $\bar{G}^0$. Also, note that $G = G^0 \cdot H$, since G_{ant} is connected and contained in H.

Let $f_1, f_2 : G \to G'$ be morphisms of algebraic groups that coincide on H. Then f_1, f_2 pull back to morphisms $f_1^0, f_2^0 : G^0 \to G'^0$ which coincide on $G_{\mathrm{ant}} \lhd G^0 \cap H$. Moreover, the common scheme-theoretic image of G_{ant} under f_1^0, f_2^0 is contained in $G'_{\mathrm{ant}} \lhd G'^0$. This yields morphisms of affine algebraic groups

$$\bar{f}_1^0, \bar{f}_2^0 : \bar{G}^0 \to G'^0/G'_{\mathrm{ant}}$$

which coincide on $\bar{G}^0 \cap \bar{H}$. Thus, $\bar{f}_1^0 = \bar{f}_2^0$, that is, the morphism of schemes

$$\varphi : G^0 \longrightarrow G'^0, \quad x \longmapsto f_1(x) f_2(x)^{-1}$$

factors through G'_{ant}. We have

$$\varphi(x\,y) = f_1(x)\, f_1(y)\, f_2(y)^{-1}\, f_2(x)^{-1}$$

identically on $G^0 \times G^0$. Since G'_{ant} is contained in the center of G'^0, it follows that φ is a morphism of algebraic groups.

As f_1 and f_2 coincide on $G_{\mathrm{ant}} \subset H$, the kernel of φ contains G_{ant}. Thus, φ factors through a morphism of algebraic groups $\psi : \bar{G}^0 \to G'_{\mathrm{ant}}$. Since $\bar{G}^0$ is affine, so is the scheme-theoretic image of ψ. Also, ψ is trivial on $\bar{G}^0 \cap \bar{H}$, an affine epimorphic subgroup of $\bar{G}^0$. Thus, ψ is trivial, that is, f_1 and f_2 coincide on G^0. Since these morphisms also coincide on H, and $G = G^0 \cdot H$, we conclude that $f_1 = f_2$.

5. Proof of Theorem 4

(i) ⇒ (ii): Recall from Theorem 2 that $G_{\mathrm{ant}} \subset H$ and $\mathcal{O}(G/H) = k$. Since G_{ant} is smooth, it is contained in H_{red}. Thus, using Theorem 2 again, it suffices to show that $\mathcal{O}(G/H_{\mathrm{red}}) = k$.

The natural map $u : G/H_{\mathrm{red}} \to G/H$ lies in a commutative square

$$\begin{array}{ccc} G \times H/H_{\mathrm{red}} & \xrightarrow{p_1} & G \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle q} \\ G/H_{\mathrm{red}} & \xrightarrow{u} & G/H, \end{array}$$

where p_1 denotes the projection, q the quotient map, and m the pull-back of the action map $G \times G/H_{\mathrm{red}} \to G/H_{\mathrm{red}}$. In fact, this square is cartesian and consists of faithfully flat morphisms (see e.g. the proof of [6, Prop. 2.8.4]). As the scheme H/H_{red} is finite and has a unique k-rational point, the map p_1 is finite and purely inseparable; thus, so is u by faithfully flat descent. Also, G/H_{red} and G/H are smooth, since so is G. Thus, the induced map on rings of rational functions

$$u^{\#} : k(G/H) \longrightarrow k(G/H_{\mathrm{red}})$$

is injective, and there exists a positive integer n (a power of the characteristic exponent of k) such that

$$k(G/H_{\mathrm{red}})^n \subset u^{\#}k(G/H).$$

Also, by normality of G/H, we have $u^{\#}\mathcal{O}(G/H) = u^{\#}k(G/H) \cap \mathcal{O}(G/H_{\mathrm{red}})$ and hence

$$\mathcal{O}(G/H_{\mathrm{red}})^n \subset u^{\#}\mathcal{O}(G/H).$$

Since $\mathcal{O}(G/H) = k$ and $\mathcal{O}(G/H_{\mathrm{red}})$ has no non-zero nilpotents, this yields the desired assertion.

(ii) $\Rightarrow$ (iii): We may replace H with H_{red}, and hence assume that H is smooth. By Lemma 7 (ii), we have $G = G^0 \cdot H$; thus, the natural map $H/H^0 \to G/G^0$ is surjective. Also, G_{ant} is connected, and contained in H by Theorem 2; hence $G_{\mathrm{ant}} \subset H^0$. So, using Theorem 2 once more, we are reduced to checking that $\mathcal{O}(G^0/H^0) = k$.

Note that

$$k = \mathcal{O}(G/H) = \mathcal{O}(G^0 \cdot H/H) \cong \mathcal{O}(G^0/G^0 \cap H).$$

Next, consider the natural map

$$\psi : G^0/H^0 \longrightarrow G^0/G^0 \cap H.$$

The finite étale group $F := (G^0 \cap H)/H^0 \subset H/H^0$ acts on G^0/H^0 by right multiplication, and ψ is the categorical quotient for that action. Thus, $\mathcal{O}(G^0/H^0)^F \cong \mathcal{O}(G^0/G^0 \cap H)$, and hence the algebra $\mathcal{O}(G^0/H^0)$ is integral over $\mathcal{O}(G^0/G^0 \cap H) = k$. As above, this implies the desired assertion.

(iii) $\Rightarrow$ (i): This follows by reverting some of the previous arguments. More specifically, we have

$$G_{\mathrm{ant}} = (G^0)_{\mathrm{ant}} \subset H^0_{\mathrm{red}} \subset H.$$

Also, $G = G^0 \cdot H_{\mathrm{red}} = G^0 \cdot H$ and hence

$$\mathcal{O}(G/H) \cong \mathcal{O}(G^0/G^0 \cap H) \cong \mathcal{O}(G^0)^{G^0 \cap H} \subset \mathcal{O}(G^0)^{H^0} = k.$$

Thus, H is epimorphic in G by Theorem 2 again.

6. Proof of Theorem 5

(i) Consider the action of G on $\mathcal{O}(G)$ via right multiplication and let $L \subset G$ be the centralizer of the subspace $\mathcal{O}(G)^H \subset \mathcal{O}(G)$. In view of [7, II.1.3.6],

L is represented by a subgroup of G that we will also denote by L. Since L acts trivially on $\mathcal{O}(G)^H$, we have $\mathcal{O}(G)^H \subset \mathcal{O}(G)^L$. On the other hand, $H \subset L$ and hence $\mathcal{O}(G)^L \subset \mathcal{O}(G)^H$. Thus, $\mathcal{O}(G)^L = \mathcal{O}(G)^H$.

We show that there exists a finite subset $F \subset \mathcal{O}(G)^H$ such that L is the centralizer $C_G(F)$. Indeed, we may find F such that $C_G(F)$ is minimal among all such centralizers. Then $C_G(F \cup \{f\}) = C_G(F)$ for any $f \in \mathcal{O}(G)^H$, and hence $C_G(F)$ centralizes the whole subspace $\mathcal{O}(G)^H$.

Choose $F = \{f_1, \ldots, f_n\} \subset \mathcal{O}(G)^H$ such that $L = C_G(F)$. Then L is the centralizer in G of $f_1 + \cdots + f_n$, viewed as a k-rational point of the G-module $\mathcal{O}(G) \oplus \cdots \oplus \mathcal{O}(G) =: n\mathcal{O}(G)$. As $f_1, \ldots, f_n$ are contained in some finite-dimensional G-submodule $V \subset n\mathcal{O}(G)$, it follows that G/L is isomorphic to a subscheme of the affine space associated with V (see [7, III.3.5.2]). In view of [9, II.5.1.2], it follows that G/L is quasi-affine. In other terms, the affinization map $\varphi_{G/L}$ is an open immersion. Since $\mathcal{O}(G/L) = \mathcal{O}(G/H)$, this yields the desired commutative triangle

$$\begin{array}{ccc} G/H & & \\ {\scriptstyle u}\downarrow & \searrow^{\varphi_{G/H}} & \\ G/L & \xrightarrow[\varphi_{G/L}]{} & X, \end{array}$$

where u denotes the natural map, and $X = \operatorname{Spec} \mathcal{O}(G/H) = \operatorname{Spec} \mathcal{O}(G/L)$.

Let K be a subgroup of G such that $K \supset H$ and G/K is quasi-affine. Then we have a commutative square of G-equivariant morphisms

$$\begin{array}{ccc} G/H & \xrightarrow{\varphi_{G/H}} & \operatorname{Spec} \mathcal{O}(G/H) = \operatorname{Spec} \mathcal{O}(G/L) \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle \varphi_v} \\ G/K & \xrightarrow{\varphi_{G/K}} & \operatorname{Spec} \mathcal{O}(G/K), \end{array}$$

where $\varphi_{G/K}$ is an open immersion. Thus, v factors through u, and hence $L \subset K$.

(ii) In view of [7, I.1.2.6], the formation of the affinization morphism commutes with arbitrary field extensions. Thus, so does the formation of L.

(iii) Consider a subgroup K of G containing H such that K/H is anti-affine. Denote by $q : G \to G/H$ the quotient map and by $x = q(e_G)$ the base point. Then the pull-back map $\mathcal{O}(G)^H \to \mathcal{O}(K)^H \cong \mathcal{O}(K/H) = k$ is identified with the homomorphism $\mathcal{O}(G/H) \to k$ given by evaluation at x. Thus, $K/H \subset G/H$ is contained in the fiber of $\varphi_{G/H}$ at x. By (i), this fiber is $L/H \subset G/H$. It follows that $K \subset L$.

We now show that L/H is anti-affine. As in the proof of Theorem 4, we have a cartesian diagram of faithfully flat morphisms

$$\begin{array}{ccc} G \times L/H & \xrightarrow{p_1} & G \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle r} \\ G/H & \xrightarrow{u} & G/L, \end{array}$$

where p_1 denotes the projection, r the quotient map, and m the pull-back of the action map $G \times G/H \to G/H$. Thus, we obtain a canonical isomorphism of sheaves on G:

$$r^*(u_*\mathcal{O}_{G/H}) \xrightarrow{\cong} (p_1)_*(m^*\mathcal{O}_{G/H}).$$

Clearly, we have $m^*\mathcal{O}_{G/H} = \mathcal{O}_{G \times L/H}$ and $r^*\mathcal{O}_{G/L} = \mathcal{O}_G$. Moreover, the natural map $\mathcal{O}_{G/L} \to u_*\mathcal{O}_{G/H}$ is an isomorphism, since $\mathcal{O}(G/L) = \mathcal{O}(G/H)$ and G/L admits a covering by open affine subschemes of the form $(G/L)_f$, where $f \in \mathcal{O}(G/L)$ (see e.g. [9, II.5.1.2]). It follows that the natural map $\mathcal{O}_G \to (p_1)_*\mathcal{O}_{G \times L/H}$ is an isomorphism as well. In particular, this yields $\mathcal{O}(G) = \mathcal{O}(G \times L/H)$, and hence $\mathcal{O}(L/H) = k$ as desired.

(iv) It suffices to show that the natural map $L^0/L^0 \cap H \to L/H$ is an isomorphism, as every homogeneous space under a connected algebraic group is geometrically irreducible (see e.g. [8, VIA.2.6.6]). The quotient $L/L^0 \cdot H$ is finite and étale (since so is L/L^0), and anti-affine (since so is L/H). Thus, this quotient consists of a unique k-rational point. Hence $L = L^0 \cdot H$; this yields the desired assertion.

(v) Let K be a subgroup of G containing H. As K is affine, we have by Theorem 2 that K/H is anti-affine if and only if H is epimorphic in K. In view of (ii), this yields the assertion.

(vi) By (ii), we may assume that k is algebraically closed. Then we have $H \subset L_{\mathrm{red}} \subset L$ and the natural map $G/L_{\mathrm{red}} \to G/L$ is finite, as shown in the proof of Theorem 4. Since G/L is quasi-affine, so is G/L_{red} in view of [9, II.5.1.2, II.5.1.12]. Thus, $L = L_{\mathrm{red}}$ by the minimality of L, i.e., L is smooth.

Example 11. Assume that k has characteristic $p > 0$. Let $Y = G/H$ be a smooth anti-affine homogeneous space, where G is affine and $H \subsetneq G$. We will construct a non-smooth anti-affine homogeneous space X under an algebraic group containing G, such that X contains Y as its largest smooth subscheme. For this, we use a process of "infinitesimal thickening" of an arbitrary homogeneous space G/H.

Let M be a finite-dimensional G-module. Viewing M as a p-Lie algebra with zero bracket and pth power map, we obtain a commutative infinitesimal algebraic group $G_p(M)$ of height 1 (see [8, VIIA.8.1.2]). The action of G on M yields an action on $G_p(M)$ by automorphisms of algebraic groups; we denote by $G_p(M) \rtimes G$ the corresponding semi-direct product.

Next, let $N \subset M$ be an H-submodule. As above, we may form the semi-direct product $G_p(N) \rtimes H$; this is a subgroup of $G_p(M) \rtimes G$. Consider the homogeneous space

$$X := G_p(M) \rtimes G/G_p(N) \rtimes H.$$

The chain of inclusions $G_p(N) \rtimes H \subset G_p(N) \rtimes G \subset G_p(M) \rtimes G$ yields a morphism

$$f : X \longrightarrow G_p(M) \rtimes G/G_p(M) \rtimes H \cong G/H = Y.$$

Moreover, f is G-equivariant and its fiber at the base point $y \in Y(k)$ is H-equivariantly isomorphic to $G_p(M)/G_p(N)$. The latter quotient group is canonically isomorphic to $G_p(M/N)$, by [8, VIIA.8.1.3]. The neutral element of $G_p(M/N)$ is fixed by H, and hence yields a section $s : Y \to X$ of $f : X \to Y$. As $G_p(M/N)$ is infinitesimal, f and s induce mutually inverse homeomorphisms of the underlying topological spaces of X and Y.

We have an isomorphism

$$\mathcal{O}(X) \cong (\mathcal{O}(G) \otimes \mathcal{O}(G_p(M/N)))^H,$$

where H acts simultaneously on $\mathcal{O}(G)$ by left multiplication, and on $\mathcal{O}(G_p(M/N))$ via its linear action on M/N. Also, recall from [8, VIIA.7.4] the canonical isomorphism

$$\mathcal{O}(G_p(M/N)) \cong \mathrm{Sym}(M/N)^*/I,$$

where $\mathrm{Sym}(M/N)^*$ denotes the symmetric algebra of the dual module of M/N, and I the ideal generated by the pth powers of all elements of $(M/N)^*$.

Assume that G is affine. By a theorem of Chevalley (see e.g. [7, II.2.3.5]), we may choose a finite-dimensional G-module M and a hyperplane $N \subset M$ such that H is the stabilizer of N for the G-action on M. In particular, N is an H-submodule of M; we denote by $L = M/N$ the quotient line. Then we have an isomorphism of H-modules

$$\mathcal{O}(G_p(M/N)) \cong \bigoplus_{i=0}^{p-1} L^{-i},$$

where L^{-i} denotes the ith tensor power of L^* (in particular, L^0 is the trivial H-module k). Denoting by $\mathcal{L}$ the G-linearized invertible sheaf on $Y = G/H$ associated with the H-module L (as in [11, I.5.8]), we then have

$$\mathcal{O}(X) \cong \bigoplus_{i=0}^{p-1} (\mathcal{O}(G) \otimes L^{-i})^H \cong \bigoplus_{i=0}^{p-1} \Gamma(Y, \mathcal{L}^{-i}).$$

Assume in addition that Y is smooth, anti-affine and non-trivial. Then the section s identifies Y to the largest smooth subscheme of X. It remains to check that X is anti-affine; for this, we show that $\Gamma(Y, \mathcal{L}^{-i}) = 0$ for all $i \geq 1$. Consider the G-module $\Gamma(Y, \mathcal{L}) = (\mathcal{O}(G) \otimes L)^H$. The exact sequence of H-modules $0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$ yields a morphism of G-modules $(\mathcal{O}(G) \otimes M)^H \to \Gamma(Y, \mathcal{L})$. Moreover, we have an isomorphism of G-modules $(\mathcal{O}(G) \otimes M)^H \cong \mathcal{O}(G/H) \otimes M = M$ in view of [11, I.3.6]. This defines a morphism of G-modules $\varphi : M \to \Gamma(Y, \mathcal{L})$, dual to the immersion of Y into the projective space of hyperplanes in M. In particular, $\varphi(N)$ is non-zero and consists of sections $\sigma \in \Gamma(Y, \mathcal{L})$ that vanish at the base point y, i.e., $\sigma_y \in \mathfrak{m}_y \mathcal{L}_y$. Choose such a section $\sigma \neq 0$ and let $\tau \in \Gamma(Y, \mathcal{L}^{-i})$. Then we have $\sigma^i \tau \in \Gamma(Y, \mathcal{O}_Y) = k$, and $\sigma^i \tau$ vanishes at y as well. Thus, $\sigma^i \tau = 0$, hence $\tau = 0$ as Y is smooth and geometrically irreducible.

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References

[1] G. M. Bergman, *Epimorphisms of Lie algebras*, unpublished manuscript available online at `https://math.berkeley.edu/~gbergman/papers/unpub/LieEpi.pdf`.

[2] F. Bien and A. Borel, *Sous-groupes épimorphiques de groupes algébriques linéaires I*, C. R. Acad. Sci. Paris Sér. I, **315** (1992), 649-653.

[3] F. Bien and A. Borel, *Sous-groupes épimorphiques de groupes algébriques linéaires II*, C. R. Acad. Sci. Paris Sér. I, **315** (1992), 1341–1346.

[4] F. Bien, A. Borel, and J. Kollár, *Rationally connected homogeneous spaces*, Invent. math. **124** (1996), 103–127.

[5] M. Brion, P. Samuel, and V. Uma, *Lectures on the structure of algebraic groups and geometric applications*, CMI Lecture Series in Mathematics **1**, Hindustan Book Agency, New Dehli, 2013.

[6] M. Brion, *Some structure theorems for algebraic groups*, Proc. Symp. Pure Math. **94**, 53–125, Amer. Math. Soc., Providence, RI, 2017.

[7] M. Demazure and P. Gabriel, *Groupes algébriques*, Masson, Paris, 1970.

[8] M. Demazure and A. Grothendieck, *Schémas en groupes (SGA 3). Tome I: Propriétés générales des schémas en groupes*, Documents Mathématiques **7**, Soc. Math. France, Paris, 2011.

[9] A. Grothendieck, *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): II. Étude globale élémentaire de quelques classes de morphismes*, Publ. Math. IHÉS **8** (1961), 5–222.

[10] F. D. Grosshans, *Algebraic homogeneous spaces and invariant theory*, Lecture Notes in Math. **1673**, Springer-Verlag, Berlin-New York, 1997.

[11] J. C. Jantzen, *Representations of algebraic groups. Second edition*, Math. Surveys Monographs **107**, Amer. Math. Soc., Providence, RI, 2003.

[12] W. Haboush and N. Lauritzen, *Varieties of unseparated flags*, in: Linear algebraic groups and their representations, Contemp. Math. **153**, 35–57, Amer. Math. Soc., Providence, RI, 1993.

[13] C. Linderholm, *A group epimorphism is surjective*, Amer. Math. Monthly **77** (1970), 176-177.

[14] Nguyêñ Quôć Thăńg and Dao Phuong Băć, *Some rationality properties of observable groups and related questions*, Ill. J. Math. **49** (2005), 431–444.

[15] D. Panyushev, *Wide subalgebras of semisimple Lie algebras*, Algebr. Represent. Theory **17** (2014), 931–944.

[16] A. Petukhov, *Geometric description of epimorphic subgroups*, Russian Math. Surveys **65** (2011), 989–990.

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