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Szegő kernel asymptotics and Kodaira embedding theorems of Levi-flat CR manifolds

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Let X be an orientable compact Levi-flat CR manifold and let L be a positive CR complex line bundle over X. We prove that certain microlocal conjugations of the associated Szegő kernel admits an asymptotic expansion with respect to high powers of L. As an application, we give a Szegő kernel proof of the Kodaira type embedding theorem on Levi-flat CR manifolds due to Ohsawa and Sibony.

1. Introduction and statement of the main results

The problem of global embedding CR manifolds is prominent in areas such as complex analysis, partial differential equations and differential geometry. A general result is the CR embedding of strictly pseudoconvex compact CR manifolds of dimension greater than five, due to Boutet de Monvel [5].

For CR manifolds which are not strictly pseudoconvex, the idea of embedding CR manifolds by means of CR sections of tensor powers L^k of a positive CR line bundle $L \to X$ was considered in [24, 25, 27, 37, 47]. This was of course inspired by Kodaira's embedding theorem.

One way to attack this problem is to produce CR sections by projecting appropriate smooth sections to the space of CR sections. So it is crucial to understand the large k behaviour of the Szegő projection Π_k, i. e. the orthogonal projection on space $H^0_b(X, L^k)$ of CR sections, and of its distributional kernel, the Szegő kernel. To study the Szegő projection it is convenient to link it to a parametrix of the $\overline{\partial}_b$-Laplacian on $(0,1)$-forms (called Kohn Laplacian). This is also the method used in [5], where the parametrix turns out to be a pseudodifferential operator of order $1/2$.

In [27], we established analogues of the holomorphic Morse inequalities of Demailly [12, 36] for CR manifolds and we deduced that the space $H^0_b(X, L^k)$ is large under the assumption that the curvature of the line bundle is adapted to the Levi form. In [25], the first author introduced a microlocal cut-off function technique and could remove the assumptions linking the curvatures of the line bundle and the Levi form under rigidity conditions on X and the line bundle. Moreover, in [24], the first author established partial Szegő kernel asymptotic expansions and Kodaira embedding theorems on CR manifolds with transversal CR S^1-action, see also [26].

All these developments need the assumptions that either the curvature of the line bundle is adapted to the Levi form or rigidity conditions on X and the line bundle. The difficulty of this kind of problem comes from the presence of positive eigenvalues of the curvature of the line bundle and negative eigenvalues of the Levi form of X. Thus, it is very interesting to consider Levi-flat CR manifolds. In this case, the eigenvalues of the Levi form are zero and we will show that it is possible to remove the assumptions linking the curvatures of the line bundle and the Levi form or the rigidity conditions on X and the line bundle.

Levi-flat CR manifolds are foliated by complex manifolds and there is a subtle interplay between the function theory on the leaves and the dynamics of the foliation. Levi-flat CR submanifolds in projective manifolds play an important role in classical complex analysis [16, 17, 41, 42] linked to the

Levi problem, foliations and dynamical systems [1, 3, 7–10, 14, 30, 35, 44–47, 49]. They admit Lefschetz pencil structures of degree k, for any k large enough, cf. [38]. The topology and dynamics of Levi-flat hypersurfaces in complex surfaces of general type was thoroughly explored in [14], where it is shown that all possible Thurston geometries except the spherical one can occur. In this context it is important to have a general criterion for the projective Levi-flat manifolds, analogue to the Kodaira embedding theorem for Kähler manifolds. This is provided by Ohsawa-Sibony theorem [47], see Theorem 1.4. A related result is the projective embedding of compact laminations [13], [19, p. 401–402]. In the program of classifying Levi-flat CR manifolds one is sometimes led to non-existence results. There are no compact Levi-flat real hypersurfaces in a Stein manifold, due to the maximum principle. On the other hand, the non-existence of smooth Levi-flat hypersurfaces in complex projective spaces $\mathbb{CP}^n$ attracted a lot of attention, cf. [35, 49]. The non-existence has been settled for $n \geq 3$ but a famous still open conjecture is whether this is true for $n = 2$.

Viewing Levi-flat CR manifolds as families of complex manifolds, we can expect analogy with classical results from complex geometry such as Kodaira embedding theorem. The natural function theoretical objects on a CR manifold are CR functions or CR sections of a bundle. Actually, Ohsawa and Sibony [47], cf. also [46], constructed a CR projective embedding of class $\mathscr{C}^\kappa$ for any $\kappa \in \mathbb{N}$ of a Levi-flat CR manifold by using $\overline{\partial}$-estimates. A natural question is whether we can improve the regularity to $\kappa = \infty$. Adachi [1] showed that the answer is no, in general. The analytic difficulty of this problem comes from the fact that the Kohn Laplacian is not hypoelliptic on Levi flat manifolds. Hypoellipticity and subelliptic estimates are used on CR manifolds with non-degenerate Levi form in order to find parametrices of the Kohn Laplacian and establish the Hodge decomposition, e. g. [5, 11, 29, 32].

In this paper, we establish a semiclassical Hodge decomposition for the Kohn Laplacian acting on powers L^k as $k \to \infty$ and we show that the composition $\Pi_k \mathcal{A}_k$ of Π_k with an appropriate pseudodifferential operator $\mathcal{A}_k$ is a semiclassical Fourier integral operator, admitting an asymptotic expansion in k (see Theorem 1.3). From this result, we can understand the large k behaviour of the Szegő projection and produce many global CR functions. As an application, we give a Szegő kernel proof of Ohsawa and Sibony's Kodaira type embedding theorem on Levi-flat CR manifolds.

We now formulate the main results. Let $(X, T^{1,0}X)$ be an orientable compact Levi-flat CR manifold of dimension $2n-1$, $n \geqslant 2$. We fix a Hermitian metric $\langle \cdot | \cdot \rangle$ on $TX \otimes_{\mathbb{R}} \mathbb{C} =: \mathbb{C}TX$ such that $T^{1,0}X$ is orthogonal to

$T^{0,1}X$. The Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $TX\otimes_{\mathbb{R}}\mathbb{C}$ induces a Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on the bundle $\Lambda^{0,q}(T^*X)$ of $(0,q)$ forms of X. We denote by dv_X the volume form on X induced by $\langle\cdot\,|\,\cdot\rangle$. Let (L,h) be a CR complex line bundle over X, where the Hermitian fiber metric on L is denoted by h. We will denote by R^L the curvature of L (see Definition 2.6). We say that L is positive if R^L_x is positive definite at every $x\in X$. Let

$$\lambda_1(x)\leq\cdots\leq\lambda_{n-1}(x), \tag{1.1}$$

be the eigenvalues of R^L_x with respect to $\langle\cdot\,|\,\cdot\rangle$ and set

$$\det R^L_x:=\lambda_1(x)\cdots\lambda_{n-1}(x). \tag{1.2}$$

For $k>0$, let (L^k,h^k) be the k-th tensor power of the line bundle (L,h). In this paper, we assume that $k\gg 1$. For $u,v\in\Lambda^{0,q}_x(T^*X)\otimes L^k_x$ we denote by $\langle\,u\,|\,v\,\rangle_{h^k}$ the induced pointwise scalar product induced by $\langle\cdot\,|\,\cdot\rangle$ and h^k. We then get natural a global L^2 inner product $(\,\cdot\,|\,\cdot\,)_k$ on $\Omega^{0,q}(X,L^k)$, $(\,\alpha\,|\,\beta\,)_k:=\int_X\langle\,\alpha\,|\,\beta\,\rangle_{h^k}\,dv_X$. Similarly, we have an L^2 inner product $(\,\cdot\,|\,\cdot\,)$ on $\Omega^{0,q}(X)$. We denote by $L^2_{(0,q)}(X,L^k)$ and $L^2_{(0,q)}(X)$ the completions of $\Omega^{0,q}(X,L^k)$ and $\Omega^{0,q}(X)$ with respect to $(\,\cdot\,|\,\cdot\,)_k$ and $(\,\cdot\,|\,\cdot\,)$, respectively. For $q=0$, we write $L^2(X):=L^2_{(0,0)}(X)$, $L^2(X,L^k):=L^2_{(0,0)}(X,L^k)$.

Let $\overline{\partial}_{b,k}:\mathscr{C}^\infty(X,L^k)\to\Omega^{0,1}(X,L^k)$ be the tangential Cauchy-Riemann operator cf. (2.11). We extend $\overline{\partial}_{b,k}$ to $L^2(X,L^k)$ by $\overline{\partial}_{b,k}:\operatorname{Dom}\overline{\partial}_{b,k}\subset L^2(X,L^k)\to L^2_{(0,1)}(X,L^k)$, $u\longmapsto\overline{\partial}_{b,k}u$, with $\operatorname{Dom}\overline{\partial}_{b,k}:=\{u\in L^2(X,L^k);\ \overline{\partial}_{b,k}u\in L^2_{(0,1)}(X,L^k)\}$, where $\overline{\partial}_{b,k}u$ is defined in the sense of distributions. The *Szegő projection*

$$\Pi_k:L^2(X,L^k)\to\operatorname{Ker}\overline{\partial}_{b,k} \tag{1.3}$$

is the orthogonal projection with respect to $(\,\cdot\,|\,\cdot\,)_k$.

The Szegő projection Π_k is not a smoothing operator. Nevertheless, our first result shows that it enjoys the following regularity property.

Theorem 1.1. *Let X be an orientable compact Levi-flat CR manifold and let (L,h) be a positive CR line bundle on X. Then for every $\ell\in\mathbb{N}_0$ there exists $N_\ell>0$ such that for every $k\geq N_\ell$, $\Pi_k(\mathscr{C}^\infty(X,L^k))$ is an infinite dimensional subspace of $\mathscr{C}^\ell(X,L^k)$ and the induced projection $\Pi_k:\mathscr{C}^\infty(X,L^k)\to\mathscr{C}^\ell(X,L^k)$ is continuous.*

The regularity statement of Theorem 1.1 is related to the regularity of the $\overline{\partial}$-Neumann problem on weakly pseudoconvex domains endowed with a positive line bundle [33, 50]. In that case one has to take high enough powers to achieve $\mathscr{C}^\ell$-regularity, too.

Let us recall now that the Szegő kernel $\Pi(x,y)$ of the boundary X of a relatively compact strictly pseudoconvex domain G is a Fourier integral operator with complex phase, by a result of Boutet de Monvel-Sjöstrand [6] (here we consider the projection on the space of CR functions or CR sections of a fixed CR line bundle). In particular, $\Pi(x,y)$ is smooth outside the diagonal $x=y$ of $X\times X$ and there is a precise description of the singularity on the diagonal $x=y$, where $\Pi(x,y)$ has a certain asymptotic expansion. More precisely, let $G=\{\rho<0\}\Subset G'$ be a strictly pseudoconvex domain in a $(n+1)$-dimensional complex manifold G', where $\rho\in\mathscr{C}^\infty(G')$ is a defining function of G. Then by taking an almost-analytic extension $\varphi=\varphi(x,y):G'\times G'\to\mathbb{C}$ of ρ with certain properties [6, (1.1)-(1.3)] we have

$$\Pi(x,y)=\int_0^\infty e^{i\varphi(x,y)t}s(x,y,t)dt+R(x,y), \tag{1.4}$$

where $s(x,y,t)\in S^n(X\times X\times\mathbb{R}_+)$ and $R(x,y)$ is a smooth function.

For a Levi-flat CR manifold we do not have such a neat characterization of the singularities of the Szegő kernel $\Pi_k(x,y)$ for fixed k. The smoothing properties of Π_k are linked to the singularities of its kernel $\Pi_k(x,y)$ and to its large k behaviour. Although it is quite difficult to describe them directly, we will show that Π_k still admits an asymptotic expansion in weak sense (that is, in Sobolev spaces, see Theorem 1.2 and Section 2.4 for an explicit example).

Let s be a local trivializing section of L on an open set $D\subset X$. We define the weight of the metric with respect to s to be the function $\phi\in\mathscr{C}^\infty(D)$ satisfying $|s|_h^2=e^{-2\phi}$. We have an isometry

$$U_{k,s}:L^2(D)\to L^2(D,L^k),\ \ u\longmapsto ue^{k\phi}s^k, \tag{1.5}$$

with inverse $U_{k,s}^{-1}:L^2(D,L^k)\to L^2(D)$, $\alpha\mapsto e^{-k\phi}s^{-k}\alpha$. The localization of Π_k with respect to the trivializing section s is given by

$$\Pi_{k,s}:L^2_{\rm comp}(D)\to L^2(D),\ \ \Pi_{k,s}=U_{k,s}^{-1}\Pi_kU_{k,s}, \tag{1.6}$$

where $L^2_{\rm comp}(D)$ is the subspace of elements of $L^2(D)$ with compact support in D. The second main result of this work shows that for $k\to\infty$, Π_k is rapidly decreasing outside the diagonal, and describes the singularities of Π_k semi-clasically in terms of an oscillatory integral.

Theorem 1.2. *Let X be an orientable compact Levi-flat CR manifold of dimension $2n-1$, $n \geq 2$. Assume that there is a positive CR line bundle L over X. Then for every $\ell \in \mathbb{N}_0$, there is $N_\ell > 0$ such that for every $k \geq N_\ell$ we have:*

(i) $\widetilde{\chi}\Pi_k\chi = O(k^{-\infty}) : \mathscr{C}^\infty(X, L^k) \to \mathscr{C}^\ell(X, L^k)$, *for all* $\chi, \widetilde{\chi} \in \mathscr{C}^\infty(X)$ *with* $\operatorname{supp}\chi \cap \operatorname{supp}\widetilde{\chi} = \emptyset$;

(ii) $\Pi_{k,s} - \mathcal{S}_k = O(k^{-\infty}) : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\ell(D)$, *where* $\mathcal{S}_k : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\infty(D)$ *is a continuous operator whose kernel satisfies*

$$\begin{aligned} \mathcal{S}_k(x,y) - \int_{\mathbb{R}} e^{ik\psi(x,y,u)} s(x,y,u,k)du \\ = O(k^{-\infty}) : H^s_{\mathrm{comp}}(D) \to H^s_{\mathrm{loc}}(D), \quad \forall s \in \mathbb{Z}, \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} &s(x,y,u,k) \sim \sum_{j=0}^{\infty} s_j(x,y,u)k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1; D \times D \times \mathbb{R}), \\ &s_0(x,x,u) = \frac{1}{2}\pi^{-n}\left|\det R^L_x\right|, \quad \forall x \in D, \quad \forall u \in \mathbb{R}, \end{aligned} \tag{1.8}$$

and the phase function $\psi \in \mathscr{C}^\infty(D \times D \times \mathbb{R})$ *satisfies* $\operatorname{Im}\psi(x,y,u) \geq 0$ *and*

$$\begin{aligned} &d_x\psi|_{(x,x,u)} = -2\mathrm{Im}\,\overline{\partial}_b\phi(x) + u\omega_0(x), \quad x \in D,\ u \in \mathbb{R}, \\ &d_y\psi|_{(x,x,u)} = 2\mathrm{Im}\,\overline{\partial}_b\phi(x) - u\omega_0(x), \quad x \in D,\ u \in \mathbb{R}, \\ &\frac{\partial\psi}{\partial u}(x,x,u) = 0 \text{ and } \psi(x,x,u) = 0, \\ &\text{if } x \neq y \text{ then } \tfrac{\partial\psi}{\partial u}(x,y,u) \neq 0 \text{ or } \psi(x,y,u) \neq 0, \end{aligned} \tag{1.9}$$

and there exists $c > 0$ such that

$$|d_y\psi(x,y,u)| \geq c\,|u|\,, \quad \forall u \in \mathbb{R}, \quad \forall (x,y) \in D \times D. \tag{1.10}$$

Here $\omega_0 \in \mathscr{C}^\infty(X, T^*X)$ *is the positive* 1*-form of unit length orthogonal to* $\Lambda^{1,0}(T^*X)$ *and* $\Lambda^{0,1}(T^*X)$*, see Definition 2.3.*

Theorem 1.2 shows that the (localized) Szegő projector is close in the semiclassical limit to an approximate Szegő projector $\mathcal{S}_k$, which has an asymptotic expansion in Sobolev spaces, given by the operator $S_k : \mathscr{C}^\infty_0(D) \to$

$\mathscr{C}^\infty(D)$ with kernel

$$S_k(x,y) = \int_{\mathbb{R}} e^{ik\psi(x,y,u)} s(x,y,u,k)du. \tag{1.11}$$

Note that integrating by parts with respect to y several times in (1.11) and using (1.10), we conclude that S_k is well-defined as a continuous operator $S_k : \mathscr{C}_0^\infty(D) \to \mathscr{C}^\infty(D)$.

For fixed $u \in \mathbb{R}$, the integrand in the formula (1.11) of S_k (hence also for $\mathcal{S}_k$ or Π_k) bears a resemblance to the Bergman kernel B_k of the k-th power of a positive line bundle L on a complex manifold (cf. [28, 48, 51], see (2.13)). Note that $B = \sum_{k\geq 0} B_k$ is basically the Szegő kernel of the strictly pseudoconvex CR manifold given by the boundary of the unit disc bundle of L^*. The kernel of B has the form (1.4) involving an integral $\int_0^\infty dt$ and the B_k are its Fourier coefficients (see [51]). In our CR Levi-flat at setting, the Π_k most resemble B_k in being semi-classical kernels (with a k in the phase) but also formally resemble B in being integrals over an additional parameter u. But the integrals over the additional parameters in (1.4) and (1.11) have completely different origins. The integral $\int_{\mathbb{R}} du$ in (1.11) arises due to the transversal direction to the leaves of the Levi foliation. This is a different kind of integral than that for B, which arrises through summation over $k \geq 0$.

For fixed k, S_k is not a FIO since the phase function $\psi(x,y,u)$ is not homogeneous of degree one with respect to u. To obtain a homogeneous FIO, we should have to sum S_k in k. Moreover, the domain of integration in (1.11) is $\mathbb{R}$, unlike (1.4), where it is $\mathbb{R}_+$. In Section 2.4 we show that the Szegő projector Π_k itself is not a FIO, in contrast to the result of Boutet de Monvel-Sjöstrand [6] for strictly pseudoconvex domains. The proof of Theorem 1.2 is also different from [6] and is based on the heat equation method of Menikoff-Sjöstrand [40]. For the precise form of $\psi(x,y,u)$ see (4.36) and (4.39). This can be compared to the form [29, Theorems 3.2, 3.4] of the phase function for the Szegő kernel on a non-degenerate CR manifold.

If M is compact complex manifold of dimension n endowed with a positive line bundle $L \to M$ then the localization of the Bergman kernel B_k corresponding to L^k has the form $B_{k,s}(z,w) = e^{ik\varphi(z,w)}b(z,w,k)$, where $b(z,w,k) \sim \sum_{j=0}^\infty k^{n-1-j} b_j(z,w)$ in $S^{n-1}_{\mathrm{loc}}(1; D\times D)$, by the works of Zelditch [51] and Shiffman-Zelditch [48], see also [28] (cf. Section 2.4). We see thus that $S_k(x,y)$ is an integrated version of the Bergman kernel on a complex manifold. This corresponds to the fact that the Levi-flat CR manifold is foliated by complex manifolds and we have a transversal direction (where there are no elliptic estimates) in which we integrate. Note that in the case of a strictly pseudoconvex CR manifold we always have a 'bad' direction for ellipticity. In our case of a Levi-flat manifold endowed with a positive line bundle we have elliptic estimates in the directions of the Levi-foliation and the 'bad' direction is the transversal one. As a consequence, as shown by (1.7), $\mathcal{S}_k(x,y)$ and hence $\Pi_{k,s}(x,y)$, admits an asymptotic expansion $S_k(x,y)+O(k^{-\infty})$ only in Sobolev spaces (see also Theorem 4.14 for the details). This is an important difference between the Levi-flat and the Kähler case.

The fact that we integrate over $\mathbb{R}$ in (1.7) prevents us from obtaining asymptotics in the $\mathscr{C}^\ell$-topology for the kernel of $\Pi_{k,s}$. However, by composing with certain semiclassical pseudo-differential operators $\mathcal{A}_k$ we obtain asymptotics in the $\mathscr{C}^\ell$-topology for the kernels of $(\Pi_{k,s}-\mathcal{S}_k)\mathcal{A}_k$ and eventually $\Pi_{k,s}\mathcal{A}_k$. The symbol of $\mathcal{A}_k$ is supported in a large interval $(-M/2,M/2)$ in the fiber direction and by taking M large enough we recover increasingly more features of Π_k. The freedom to choose these operators and the constant M will be crucial for proving the embedding Theorem 1.4 (e. g. in (7.1)).

Let $\mathcal{A}_k$ be a properly supported semi-classical pseudodifferential operator on D of order 0 and classical symbol (see Definition 2.2)

$$\begin{aligned}
&\alpha(x,\eta,k)\sim\sum_{j=0}^{\infty}k^{-j}\alpha_j(x,\eta)\ \text{ in } S^0_{\mathrm{loc}}(1,T^*D),\\
&\alpha(x,\eta,k)=0,\ \alpha_j(x,\eta)=0,\ j=0,1,2,\ldots,\\
&\text{for } |\eta|\geq \tfrac{1}{2}M, \text{ for some } M>0.
\end{aligned}\tag{1.12}$$

Note that $\mathcal{A}_k$ is smoothing for each k. A semi-classical pseudodifferential operator with these properties will be called *good.*

Theorem 1.3. *Let X be an orientable compact Levi-flat CR manifold of dimension $2n-1$, $n\geq 2$. Assume that there is a positive CR line bundle L over X. Assume that $\mathcal{A}_k$ is a good semi-classical pseudodifferential operator on D. Then for every $\ell\in\mathbb{N}_0$, there is $N_\ell>0$ such that for every $k\geq N_\ell$, $(\Pi_{k,s}\mathcal{A}_k)(\cdot\,,\cdot)\in\mathscr{C}^\ell(D\times D)$ and*

$$\begin{aligned}
(\Pi_{k,s}\mathcal{A}_k)(x,y)\equiv\int_{\mathbb{R}}&e^{ik\psi(x,y,u)}a(x,y,u,k)du\\
&\mod O(k^{-\infty})\ \ \textit{in}\ \mathscr{C}^\ell(D\times D),
\end{aligned}\tag{1.13}$$

where

(1.14)

$$a(x,y,u,k) \sim \sum_{j=0}^{\infty} a_j(x,y,u)k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1; D \times D \times (-M,M)),$$

$$a(x,y,u,k), a_j(x,y,u) \in \mathscr{C}^\infty_0(D \times D \times (-M,M)), \quad j = 0,1,2,\ldots,$$

$$a_0(x,x,u) = \frac{1}{2}\pi^{-n}\left|\det R^L_x\right| \alpha_0\big(x, u\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)\big), \quad x \in D,\ |u| < M,$$

and $\psi \in \mathscr{C}^\infty(D \times D \times \mathbb{R})$ *is as in Theorem 1.2.*

For more results and references about the singularities of the Szegő kernel and embedding of CR manifolds we refer to [29].

As an application of Theorem 1.1 and Theorem 1.3, we show that by projecting appropriate sections through Π_k we obtain CR sections which separate points and tangent vectors. Hence we give a Szegő kernel proof of the following result due to Ohsawa and Sibony [46, 47].

Theorem 1.4. *Let X be an orientable compact Levi-flat CR manifold of dimension $2n-1$, $n \geq 2$. Assume that there is a positive CR line bundle L over X. Then, for every $\ell \in \mathbb{N}$ there is a $M_\ell > 0$ such that for every $k \geq M_\ell$, we can find N_k CR sections $s_0, s_1, \ldots, s_{N_k} \in \mathscr{C}^\ell(X, L^k)$, such that the map $X \ni x \mapsto [s_0(x), s_1(x), \ldots, s_{N_k}(x)] \in \mathbb{CP}^{N_k}$ is an embedding.*

Analytic proofs of the Kodaira embedding theorem for Kähler and symplectic manifolds, based on the Bergman/Szegő asymptotics, were given in [4, 36, 48, 51] (see [24, 26] for the Kodaira embedding of CR manifolds). Let us briefly describe the idea of the proof of Theorem 1.4. Using the fact that $\Pi_{k,s}\mathcal{A}_k$ is a semi-classical FIO and the freedom to choose $\mathcal{A}_k$, we show in Lemma 7.3 that for k large enogh, for every $\ell \in \mathbb{N}$ the $\mathscr{C}^\ell$ CR sections of L^k give local coordinates at all points of X. Hence we find a $\mathscr{C}^\ell$ CR immersion $\Phi_k : X \to \mathbb{CP}^N$. In contrast to the Kähler or symplectic case we do not show that Φ_k is injective. Rather, we use the fact that Φ_k separates points in the neighborhood of the diagonal in $X \times X$ and construct (by using Theorems 1.2 and 1.3) another $\mathscr{C}^\ell$ CR map $\Psi_m : X \to \mathbb{CP}^{N'}$ given by sections of a high power L^m, which separates points outside a certain distance of the diagonal. Therefore, the map $(\Phi_k, \Psi_m) : X \to \mathbb{CP}^N \times \mathbb{CP}^{N'}$ is injective and hence a $\mathscr{C}^\ell$ embedding, which composed with the Segre embedding (7.28) yields an embedding X to $\mathbb{CP}^{(N+1)(N'+1)-1}$.

The paper is organized as follows. In Section 2 we collect some notations, terminology, definitions and statements we use throughout. In Section 3, we give an explicit formula for the semi-classical Kohn Laplacian $\Box^{(q)}_{b,k}$ in local coordinates and we determine the characteristic manifold for $\Box^{(q)}_{b,k}$. In Section 4 we exhibit a semi-classical Hodge decomposition for $\Box^{(q)}_{b,k}$. In Section 5, we establish the regularity of the Szegő projection and we prove Theorem 1.1. In Section 6, by using the semi-classical Hodge decomposition theorem established in Section 4 and the regularity for the Szegő projection, we prove Theorem 1.2 and Theorem 1.3. In Section 7, we prove Theorem 1.4.

2. Preliminaries

In this section we introduce useful notions from semi-classical analysis and CR geometry. We then present background and examples of Levi-flat CR manifolds. Finally, we treat an explicit example of Szegő kernel of a positive line bundle.

2.1. Definitions and notations from semi-classical analysis

We use the following notations: $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{R}$ is the set of real numbers, $\overline{\mathbb{R}}_+:=\{x\in\mathbb{R};\,x\geq 0\}$. For a multiindex $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}_0^n$ we set $|\alpha|=\alpha_1+\cdots+\alpha_n$. For $x=(x_1,\ldots,x_n)$ we write

$$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n},\quad \partial_{x_j}=\frac{\partial}{\partial x_j}\,,\quad \partial^\alpha_x=\partial^{\alpha_1}_{x_1}\cdots\partial^{\alpha_n}_{x_n}=\frac{\partial^{|\alpha|}}{\partial x^\alpha}\,.$$

Let $z=(z_1,\ldots,z_n)$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\ldots,n$, be coordinates of $\mathbb{C}^n$. We write

$$z^\alpha=z_1^{\alpha_1}\cdots z_n^{\alpha_n}\,,\quad \overline{z}^\alpha=\overline{z}_1^{\alpha_1}\cdots\overline{z}_n^{\alpha_n}\,,\quad \partial_{z_j}=\frac{\partial}{\partial z_j}=\frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}}-i\frac{\partial}{\partial x_{2j}}\Big)\,,$$
$$\partial_{\overline{z}_j}=\frac{\partial}{\partial\overline{z}_j}=\frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}}+i\frac{\partial}{\partial x_{2j}}\Big)\,,\quad \partial^\alpha_z=\partial^{\alpha_1}_{z_1}\cdots\partial^{\alpha_n}_{z_n}=\frac{\partial^{|\alpha|}}{\partial z^\alpha}\,,$$
$$\partial^\alpha_{\overline{z}}=\partial^{\alpha_1}_{\overline{z}_1}\cdots\partial^{\alpha_n}_{\overline{z}_n}=\frac{\partial^{|\alpha|}}{\partial\overline{z}^\alpha}\,.$$

Let M be a $\mathscr{C}^\infty$ orientable paracompact manifold. We let TM and T^*M denote the tangent bundle of M and the cotangent bundle of M respectively. The complexified tangent bundle of M and the complexified cotangent bundle of M will be denoted by $\mathbb{C}TM$ or $TM\otimes_{\mathbb{R}}\mathbb{C}$ and $\mathbb{C}T^*M$ or $T^*M\otimes_{\mathbb{R}}\mathbb{C}$ respectively. We denote by $\langle\,\cdot\,,\cdot\,\rangle$ the pointwise duality between TM and T^*M. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $TM\otimes_{\mathbb{R}}\mathbb{C}\times T^*M\otimes_{\mathbb{R}}\mathbb{C}$.

Let E be a $\mathscr{C}^\infty$ vector bundle over M. The fiber of E at $x \in M$ will be denoted by E_x. Let F be another vector bundle over M. We write $F \boxtimes E^*$ to denote the vector bundle over $M \times M$ with fiber over $(x, y) \in M \times M$ consisting of the linear maps from E_x to F_y.

Let $Y \subset M$ be an open set and take any L^2 inner product on $\mathscr{C}_0^\infty(Y, E)$. By using this L^2 inner product, in this paper, we will consider a distribution section of E over Y is a continuous linear form on $\mathscr{C}_0^\infty(Y, E)$. From now on, let $\mathscr{D}'(Y, E)$ denote the space of distribution sections of E over Y and let $\mathscr{E}'(Y, E)$ be the subspace of $\mathscr{D}'(Y, E)$ whose elements have compact support in Y. For $m \in \mathbb{R}$, we let $H^m(Y, E)$ denote the Sobolev space of order m of sections of E over Y. Put $H^m_{\rm loc}(Y, E) = \big\{u \in \mathscr{D}'(Y, E);\ \varphi u \in H^m(Y, E),\ \forall \varphi \in \mathscr{C}_0^\infty(Y)\big\}$, $H^m_{\rm comp}(Y, E) = H^m_{\rm loc}(Y, E) \cap \mathscr{E}'(Y, E)$.

The Schwartz kernel theorem asserts that for any continuous linear operator

$$A : \mathscr{C}_0^\infty(M, E) \to \mathscr{D}'(M, F)$$

there exists a unique distribution $A(\cdot, \cdot) \in \mathscr{D}'(M \times M, F \boxtimes E^*)$ such that $(Au, v) = (A(\cdot, \cdot), v \otimes u)$ for any $u \in \mathscr{C}_0^\infty(M, E)$, $v \in \mathscr{C}_0^\infty(M, F^*)$ (see [20, Theorems 5.2.1, 5.2.6], [36, Thorem B.2.7]). The distribution $A(\cdot, \cdot)$ is called the Schwartz distribution kernel of A. We say that A is properly supported if the canonical projections on the two factors restricted to $\operatorname{supp} A(\cdot, \cdot) \subset M \times M$ are proper. If $A(\cdot, \cdot) \in \mathscr{C}^\infty(M \times M, F \boxtimes E^*)$, we say that A is a *smoothing operator* and we write $A \equiv 0$. Furthermore, A is smoothing if and only if for all $N \geq 0$ and $s \in \mathbb{R}$, $A : H^s_{\rm comp}(M, E) \to H^{s+N}_{\rm loc}(M, F)$ is continuous.

Let W_1, W_2 be open sets in $\mathbb{R}^N$ and let E and F be complex Hermitian vector bundles over W_1 and W_2. Let $s, s' \in \mathbb{R}$ and $n_0 \in \mathbb{R}$. For a k-dependent continuous function $F_k : H^s_{\rm comp}(W_1, E) \to H^{s'}_{\rm loc}(W_2, F)$ we write $F_k = O(k^{n_0}) : H^s_{\rm comp}(W_1, E) \to H^{s'}_{\rm loc}(W_2, F)$, if for any $\chi_0 \in \mathscr{C}^\infty(W_2), \chi_1 \in \mathscr{C}_0^\infty(W_1)$, there is a positive constant $c > 0$ independent of k, such that $\|(\chi_0 F_k \chi_1)u\|_{s'} \leq ck^{n_0} \|u\|_s$, $\forall u \in H^s_{\rm loc}(W_1, E)$, where $\|\cdot\|_s$ denotes the usual Sobolev norm of order s. We write $F_k = O(k^{-\infty}) : H^s_{\rm comp}(W_1, E) \to H^{s'}_{\rm loc}(W_2, F)$, if $F_k = O(k^{-N}) : H^s_{\rm comp}(W_1, E) \to H^{s'}_{\rm loc}(W_2, F)$, for every $N > 0$. Similarly, let $\ell \in \mathbb{N}$, for a k-dependent continuous function $G_k : \mathscr{C}_0^\infty(W_1, E) \to \mathscr{C}^\ell(W_2, F)$ we write $G_k = O(k^{-\infty}) : \mathscr{C}_0^\infty(W_1, E) \to \mathscr{C}^\ell(W_2, F)$, if for any $\chi_0 \in \mathscr{C}^\infty(W_2), \chi_1 \in \mathscr{C}_0^\infty(W_1)$ and $N > 0$, there are positive constants $c > 0$ and $M \in \mathbb{N}_0$ independent of k, such that $\|(\chi_0 G_k \chi_1)u\|_{\mathscr{C}^\ell(W_2, F)} \leq ck^{-N} \|u\|_{\mathscr{C}^M(W_1, E)}$, $\forall u \in \mathscr{C}_0^\infty(W_1, E)$.

A k-dependent continuous operator $A_k : \mathscr{C}_0^\infty(W_1, E) \to \mathscr{D}'(W_2, F)$ is called k-negligible on $W_2 \times W_1$ if for k large enough A_k is smoothing and for any $K \Subset W_2 \times W_1$, any multi-indices α, β and any $N \in \mathbb{N}$ there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$\left|\partial_x^\alpha \partial_y^\beta A_k(x,y)\right| \leq C_{K,\alpha,\beta,N} k^{-N}, \text{ on } K. \tag{2.1}$$

Let $C_k : \mathscr{C}_0^\infty(W_1, E) \to \mathscr{D}'(W_2, F)$ be another k-dependent continuous operator. We write $A_k \equiv C_k \mod O(k^{-\infty})$ (on $W_2 \times W_1$) or $A_k(x,y) \equiv C_k(x,y) \mod O(k^{-\infty})$ (on $W_2 \times W_1$) if $A_k - C_k$ is k-negligible on $W_2 \times W_1$.

Similarly, for $\ell \in \mathbb{N}_0$, $A_k : \mathscr{C}_0^\infty(W_1, E) \to \mathscr{D}'(W_2, F)$ is called k-negligible in the $\mathscr{C}^\ell$ norm on $W_2 \times W_1$ if $A_k(x,y) \in \mathscr{C}^\ell(W_2 \times W_1, E_y \boxtimes F_x)$ for k large and (2.1) holds for multi-indices α, β with $|\alpha| + |\beta| \leq \ell$.

Let $C_k : \mathscr{C}_0^\infty(W_1, E) \to \mathscr{D}'(W_2, F)$ be another k-dependent continuous operator. We write $A_k \equiv C_k \mod O(k^{-\infty})$ in the $\mathscr{C}^\ell$ norm (on $W_2 \times W_1$) or $A_k(x,y) \equiv C_k(x,y) \mod O(k^{-\infty})$ in $\mathscr{C}^\ell$ norm (on $W_2 \times W_1$) if $A_k - C_k$ is k-negligible in $\mathscr{C}^\ell$ norm on $W_2 \times W_1$.

Let $B_k : L^2(X, L^k) \to L^2(X, L^k)$ be a continuous operator. Let s, s_1 be local trivializing sections of L on open sets $D_0 \Subset M$, $D_1 \Subset M$ respectively, $|s|_h^2 = e^{-2\phi}$, $|s_1|_h^2 = e^{-2\phi_1}$. The localized operator (with respect to the trivializing sections s and s_1) of B_k is given by

$$\begin{aligned} &B_{k,s,s_1} : L^2(D_1) \cap \mathscr{E}'(D_1) \to L^2(D), \\ &u \longmapsto e^{-k\phi} s^{-k} B_k(s_1^k e^{k\phi_1} u) = U_{k,s}^{-1} B_k U_{k,s_1}, \end{aligned} \tag{2.2}$$

and let $B_{k,s,s_1}(x,y) \in \mathscr{D}'(D \times D_1)$ be the distribution kernel of B_{k,s,s_1}. We write $B_k = O(k^{n_0}) : H^s(X, L^k) \to H^{s'}(X, L^k)$, $n_0 \in \mathbb{R}$, if for all local trivializing sections s, s_1 on D and D_1 respectively, we have $B_{k,s,s_1} = O(k^{n_0}) : H^s_{\mathrm{comp}}(D_1) \to H^{s'}_{\mathrm{loc}}(D)$. We write $B_k = O(k^{-\infty}) : H^s(X, L^k) \to H^{s'}(X, L^k)$, $n_0 \in \mathbb{R}$, if for all local trivializing sections s, s_1 on D and D_1 respectively, we have $B_{k,s,s_1} = O(k^{-\infty}) : H^s_{\mathrm{comp}}(D_1) \to H^{s'}_{\mathrm{loc}}(D)$. Fix $\ell \in \mathbb{N}$. We write $B_k = O(k^{-\infty}) : \mathscr{C}^\infty(X, L^k) \to \mathscr{C}^\ell(X, L^k)$, if for all local trivializing sections s, s_1 on D and D_1 respectively, we have $B_{k,s,s_1} = O(k^{-\infty}) : \mathscr{C}_0^\infty(D_1) \to \mathscr{C}^\ell(D)$. We recall semi-classical symbol spaces (see Dimassi-Sjöstrand [15, Chapter 8]):

Definition 2.1. Let W be an open set in $\mathbb{R}^N$. Let

$$S(1; W) := \left\{ a \in \mathscr{C}^\infty(W) \,|\, \forall \alpha \in \mathbb{N}_0^N : \sup_{x \in W} |\partial^\alpha a(x)| < \infty \right\},$$

$$S^0_{\rm loc}(1;W) := \Big\{(a(\cdot,k))_{k\in\mathbb{N}} \,|\, \forall\alpha\in\mathbb{N}_0^N, \forall\chi\in\mathscr{C}_0^\infty(W):\\ \sup_{k\in\mathbb{N}}\sup_{x\in W}|\partial^\alpha a(x,k)|<\infty\Big\}.$$

For $m\in\mathbb{R}$ let $S^m_{\rm loc}(1;W)=\big\{(a(\cdot,k))_{k\in\mathbb{N}}\,|\,(k^{-m}a(\cdot,k))\in S^0_{\rm loc}(1;W)\big\}$. So $a(\cdot,k))\in S^m_{\rm loc}(1;W)$ if for every $\alpha\in\mathbb{N}_0^N$ and $\chi\in\mathscr{C}_0^\infty(W)$, there exists $C_\alpha>0$, such that $|\partial^\alpha(\chi a(\cdot,k))|\leq C_\alpha k^m$ on W.

Consider a sequence $a_j\in S^{m_j}_{\rm loc}(1;W)$, $j\in\mathbb{N}_0$, where $m_j\searrow-\infty$, and let $a\in S^{m_0}_{\rm loc}(1;W)$. We say that $a(\cdot,k)\sim\sum\limits_{j=0}^\infty a_j(\cdot,k)$, in $S^{m_0}_{\rm loc}(1;W)$, if for every $\ell\in\mathbb{N}_0$ we have $a-\sum_{j=0}^\ell a_j\in S^{m_{\ell+1}}_{\rm loc}(1;W)$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\rm loc}(1;W)=S^{-\infty}_{\rm loc}(1;W):=\cap_m S^m_{\rm loc}(1;W)$.

We say that $a(\cdot,k)\in S^m_{\rm loc}(1;W)$ is a classical symbol on W of order m if

$$a(\cdot,k)\sim\sum_{j=0}^\infty k^{m-j}a_j \text{ in } S^{m_0}_{\rm loc}(1;W),\quad a_j(x)\in S_{\rm loc}(1),\ j=0,1,\ldots. \tag{2.3}$$

The set of all classical symbols on W of order m_0 is denoted by $S^{m_0}_{{\rm loc},{\rm cl}}(1;W)$ $=S^{m_0}_{{\rm loc},{\rm cl}}(1;W)$.

Definition 2.2. Let W be an open set in $\mathbb{R}^N$. A semi-classical pseudodifferential operator on W of order m and classical symbol is a k-dependent continuous operator $A_k:\mathscr{C}_0^\infty(W)\to\mathscr{C}^\infty(W)$ such that the distribution kernel $A_k(x,y)$ is given by the oscillatory integral

$$\begin{aligned}A_k(x,y)&\equiv\frac{k^N}{(2\pi)^N}\int e^{ik\langle x-y,\eta\rangle}a(x,y,\eta,k)d\eta \mod O(k^{-\infty}),\\ &a(x,y,\eta,k)\in S^m_{{\rm loc},{\rm cl}}(1;W\times W\times\mathbb{R}^N).\end{aligned} \tag{2.4}$$

We shall identify A_k with $A_k(x,y)$. It is clear that A_k has a unique continuous extension $A_k:\mathscr{E}'(W)\to\mathscr{D}'(W)$. Moreover, it is well-known [18] that there is a symbol

$$\alpha(x,\eta,k)\in S^m_{{\rm loc},{\rm cl}}(1;W\times\mathbb{R}^N)=S^m_{{\rm loc},{\rm cl}}(1;T^*W) \tag{2.5}$$

independ on y such that

$$A_k(x,y)\equiv\frac{k^N}{(2\pi)^N}\int e^{ik\langle x-y,\eta\rangle}\alpha(x,\eta,k)d\eta \mod O(k^{-\infty}). \tag{2.6}$$

2.2. CR manifolds and bundles

A Cauchy-Riemann (CR) manifold (of hypersurface type) is a pair $(X, T^{1,0}X)$ where X is a smooth manifold of dimension $2n-1$, $n \geqslant 2$, and $T^{1,0}X$ is a sub-bundle of the complexified tangent bundle $\mathbb{C}TX := \mathbb{C} \otimes TX$, of rank $(n-1)$, such that $T^{1,0}X \cap \overline{T^{1,0}X} = \{0\}$ and the set of smooth sections of $T^{1,0}X$ is closed under the Lie bracket. We call $T^{1,0}X$ the CR structure of X and we denote $T^{0,1}X := \overline{T^{1,0}X}$.

We say that $(X, T^{1,0}X)$ is a *Levi-flat CR manifold* if the set of smooth sections of $T^{1,0}X \oplus T^{0,1}X$ is closed under the Lie bracket. If X is Levi-flat, there exists a smooth foliation of X, of real codimension one and whose leaves are complex manifolds: it is obtained by integrating the distribution $(T^{1,0}X \oplus T^{0,1}X) \cap TX$.

In this paper, we assume throughout that X is an orientable Levi-flat manifold.

Fix a smooth Hermitian metric $\langle \cdot | \cdot \rangle$ on $TX \otimes_{\mathbb{R}} \mathbb{C}$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X$ and $\langle u | v \rangle$ is real if u, v are real tangent vectors. Then locally there is a real non-vanishing vector field T of length one which is pointwise orthogonal to $T^{1,0}X \oplus T^{0,1}X$. T is unique up to the choice of sign. For $u \in TX \otimes_{\mathbb{R}} \mathbb{C}$, we write $|u|^2 := \langle u | u \rangle$. Denote by $\Lambda^{1,0}(T^*X)$ and $\Lambda^{0,1}(T^*X)$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. They can be identified with subbundles of the complexified cotangent bundle $T^*X \otimes_{\mathbb{R}} \mathbb{C}$.

Define the vector bundle of $(0,q)$-forms by $\Lambda^{0,q}(T^*X) := \Lambda^q(\Lambda^{0,1}(T^*X))$. The Hermitian metric $\langle \cdot | \cdot \rangle$ on $TX \otimes_{\mathbb{R}} \mathbb{C}$ induces, by duality, a Hermitian metric on $TX \otimes_{\mathbb{R}} \mathbb{C}$ and also on the bundles of $(0,q)$ forms $\Lambda^{0,q}(T^*X)$, $q = 0, 1, \ldots, n-1$. We shall also denote all these induced metrics by $\langle \cdot | \cdot \rangle$. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $\Lambda^{0,q}(T^*X)$ over D and let $\Omega_0^{0,q}(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D. Similarly, if E is a vector bundle over D, then we let $\Omega^{0,q}(D, E)$ denote the space of smooth sections of $\Lambda^{0,q}(T^*X) \otimes E$ over D and let $\Omega_0^{0,q}(D, E)$ be the subspace of $\Omega^{0,q}(D, E)$ whose elements have compact support in D.

Locally we can choose an orthonormal frame $\omega_1, \ldots, \omega_{n-1}$ of the bundle $\Lambda^{1,0}(T^*X)$. Then $\overline{\omega}_1, \ldots, \overline{\omega}_{n-1}$ is an orthonormal frame of the bundle $\Lambda^{0,1}(T^*X)$. The real $(2n-2)$-form $\omega = i^{n-1}\omega_1 \wedge \overline{\omega}_1 \wedge \cdots \wedge \omega_{n-1} \wedge \overline{\omega}_{n-1}$ is independent of the choice of the orthonormal frame. Thus ω is globally defined. Locally there is a real 1-form ω_0 of length one which is orthogonal to $\Lambda^{1,0}(T^*X) \oplus \Lambda^{0,1}(T^*X)$. The form ω_0 is unique up to the choice of sign. Since X is orientable, there is a nowhere vanishing $(2n-1)$ form Q on X. Thus, ω_0 can be specified uniquely by requiring that $\omega \wedge \omega_0 = fQ$, where f is a positive function. Therefore ω_0, so chosen, is globally defined.

Definition 2.3. We call ω_0 the positive 1-form of unit length orthogonal to $\Lambda^{1,0}(T^*X)$ and $\oplus\Lambda^{0,1}(T^*X)$.

We choose a vector field T so that

$$|T| = 1\,, \quad \langle\, T\,,\, \omega_0\,\rangle = -1\,. \tag{2.7}$$

Therefore T is uniquely determined. We call T the uniquely determined global real vector field. We have the pointwise orthogonal decompositions:

$$\begin{aligned} T^*X \otimes_{\mathbb{R}} \mathbb{C} &= \Lambda^{1,0}(T^*X) \oplus \Lambda^{0,1}(T^*X) \oplus \mathbb{C}\omega_0, \\ TX \otimes_{\mathbb{R}} \mathbb{C} &= T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}T. \end{aligned} \tag{2.8}$$

Let $\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. Let $U \subset X$ be an open set. We say that a function $u \in \mathscr{C}^\infty(U)$ is Cauchy-Riemann (CR for short) (on U) if $\overline{\partial}_b u = 0$.

Definition 2.4. Let L be a complex line bundle over a CR manifold X. We say that L is a Cauchy-Riemann (CR for short) (complex) line bundle over X if its transition functions are CR.

Definition 2.5. The *Szegő kernel* of the pair (X, L^k) is the Schwartz distribution kernel $\Pi_k(\cdot,\cdot) \in \mathscr{D}'(X \times X, L^k \boxtimes (L^k)^*)$ of the Szegő projection Π_k given by (1.3).

If X is Levi-flat, then the restriction a CR line bundle to any leaf Y of the Levi-foliation is a holomorphic line bundle.

From now on, we let (L, h) be a CR line bundle over X, where the Hermitian fiber metric on L is denoted by h. We will denote by ϕ the local weights of the Hermitian metric. More precisely, if s is a local trivializing section of L on an open subset $D \subset X$, then the local weight of h with respect to s is the function $\phi \in \mathscr{C}^\infty(D, \mathbb{R})$ for which

$$|s(x)|^2_h = e^{-2\phi(x)}\,, \quad x \in D. \tag{2.9}$$

Definition 2.6. Let s be a local trivializing section of L on an open subset $D \subset X$ and ϕ the corresponding local weight as in (2.9). For $p \in D$, we define the Hermitian quadratic form M^ϕ_p on $T^{1,0}_pX$ by

$$M^\phi_p(U,V)=\Big\langle U\wedge\overline{V}, d\big(\overline{\partial}_b\phi-\partial_b\phi\big)(p)\Big\rangle,\ \ U,V\in T^{1,0}_pX, \tag{2.10}$$

where d is the usual exterior derivative and $\overline{\partial_b\phi}=\overline{\partial}_b\overline{\phi}$. Since X is Levi-flat, the definition of M^ϕ_p does not depend on the choice of local trivializations (see [27, Proposition 4.2]). Hence there exists a smooth section R^L of the bundle of Hermitian forms on $T^{1,0}X$ such that $R^L|_D=M^\phi$. We call R^L the curvature of (L,h). We say that (L,h), or R^L, is positive if R^L_x is positive definite, for every $x\in X$. We say that L is a positive CR line bundle over X if there is a Hermitian fiber metric h on L such that the induced curvature R^L is positive.

In this paper, we assume that L is a positive CR line bundle over a Levi-flat CR manifold X and we fix a Hermitian fiber metric h of L such that the induced curvature R^L is positive. Note that a positive line bundle (L,h) in the sense of Definition 2.6 is positive along the leaves of the Levi-foliation: its restriction $(L,h)|_Y$ to any leaf Y is positive (that is, the curvature of the associated Chern connection is positive).

Let L^k, $k>0$, be the k-th tensor power of the line bundle L. The Hermitian fiber metric on L induces a Hermitian fiber metric on L^k that we shall denote by h^k. If s is a local trivializing section of L then s^k is a local trivializing section of L^k. We write $\overline{\partial}_{b,k}$ to denote the tangential Cauchy-Riemann operator acting on forms with values in L^k, defined locally by

$$\overline{\partial}_{b,k}:\Omega^{0,q}(X,L^k)\to\Omega^{0,q+1}(X,L^k)\,,\quad \overline{\partial}_{b,k}(s^ku):=s^k\overline{\partial}_bu, \tag{2.11}$$

where s is a local trivialization of L on an open subset $D\subset X$ and $u\in\Omega^{0,q}(D)$.

2.3. Background on Levi-flat CR manifolds and examples

Originally, Levi-flat CR manifolds first arose as Levi-flat real hypersurfaces in the study of the Levi problem, which asks the characterization of a domain of holomorphy by Levi pseudoconvexity of its boundary. While the Levi problem has an affirmative answer for domains in $\mathbb{C}^n$ (by the works of Oka, Bremmerman, Norguet) or $\mathbb{CP}^n$ (by results of Fujita and Takeuchi), Grauert [16] pointed out that some domains with Levi-flat boundary give counterexamples to the Levi problem (see also [17, 41]). These domains do not possess any non-constant holomorphic functions but they are typically endowed with a positive and ample line bundle, so the relevant function theory here deals with sections of positive line bundles and meromorphic functions, see e. g. [17]. From an analytic point of view this leads to the study of $\overline{\partial}$-Neumann problem in this situation [33, 50].

On the other hand, if we look upon Levi-flat CR manifolds intrinsically, the function theory should deal with CR functions or sections, that is, functions or sections which are holomorphic along the leaves of the Levi foliation. By a theorem of Inaba [30, Theorem 1], every continuous CR function on a compact Levi-flat CR manifold is constant along leaves of the Levi foliation. If the foliation has dense leaves, it follows that continuous CR functions are constant. Hence, as in the case of compact complex manifolds, we are led to perform function theory with sections of positive line bundles. The study of CR meromorphic functions on compact Levi-flat CR manifolds can also be seen as an alternative generalization of function theory on compact compact complex manifolds (the leaves of the foliation).

We present here a list of interesting Levi-flat manifolds carrying a positive line bundle.

(i) Linear hypersurfaces in tori. Let $n \geq 2$ and let Γ be the lattice in $\mathbb{C}^n$ generated by $\mathbb{R}$-linearly independent vectors $w_j = (w_{j1}, \ldots, w_{jn})$, $j = 1, \ldots, 2n$, where $w_1 = (1, 0, \ldots, 0)$ and $\operatorname{Re} w_{j1} = 0$ for $j = 2, \ldots, 2n$. Let T^n be the torus $\mathbb{C}^n/\Gamma$ and let $\pi : \mathbb{C}^n \to T^n$ be the natural map. For $c \in \mathbb{R}$ set $X_c = \pi\big(\{z \in \mathbb{C}^n : \operatorname{Re} z_1 = c\}\big)$. Then X_c is a compact Levi-flat hypersurface in T^n. If T^n is projective, X_c carries a positive CR line bundle obtained by restriction of a positive holomorphic bundle on T^n.

This construction was used by Grauert in order to give an example of a pseudoconvex domain that is not holomorphically convex, see [16], [41, p. 387]. Namely, let $U \subset \mathbb{C}^n$ be defined by $0 < \operatorname{Re} z_1 < 1$ and let $D = \pi(U)$. Then every holomorphic function on D is constant.

(ii) Grauert tubes in topologically trivial holomorphic line bundles. Let M be a compact projective manifold and $\pi : F \to M$ a topologically trivial holomorphic line bundle. There exists a finite open covering (U_α) of M and holomorphic frames e_α over U_α with $e_\beta = g_{\alpha\beta} e_\alpha$ on $U_\alpha \cap U_\beta$ for holomorphic transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ such that $|g_{\alpha\beta}| \equiv 1$. We define a Hermitian metric h on F by setting $|e_\alpha|_h = 1$. Then $X_c = \{v \in F : |v|_h = c\}$, $c > 0$, is a real analytic Levi-flat hypersurface in F, cf. [17, Satz 2]. If $L \to M$ is a positive line bundle, then $\pi^* L|_{X_c}$ is a positive CR line bundle. The Levi foliation of X_c has dense leaves if and only if all tensor powers F^k for $k \neq 0$ are holomorphically non-trivial.

Again, this construction is related to the Levi problem for pseudoconvex domains. Grauert [17] showed that $D_c = \{v \in F : |v|_h < c\}$, for $c > 0$, are meromorphically convex but not holomorphically convex domains.

(iii) Circle bundles over projective manifolds. Let M be a projective compact manifold. Let $\pi : \mathcal{D} \to M$ be a holomorphic fiber bundle over M with fiber the unit disc $\mathbb{D} \subset \mathbb{C}$. It can be easily seen that holomorphic trivializations form a trivializing cover, that is, the transition functions are locally constant. The disc bundle is thus isomorphic to a bundle of the form $\mathcal{D}_\rho := M \times_\rho \mathbb{D} := \widetilde{M} \times \mathbb{D}/\!\sim$, where $\rho : \pi_1(M) \to \mathrm{Aut}(\mathbb{D})$ is a group homomorphism, $\widetilde{M}$ is the universal cover of M and the relation equivalence $\sim$ is given by $(x, \zeta) \sim (\gamma x, \rho(\gamma)\zeta)$, for $x \in \widetilde{M}$, $\zeta \in \mathbb{D}$ and $\gamma \in \pi_1(M)$. Since $\mathrm{Aut}(\mathbb{D})$ is a group of biholomorphisms of $\mathbb{D}$ consisting of Möbius transformations preserving $\mathbb{D}$, acting on $\mathbb{CP}^1$ and fixing the unit circle $\mathbb{S}^1 = \partial\mathbb{D}$, it follows that a holomorphic disc bundle is canonically embedded in the complex manifold $N_\rho := M \times_\rho \mathbb{CP}^1 \to M$, and the boundary of $\mathcal{D}_\rho$ in N_ρ is a compact Levi-flat CR manifold $X_\rho = M \times_\rho \partial\mathbb{D}$. Note that N_ρ is a projective manifold by [31, Theorem 8], so any projective embedding of N_ρ induces a positive CR line bundle on X_ρ.

Other positive CR line bundles over X_ρ are given by the pullback $\pi^* L|_{X_\rho}$ of any positive line bundle $L \to M$. It was shown in [1, Main Theorem] that if M is a compact Riemann surface, $\pi^* L|_{X_\rho}$ is not C^∞ ample if $\mathcal{D}_\rho$ has a unique non-holomorphic harmonic section h with $\mathrm{rank}_{\mathbb{R}}\, dh = 2$ on an open dense set. A concrete example when the latter situation occurs is obtained by taking M to be a hyperbolic compact Riemann surface, regarding $\pi_1(M) \subset \mathrm{Aut}(\mathbb{D})$ as a Fuchsian representation and taking a non-trivial quasiconformal deformation $\rho : \pi_1(M) \to \mathrm{Aut}(\mathbb{D})$ of Γ, see [1].

The present construction was used in [14, Section 2] in order to construct Levi-flat hypersurfaces with nontrivial Euler class in complex surfaces of general type.

A generalization, particularly relevant in the context of the Ohsawa-Sibony embedding theorem, is the following. Let $\rho : \pi_1(M) \to \mathrm{Diff}(\mathbb{S}^1)$ be a group homomorphism, whose image is not necessarily contained in the Möbius transformation group. Then $X_\rho = M \times_\rho \mathbb{S}^1$ is Levi-flat and if $\pi : X_\rho \to M$ is the canonical projection and if $L \to M$ is positive, then $\pi^* L$ is a positive CR line bundle on X_ρ. Theorem 1.4 gives a realization of these X_ρ as $\mathscr{C}^\ell$ CR submanifolds in complex projective space for arbitrary large ℓ, while it is not clear a priori whether we can construct its filling $\mathcal{D}_\rho$ and its ambient N_ρ. Actually, for some special choice of M and ρ, it can be shown that X_ρ cannot be realized as a $\mathscr{C}^\infty$ Levi-flat real hypersurface, see

[3, 30]. For example, there does not exist a $\mathscr{C}^\infty$ Levi-flat hypersurface X in a two-dimensional complex manifold such that the Levi foliation of X is homeomorphic to Reeb's foliation of $\mathbb{S}^3$. An open question is whether such Levi-flat manifolds X_ρ can be realized as $\mathscr{C}^\ell$ Levi-flat real hypersurfaces for some finite $\ell \in \mathbb{N}$.

(iv) Levi-flat boundaries of Stein domains. In the examples (i) and (ii), Grauert constructed Levi-flat hypersurfaces bounding pseudoconvex non-Stein domains. Nemirovski [42] constructed examples of compact complex surfaces which contain a smooth Levi-flat hypersurface splitting the surface in two Stein domains. This construction admits a generalization to complex manifolds of arbitrary dimension as noted in [42], [45, p. 168].

Consider a holomorphic $\mathbb{C}^*$-bundle $B \to S$ where S is a projective manifold and the action of $\mathbb{Z}$ generated by $(w, z) \to (w, 2z)$ in terms of the local coordinate w of S and the fiber coordinate z. Then, for any meromorphic section s of the associated $\widehat{\mathbb{C}}$-bundle associated to B such that its zeros and poles are mutually disjoint and of order one, a Levi flat hypersurface X in a torus bundle $B/\mathbb{Z} \to S$ is obtained as the closure of the union of $\mathbb{R}^* s(x)/\mathbb{Z}$, where x runs through the complement of $s^{-1}(0) \cup s^{-1}(\infty)$. If $S \setminus s^{-1}(0) \cup s^{-1}(\infty)$ is Stein, X bounds an annulus bundle over a Stein manifold which is Stein (since holomorphic fiber bundles over Stein manifolds with one-dimensional Stein fibers are Stein). If the torus bundle $B/\mathbb{Z}$ is projective, then X carries a positive line bundle.

(v) Fibered Levi-flats in singular holomorphic fibrations. Such a fibration stands for a holomorphic map $f : B \to S$ where B is a complex surface and S is a compact Riemann surface. The fibers are not necessarily connected. Let $\{p_1, \dots, p_n\}$ be the singular values of f. A fibered Levi-flat hypersurface in B has the form $f^{-1}(\gamma)$, where $\gamma \subset S \setminus \{p_1, \dots, p_n\}$ is a simple closed path. In [14, Section 2] examples of fibered Levi-flat hypersurfaces are given, which carry the geometry of $\mathbb{R}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, or Sol. In particular, $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ are carried by fibered Levi-flat hypersurfaces in surfaces of general type.

(vi) Levi-flat hypersurfaces in two dimensional tori an Kummer surfaces. For these examples we refer to [43, 44].

(vii) Taut Levi-flat foliations. Let X be a Levi-flat CR 3-manifold. The Levi-foliation $\mathcal{F}$ is called taut if if there exists a C^1 embedded circle (called transversal) in X which transversely intersects every leaf of $\mathcal{F}$, cf. [8, Section 4.4]. By results of Sullivan and Rummler [8, Theorem 4.31], this is equivalent to the fact that X admits a C^2 Riemannian metric for which leaves of

$\mathcal{F}$ are minimal surfaces. Using this characterization one shows [38, Lemma 13]:

Proposition 2.7. *A compact Levi-flat CR 3-manifold possesses a smooth CR line bundle which is positive along leaves if and only if the Levi foliation is taut.*

Indeed, if X possesses a positive CR line bundle then the Ohsawa-Sibony embedding theorem implies that X can be CR embedded in a complex projective space by a C^2 map. We obtain thus a C^2 Riemannian metric on X by pulling back the Fubini-Study metric. Then, any leaf of $\mathcal{F}$ is minimal since any complex submanifold in a Kähler manifold is minimal. Conversely, if X is taut, by smoothing a closed transversal and regarding its intersection with the leaves of $\mathcal{F}$ as a divisor, we can construct a smooth positive CR line bundle on X.

(viii) Positive normal bundle. An important CR line bundle on a Levi-flat CR manifold is the normal line bundle $N_{\mathcal{F}}$ to the Levi foliation $\mathcal{F}$, cf. [2, Definition 2.15], [47, p. 89]. Brunella [7] observed that the positivity of $N_{\mathcal{F}}$ implies convexity properties of the complement of a Levi-flat hypersurface in a complex manifold (see [2] for the converse and the relation to the Diederich-Fornaess exponent). Explicit examples of Levi-flat CR manifold with positive normal line bundle can be found in [2, Example 4.5], [7, Example 4.2]. In [10, Théorème 2.2.3] the following general result is proved for three dimensional compact Levi-flat manifolds: if the Levi foliation $\mathcal{F}$ has no invariant transverse measure then $N_{\mathcal{F}}$ is positive.

Let us finally note that if X is a Levi-flat CR manifold and M is a projective manifold, and $L \to X$, $E \to M$ are positive line bundles, then $X \times M$ is a Levi-flat CR manifold possessing the positive line bundle $L \boxtimes E \to X \times M$. We can also construct examples of Levi-flat CR manifolds possessing a positive line bundle by taking Galois coverings or quotients by discrete groups of a given Levi-flat manifold with positive line bundle.

2.4. An explicit example of Szegő kernel

Let (L, h^L) be a holomorphic line bundle over a compact complex manifold M of dimension $n-1$, where h^L is a Hermitian fiber metric of L. Let R^L be the curvature induced by h^L and we assume that $iR^L > 0$ on M. Consider $X := M \times S^1$. We will identify S^1 with $(-\pi, \pi]$. Then, X is a Levi-flat CR manifold and the pull-back of (L, h^L) is a positive CR line bundle over X, denoted also (L, h^L). In this simple example, we will give an explicit formula for the phase function $\psi(x,y,u)$ and we will see that $\psi(x,y,u)$ fails to be positively homogeneous in u and Π_k is not a Fourier integral operator.

Fix $k>0$. Taking a Hermitian metric on $T^{1,0}M$ with volume form dv_M and the metric $d\theta$ on S^1, we endow X with the product Hermitian metric whose volume form is $dv_X = dv_M \wedge d\theta$. We then get natural L^2 inner products $(\,\cdot\,|\,\cdot\,)_k$ on $L^2(M,L^k)$ and $L^2(X,L^k)$. Let $B_k : L^2(M,L^k) \to \operatorname{Ker}\overline{\partial}$ be the orthogonal projection (Bergman projection). For $f \in L^2(X,L^k)$ we have the Fourier decomposition $f = \sum_{m\in\mathbb{Z}} e^{im\theta} f_m$ with $f_m \in L^2(M,L^k)$, for $m \in \mathbb{Z}$. We can check that the Szegő projection Π_k is given by

$$\Pi_k : L^2(X,L^k) \to \operatorname{Ker}\overline{\partial}_b, \quad f = \sum_{m\in\mathbb{Z}} e^{im\theta} f_m \longmapsto \sum_{m\in\mathbb{Z}} e^{im\theta} B_k f_m. \tag{2.12}$$

We now study the distribution kernel of Π_k. Let s be a local trivializing section of L on an open set $D \subset M$, $|s|^2_{h^L} = e^{-2\phi}$, and let $B_{k,s}$ be the localization of B_k with respect to the trivializing section s (see (1.6)). We write $x = (z, x_{2n-1})$, $y = (w, y_{2n-1})$, to denote the coordinates of $M \times S^1$, where $z = (z_1,\ldots,z_{n-1})$, $w = (w_1,\ldots,w_{n-1})$, denote coordinates on M and x_{2n-1}, y_{2n-1}, coordinates on S^1. By the works of Zelditch [51] and Shiffman-Zelditch [48], see also [28], we know that the kernel $B_{k,s}(z,w)$ of $B_{k,s}$ has the form

$$B_{k,s}(z,w) = e^{ik\varphi(z,w)} b(z,w,k) \text{ on } D \times D, \tag{2.13}$$

where

$$\begin{aligned}
&\varphi(z,w) \in \mathscr{C}^\infty(D\times D), \quad \operatorname{Im}\varphi(z,w) \approx |z-w|^2,\\
&b(z,w,k) \sim \sum_{j=0}^{\infty} k^{n-1-j} b_j(z,w)
\end{aligned}$$

in $S^{n-1}_{\mathrm{loc}}(1; D\times D)$ (see Definition 2.1). From (2.13) and (2.12), for any $f \in \mathscr{C}^\infty_0(D \times (-\pi,\pi])$, we have

$$\begin{aligned}
&(\Pi_{k,s} f)(x)\\
&= \sum_{m\in\mathbb{Z}} e^{imx_{2n-1}} \int_M \int_{-\pi}^{\pi} e^{ik\varphi(z,w)} b(z,w,k) e^{-imy_{2n-1}} f(w,y_{2n-1})\, dy_{2n-1}\, dv_M(w)\\
&= \int_M e^{ik\varphi(z,w)} b(z,w,k) f(w,x_{2n-1})\, dv_M(w)
\end{aligned} \tag{2.14}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_M \int_{-\pi}^{\pi}\int_{\mathbb{R}} e^{ik\varphi(z,w)+i\langle x_{2n-1}-y_{2n-1},\eta\rangle} \\
&\qquad\qquad \times b(z,w,k) f(w,y_{2n-1})\, d\eta\, dy_{2n-1}\, dv_M(w) \\
&= \frac{1}{2\pi}\int_M \int_{-\pi}^{\pi}\int_{\mathbb{R}} e^{ik(\varphi(z,w)+\langle x_{2n-1}-y_{2n-1},u\rangle)} \\
&\qquad\qquad \times kb(z,w,k) f(w,y_{2n-1})\, du\, dy_{2n-1}\, dv_M(w) \\
&= \frac{1}{2\pi}\int_M \int_{-\pi}^{\pi} \Pi_{k,s}(x,y) f(y)\, dv_X(y),
\end{aligned}
$$

where

$$
\Pi_{k,s}(x,y) = \int_{\mathbb{R}} e^{ik\psi(x,y,u)} s(x,y,u,k)\, du \tag{2.15}
$$

with

$$
\begin{aligned}
\psi(x,y,u) &= \varphi(z,w) + \left\langle x_{2n-1}-y_{2n-1}, u \right\rangle, \\
s(x,y,u,k) &= \frac{1}{2\pi} kb(z,w,k).
\end{aligned} \tag{2.16}
$$

Formulas (2.15) and (2.16) show that Π_k is not a Fourier integral operator with complex phase. The phase function $\psi(x,y,u)$ in (1.7) fails to be positively homogeneous of degree 1 with respect to u. Note also that (2.15) and (2.16) exhibit the Szegő kernel in the form given in Theorem 1.2.

3. The semi-classical Kohn Laplacian

In this section we introduce the Kohn Laplacian $\Box^{(q)}_{b,k}$ acting on sections of L^k and we determine its local form $\Box^{(q)}_{s,k}$ with respect to a frame s and its characteristic manifold. We show that the standard symplectic form of the cotangent bundle is non-degenerate on the characteristic manifold. This will be used in running the heat equation method in Section 4, for solving the eikonal equation (4.9) (see Theorems 4.3, 4.4, 4.5).

We start with some notations. For $v \in \Lambda^{0,q}(T^*X)$ we denote by $v\wedge : \Lambda^{0,\bullet}(T^*X) \to \Lambda^{0,\bullet+q}(T^*X)$ the exterior multiplication by v and let $v^{\wedge,*} : \Lambda^{0,\bullet}(T^*X) \to \Lambda^{0,\bullet-q}(T^*X)$ be the adjoint of $v\wedge$ with respect to $\langle\cdot\,|\,\cdot\rangle$. Hence, $\langle v\wedge u\,|\,g\rangle = \langle u\,|\,v^{\wedge,*}g\rangle$, for all $u \in \Lambda^{0,p}(T^*X)$, $g \in \Lambda^{0,p+q}(T^*X)$.

For any $r=0,1,\ldots,n-2$, we denote by $\overline{\partial}^*_{b,k} : \operatorname{Dom}\overline{\partial}^*_{b,k} \subset L^2_{(0,r+1)}(X,L^k) \to L^2_{(0,r)}(X,L^k)$ the Hilbert space adjoint of $\overline{\partial}_{b,k}$ with respect to $(\cdot\,|\,\cdot)_k$. Let

$\Box^{(q)}_{b,k}$ denote the (Gaffney extension of the) *Kohn Laplacian* given by

$$\begin{aligned}\operatorname{Dom}\Box^{(q)}_{b,k} = \{u \in \operatorname{Dom}\overline{\partial}_{b,k} \cap \operatorname{Dom}\overline{\partial}^*_{b,k} \subset L^2_{(0,q)}(X, L^k);\\ \overline{\partial}_{b,k}u \in \operatorname{Dom}\overline{\partial}^*_{b,k},\ \overline{\partial}^*_{b,k}u \in \operatorname{Dom}\overline{\partial}_{b,k}\}\,,\end{aligned} \tag{3.1}$$

and $\Box^{(q)}_{b,k}u = \overline{\partial}_{b,k}\overline{\partial}^*_{b,k}u + \overline{\partial}^*_{b,k}\overline{\partial}_{b,k}u$ for $s \in \operatorname{Dom}\Box^{(q)}_{b,k}$. Note that $\operatorname{Ker}\Box^{(0)}_{b,k} = \operatorname{Ker}\overline{\partial}_{b,k}$. By a result of Gaffney [36, Proposition 3.1.2], $\Box^{(q)}_{b,k}$ is a positive self-adjoint operator.

Let s be a local trivializing of L on an open subset $D \subset X$. By using the map (1.5) we have define localizations $\overline{\partial}_{s,k}$ of $\overline{\partial}_{b,k}$, $\overline{\partial}^*_{s,k}$ of $\overline{\partial}^*_{b,k}$ and $\Box^{(q)}_{s,k}$ of $\Box^{(q)}_{b,k}$ with respect to s through unitary identifications:

$$\begin{cases}\mathscr{C}^\infty_0(D, \Lambda^{0,q}(T^*X)) \longleftrightarrow \mathscr{C}^\infty_0(D, L^k \otimes \Lambda^{0,q}(T^*X))\\ \qquad u \longleftrightarrow \widetilde{u} = U_{k,s}u, \quad u = U^{-1}_{k,s}\widetilde{u},\\ \qquad \overline{\partial}_{s,k} \longleftrightarrow \overline{\partial}_{b,k}, \quad \overline{\partial}_{s,k}u = U^{-1}_{k,s}\overline{\partial}_{b,k}U_{k,s},\\ \qquad \overline{\partial}^*_{s,k} \longleftrightarrow \overline{\partial}^*_{b,k}, \quad \overline{\partial}^*_{s,k}u = U^{-1}_{k,s}\overline{\partial}^*_{b,k}U_{k,s},\\ \qquad \Box^{(q)}_{s,k} \longleftrightarrow \Box^{(q)}_{b,k}, \quad \Box^{(q)}_{s,k}u = U^{-1}_{k,s}\Box^{(q)}_{b,k}U_{k,s}.\end{cases} \tag{3.2}$$

It is easy to see that

$$\overline{\partial}_{s,k} = \overline{\partial}_b + k(\overline{\partial}_b\phi)\wedge\,, \quad \overline{\partial}^*_{s,k} = \overline{\partial}^*_b + k(\overline{\partial}_b\phi)^{\wedge,*} \tag{3.3}$$

where $\overline{\partial}^*_b : \Omega^{0,q+1}(X) \to \Omega^{0,q}(X)$ is the formal adjoint of $\overline{\partial}_b$ with respect to $(\,\cdot\,|\,\cdot\,)$, and

$$\Box^{(q)}_{s,k} = \overline{\partial}_{s,k}\overline{\partial}^*_{s,k} + \overline{\partial}^*_{s,k}\overline{\partial}_{s,k}. \tag{3.4}$$

The operator $\Box^{(q)}_{s,k}$ will be called the *localized Kohn Laplacian.*

Let us choose a smooth orthonormal frame $\{e_j\}_{j=1}^{n-1}$ for $\Lambda^{0,1}(T^*X)$ on D. Let $\{Z_j\}_{j=1}^{n-1}$ denote the dual frame of $T^{0,1}X$. Let Z^*_j be the formal adjoint of Z_j with respect to $(\,\cdot\,|\,\cdot\,)$, $j = 1, \ldots, n-1$, that is, $(Z_jf \mid h) = (f \mid Z^*_jh), f, h \in \mathscr{C}^\infty_0(D)$.

Proposition 3.1 ([24, Proposition 3.1]). *With the notations used before, using the identification* (3.2)*, we can identify the Kohn Laplacian* $\Box^{(q)}_{b,k}$

with

$$\begin{aligned}\Box^{(q)}_{s,k} &= \overline{\partial}_{s,k}\overline{\partial}^*_{s,k} + \overline{\partial}^*_{s,k}\overline{\partial}_{s,k} \\ &= \sum_{j=1}^{n-1}(Z^*_j + k\overline{Z}_j(\phi))(Z_j + kZ_j(\phi)) \\ &\quad + \sum_{j,t=1}^{n-1} e_j \wedge e_t^{\wedge,*} \circ [Z_j + kZ_j(\phi), Z^*_t + k\overline{Z}_t(\phi)] \\ &\quad + \varepsilon(Z + kZ(\phi)) + \varepsilon(Z^* + k\overline{Z}(\phi)) + f,\end{aligned} \tag{3.5}$$

where $\varepsilon(Z + kZ(\phi))$ denotes remainder terms of the form $\sum a_j(Z_j + kZ_j(\phi))$ with a_j smooth, matrix-valued and independent of k, for all j, and similarly for $\varepsilon(Z^* + k\overline{Z}(\phi))$ and f is a smooth function independent of k.

Note that the bracket in (3.5) is the commutator of $Z_j + kZ_j(\phi)$ and $Z^*_t + k\overline{Z}_t(\phi)$, $Z_j + kZ_j(\phi)(Z^*_t + k\overline{Z}_t(\phi))$ is a vector field plus a function.

Until further notice, we work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on D. Let $\xi = (\xi_1, \ldots, \xi_{2n-1})$ denote the dual variables of x. Then (x, ξ) are local coordinates of the cotangent bundle T^*D. Let $q_j(x, \xi)$ be the semi-classical principal symbol of $Z_j + kZ_j(\phi)$, $j = 1, \ldots, n-1$. If $r_j(x, \xi)$ denotes the principal symbol of Z_j, then $q_j(x, \xi) = r_j(x, \xi) + Z_j(\phi)$. The semi-classical principal symbol of $\Box^{(q)}_{s,k}$ is given by

$$p_0 = \sum_{j=1}^{n-1} \overline{q}_j q_j. \tag{3.6}$$

The characteristic manifold Σ of $\Box^{(q)}_{s,k}$ is

$$\begin{aligned}\Sigma &= \{(x, \xi) \in T^*D;\ p_0(x, \xi) = 0\} \\ &= \{(x, \xi) \in T^*D; \\ &\qquad q_1(x, \xi) = \cdots = q_{n-1}(x, \xi) = \overline{q}_1(x, \xi) = \cdots = \overline{q}_{n-1}(x, \xi) = 0\}.\end{aligned} \tag{3.7}$$

From (3.7), we see that p_0 vanishes to second order at Σ.

Proposition 3.2. *We have*

$$\Sigma = \left\{(x, \xi) \in T^*D;\ \xi = \lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x), \lambda \in \mathbb{R}\right\}. \tag{3.8}$$

We refer the reader to [24, Proposition 3.2] for the proof of Proposition 3.2.

Let $\sigma = d\xi \wedge dx$ denote the canonical two form on T^*D. We are interested in whether σ is non-degenerate at $\rho \in \Sigma$. We recall that σ is non-degenerate at $\rho \in \Sigma$ if $\sigma(u, v) = 0$ for all $v \in T_\rho\Sigma \otimes_{\mathbb{R}} \mathbb{C}$, where $u \in T_\rho\Sigma \otimes_{\mathbb{R}} \mathbb{C}$, then $u = 0$. From now on, for any $f \in \mathscr{C}^\infty(T^*D, \mathbb{C})$, we write H_f to denote the Hamilton field of f. That is, in local symplectic coordinates (x, ξ),

$$H_f = \sum_{j=1}^{2n-1} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

For $f, g \in \mathscr{C}^\infty(T^*D, \mathbb{C})$, $\{f, g\}$ denotes the Poisson bracket of f and g. We recall that

$$\{f, g\} = \sum_{s=1}^{2n-1} \left(\frac{\partial f}{\partial \xi_s} \frac{\partial g}{\partial x_s} - \frac{\partial f}{\partial x_s} \frac{\partial g}{\partial \xi_s} \right).$$

First, we need the following.

Lemma 3.3. *For $\rho = (p, \lambda_0\omega_0(p) - 2\mathrm{Im}\, \overline{\partial}_b\phi(p)) \in \Sigma$, we have*

$$\sigma(H_{q_j}, H_{q_t})|_\rho = 0, \quad j, t = 1, \ldots, n-1, \tag{3.9}$$

$$\sigma(H_{\overline{q}_j}, H_{\overline{q}_t})|_\rho = 0, \quad j, t = 1, \ldots, n-1, \tag{3.10}$$

and

$$\begin{aligned} \sigma(H_{\overline{q}_j}, H_{q_t})|_\rho &= i\langle[\overline{Z}_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p)\rangle \\ &\quad - i(\overline{Z}_j Z_t + Z_t\overline{Z}_j)\phi(p), \quad j, t = 1, \ldots, n-1, \end{aligned} \tag{3.11}$$

where Z_j are as in (3.5) *and q_j is the semi-classical principal symbol of $Z_j + kZ_j(\phi)$, $j = 1, \ldots, n-1$.*

Proof. We write $\rho = (p, \xi_0)$. It is straightforward to see that

$$\sigma(H_{q_j}, H_{q_t})|_\rho = \{q_j, q_t\}(\rho) = -\langle[Z_j, Z_t](p), \xi_0\rangle + i[Z_j, Z_t]\phi(p). \tag{3.12}$$

We have

$$\begin{aligned} \langle[Z_j, Z_t](p), \xi_0\rangle &= \langle[Z_j, Z_t](p), \lambda_0\omega_0(p) - 2\mathrm{Im}\, \overline{\partial}_b\phi(p)\rangle \\ &= \lambda_0\langle[Z_j, Z_t](p), \omega_0(p)\rangle \\ &\quad + i\langle[Z_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p)\rangle. \end{aligned} \tag{3.13}$$

Since $[Z_j, Z_t](p) \in T_p^{0,1}X$, we have

$$\langle[Z_j, Z_t](p), \omega_0(p)\rangle = 0 \tag{3.14}$$

and

$$\langle [Z_j, Z_t](p), \partial_b\phi(p)\rangle = 0. \tag{3.15}$$

Thus,

$$\langle [Z_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p)\rangle = \langle [Z_j, Z_t](p), \overline{\partial}_b\phi(p)\rangle = [Z_j, Z_t]\phi(p). \tag{3.16}$$

From (3.13), (3.14) and (3.16), we get

$$\langle [Z_j, Z_t](p), \xi_0\rangle = i[Z_j, Z_t]\phi(p).$$

Combining this with (3.12), we get (3.9). The proof of (3.10) is the same.

As in (3.12), it is straightforward to see that

$$\begin{aligned} \sigma(H_{\overline{q}_j}, H_{q_t})|_\rho &= \left\{\overline{q}_j, q_t\right\}(\rho) \\ &= \langle [\overline{Z}_j, Z_t](p), \xi_0\rangle - i(\overline{Z}_j Z_t + Z_t\overline{Z}_j)\phi(p), \end{aligned} \tag{3.17}$$

where $j, t = 1, \ldots, n-1$. We have

$$\begin{aligned} \langle [\overline{Z}_j, Z_t](p), \xi_0\rangle &= \langle [\overline{Z}_j, Z_t](p), \lambda_0\omega_0(p) - 2\mathrm{Im}\,\overline{\partial}_b\phi(p)\rangle \\ &= \lambda_0\langle [\overline{Z}_j, Z_t](p), \omega_0(p)\rangle \\ &\quad + i\langle [\overline{Z}_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p)\rangle. \end{aligned} \tag{3.18}$$

Since X is Levi-flat, $\lambda_0\langle [\overline{Z}_j, Z_t](p), \omega_0(p)\rangle = 0$ and hence

$$\langle [\overline{Z}_j, Z_t](p), \xi_0\rangle = i\langle [\overline{Z}_j, Z_t](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p)\rangle. \tag{3.19}$$

Combining (3.19) with (3.17), (3.11) follows. □

We need the following.

Lemma 3.4 ([27, Lemma 4.1]). *For any $U, V \in T^{1,0}_pX$, pick $\mathcal{U}, \mathcal{V} \in \mathscr{C}^\infty(D, T^{1,0}X)$ such that $\mathcal{U}(p) = U$, $\mathcal{V}(p) = V$. Then,*

$$R^L_p(U, \overline{V}) = -\big\langle \big[\mathcal{U}, \overline{\mathcal{V}}\big](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p)\big\rangle + \big(\mathcal{U}\overline{\mathcal{V}} + \overline{\mathcal{V}}\mathcal{U}\big)\phi(p). \tag{3.20}$$

Now, we can prove:

Theorem 3.5. *σ is non-degenerate at every point of Σ.*

Proof. Note that

$$\Sigma = \left\{(x,\xi) \in T^*D;\, q_j(x,\xi) = \overline{q}_j(x,\xi) = 0, \quad j = 1,\ldots,n-1\right\}.$$

Let $\mathbb{C}T_\rho\Sigma$ and $\mathbb{C}T_\rho(T^*D)$ be the complexifications of $T_\rho\Sigma$ and $T_\rho(T^*D)$ respectively. Let $T_\rho\Sigma^\perp$ be the orthogonal to $\mathbb{C}T_\rho\Sigma$ in $\mathbb{C}T_\rho(T^*D)$ with respect to the canonical two form σ. We notice that $\dim_{\mathbb{C}} T_\rho\Sigma^\perp = 2n-2$. It is easy to check that

$$\sigma(v, H_{q_j})|_\rho = \langle dq_j(\rho), v\rangle, \quad \sigma(v, H_{\overline{q}_j})|_\rho = \langle d\overline{q}_j(\rho), v\rangle,$$

$j = 1,\ldots,n-1$, $v \in \mathbb{C}T_\rho(T^*D)$. Thus, if $v \in \mathbb{C}T_\rho\Sigma$, we get $\sigma(H_{q_j}, v)|_\rho = 0$, $\sigma(H_{\overline{q}_j}, v)|_\rho = 0$, $j = 1,\ldots,n-1$. We conclude that $H_{q_1},\ldots,H_{q_{n-1}}, H_{\overline{q}_1},\ldots, H_{\overline{q}_{n-1}}$ is a basis for $T_\rho\Sigma^\perp$.

Let $\nu \in \mathbb{C}T_\rho\Sigma \bigcap T_\rho\Sigma^\perp$. We write $\nu = \sum_{j=1}^{n-1}(\alpha_j H_{q_j}(\rho) + \beta_j H_{\overline{q}_j}(\rho))$. Since $\nu \in \mathbb{C}T_\rho\Sigma$, we have

$$\sigma(\nu, H_{q_t})|_\rho = \sigma(\nu, H_{\overline{q}_t})|_\rho = 0,$$

$t = 1,\ldots,n-1$. In view of (3.9), (3.10), (3.11) and (3.20), we see that

$$\begin{aligned}\sigma(\nu, H_{q_t})|_\rho &= \sum_{j=1}^{n-1}\beta_j\Big(-iR_p^L(\overline{Z}_j, Z_t)\Big)\\ &= -iR_p^L(Y, Z_t) = 0,\end{aligned} \tag{3.21}$$

for all $t = 1,\ldots,n-1$, where $Y = \sum_{j=1}^{n-1}\beta_j\overline{Z}_j(p) \in T_p^{1,0}X$. Since R_p^L is non-degenerate, we get $Y = 0$. Thus, $\beta_j = 0$, $j = 1,\ldots,n-1$. Similarly, we can repeat the process above to show that $\alpha_j = 0$, $j = 1,\ldots,n-1$. We conclude that $\mathbb{C}T_\rho\Sigma \bigcap T_\rho\Sigma^\perp = 0$. Hence σ is non-degenerate at ρ. The theorem follows. □

4. Semi-classical Hodge decomposition for the localized Kohn Laplacian

In this section, we will apply the method introduced in [24] to establish semi-classical Hodge decomposition theorems for $\Box_{s,k}^{(0)}$, based on the heat equation method of Menikoff-Sjöstrand [40]. We first add one extra variable to the local $(2n-1)$ coordinates on X and introduce the operator $\Box_s^{(q)}$ acting in $2n$ variables and linked to the localized Kohn Laplacian $\Box_{s,k}^{(q)}$ by (4.4). We use the heat equation method [40], [22, Proposition 6.5], to construct a parametrix for $\Box_s^{(0)}$ in Theorem 4.8. The corresponding Szegő operator S in that Theorem (cf. (4.29)) turns out to be a complex Fourier integral operator cf. Theorem 4.9 with phase function Φ. Returning to $\Box_{s,k}^{(q)}$ this yields the semiclassical Hodge decomposition by Theorem 4.13, with Szegő operators $\mathcal{S}_k$ having an expansion in Sobolev spaces cf. Theorem 4.14 given by a kernel with phase function ψ given by the restriction of Φ. We then refine the result to show that composing with certain pseudodifferential operators $\mathcal{A}_k$ we obtain an expansion of $\mathcal{S}_k\mathcal{A}_k$ in the $\mathscr{C}^\infty$ topology and calculate its leading term (Theorems 4.15 and 4.17).

4.1. The heat equation for the local operator $\Box_s^{(0)}$

Let Ω be an open set in $\mathbb{R}^N$ and let f, g be positive continuous functions on Ω. We write $f \asymp g$ if for every compact set $K \subset \Omega$ there is a constant $c_K > 0$ such that $f \leq c_K g$ and $g \leq c_K f$ on K.

Let s be a local trivializing section of L on an open subset $D \Subset X$ and $|s|_h^2 = e^{-2\phi}$. In this section, we work with some real local coordinates $x = (x_1, \ldots, x_{2n-1})$ defined on D. We write $\xi = (\xi_1, \ldots, \xi_{2n-1})$ or $\eta = (\eta_1, \ldots, \eta_{2n-1})$ to denote the dual coordinates of x. We consider the domain $\widehat{D} := D \times \mathbb{R}$. We write $\widehat{x} := (x, x_{2n}) = (x_1, x_2, \ldots, x_{2n-1}, x_{2n})$ to denote the coordinates of $D \times \mathbb{R}$, where x_{2n} is the coordinate of $\mathbb{R}$. We write $\widehat{\xi} := (\xi, \xi_{2n})$ or $\widehat{\eta} := (\eta, \eta_{2n})$ to denote the dual coordinates of $\widehat{x}$, where ξ_{2n} and η_{2n} denote the dual coordinate of x_{2n}. We shall use the following notations: $\langle x, \eta\rangle := \sum_{j=1}^{2n-1} x_j\eta_j$, $\langle x, \xi\rangle := \sum_{j=1}^{2n-1} x_j\xi_j$, $\langle \widehat{x}, \widehat{\eta}\rangle := \sum_{j=1}^{2n} x_j\eta_j$, $\langle \widehat{x}, \widehat{\xi}\rangle := \sum_{j=1}^{2n} x_j\xi_j$.

Let $\Lambda^{0,q}(T^*\widehat{D})$ be the bundle with fiber

$$\Lambda^{0,q}_{\widehat{x}}(T^*\widehat{D}) := \left\{u \in \Lambda^{0,q}(T^*X);\ \widehat{x} = (x, x_{2n})\right\}$$

at $\widehat{x} \in \widehat{D}$. From now on, for every point $\widehat{x} = (x, x_{2n}) \in \widehat{D}$, we identify $\Lambda^{0,q}_{\widehat{x}}(T^*\widehat{D})$ with $\Lambda^{0,q}_x(T^*X)$. Let $\langle \cdot \,|\, \cdot \rangle$ be the Hermitian metric on $T^*\widehat{D} \otimes_{\mathbb{R}} \mathbb{C}$ given by $\langle \widehat{\xi} \,|\, \widehat{\eta} \rangle = \langle \xi \,|\, \eta \rangle + \xi_{2n}\overline{\eta_{2n}}$, $(\widehat{x}, \widehat{\xi}), (\widehat{x}, \widehat{\eta}) \in T^*\widehat{D} \otimes_{\mathbb{R}} \mathbb{C}$. Let $\Omega^{0,q}(\widehat{D})$ denote the space of smooth sections of $\Lambda^{0,q}(T^*\widehat{D})$ over $\widehat{D}$ and put $\Omega^{0,q}_0(\widehat{D}) := \Omega^{0,q}(\widehat{D}) \cap \mathscr{E}'(\widehat{D}, \Lambda^{0,q}(T^*\widehat{D}))$. Using $ku(x) = e^{-ikx_{2n}}\left(-i\frac{\partial}{\partial x_{2n}}\left(e^{ikx_{2n}}u\right)(x)\right)$, $u \in \Omega^{0,q}(D)$, we consider the following operators

$$\begin{aligned}
&\overline{\partial}_s : \Omega^{0,r}(\widehat{D}) \to \Omega^{0,r+1}(\widehat{D}), \quad \overline{\partial}_{s,k}u = e^{-ikx_{2n}}\overline{\partial}_s(ue^{ikx_{2n}}), \quad u \in \Omega^{0,r}(D),\\
&\overline{\partial}^*_s : \Omega^{0,r+1}(\widehat{D}) \to \Omega^{0,r}(\widehat{D}), \quad \overline{\partial}^*_{s,k}u = e^{-ikx_{2n}}\overline{\partial}^*_s(ue^{ikx_{2n}}), \quad u \in \Omega^{0,r+1}(D),
\end{aligned} \tag{4.1}$$

where $r = 0, 1, \ldots, n-1$ and $\overline{\partial}_{s,k}$, $\overline{\partial}^*_{s,k}$ are given by (3.2). From (3.3) it is easy to see that

$$\begin{aligned}\overline{\partial}_s &= \sum_{j=1}^{n-1}\left(e_j \wedge \left(Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}\right) + (\overline{\partial}_b e_j)\wedge e_j^{\wedge,*}\right),\\ \overline{\partial}^*_s &= \sum_{j=1}^{n-1}\left(e_j^{\wedge,*}\left(Z^*_j - i\overline{Z}_j(\phi)\frac{\partial}{\partial x_{2n}}\right) + e_j \wedge (\overline{\partial}_b e_j)^{\wedge,*}\right),\end{aligned} \tag{4.2}$$

where $Z_1, \ldots, Z_{n-1}$, $Z^*_1, \ldots, Z^*_{n-1}$ and $e_1, \ldots, e_{n-1}$ are as in Proposition 3.1. Put

$$\Box^{(q)}_s := \overline{\partial}_s\overline{\partial}^*_s + \overline{\partial}^*_s\overline{\partial}_s : \Omega^{0,q}(\widehat{D}) \to \Omega^{0,q}(\widehat{D}). \tag{4.3}$$

From (4.1), we have

$$\Box^{(q)}_{s,k}u = e^{-ikx_{2n}}\Box^{(q)}_s(ue^{ikx_{2n}}), \quad \forall u \in \Omega^{0,q}(D), \tag{4.4}$$

where $\Box^{(q)}_{s,k}$ is given by (3.2). Let $u \in \Omega^{0,q}_0(\widehat{D})$. Note that

$$\begin{aligned}k\int e^{-ikx_{2n}}u(x)dx_{2n} &= \int i\frac{\partial}{\partial x_{2n}}(e^{-ikx_{2n}})u(x)dx_{2n}\\ &= \int e^{-ikx_{2n}}\left(-i\frac{\partial u}{\partial x_{2n}}(x)\right)dx_{2n}.\end{aligned}$$

From this observation and the explicit formulas for $\overline{\partial}_{s,k}$, $\overline{\partial}^*_{s,k}$, $\overline{\partial}_s$ and $\overline{\partial}^*_s$ (see (3.3) and (4.2)), we conclude that

$$\Box^{(q)}_{s,k}\int e^{-ikx_{2n}}u(x)dx_{2n} = \int e^{-ikx_{2n}}(\Box^{(q)}_s u)(x)dx_{2n}, \quad u \in \Omega^{0,q}_0(\widehat{D}). \tag{4.5}$$

As in Proposition 4.1 in [24], we have:

Proposition 4.1. *With the notations used before, we have*

$$\begin{aligned}\Box_s^{(q)} &= \overline{\partial}_s\overline{\partial}_s^* + \overline{\partial}_s^*\overline{\partial}_s \\ &= \sum_{j=1}^{n-1}\left(Z_j^* - i\overline{Z}_j(\phi)\frac{\partial}{\partial x_{2n}}\right)\left(Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}\right) \\ &\quad + \sum_{j,t=1}^{n-1} e_j\wedge e_t^{\wedge,*}\left[Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}, Z_t^* - i\overline{Z}_t(\phi)\frac{\partial}{\partial x_{2n}}\right] \\ &\quad + \varepsilon\left(Z - iZ(\phi)\frac{\partial}{\partial x_{2n}}\right) + \varepsilon\left(Z^* - i\overline{Z}(\phi)\frac{\partial}{\partial x_{2n}}\right) \\ &\quad + \textit{zero order terms},\end{aligned} \tag{4.6}$$

where $\varepsilon(Z - iZ(\phi)\frac{\partial}{\partial x_{2n}})$ *denotes remainder terms of the form* $\sum a_j(Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}})$ *with* a_j *smooth, matrix-valued, for all* j, *and similarly for* $\varepsilon(Z^* - i\overline{Z}(\phi)\frac{\partial}{\partial x_{2n}})$.

In this paper, we will only consider $q = 0$. Consider the following problem for the heat equation

$$\begin{cases}(\partial_t + \Box_s^{(0)})u(t,\widehat{x}) = 0 & \text{in } \mathbb{R}_+\times\widehat{D}, \\ u(0,\widehat{x}) = v(\widehat{x}).\end{cases} \tag{4.7}$$

Definition 4.2. We say that $a(t,\widehat{x},\widehat{\eta})\in\mathscr{C}^\infty(\overline{\mathbb{R}}_+\times T^*\widehat{D})$ is quasi-homogeneous of degree j if $a(t,\widehat{x},\lambda\widehat{\eta}) = \lambda^j a(\lambda t,\widehat{x},\widehat{\eta})$ for all $\lambda > 0$, $|\widehat{\eta}|\ge 1$. We say that $b(\widehat{x},\widehat{\eta})\in\mathscr{C}^\infty(T^*\widehat{D})$ is positively homogeneous of degree j if $b(\widehat{x},\lambda\widehat{\eta}) = \lambda^j b(\widehat{x},\widehat{\eta})$ for all $\lambda > 0$, $|\widehat{\eta}|\ge 1$.

We look for an approximate solution of (4.7) of the form $u(t,\widehat{x}) = A(t)v(\widehat{x})$,

$$A(t)v(\widehat{x}) = \frac{1}{(2\pi)^{2n}}\iint e^{i(\Psi(t,\widehat{x},\widehat{\eta}) - \langle\widehat{y},\widehat{\eta}\rangle)}a(t,\widehat{x},\widehat{\eta})v(\widehat{y})d\widehat{y}d\widehat{\eta} \tag{4.8}$$

where formally $a(t,\widehat{x},\widehat{\eta}) \sim \sum_{j=0}^{\infty} a_j(t,\widehat{x},\widehat{\eta})$, $a_j(t,\widehat{x},\widehat{\eta})\in\mathscr{C}^\infty(\overline{\mathbb{R}}_+\times T^*\widehat{D})$, $a_j(t,\widehat{x},\widehat{\eta})$ is a quasi-homogeneous function of degree $-j$. The phase $\Psi(t,\widehat{x},\widehat{\eta})$

should solve the eikonal equation

$$\frac{\partial \Psi}{\partial t} - i\widehat{p}_0(\widehat{x}, \Psi'_{\widehat{x}}) = O(|\mathrm{Im}\,\Psi|^N), \forall N \geq 0, \quad \Psi|_{t=0} = \langle \widehat{x}, \widehat{\eta} \rangle \tag{4.9}$$

with $\mathrm{Im}\,\Psi \geq 0$, where $\widehat{p}_0$ denotes the principal symbol of $\Box_s^{(0)}$. From (4.6), we have

$$\widehat{p}_0 = \sum_{j=1}^{n-1} \overline{\widehat{q}}_j \widehat{q}_j, \tag{4.10}$$

where $\widehat{q}_j$ is the principal symbol of $Z_j - iZ_j(\phi)\frac{\partial}{\partial x_{2n}}$, $j = 1, \ldots, n-1$. The characteristic manifold $\widehat{\Sigma}$ of $\Box_s^{(0)}$ is given by

$$\widehat{\Sigma} = \Big\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\ \widehat{q}_1(\widehat{x}, \widehat{\xi}) = \cdots = \widehat{q}_{n-1}(\widehat{x}, \widehat{\xi}) = \overline{\widehat{q}}_1(\widehat{x}, \widehat{\xi}) = \cdots = \overline{\widehat{q}}_{n-1}(\widehat{x}, \widehat{\xi}) = 0 \Big\}. \tag{4.11}$$

From (4.11), we see that $\widehat{p}_0$ vanishes to second order at $\widehat{\Sigma}$. Let $\widehat{\sigma}$ denote the canonical two form on $T^*\widehat{D}$. As in Proposition 3.2 and Theorem 3.5, we have

Theorem 4.3. *With the notations used above, we have*

$$\widehat{\Sigma} = \Big\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\ \widehat{\xi} = (\lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n}, \xi_{2n}), \lambda \in \mathbb{R} \Big\}. \tag{4.12}$$

Put

$$\begin{aligned} \widehat{\Sigma}_+ &= \Big\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\ \widehat{\xi} = (\lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n}, \xi_{2n}), \lambda \in \mathbb{R}, \xi_{2n} > 0 \Big\}, \\ \widehat{\Sigma}_- &= \Big\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\ \widehat{\xi} = (\lambda\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)\xi_{2n}, \xi_{2n}), \lambda \in \mathbb{R}, \xi_{2n} < 0 \Big\}. \end{aligned} \tag{4.13}$$

Then, $\widehat{\sigma}$ is non-degenerate at every point of $\widehat{\Sigma}_+ \cup \widehat{\Sigma}_-$.

Consider the conic open set of $T^*\widehat{D}$ defined by

$$U = \Big\{ (\widehat{x}, \widehat{\xi}) \in T^*\widehat{D};\ \widehat{\xi} = (\xi, \xi_{2n}), \xi_{2n} > 0 \Big\}. \tag{4.14}$$

Until further notice, we work in U. Since $\widehat{\sigma}$ is non-degenerate at each point of $U \cap \widehat{\Sigma} = \widehat{\Sigma}_+$, (4.9) can be solved with $\operatorname{Im}\Psi \geq 0$ on U. More precisely, we have the following.

Theorem 4.4. *There exists $\Psi(t, \widehat{x}, \widehat{\eta}) \in \mathscr{C}^\infty(\overline{\mathbb{R}}_+ \times U)$ such that $\Psi(t, \widehat{x}, \widehat{\eta})$ is quasi-homogeneous of degree 1 and $\operatorname{Im}\Psi \geq 0$ and such that (4.9) holds where the error term is uniform on every set of the form $[0, T] \times K$ with $T > 0$ and $K \subset U$ compact. Furthermore, Ψ is unique up to a term which is $O(|\operatorname{Im}\Psi|^N)$ locally uniformly for every N and*

$$\begin{aligned} \Psi(t, \widehat{x}, \widehat{\eta}) &= \langle \widehat{x}, \widehat{\eta} \rangle \ \text{on } \widehat{\Sigma}_+, \\ d_{\widehat{x},\widehat{\eta}}(\Psi - \langle \widehat{x}, \widehat{\eta} \rangle) &= 0 \ \text{on } \widehat{\Sigma}_+. \end{aligned} \tag{4.15}$$

Moreover, we have

$$\operatorname{Im}\Psi(t, \widehat{x}, \widehat{\eta}) \asymp \left(|\widehat{\eta}| \frac{t\,|\widehat{\eta}|}{1 + t\,|\eta|} \right) \left(\operatorname{dist}\left((\widehat{x}, \frac{\widehat{\eta}}{|\widehat{\eta}|}), \widehat{\Sigma}_+ \right) \right)^2, \quad t \geq 0, \quad (\widehat{x}, \widehat{\eta}) \in U. \tag{4.16}$$

Furthermore, we can take $\Psi(t, \widehat{x}, \widehat{\eta})$ so that

$$\Psi(t, \widehat{x}, \widehat{\eta}) = \Psi(t, (x, 0), \widehat{\eta}) + x_{2n}\eta_{2n}. \tag{4.17}$$

Theorem 4.5. *There exists a function $\Psi(\infty, \widehat{x}, \widehat{\eta}) \in \mathscr{C}^\infty(U)$ with a uniquely determined Taylor expansion at each point of $\widehat{\Sigma}_+$ such that $\Psi(\infty, \widehat{x}, \widehat{\eta})$ is positively homogeneous of degree 1 and for every compact set $K \subset U$ there is a $c_K > 0$ such that*

$$\begin{aligned} &\operatorname{Im}\Psi(\infty, \widehat{x}, \widehat{\eta}) \geq c_K\,|\widehat{\eta}| \left(\operatorname{dist}\left((\widehat{x}, \frac{\widehat{\eta}}{|\widehat{\eta}|}), \widehat{\Sigma}_+ \right) \right)^2, \\ &d_{\widehat{x},\widehat{\eta}}(\Psi(\infty, \widehat{x}, \widehat{\eta}) - \langle \widehat{x}, \widehat{\eta} \rangle) = 0 \ \text{on } \widehat{\Sigma}_+. \end{aligned}$$

If $\lambda \in C(U)$, $\lambda > 0$ and $\lambda(\widehat{x}, \widehat{\xi}) < \min \lambda_j(\widehat{x}, \widehat{\xi})$, for all $(\widehat{x}, \widehat{\xi}) = (\widehat{x}, (\lambda\omega_0(x) - 2\operatorname{Im}\overline{\partial}_b\phi(x)\xi_{2n}, \xi_{2n})) \in \widehat{\Sigma}_+$, where $\lambda_j(\widehat{x}, \widehat{\xi})$ are the eigenvalues of the Hermitian quadratic form $\xi_{2n}R^L_x$, then the solution $\Psi(t, \widehat{x}, \widehat{\eta})$ of (4.9) can be chosen so that for every compact set $K \subset U$ and all indices α, β, γ, there is a constant $c_{\alpha,\beta,\gamma,K} > 0$ such that

$$\left| \partial^\alpha_{\widehat{x}} \partial^\beta_{\widehat{\eta}} \partial^\gamma_t (\Psi(t, \widehat{x}, \widehat{\eta}) - \Psi(\infty, \widehat{x}, \widehat{\eta})) \right| \leq c_{\alpha,\beta,\gamma,K} e^{-\lambda(\widehat{x},\widehat{\eta})t} \ \text{on } \overline{\mathbb{R}}_+ \times K. \tag{4.18}$$

For the proofs of Theorem 4.4 and Theorem 4.5, we refer to Menikoff-Sjöstrand [40], [22] and [24, Section 4.1].

From now on, we assume that $\Psi(t,\widehat{x},\widehat{\eta})$ has the form (4.17) and hence

$$\Psi(\infty,\widehat{x},\widehat{\eta})=\Psi(\infty,(x,0),\widehat{\eta})+x_{2n}\eta_{2n}. \tag{4.19}$$

We let the full symbol of $\Box_s^{(0)}$ be $\sum_{j=0}^{2}\widehat{p}_j(\widehat{x},\widehat{\xi})$, where $\widehat{p}_j(\widehat{x},\widehat{\xi})$ is positively homogeneous of order $2-j$. We apply $\partial_t+\Box_s^{(0)}$ formally under the integral in (4.8) and then introduce the asymptotic expansion of $\Box_s^{(0)}(ae^{i\Psi})$. Setting $(\partial_t+\Box_s^{(0)})(ae^{i\Psi})\sim 0$ and regrouping the terms according to the degree of quasi-homogeneity, we obtain for each N the transport equations

$$\begin{cases}T(t,\widehat{x},\widehat{\eta},\partial_t,\partial_{\widehat{x}})a_0=O(\lvert\operatorname{Im}\Psi\rvert^N),\\ T(t,\widehat{x},\widehat{\eta},\partial_t,\partial_{\widehat{x}})a_j+R_j(t,\widehat{x},\widehat{\eta},a_0,\ldots,a_{j-1})=O(\lvert\operatorname{Im}\Psi\rvert^N)\,.\end{cases} \tag{4.20}$$

Here

$$T(t,\widehat{x},\widehat{\eta},\partial_t,\partial_{\widehat{x}})=\partial_t-i\sum_{j=1}^{2n}\frac{\partial\widehat{p}_0}{\partial\xi_j}(\widehat{x},\Psi'_{\widehat{x}})\frac{\partial}{\partial x_j}+q(t,\widehat{x},\widehat{\eta}),$$

where

$$q(t,\widehat{x},\widehat{\eta})=\widehat{p}_1(\widehat{x},\Psi'_{\widehat{x}})+\frac{1}{2i}\sum_{j,t=1}^{2n}\frac{\partial^2\widehat{p}_0(\widehat{x},\Psi'_{\widehat{x}})}{\partial\xi_j\partial\xi_t}\frac{\partial^2\Psi(t,\widehat{x},\widehat{\eta})}{\partial x_j\partial x_t}$$

and R_j is a linear differential operator acting on $a_0,a_1,\ldots,a_{j-1}$. We note that $q(t,\widehat{x},\widehat{\eta})\to q(\infty,\widehat{x},\widehat{\eta})$ as $t\to\infty$, exponentially fast in the sense of (4.18) and the same is true for the coefficients of R_j, for all j.

Following [24], we can solve the transport equations (4.20). To state the results precisely, we pause and introduce some symbol spaces.

Definition 4.6. Let $\mu\geq 0$ be a non-negative constant. We say that $a\in\widehat{S}^m_\mu(\overline{\mathbb{R}}_+\times U)$ if $a\in\mathscr{C}^\infty(\overline{\mathbb{R}}_+\times U)$ and for all indices $\alpha,\beta\in\mathbb{N}_0^{2n}$, $\gamma\in\mathbb{N}_0$, every compact set $K\Subset\widehat{D}$, there exists a constant $c>0$ such that

$$\left|\partial_t^\gamma\partial_{\widehat{x}}^\alpha\partial_{\widehat{\eta}}^\beta a(t,\widehat{x},\widehat{\eta})\right|\leq ce^{-t\mu\lvert\eta_{2n}\rvert}(1+\lvert\eta\rvert)^{m+\gamma-\lvert\beta\rvert},\quad \widehat{x}\in K,\ (\widehat{x},\widehat{\eta})\in U.$$

Put $\widehat{S}^{-\infty}_\mu(\overline{\mathbb{R}}_+\times U):=\bigcap_{m\in\mathbb{R}}\widehat{S}^m_\mu(\overline{\mathbb{R}}_+\times U)$. Let $a_j\in\widehat{S}^{m_j}_\mu(\overline{\mathbb{R}}_+\times U)$, $j\in\mathbb{N}_0$, with $m_j\to-\infty$, $j\to\infty$. Then there exists $a\in\widehat{S}^{m_0}_\mu(\overline{\mathbb{R}}_+\times U)$, unique modulo $\widehat{S}^{-\infty}_\mu(\overline{\mathbb{R}}_+\times U)$, such that $a-\sum_{j=0}^{k-1}a_j\in\widehat{S}^{m_k}_\mu(\overline{\mathbb{R}}_+\times U)$ for $k\in\mathbb{N}_0$. If a and a_j have the properties above, we write $a\sim\sum_{j=0}^{\infty}a_j$ in $\widehat{S}^{m_0}_\mu(\overline{\mathbb{R}}_+\times U)$. Following the proof of [24, Theorem 4.15] we get:

Theorem 4.7. *We can find solutions* $a_j(t,\widehat{x},\widehat{\eta})\in\widehat{S}^{-j}_0(\overline{\mathbb{R}}_+\times U)$, $j=0,1,\ldots$ *of the system* (4.20), *where* $a_j(t,\widehat{x},\widehat{\eta})$ *is a quasi-homogeneous function of degree* $-j$, *for each* j, *with*

$$a_0(0,\widehat{x},\widehat{\eta})=1 \text{ on } U,\quad a_j(t,\widehat{x},\widehat{\eta})=0 \text{ on } U,\ j=1,2,\ldots, \tag{4.21}$$

$$\begin{aligned}&a_j(t,\widehat{x},\widehat{\eta})-a_j(\infty,\widehat{x},\widehat{\eta})\in\widehat{S}^{-j}_\mu(\overline{\mathbb{R}}_+\times U),\quad j=0,1,2,\ldots,\\&a_0(\infty,\widehat{x},\widehat{\eta})\neq 0,\quad \forall(\widehat{x},\widehat{\eta})\in\widehat{\Sigma}_+,\end{aligned} \tag{4.22}$$

where $\mu>0$ *is a constant and* $a_j(\infty,\widehat{x},\widehat{\eta})\in\mathscr{C}^\infty(U)$, $j=0,1,\ldots$, $a_j(\infty,\widehat{x},\widehat{\eta})$ *is a positively homogeneous function of degree* $-j$, *for each* j.

Let $m\in\mathbb{R}$, $0\leq\rho,\delta\leq 1$. For a conic open subset Γ of $T^*\widehat{D}$, let $S^m_{\rho,\delta}(\Gamma)$ denote the Hörmander symbol space on Γ of order m type (ρ,δ) (see [18, Definition 1.1]) and let $S^m_{\rm cl}(\Gamma)$ denote the space of classical symbols on Γ of order m (see [18, p. 35]). Let $B\subset D$ be an open set. Let $L^m_{\frac{1}{2},\frac{1}{2}}(B)$ and $L^m_{\rm cl}(B)$ denote the space of pseudodifferential operators on B of order m type $(\frac{1}{2},\frac{1}{2})$ and the space of classical pseudodifferential operators on B of order m. The classical result of Calderon and Vaillancourt [21, Theorem 18.6.6] tells us that for any $A\in L^m_{\frac{1}{2},\frac{1}{2}}(B)$,

$$A: H^s_{\rm comp}(B)\to H^{s-m}_{\rm loc}(B) \text{ is continuous, for every } s\in\mathbb{R}. \tag{4.23}$$

We return to our situation. For $j\in\mathbb{N}_0$, let $a_j(t,\widehat{x},\widehat{\eta})\in\widehat{S}^{-j}_0(\overline{\mathbb{R}}_+\times U)$ and $a_j(\infty,\widehat{x},\widehat{\eta})\in\mathscr{C}^\infty(U)$ be as in Theorem 4.7. Let

$$\begin{aligned}&a(\infty,\widehat{x},\widehat{\eta})\sim\sum_{j=0}^\infty a_j(\infty,\widehat{x},\widehat{\eta}) \text{ in } S^0_{1,0}(U),\\&a(t,\widehat{x},\widehat{\eta})\sim\sum_{j=0}^\infty a_j(t,\widehat{x},\widehat{\eta}) \text{ in } \widehat{S}^0_0(\overline{\mathbb{R}}_+\times U),\\&a(t,\widehat{x},\widehat{\eta})-a(\infty,\widehat{x},\widehat{\eta})\in\widehat{S}^0_\mu(\overline{\mathbb{R}}_+\times U),\quad \mu>0.\end{aligned} \tag{4.24}$$

Take $\alpha(\eta_{2n})\in\mathscr{C}^\infty(\mathbb{R})$ with $\alpha(\eta_{2n})=1$ if $\eta_{2n}\leq\frac{1}{2}$, $\alpha(\eta_{2n})=0$ if $\eta_{2n}\geq 1$. Choose $\chi\in\mathscr{C}^\infty_0(\mathbb{R}^{2n})$ so that $\chi(\widehat{\eta})=1$ when $|\widehat{\eta}|<1$ and $\chi(\widehat{\eta})=0$ when $|\widehat{\eta}|>2$. For $\varepsilon>0$, put

$$\begin{aligned}G_\varepsilon(\widehat{x},\widehat{y})=\frac{1}{(2\pi)^{2n}}\int\Big(\int_0^\infty&\big(e^{i(\Psi(t,\widehat{x},\widehat{\eta})-\langle\widehat{y},\widehat{\eta}\rangle)}a(t,\widehat{x},\widehat{\eta})\\&-e^{i(\Psi(\infty,\widehat{x},\widehat{\eta})-\langle\widehat{y},\widehat{\eta}\rangle)}a(\infty,\widehat{x},\widehat{\eta})\big)(1-\chi(\widehat{\eta}))\chi(\varepsilon\widehat{\eta})(1-\alpha(\eta_{2n}))dt\Big)d\widehat{\eta}.\end{aligned}$$

By Chapter 5 in part I of [22], we have for any $u \in \mathscr{C}_0^\infty(\widehat{D})$,

$$\lim_{\varepsilon\to 0}\int G_\varepsilon(\widehat{x},\widehat{y})u(\widehat{y})d\widehat{y} \in \mathscr{C}^\infty(\widehat{D}),$$

and the operator $G : \mathscr{C}_0^\infty(\widehat{D}) \to \mathscr{C}^\infty(\widehat{D})$, $u \mapsto \lim_{\varepsilon\to 0}\int G_\varepsilon(\widehat{x},\widehat{y})u(\widehat{y})dy$, is continuous, has a unique continuous extension: $G : \mathscr{E}'(\widehat{D}) \to \mathscr{D}'(\widehat{D})$ and $G \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\widehat{D})$ with symbol

$$q(\widehat{x},\widehat{\eta}) = \int_0^\infty \Big(e^{i(\Psi(t,\widehat{x},\widehat{\eta})-\langle\widehat{x},\widehat{\eta}\rangle)}a(t,\widehat{x},\widehat{\eta}) - e^{i(\Psi(\infty,\widehat{x},\widehat{\eta})-\langle\widehat{x},\widehat{\eta}\rangle)}a(\infty,\widehat{x},\widehat{\eta})\Big)dt \\ \times (1-\alpha(\eta_{2n}))$$

in $S^{-1}_{\frac{1}{2},\frac{1}{2}}(T^*\widehat{D})$. We denote

$$G(\widehat{x},\widehat{y}) = \frac{1}{(2\pi)^{2n}}\int\Big(\int_0^\infty \big(e^{i(\Psi(t,\widehat{x},\widehat{\eta})-\langle\widehat{y},\widehat{\eta}\rangle)}a(t,\widehat{x},\widehat{\eta}) \\ - e^{i(\Psi(\infty,\widehat{x},\widehat{\eta})-\langle\widehat{y},\widehat{\eta}\rangle)}a(\infty,\widehat{x},\widehat{\eta})\big)(1-\chi(\widehat{\eta}))(1-\alpha(\eta_{2n}))dt\Big)d\widehat{\eta}. \tag{4.25}$$

Similarly, for $\varepsilon > 0$, put

$$S_\varepsilon(\widehat{x},\widehat{y}) = \frac{1}{(2\pi)^{2n}}\int e^{i(\Psi(\infty,\widehat{x},\widehat{\eta})-\langle\widehat{y},\widehat{\eta}\rangle)}a(\infty,\widehat{x},\widehat{\eta})(1-\chi(\widehat{\eta}))\chi(\varepsilon\widehat{\eta})(1-\alpha(\eta_{2n}))d\widehat{\eta}.$$

By [22, Chapter 5, part I]) we have for $u \in \mathscr{C}_0^\infty(\widehat{D})$, $\lim_{\varepsilon\to 0}\int S_\varepsilon(\widehat{x},\widehat{y})u(\widehat{y})d\widehat{y} \in \mathscr{C}^\infty(\widehat{D})$, the operator

$$S : \mathscr{C}_0^\infty(\widehat{D}) \to \mathscr{C}^\infty(\widehat{D}), \quad u \mapsto \lim_{\varepsilon\to 0}\int S_\varepsilon(\widehat{x},\widehat{y})u(\widehat{y})dy, \tag{4.26}$$

is continuous, has a unique continuous extension: $S : \mathscr{E}'(\widehat{D}) \to \mathscr{D}'(\widehat{D})$ and $S \in L^0_{\frac{1}{2},\frac{1}{2}}(\widehat{D})$ with symbol $s(\widehat{x},\widehat{\eta}) = e^{i(\Psi(\infty,\widehat{x},\widehat{\eta})-\langle\widehat{x},\widehat{\eta}\rangle)}a(\infty,\widehat{x},\widehat{\eta})(1-\alpha(\eta_{2n})) \in S^0_{\frac{1}{2},\frac{1}{2}}(T^*\widehat{D})$. We denote

$$S(\widehat{x},\widehat{y}) = \frac{1}{(2\pi)^{2n}}\int e^{i(\Psi(\infty,\widehat{x},\widehat{\eta})-\langle\widehat{y},\widehat{\eta}\rangle)}a(\infty,\widehat{x},\widehat{\eta})\big(1-\chi(\widehat{\eta})\big)\big(1-\alpha(\eta_{2n})\big)d\widehat{\eta}. \tag{4.27}$$

Put

$$\widetilde{I} = (2\pi)^{-2n}\int e^{i\langle\widehat{x}-\widehat{y},\widehat{\eta}\rangle}\big(1-\alpha(\eta_{2n})\big)d\widehat{\eta}. \tag{4.28}$$

We can repeat the proof of [22, Proposition 6.5] with minor changes and obtain:

Theorem 4.8. *With the notations used above, we have*

$$S+\Box^{(0)}_s\circ G\equiv\widetilde{I}\ \ \text{on}\ \widehat{D},\quad \overline{\partial}_s\circ S\equiv 0\ \ \text{on}\ \widehat{D},\quad \Box^{(0)}_s\circ S\equiv 0\ \ \text{on}\ \widehat{D}. \tag{4.29}$$

The next result follows from the complex stationary phase formula [39] with essentially the same proof as of [24, Theorem 4.29].

Theorem 4.9. *With the notations and assumptions above, let* $S=S(\widehat{x},\widehat{y})\in L^0_{\frac12,\frac12}(\widehat{D})$ *be as in Theorem 4.8. Then, on* $\widehat{D}$*, we have*

$$S(\widehat{x},\widehat{y})\equiv\int_{u\in\mathbb{R},t\in\mathbb{R}_+}e^{i\Phi(\widehat{x},\widehat{y},u,t)}b(\widehat{x},\widehat{y},u,t)(1-\alpha(t))dudt \tag{4.30}$$

with symbol

$$\begin{aligned}&b(\widehat{x},\widehat{y},u,t)\sim\sum_{j=0}^{\infty}b_j(\widehat{x},\widehat{y},u,t)\ \text{in}\ S^{n-1}_{1,0}(\widehat{D}\times\widehat{D}\times\mathbb{R}\times\mathbb{R}_+),\\&b_j(\widehat{x},\widehat{y},\lambda u,\lambda t)=\lambda^{n-1-j}b_j(\widehat{x},\widehat{y},u,t),\\&\forall(\widehat{x},\widehat{y},u,t)\in\widehat{D}\times\widehat{D}\times\mathbb{R}\times\mathbb{R}_+,\quad\lambda\geq1,\quad\forall j,\\&b_0(\widehat{x},\widehat{x},u,t)\neq0,\quad\forall(\widehat{x},\widehat{y},u,t)\in\widehat{D}\times\widehat{D}\times\mathbb{R}\times\mathbb{R}_+,\quad\lambda\geq1,\end{aligned} \tag{4.31}$$

and phase function

$$\begin{aligned}&\Phi(\widehat{x},\widehat{y},u,t)=(x_{2n}-y_{2n})t+\varphi(x,y,u,t),\\&\varphi(x,y,u,t)\in\mathscr{C}^\infty(D\times D\times\mathbb{R}\times\mathbb{R}_+),\\&\varphi(x,y,\lambda u,\lambda t)=\lambda\varphi(x,y,u,t),\\&\forall(x,y,u,t)\in D\times D\times\mathbb{R}\times\mathbb{R}_+,\quad\lambda\geq1,\\&\operatorname{Im}\varphi(x,y,u,t)\geq0,\quad\varphi(x,x,u,t)=0,\quad\forall x\in D,\quad u\in\mathbb{R},\quad t\in\mathbb{R}_+,\\&d_x\varphi|_{(x,x,u,t)}=-2t\operatorname{Im}\overline{\partial}_b\phi(x)+u\omega_0(x),\quad\forall x\in D,\quad u\in\mathbb{R},\quad t\in\mathbb{R}_+,\\&d_y\varphi|_{(x,x,u,t)}=2t\operatorname{Im}\overline{\partial}_b\phi(x)-u\omega_0(x),\quad\forall x\in D,\quad u\in\mathbb{R},\quad t\in\mathbb{R}_+,\\&\tfrac{\partial\varphi}{\partial u}(x,y,u,t)=0\ \text{and}\ \tfrac{\partial\varphi}{\partial t}(x,y,u,t)=0\ \text{if and only if}\ x=y.\end{aligned} \tag{4.32}$$

We can repeat the method in [24, Section 4.4] with minor changes to compute the tangential Hessian of the phase function $\varphi(x,y,u,t)$. This will yield theTaylor expansion of the phase function ψ from Theorems 1.2 and 1.3, see Theorem 4.10. Since the computation is simpler we therefore omit the details. We only state the result. Fix $p \in D$ and let $\overline{Z}_1, \ldots, \overline{Z}_{n-1}$ be an orthonormal frame of $T^{1,0}_x X$ varying smoothly with x in a neighbourhood of p, for which the Hermitian quadratic form R^L_x is diagonalized at $x = p$. Let s be a local trivializing section of L and let $x = (x_1, \ldots, x_{2n-1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, be local coordinates of X defined in some small neighbourhood of p such that

$$
\begin{aligned}
& x(p) = 0,\ \omega_0(p) = dx_{2n-1},\ T(p) = -\frac{\partial}{\partial x_{2n-1}}, \\
& \Big\langle \frac{\partial}{\partial x_j}(p) \,|\, \frac{\partial}{\partial x_t}(p) \Big\rangle = 2\delta_{j,t}, \quad j, t = 1, \ldots, 2n-2, \\
& \overline{Z}_j(x) = \frac{\partial}{\partial z_j} + i \sum_{t=1}^{n-1} \tau_{j,t} \overline{z}_t \frac{\partial}{\partial x_{2n-1}} \\
& \qquad + c_j x_{2n-1} \frac{\partial}{\partial x_{2n-1}} + O(|x|^2), \quad j = 1, \ldots, n-1, \\
& \phi(x) = \beta x_{2n-1} + \sum_{j=1}^{n-1} \big(\alpha_j z_j + \overline{\alpha}_j \overline{z}_j\big) \\
& \qquad + \frac{1}{2} \sum_{l,t=1}^{n-1} \mu_{t,l} z_t \overline{z}_l + \sum_{l,t=1}^{n-1} \big(a_{l,t} z_l z_t + \overline{a}_{l,t} \overline{z}_l \overline{z}_t\big) \\
& \qquad + \sum_{j=1}^{n-1} \big(d_j z_j x_{2n-1} + \overline{d}_j \overline{z}_j x_{2n-1}\big) + O(|x_{2n-1}|^2) + O(|x|^3),
\end{aligned}
\tag{4.33}
$$

where $\beta \in \mathbb{R}$, $\tau_{j,t}$, c_j, α_j, $\mu_{j,t}$, $a_{j,t}$, d_j are complex numbers, $\mu_{j,t} = \overline{\mu}_{t,j}$, $\tau_{j,t} + \overline{\tau}_{t,j} = 0$, $j, t = 1, \ldots, n-1$. We define now the phase function

$$
\psi(x, y, u) := \varphi(x, y, u, 1). \tag{4.34}
$$

Note that $\varphi(x, y, u, t)$ is positively homogeneous of degree 1 with respect to (u, t) but $\psi(x, y, u)$ fails to be positively homogeneous of degree 1 with respect to u. We work in local coordinates as in (4.33) and for $x = (x_1, \ldots, x_{2n-1})$ we denote $x' = (x_1, \ldots, x_{2n-2})$, $|x'|^2 = \sum_{j=1}^{2n-2} |x_j|^2$.

Theorem 4.10. *There exists a neighborhood D_0 of p and $c>0$ such that for all $(x,y,u)\in D_0\times D_0\times\mathbb{R}$ we have*

$$\begin{aligned}&\operatorname{Im}\psi(x,y,u)\geq c\left|x'-y'\right|^2,\\&\operatorname{Im}\psi(x,y,u)+\left|\frac{\partial\psi}{\partial u}(x,y,u)\right|\geq c(|x_{2n-1}-y_{2n-1}|+\left|x'-y'\right|^2)\end{aligned}\tag{4.35}$$

Moreover, there exists a function $f\in\mathscr{C}^\infty(D_0)$, $f(0,0,u)=0$ for $u\in\mathbb{R}$, such that

$$\begin{aligned}&\psi(x,y,u)\\&=-i\sum_{j=1}^{n-1}\alpha_j(z_j-w_j)+i\sum_{j=1}^{n-1}\overline{\alpha}_j(\overline{z}_j-\overline{w}_j)+u(x_{2n-1}-y_{2n-1})\\&\quad-\frac{i}{2}\sum_{j,l=1}^{n-1}(a_{l,j}+a_{j,l})(z_jz_l-w_jw_l)+\frac{i}{2}\sum_{j,l=1}^{n-1}(\overline{a}_{l,j}+\overline{a}_{j,l})(\overline{z}_j\overline{z}_l-\overline{w}_j\overline{w}_l)\\&\quad+\frac{1}{2}\sum_{j,l=1}^{n-1}iu(\overline{\tau}_{l,j}-\tau_{j,l})(z_j\overline{z}_l-w_j\overline{w}_l)\\&\quad+\sum_{j=1}^{n-1}(-ic_j\beta-uc_j-id_j)(z_jx_{2n-1}-w_jy_{2n-1})\\&\quad+\sum_{j=1}^{n-1}(i\overline{c}_j\beta-u\overline{c}_j+i\overline{d}_j)(\overline{z}_jx_{2n-1}-\overline{w}_jy_{2n-1})\\&\quad-\frac{i}{2}\sum_{j=1}^{n-1}\lambda_j(z_j\overline{w}_j-\overline{z}_jw_j)+\frac{i}{2}\sum_{j=1}^{n-1}\lambda_j\left|z_j-w_j\right|^2\\&\quad+(x_{2n-1}-y_{2n-1})f(x,y,u)+O(|(x,y)|^3),\end{aligned}\tag{4.36}$$

where $\lambda_j=\lambda_j(p)>0$, $j=1,\dots,n-1$, are the eigenvalues of R^L_p with respect to $\langle\cdot|\cdot\rangle$.

The form of ψ should be compared to the form [29, Theorems 3.2, 3.4] of the phase function for the Szegő kernel on a non-degenerate CR manifold.

Remark 4.11. The phase function $\Phi(\widehat{x},\widehat{y},u,t)$ has the following properties: there is a

$$h(\widehat{x},\widehat{y},u,t)\in\mathscr{C}^\infty(\hat{D}\times\hat{D}\times\mathbb{R}\times\mathbb{R}_+,\Lambda^{0,1}(T^*\hat{D}))$$

such that

$$\begin{aligned}(4.37)\qquad &\overline{\partial}_s\Phi(\widehat{x},\widehat{y},u,t)-h(\widehat{x},\widehat{y},u,t)\Phi(\widehat{x},\widehat{y},u,t)\\ &\text{vanishes to infinite order at } \widehat{x}=\widehat{y},\\ &\operatorname{Im}\Phi(\widehat{x},\widehat{y},u,t)\approx t\,|z-w|^2.\end{aligned}$$

The phase function Φ is not unique. Any complex phase function $\Phi_1(\widehat{x},\widehat{y},u,t)$ satisfying (4.37) (4.32) and (4.36), is equivalent to Φ in the sense of Melin-Sjöstrand [39]. From this observation, given $p\in D$, if we take local coordinates x and local holomorphic trivializing section s, $|s|^2_{h^L}=e^{-2\phi}$ such that (4.33) holds, then near p, we can take $\Phi(\widehat{x},\widehat{y},u,t)$ so that for every $N\in\mathbb{N}$,

$$\begin{aligned}(4.38)\qquad \Phi(\widehat{x},\widehat{y},u,t)&=t(x_{2n}-y_{2n})+u(x_{2n-1}-y_{2n-1})+it(\phi(x)+\phi(y))\\ &\quad-it\Bigg(\sum_{|\alpha|+|\beta|\le N}\frac{\partial^{|\alpha|+|\beta|}\phi}{\partial z^\alpha\partial\overline{z}^\beta}(0,x_{2n-1})\frac{z^\alpha}{\alpha!}\frac{\overline{w}^\beta}{\beta!}\\ &\qquad+\sum_{|\alpha|+|\beta|\le N}\frac{\partial^{|\alpha|+|\beta|}\phi}{\partial z^\alpha\partial\overline{z}^\beta}(0,y_{2n-1})\frac{z^\alpha}{\alpha!}\frac{\overline{w}^\beta}{\beta!}\Bigg)\\ &\quad+O(|z-w|^{N+1}).\end{aligned}$$

From (4.38), we have for every $N\in\mathbb{N}$,

$$\begin{aligned}(4.39)\qquad \psi(x,y,u)&=u(x_{2n-1}-y_{2n-1})+i(\phi(x)+\phi(y))\\ &\quad-i\Bigg(\sum_{|\alpha|+|\beta|\le N}\frac{\partial^{|\alpha|+|\beta|}\phi}{\partial z^\alpha\partial\overline{z}^\beta}(0,x_{2n-1})\frac{z^\alpha}{\alpha!}\frac{\overline{w}^\beta}{\beta!}\\ &\qquad+\sum_{|\alpha|+|\beta|\le N}\frac{\partial^{|\alpha|+|\beta|}\phi}{\partial z^\alpha\partial\overline{z}^\beta}(0,y_{2n-1})\frac{z^\alpha}{\alpha!}\frac{\overline{w}^\beta}{\beta!}\Bigg)\\ &\quad+O(|z-w|^{N+1}).\end{aligned}$$

4.2. Semi-classical Hodge decomposition for $\Box^{(0)}_{s,k}$

In this section we apply Theorem 4.8 and Theorem 4.9 to describe the semi-classical Hodge theory for $\Box^{(0)}_{s,k}$. In particular we define the approximate Szegő projector $\mathcal{S}_k$ which appears in Theorem 1.2 and study its kernel.

Let s be a local trivializing section of L on an open subset $D\subset X$ and $|s|^2_h=e^{-2\phi}$. Let $\chi(x_{2n}),\chi_1(x_{2n})\in\mathscr{C}^\infty_0(\mathbb{R})$, $\chi,\chi_1\ge 0$. We assume that $\chi_1=$

1 on $\operatorname{supp}\chi$. We take χ so that $\int \chi(x_{2n})dx_{2n} = 1$. Put

$$\chi_k(x_{2n}) = e^{ikx_{2n}}\chi(x_{2n}). \tag{4.40}$$

We say that a sequence (g_k) in $\mathbb{C}$ is rapidly decreasing and write $g_k = O(k^{-\infty})$ if for every $N > 0$, there exists $C_N > 0$ independent of k such that for all k we have $|g_k| \leq C_N k^{-N}$.

Proposition 4.12. *Let* $\widetilde{I} = (2\pi)^{-2n}\int e^{i\langle\widehat{x}-\widehat{y},\widehat{\eta}\rangle}(1-\alpha(\eta_{2n}))d\widehat{\eta}$ *be as in* (4.28). *Let* $\widetilde{I}_k$ *be the continuous operator* $\mathscr{C}_0^\infty(D) \to \mathscr{C}^\infty(D)$ *given by*

$$\widetilde{I}_k : \mathscr{C}_0^\infty(D) \to \mathscr{C}^\infty(D),\ \ f \longmapsto \int e^{-ikx_{2n}}\chi_1(x_{2n})\widetilde{I}(\chi_k f)(\widehat{x})dx_{2n}.$$

Then, $\widetilde{I}_k = (1+g_k)I$ *on* $\mathscr{C}_0^\infty(D)$, *where* I *is the identity map on* $\mathscr{C}_0^\infty(D)$ *and* (g_k) *is a rapidly decreasing sequence.*

Proof. It is easy to see that

$$I = (2\pi)^{-2n}\int e^{i\langle\widehat{x}-\widehat{y},\widehat{\eta}\rangle-ik(x_{2n}-y_{2n})}\chi_1(x_{2n})\chi(y_{2n})\,d\widehat{\eta}\,dy_{2n}\,dx_{2n}\ \text{ on } \mathscr{C}_0^\infty(D).$$

From this observation, we can check that $\widetilde{I}_k = (1+g_k)I$ where

$$g_k = -(2\pi)^{-2n}\int e^{i\langle x_{2n}-y_{2n},\eta_{2n}-k\rangle}\alpha(\eta_{2n})\chi_1(x_{2n})\chi(y_{2n})d\eta_{2n}\,dy_{2n}\,dx_{2n}. \tag{4.41}$$

Since $\alpha(\eta_{2n}) = 0$ if $\eta \geq 1$, we can integrate by parts in (4.41) with respect to y_{2n} several times and conclude that $g_k = O(k^{-\infty})$. ∎

Let $S \in L^0_{\frac{1}{2},\frac{1}{2}}(\widehat{D})$ and $G \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(\widehat{D})$ be as in Theorem 4.8. For $s \in \mathbb{N}_0$ define

$$\mathcal{S}_k : H^s_{\mathrm{comp}}(D) \to H^s_{\mathrm{loc}}(D),\ \ f \longmapsto \frac{1}{1+g_k}\int e^{-ikx_{2n}}\chi_1(x_{2n})S(\chi_k f)(\widehat{x})dx_{2n}, \tag{4.42}$$

$$\mathcal{G}_k : H^s_{\mathrm{loc}}(D) \to H^{s+1}_{\mathrm{loc}}(D),\ \ f \longmapsto \frac{1}{1+g_k}\int e^{-ikx_{2n}}\chi_1(x_{2n})G(\chi_k f)(\widehat{x})dx_{2n}. \tag{4.43}$$

The operator $\mathcal{S}_k$ is the approximate Szegő projector and $\mathcal{G}_k$ is the corresponding Green operator. From (4.42), (4.43) and the fact that $S : H^s_{\mathrm{comp}}(\widehat{D})$

$\to H^s_{\rm loc}(\widehat{D})$ is continuous for every $s \in \mathbb{R}$, $G : H^s_{\rm comp}(\widehat{D}) \to H^{s+1}_{\rm loc}(\widehat{D})$ is continuous for every $s \in \mathbb{R}$, it is straightforward to check that

$$\begin{aligned} &\mathcal{S}_k = O(k^s) : H^s_{\rm comp}(D) \to H^s_{\rm loc}(D), \quad \forall s \in \mathbb{N}_0, \\ &\mathcal{G}_k = O(k^s) : H^s_{\rm comp}(D) \to H^{s+1}_{\rm loc}(D), \quad \forall s \in \mathbb{N}_0. \end{aligned} \tag{4.44}$$

Repeating the proof of [24, Theorem 5.4] by making use of Proposition 4.12 we get the semiclassical Hodge theory for the localized Kohn laplacian $\Box^{(0)}_{s,k}$:

Theorem 4.13. *Let s be a local trivializing section of L on an open subset $D \subset X$ and $|s|^2_h = e^{-2\phi}$. Let $\mathcal{S}_k$ and $\mathcal{G}_k$ be as in* (4.42), (4.43) *respectively. Then,*

$$\begin{aligned} &\mathcal{S}^*_k, \mathcal{S}_k = O(k^s) : H^s_{\rm comp}(D) \to H^s_{\rm loc}(D), \quad \forall s \in \mathbb{Z}, \\ &\mathcal{G}^*_k, \mathcal{G}_k = O(k^s) : H^s_{\rm comp}(D) \to H^{s+1}_{\rm loc}(D), \quad \forall s \in \mathbb{Z}, \end{aligned} \tag{4.45}$$

and we have on D,

$$\overline{\partial}_{s,k}\mathcal{S}_k \equiv 0 \mod O(k^{-\infty}), \tag{4.46}$$

$$\Box^{(0)}_{s,k}\mathcal{S}_k \equiv 0, \quad \mathcal{S}^*_k\Box^{(0)}_{s,k} \equiv 0 \mod O(k^{-\infty}), \tag{4.47}$$

$$\mathcal{S}_k + \Box^{(0)}_{s,k}\mathcal{G}_k = I, \tag{4.48}$$

$$\mathcal{G}^*_k\Box^{(0)}_{s,k} + \mathcal{S}^*_k = I, \tag{4.49}$$

where $\mathcal{S}^*_k$, $\mathcal{G}^*_k$ are the formal adjoints of $\mathcal{S}_k$, $\mathcal{G}_k$ with respect to $(\cdot\,|\,\cdot)$ respectively and $\Box^{(0)}_{s,k}$ is given by (3.2).

We study further the kernel of the approximate Szegő projector.

Theorem 4.14. *Let ψ be the phase function* (4.34). *There exists a symbol*

$$\begin{aligned} &s(x,y,u,k) \in S^n_{\rm loc,cl}(1; D \times D \times \mathbb{R}), \\ &s(x,y,u,k) \sim \sum_{j=0}^{\infty} s_j(x,y,u)k^{n-j} \text{ in } S^n_{\rm loc}(1; D \times D \times \mathbb{R}), \end{aligned} \tag{4.50}$$

such that the operator S_k with kernel

$$S_k(x,y) = \int_{\mathbb{R}} e^{ik\psi(x,y,u)} s(x,y,u,k)du,$$

satisfies

$$\mathcal{S}_k(x,y) - S_k(x,y) = O(k^{-\infty}) : H^s_{\rm comp}(D) \to H^s_{\rm loc}(D), \ \forall s \in \mathbb{Z}. \tag{4.51}$$

Proof. From the definition (4.42) of $\mathcal{S}_k$ and Theorem 4.9, we see that the distribution kernel of $\mathcal{S}_k$ is given by

$$\begin{aligned}(4.52)\qquad \mathcal{S}_k(x,y) &\equiv \int\limits_{t\in\mathbb{R}_+} e^{i\Phi(\widehat{x},\widehat{y},u,t)-ikx_{2n}+iky_{2n}}\\ &\qquad \times b(\widehat{x},\widehat{y},u,t)\chi_1(x_{2n})\chi(y_{2n})(1-\alpha(t))dx_{2n}dtdy_{2n}du\\ &\equiv \int\limits_{\substack{u\in\mathbb{R}\\ \sigma\in\mathbb{R}_+}} e^{ik\varrho(x,y,u,\sigma)}k^2\sigma b(\widehat{x},\widehat{y},k\sigma u,k\sigma)\\ &\qquad \times \chi_1(x_{2n})\chi(y_{2n})(1-\alpha(k\sigma))dx_{2n}d\sigma dy_{2n}du,\end{aligned}$$

$\bmod O(k^{-\infty})$, where

$$\varrho(x,y,u,\sigma)=\sigma\psi(x,y,u)+(x_{2n}-y_{2n})(\sigma-1),$$

and the integrals above are defined as oscillatory integrals. Let $\gamma(\sigma)\in\mathscr{C}_0^\infty(\mathbb{R}_+)$ with $\gamma(\sigma)=1$ in some small neighbourhood of 1. Denote by $I_0(x,y)$ the integral

$$\begin{aligned}(4.53)\qquad &\int\limits_{\sigma\geq 0} e^{ik\varrho(x,y,u,\sigma)}\gamma(\sigma)k^2\sigma b(\widehat{x},\widehat{y},k\sigma u,k\sigma)\\ &\qquad \times(1-\alpha(k\sigma))\chi_1(x_{2n})\chi(y_{2n})dx_{2n}d\sigma dy_{2n}du,\end{aligned}$$

and by $I_1(x,y)$ the integral

$$\begin{aligned}(4.54)\qquad &\int\limits_{\sigma\geq 0} e^{ik\varrho(x,y,u,\sigma)}(1-\gamma(\sigma))k^2\sigma b(\widehat{x},\widehat{y},k\sigma u,k\sigma)\\ &\qquad \times(1-\alpha(k\sigma))\chi_1(x_{2n})\chi(y_{2n})dx_{2n}d\sigma dy_{2n}du.\end{aligned}$$

Then,

$$(4.55)\qquad \mathcal{S}_k(x,y)\equiv I_0(x,y)+I_1(x,y)\mod O(k^{-\infty}).$$

First, we study $I_1(x,y)$. Note that when $\sigma\neq 1$, $d_{y_{2n}}\big(\sigma\psi(x,y,u)+(x_{2n}-y_{2n})(\sigma-1)\big)=1-\sigma\neq 0$. Thus, we can integrate by parts in y_{2n} several times and get that

$$I_1 = O(k^{-\infty}) : H^s_{\rm comp}(D) \to H^s_{\rm loc}(D), \quad \forall s \in \mathbb{Z}. \tag{4.56}$$

Next, we study the kernel $I_0(x,y)$. We may assume that $b(\widehat{x},\widehat{y},k\sigma u,k\sigma)$ is supported in some small neighbourhood of $\widehat{x}=\widehat{y}$. We want to apply the stationary phase method of Melin and Sjöstrand [39, p. 148] to carry out the $dx_{2n}d\sigma$ integration in (4.53). Put

$$\Theta(\widehat{x},\widehat{y},\sigma) := \sigma\psi(x,y,u) + (x_{2n}-y_{2n})(\sigma-1).$$

We first notice that $d_\sigma\Theta(\widehat{x},\widehat{y},\sigma)|_{\widehat{x}=\widehat{y}}=0$ and $d_{x_{2n}}\Theta(\widehat{x},\widehat{y},\sigma)|_{\sigma=1}=0$. Thus, $x=y$ and $\sigma=1$ are real critical points. Furthermore, we have

$$\begin{aligned}&\Theta''_{\sigma\sigma}(\widehat{x},\widehat{x},1)=0, \quad \Theta''_{x_{2n}\sigma}(\widehat{x},\widehat{x},1)=1,\\ &\Theta''_{\sigma x_{2n}}(\widehat{x},\widehat{x},1)=1, \quad \Theta''_{x_{2n}x_{2n}}(\widehat{x},\widehat{x},1)=0.\end{aligned}$$

Thus, the Hessian of $\Theta(\widehat{x},\widehat{y},\sigma)$ with respect to (σ,x_{2n}) at $\widehat{x}=\widehat{y}$, $\sigma=1$, is given by

$$\begin{pmatrix} \Theta''_{\sigma\sigma}(\widehat{x},\widehat{x},1) & \Theta''_{x_{2n}\sigma}(\widehat{x},\widehat{x},1) \\ \Theta''_{\sigma x_{2n}}(\widehat{x},\widehat{x},1) & \Theta''_{x_{2n}x_{2n}}(\widehat{x},\widehat{x},1)\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

Thus, $\Theta(\widehat{x},\widehat{y},\sigma)$ is a non-degenerate complex valued phase function in the sense of Melin-Sjöstrand [39]. Let $\widetilde{\Theta}(\widetilde{\widehat{x}},\widetilde{\widehat{y}},\widetilde{\sigma}) := \widetilde{\psi}(\widetilde{x},\widetilde{y},u)\widetilde{\sigma} + (\widetilde{x}_{2n}-\widetilde{y}_{2n})(\widetilde{\sigma}-1)$ be an almost analytic extension of $\Theta(\widehat{x},\widehat{y},\widehat{\sigma})$, where $\widetilde{\psi}(\widetilde{x},\widetilde{y},u)$ is an almost analytic extension of $\psi(x,y,u)$. Here we fix u. We can check that given y_{2n} and (x,y), $\widetilde{x}_{2n}=y_{2n}-\psi(x,y,u)$, $\widetilde{\sigma}=1$ are the solutions of $\frac{\partial\widetilde{\Theta}}{\partial\widetilde{\sigma}}=0$, $\frac{\partial\widetilde{\Theta}}{\partial\widetilde{x}_{2n}}=0$. From this and by the stationary phase formula of Melin-Sjöstrand [39], we get

$$\begin{aligned} I_0(x,y) - \int e^{ik\psi(x,y,u)}s(x,y,u,k)du \\ = O(k^{-\infty}) : H^s_{\rm comp}(D) \to H^s_{\rm loc}(D), \quad \forall s\in\mathbb{Z},\end{aligned} \tag{4.57}$$

where $s(x,y,u,k) \in S^n_{{\rm loc},{\rm cl}}(1, D\times D\times\mathbb{R})$,

$$s(x,y,u,k) \sim \sum_{j=0}^{\infty} s_j(x,y,u)k^{n-j} \text{ in } S^n_{\rm loc}(1, D\times D\times\mathbb{R}),$$

with $s_j(x,y,u)\in\mathscr{C}^\infty(D\times D\times\mathbb{R})$, $j=0,1,2,\ldots$. From (4.56), (4.57) and (4.55), the theorem follows. $\square$

We show now that the composition of $\mathcal{S}_k$ with a classical semi-classical pseudodifferential has an asymptotic expansion and calculate the leading term. From Theorem 4.14 and the stationary phase method of Melin and Sjöstrand, we deduce:

Theorem 4.15. *Let $\mathcal{A}_k$ be a properly supported classical semi-classical pseudodifferential operator on D of order 0 as in (2.6) and (2.5) with symbol $\beta \in S^0_{\mathrm{loc},\mathrm{cl}}(1; T^*D)$ such that $\beta(x,\eta,k) = 0$ if $|\eta| \geq \frac{1}{2}M$, for some large $M > 0$. We have*

$$(\mathcal{S}_k \circ \mathcal{A}_k)(x,y) \equiv \int e^{ik\psi(x,y,u)} a(x,y,u,k) du \mod O(k^{-\infty}), \tag{4.58}$$

where

(4.59)
$$\begin{aligned}
& a(x,y,u,k) \in \mathscr{C}^\infty_0(D \times D \times (-M,M)) \cap S^n_{\mathrm{loc},\mathrm{cl}}(1; D \times D \times (-M,M)), \\
& a(x,y,u,k) \sim \sum_{j=0}^{\infty} a_j(x,y,u) k^{n-j} \ \textit{in}\ S^n_{\mathrm{loc}}(1; D \times D \times -(M,M)), \\
& a_j(x,y,u) \in \mathscr{C}^\infty_0(D \times D \times (-M,M)), \quad j = 0,1,2,\ldots,
\end{aligned}$$

and $\psi(x,y,u) = \varphi(x,y,u,1)$, $\varphi(x,y,u,t)$ is as in Theorem 4.9.

Recall that $\mathcal{A}_k$ is called properly supported if the restrictions of the projections $(x,y) \mapsto x$ and $(x,y) \mapsto y$ to $\operatorname{supp} \mathcal{A}_k(\cdot,\cdot) \subset X \times X$ are proper. Let

$$\mathcal{A}_k \equiv \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik\langle x-y,\eta\rangle} \beta(x,\eta,k) d\eta \mod O(k^{-\infty})$$

be as in Theorem 4.15. Put

$$\beta(x,\eta,k) \sim \sum_{j=0}^{\infty} \beta_j(x,\eta) k^{-j}, \ \ \beta_j(x,\eta) \in \mathscr{C}^\infty(T^*D), \quad j = 0,1,2,\ldots. \tag{4.60}$$

From the last formula of (4.31), it is straightforward to see that

$$a_0(x,x,u) \neq 0 \ \text{ if } \beta_0(x, u\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)) \neq 0, \tag{4.61}$$

where $a_0(x,y,u)$ is as in (4.59). In the rest of this section, we will calculate $a_0(x,x,u)$.

Fix $D_0 \Subset D$ and let $\chi, \widehat{\chi} \in \mathscr{C}^\infty_0(D,[0,1])$, $\chi = \widehat{\chi} = 1$ on D_0 and $\chi = 1$ on some neighbourhood of $\operatorname{supp} \widehat{\chi}$.

Lemma 4.16. *With the notations above, we have*

$$(\widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k^*\chi)(\chi\mathcal{S}_k\mathcal{A}_k\widehat{\chi}) \equiv \widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k\mathcal{A}_k\widehat{\chi} \mod O(k^{-\infty}), \tag{4.62}$$

where $\mathcal{A}_k^*$ is the formal adjoint of $\mathcal{A}_k$.

Proof. From (4.49), we have

$$\widehat{\chi}\mathcal{A}_k^*\mathcal{G}_k^*\Box_{s,k}^{(0)}\chi + \widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k^*\chi = \widehat{\chi}\mathcal{A}_k^*\chi. \tag{4.63}$$

From (4.63), we have

$$\widehat{\chi}\mathcal{A}_k^*\mathcal{G}_k^*\Box_{s,k}^{(0)}\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi} + \widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k^*\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi} = \widehat{\chi}\mathcal{A}_k^*\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi}. \tag{4.64}$$

From (4.58), it is not difficult to check that $\mathcal{S}_k\mathcal{A}_k$ is k-negligible away the diagonal. From this observation, (4.45) and (4.47), we conclude that

$$\widehat{\chi}\mathcal{A}_k^*\mathcal{G}_k^*\Box_{s,k}^{(0)}\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi} \equiv 0 \mod O(k^{-\infty}). \tag{4.65}$$

From (4.65) and (4.64), we get

$$\widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k^*\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi} \equiv \widehat{\chi}\mathcal{A}_k^*\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi} \mod O(k^{-\infty}). \tag{4.66}$$

Again, since $\mathcal{S}_k\mathcal{A}_k$ is k-negligible away the diagonal, we deduce that

$$\widehat{\chi}\mathcal{A}_k^*\chi^2\mathcal{S}_k\mathcal{A}_k\widehat{\chi} \equiv \widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k\mathcal{A}_k\widehat{\chi} \mod O(k^{-\infty}). \tag{4.67}$$

From (4.66) and (4.67), we get (4.62). ☐

From (4.62), (4.58) and the complex stationary phase formula of Melin-Sjöstrand [39], we have mod $O(k^{-\infty})$,

$$\begin{aligned}\big((\widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k^*\chi)(\chi\mathcal{S}_k\mathcal{A}_k\widehat{\chi})\big)(x,y) &\equiv (\widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k\mathcal{A}_k\widehat{\chi})(x,y)\\ &\equiv \int e^{ik\psi(x,y,u)}g(x,y,u,k)du,\end{aligned} \tag{4.68}$$

where

(4.69)

$$\begin{aligned}&g(x,y,u,k) \in \mathscr{C}_0^\infty(D\times D\times(-M,M)) \cap S^n_{\mathrm{loc,cl}}(1;D\times D\times(-M,M)),\\ &g(x,y,u,k) \sim \sum_{j=0}^{\infty} g_j(x,y,u)k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1;D\times D\times\mathbb{R}),\\ &g_j(x,y,u) \in \mathscr{C}_0^\infty(D\times D\times(-M,M)),\quad j=0,1,2,\ldots,\end{aligned}$$

and

$$g_0(x,x,u) = a_0(x,x,u)\overline{\beta_0}(x, u\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)),\quad \forall (x,x,u) \in D_0 \times D_0 \times (-M,M). \tag{4.70}$$

On the other hand, we can repeat the procedure of Section 5 in [24] (see the discussion after Theorem 5.6 in [24]) and deduce that

$$\big((\widehat{\chi}\mathcal{A}_k^*\mathcal{S}_k^*\chi)(\chi\mathcal{S}_k\mathcal{A}_k\widehat{\chi})\big)(x,y) \equiv \int e^{ik\psi_1(x,y,u)}h(x,y,u,k)du \mod O(k^{-\infty}) \tag{4.71}$$

with

(4.72)
$$\begin{aligned}
&h(x,y,u,k) \in S^n_{\mathrm{loc},\mathrm{cl}}(1, D\times D\times(-M,M)) \cap \mathscr{C}_0^\infty(D\times D\times(-M,M)),\\
&h(x,y,u,k) \sim \sum_{j=0}^\infty h_j(x,y,u)k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1, D\times D\times(-M,M)),\\
&h_j(x,y,u) \in \mathscr{C}_0^\infty(D\times D\times(-M,M)),\quad j=0,1,2,\ldots,
\end{aligned}$$

(4.73)
$$\begin{aligned}
&h_0(x,x,u) = 2\pi^n \left|\det R_x^L\right|^{-1} |a_0(x,x,u)|^2, \quad \forall (x,x,u) \in D_0\times D_0\times(-M,M),\\
&g_0(x,x,u) = h_0(x,x,u), \quad \forall (x,x,u) \in D\times D\times(-M,M),
\end{aligned}$$

and for all $(x,x,u) \in D\times D\times(-M,M)$, we have

(4.74)
$$\begin{aligned}
&\psi_1(x,x,u) = 0,\quad d_x\psi_1(x,x,u) = d_x\psi(x,x,u),\quad d_y\psi_1(x,x,u) = d_y\psi(x,x,u),\\
&\mathrm{Im}\,\psi_1(x,y,u) \geq 0,\quad \forall (x,y,u) \in D\times D\times(-M,M).
\end{aligned}$$

From (4.73) and (4.70), we get for all $(x,x,u) \in D_0\times D_0\times(-M,M)$,

$$a_0(x,x,u)\overline{\beta_0}(x, u\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)) = 2\pi^n \left|\det R_x^L\right|^{-1} |a_0(x,x,u)|^2. \tag{4.75}$$

If the quantity $\overline{\beta_0}(x, u\omega_0(x) - 2\mathrm{Im}\,\overline{\partial}_b\phi(x)) = 0$, we get $a_0(x,x,u)=0$. If this quantity doesn't vanish, in view of (4.61), we know that $a_0(x,x,u) \neq 0$. From this observation and (4.75), we obtain:

Theorem 4.17. *For $a_0(x,y,u)$ in* (4.59),

$$a_0(x,x,u)=\frac{1}{2}\pi^{-n}\left|\det R^L_x\right|\beta_0\big(x,u\omega_0(x)-2\mathrm{Im}\,\overline{\partial}_b\phi(x)\big),$$
$$(x,x,u)\in D\times D\times(-M,M),$$

where $\beta_0(x,\eta)\in\mathscr{C}^\infty(T^*D)$ is as in (4.60) *and* $\det R^L_x$ *as in* (1.2).

Remark 4.18. It should be noticed that by using the complex stationary phase formula of Melin-Sjöstrand and the method in [23], we can write down a general recurrsion relation for the symbols $a_j(x,y,u)$ and $\beta_j(x,\eta)$. We only calculate the leading term $a_0(x,x,u)$ in this paper.

5. Regularity of the Szegő projection Π_k

In this section, we will prove Theorem 1.1. For this purpose we first establish the spectral gap for the Kohn Laplacian $\Box^{(1)}_{b,k}$ and then Sobolev estimates for the associated Green operator and finally for Π_k.

We start with a local form of the spectral gap estimate for $(0,1)$-forms.

Lemma 5.1. *Let s be a local trivializing section of L on an open set $D\subset X$. Then, there is a constant $C>0$ independent of k such that*

$$\left\|\overline{\partial}_{b,k}u\right\|^2_k+\|\overline{\partial}^*_{b,k}u\|^2_k\geq\Big(Ck-\frac{1}{C}\Big)\|u\|^2_k\,,\ \text{for all } u\in\Omega^{0,1}_0(D,L^k).$$

Proof. Let $u\in\Omega^{0,1}_0(D,L^k)$. Put $u=s^k\widehat{u}$, $\widehat{u}\in\Omega^{0,1}_0(D)$. In view of (3.2), we have

$$\Box^{(1)}_{b,k}u=e^{k\phi}s^k\Box^{(1)}_{s,k}(e^{-k\phi}\widehat{u}).\tag{5.1}$$

Put $\widehat{u}=\sum\limits_{j=1}^{n-1}\widehat{u}_je_j$, where $e_1,\ldots,e_{n-1}\in\Lambda^{0,1}(T^*X)$ is as in Proposition 3.1. From (3.5), we have

$$\begin{aligned}(\Box^{(1)}_{s,k}(e^{-k\phi}\widehat{u}\,)\,|\,e^{-k\phi}\widehat{u}\,)&=\sum_{j=1}^{n-1}\left\|(Z_j+kZ_j(\phi))(e^{-k\phi}\widehat{u})\right\|^2\\&\quad+\sum_{j,t=1}^{n-1}(\,[Z_j+kZ_j(\phi),-\overline{Z}_t+k\overline{Z}_t(\phi)](e^{-k\phi}\widehat{u}_t)\,|\,e^{-k\phi}\widehat{u}_j\,)\\&\quad+((\varepsilon(Z+kZ(\phi))+\varepsilon(Z^*+k\overline{Z}(\phi)))(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u})\\&\quad+(fe^{-k\phi}\widehat{u}\,|\,e^{-k\phi}\widehat{u}).\end{aligned}\tag{5.2}$$

Here we use the same notations as in Proposition 3.1. Fix $j,t=1,2,\ldots,n-1$. Put

$$[Z_j - \overline{Z}_t] = \sum_{s=1}^{n-1}(a_s^{j,t}Z_s - b_s^{j,t}\overline{Z}_s)\,, \quad a_s^{j,t}, b_s^{j,t} \in \mathscr{C}^\infty(D).$$

Recall than by [27, Lemma 4.1], for any $U,V\in T_p^{1,0}X$ and any $\mathcal{U},\mathcal{V}\in C^\infty(D,T^{1,0}X)$ that satisfy $\mathcal{U}(p)=U$, $\mathcal{V}(p)=V$, we have

$$\begin{aligned}R_p^L(U,V) &= M_p^\phi(U,V)\\ &= -\big\langle \big[\,\mathcal{U},\overline{\mathcal{V}}\,\big](p), \overline{\partial}_b\phi(p) - \partial_b\phi(p)\big\rangle + \big(\mathcal{U}\overline{\mathcal{V}} + \overline{\mathcal{V}}\mathcal{U}\big)\phi(p).\end{aligned} \tag{5.3}$$

By using (5.3) we obtain

$$\begin{aligned}&[Z_j + kZ_j(\phi), -\overline{Z}_t + k\overline{Z}_t(\phi)]\\ &= \sum_{s=1}^{n-1}(a_s^{j,t}Z_s - b_s^{j,t}\overline{Z}_s) + k(Z_j\overline{Z}_t + \overline{Z}_tZ_j)(\phi)\\ &= \sum_{s=1}^{n-1}(a_s^{j,t}(Z_s + kZ_s(\phi)) + b_s^{j,t}(-\overline{Z}_s + k\overline{Z}_s(\phi)))\\ &\quad - k\big\langle\,[Z_j - \overline{Z}_t]\,, \overline{\partial}_b\phi - \partial_b\phi\,\big\rangle + k(Z_j\overline{Z}_t + \overline{Z}_tZ_j)(\phi)\\ &= \varepsilon(Z + kZ(\phi)) + \varepsilon(-\overline{Z} + k\overline{Z}(\phi)) + kR_x^L(\overline{Z}_t, Z_j).\end{aligned} \tag{5.4}$$

From (5.4) and (5.2), we get

$$\begin{aligned}&\big(\Box_{s,k}^{(1)}(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u}\,\big)\\ &= \sum_{j=1}^{n-1}\Big\|(Z_j + kZ_j(\phi))(e^{-k\phi}\widehat{u})\Big\|^2\\ &\quad + k\sum_{j,t=1}^{n-1}\big(\,R_x^L(\overline{Z}_t, Z_j)(e^{-k\phi}\widehat{u}_t)\,|\,e^{-k\phi}\widehat{u}_j\,\big)\\ &\quad + ((\varepsilon(Z + kZ(\phi)) + \varepsilon(Z^* + k\overline{Z}(\phi)))(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u})\\ &\quad + (\widetilde{f}e^{-k\phi}\widehat{u}\,|\,e^{-k\phi}\widehat{u}),\end{aligned} \tag{5.5}$$

where $\widetilde{f}$ is a smooth function independent of k. Since $R^L>0$, from (5.5), it is not difficult to see that

$$\Big(\Box_{s,k}^{(1)}(e^{-k\phi}\widehat{u})\,|\,e^{-k\phi}\widehat{u}\Big) \geq \bigg(\widetilde{C}k - \frac{1}{\widetilde{C}}\bigg)\Big\|e^{-k\phi}\widehat{u}\Big\|^2, \tag{5.6}$$

where $\widetilde{C} > 0$ is a constant independent of k and u. From (5.1), we can check that

$$\big(\Box^{(1)}_{s,k}(e^{-k\phi}\widehat{u}) \,|\, e^{-k\phi}\widehat{u}\big) = \big(\Box^{(1)}_{b,k}u \,|\, u\big)_k = \|\overline{\partial}_{b,k}u\|^2_k + \|\overline{\partial}^*_{b,k}\widehat{u}\|^2_k.$$

Moreover, it is clearly that $\|u\|_k = \|e^{-k\phi}\widehat{u}\|$. From this observation and (5.6), the lemma follows. $\Box$

Ohsawa and Sibony [47] established analogues of the Nakano and Akizuki vanishing theorems for Levi flat CR manifolds. The following result can be seen as an analogue of the spectral gap and Kodaira-Serre vanishing theorem [36, Theorems 1.5.5-6].

Theorem 5.2. *There is a constant $C_0 > 0$ independent of k such that*

$$\left\|\overline{\partial}_{b,k}u\right\|^2_k + \|\overline{\partial}^*_{b,k}u\|^2_k \geq \left(C_0k - \frac{1}{C_0}\right)\|u\|^2_k\,,$$
$$\forall u \in \operatorname{Dom}\overline{\partial}_{b,k} \cap \operatorname{Dom}\overline{\partial}^*_{b,k} \subset L^2_{(0,1)}(X, L^k).$$

Hence, for k large, $\operatorname{Ker}\Box^{(1)}_{b,k} = \{0\}$ *and* $\Box^{(1)}_{b,k}$ *has L^2 closed range.*

From Theorem 5.2, we deduce that $\Box^{(1)}_{b,k}$ is injective for large k so we can consider the Green operator $N^{(1)}_k : L^2_{(0,1)}(X, L^k) \to \operatorname{Dom}\Box^{(1)}_{b,k}$, which is the inverse of $\Box^{(1)}_{b,k}$. We have

$$\Box^{(1)}_{b,k}N^{(1)}_k = I \text{ on } L^2_{(0,1)}(X), \quad N^{(1)}_k\Box^{(1)}_{b,k} = I \text{ on } \operatorname{Dom}\Box^{(1)}_{b,k}. \tag{5.7}$$

Proof. We first claim that there is a constant $C_0 > 0$ independent of k such that

$$\left\|\overline{\partial}_{b,k}u\right\|^2_k + \|\overline{\partial}^*_{b,k}u\|^2_k \geq \left(C_0k - \frac{1}{C_0}\right)\|u\|^2_k\,, \quad \forall u \in \Omega^{0,1}(X, L^k). \tag{5.8}$$

Let $X = \bigcup_{j=1}^N D_j$, where $D_j \subset X$ is an open set with $L|_{D_j}$ is trivial. Take $\chi_j \in \mathscr{C}^\infty_0(D_j, [0,1])$, $j = 1, \ldots, N$, with $\sum\limits_{j=1}^N \chi_j = 1$ on X. Let $u \in \Omega^{0,1}(D, L^k)$.

From Lemma 5.1, we see that for every $j = 1, 2, \ldots, N$, we can find a constant $C_j > 0$ independent of k and u such that

$$\left\|\overline{\partial}_{b,k}(\chi_j u)\right\|_k^2 + \|\overline{\partial}^*_{b,k}(\chi_j u)\|_k^2 \geq \left(C_j k - \frac{1}{C_j}\right) \|\chi_j u\|_k^2 . \tag{5.9}$$

It is easy to see that

$$\begin{aligned} \left\|\overline{\partial}_{b,k}(\chi_j u)\right\|_k^2 + \|\overline{\partial}^*_{b,k}(\chi_j u)\|_k^2 &\leq \left\|\chi_j\overline{\partial}_{b,k} u\right\|_k^2 + \|\chi_j\overline{\partial}^*_{b,k} u\|_k^2 + M_j \|u\|_k^2 \\ &\leq \left\|\overline{\partial}_{b,k} u\right\|_k^2 + \|\overline{\partial}^*_{b,k} u\|_k^2 + M_j \|u\|_k^2 , \end{aligned} \tag{5.10}$$

where $M_j > 0$ is a constant independent of k and u. From (5.10) and (5.9), we get

$$\begin{aligned} N\Big(\left\|\overline{\partial}_{b,k} u\right\|_k^2 + \|\overline{\partial}^*_{b,k} u\|_k^2\Big) &\geq \sum_{j=1}^{N} \left(\left(C_j k - \frac{1}{C_j}\right) \|\chi_j u\|_k^2 - M_j \|u\|_k^2\right) \\ &\geq \left(ck - \frac{1}{c}\right) \|u\|_k^2 , \end{aligned} \tag{5.11}$$

where $c > 0$ is a constant independent of k. From (5.11), the claim (5.8) follows.

Now, let $u \in \operatorname{Dom}\overline{\partial}_{b,k} \cap \operatorname{Dom}\overline{\partial}^*_{b,k}$. From Friedrichs' Lemma (see Appendix D in [11]), we can find $u_j \in \Omega^{0,1}(X, L^k)$, $j = 1, 2, \ldots$, with $u_j \to u$ in $L^2_{(0,1)}(X, L^k)$, $\overline{\partial}_{b,k} u_j \to \overline{\partial}_{b,k} u$ in $L^2_{(0,2)}(X, L^k)$ and $\overline{\partial}^*_{b,k} u_j \to \overline{\partial}^*_{b,k} u$ in $L^2(X, L^k)$. From (5.8), we have

$$\begin{aligned} \left\|\overline{\partial}_{b,k} u\right\|_k^2 + \|\overline{\partial}^*_{b,k} u\|_k^2 &= \lim_{j\to\infty}\Big(\left\|\overline{\partial}_{b,k} u_j\right\|_k^2 + \|\overline{\partial}^*_{b,k} u_j\|_k^2\Big) \\ &\geq \left(C_0 k - \frac{1}{C_0}\right) \lim_{j\to\infty} \|u_j\|_k^2 = \left(C_0 k - \frac{1}{C_0}\right) \|u\|_k^2 . \end{aligned}$$

The theorem follows. □

We pause and introduce some notations. Let s be a local trivializing section of L on an open set $D \subset X$, $|s|_h^2 = e^{-2\phi}$. Let $u \in \Omega_0^{0,q}(D, L^k)$. On D, we write $u = s^k \widetilde{u}$, $\widetilde{u} \in \Omega_0^{0,q}(D)$. For every $m \in \mathbb{N}_0$, define

$$\|u\|_{m,k}^2 := \sum_{|\alpha| \leq m, \alpha \in \mathbb{N}_0^{2n-1}} \int \left|\partial_x^\alpha(\widetilde{u} e^{-k\phi})\right|^2 dv_X.$$

By using a partition of unity, we can define $\|u\|^2_{m,k}$ for all $u \in \Omega^{0,q}(X, L^k)$ in the standard way. We call $\|\cdot\|_{m,k}$ the Sobolev norm of order m with respect to h^k. We will need the following.

Proposition 5.3 ([47, Proposition 1]). *For every $m \in \mathbb{N}_0$ there is $N_m > 0$ such that for every $k \geq N_m$,*

$$\|\overline{\partial}^*_{b,k} u\|_{m,k} \leq k^{M(m)} \|\Box^{(1)}_{b,k} u\|_{m,k}, \quad u \in \Omega^{0,1}(X, L^k), \tag{5.12}$$

where $M(m) > 0$ is a constant independent of k and u.

Theorem 5.4. *For every $m \in \mathbb{N}$, there exist $N_m > 0$ and $M(m) > 0$ such that for every $k \geq N_m$,*

$$\begin{aligned} &\overline{\partial}^*_{b,k} N^{(1)}_k : \Omega^{0,1}(X, L^k) \to H^m(X, L^k), \\ &\|\overline{\partial}^*_{b,k} N^{(1)}_k u\|_{m,k} \leq k^{M(m)} \|u\|_{m,k}\,, \ u \in \Omega^{0,1}(X, L^k). \end{aligned} \tag{5.13}$$

Proof. The theorem essentially follows from Proposition 5.3 and the elliptic regularization method introduced by Kohn-Nirenberg [11, p. 102], [34, p. 449]. Namely, for every $\varepsilon > 0$, consider the operator $\Box^{(1)}_{\varepsilon,k} := \Box^{(1)}_{b,k} + \varepsilon T^* T$, where T is defined in (2.7) and T^* is its formal adjoint with respect to $(\,\cdot\,|\,\cdot\,)_k$. Fix $m \in \mathbb{N}$. From Theorem 5.2 and Proposition 5.3, there is a $N_m > 0$ such that for every $k \geq N_m$,

$$\begin{aligned} &\|u\|^2_k \leq (\,\Box^{(1)}_{b,k} u \,|\, u\,)_k, \quad \forall u \in \Omega^{0,1}(X, L^k), \\ &\|u\|_{\ell,k} \leq k^{M(m)} \|\Box^{(1)}_{b,k} u\|_{\ell,k}, \quad \forall u \in \Omega^{0,1}(X, L^k), \quad \forall \ell \in \mathbb{N}_0, \quad \ell \leq m, \end{aligned} \tag{5.14}$$

where $M(m) > 0$ is a constant independent of k and u.

Take $g \in \Omega^{0,1}(X, L^k)$ and put $N^{(1)}_k g = v$. We have $\Box^{(1)}_{b,k} v = g$. From (5.14), it is easy to see that for every $k \geq N_m$ and every $\varepsilon > 0$, $\Box^{(1)}_{\varepsilon,k}$ is injective and has range $L^2_{(0,1)}(X, L^k)$. Now, we assume that $k \geq N_m$. For every $\varepsilon > 0$, we can find $v_\varepsilon \in \Omega^{0,1}(X, L^k)$ such that $\Box^{(1)}_{\varepsilon,k} v_\varepsilon = g$. Moreover, from (5.14) and the proof of Proposition 5.3 (see also [47, Proposition 1]), it is straightforward to see that for every $\varepsilon > 0$,

$$\begin{aligned} &\|v_\varepsilon\|_k \leq \|g\|_k\,, \quad \|\overline{\partial}_{b,k} v_\varepsilon\|_k \leq \|g\|_k\,, \\ &\|\overline{\partial}^*_{b,k} v_\varepsilon\|_{\ell,k} \leq k^{M(m)} \|g\|_{\ell,k}\,, \quad \forall \ell \in \mathbb{N}_0, \quad \ell \leq m. \end{aligned} \tag{5.15}$$

From (5.15), we can find $\varepsilon_j \searrow 0$ such that $v_{\varepsilon_j} \to \widetilde{v}$ in $L^2_{(0,1)}(X, L^k)$ as $j \to \infty$, $\overline{\partial}_{b,k} v_{\varepsilon_j} \to \overline{\partial}_{b,k} \widetilde{v}$ in $L^2_{(0,2)}(X, L^k)$, $\overline{\partial}^*_{b,k} v_{\varepsilon_j} \to \overline{\partial}^*_{b,k} \widetilde{v}$ in $H^\ell(X, L^k)$,

$\forall \ell \in \mathbb{N}_0$, $\ell \leq m$, and $\Box^{(1)}_{b,k}\widetilde{v} = g$ in the sense of distributions. Since $\overline{\partial}_{b,k}\widetilde{v} \in L^2_{(0,2)}(X, L^k)$, $\overline{\partial}^*_{b,k}\widetilde{v} \in H^1(X, L^k)$, we have $\widetilde{v} \in \operatorname{Dom}\overline{\partial}_{b,k} \cap \operatorname{Dom}\overline{\partial}^*_{b,k}$, $\overline{\partial}^*_{b,k}\widetilde{v} \in \operatorname{Dom}\overline{\partial}_{b,k}$. Note that $\overline{\partial}^*_{b,k}\overline{\partial}_{b,k}\widetilde{v} = g - \overline{\partial}_{b,k}\overline{\partial}^*_{b,k}\widetilde{v} \in L^2_{(0,1)}(X, L^k)$. From this observation, we can check that $\overline{\partial}_{b,k}\widetilde{v} \in \operatorname{Dom}\overline{\partial}^*_{b,k}$. Thus, $\widetilde{v} \in \operatorname{Dom}\Box^{(1)}_{b,k}$. Since $\Box^{(1)}_{b,k}\widetilde{v} = g = \Box^{(1)}_{b,k}v$ and $\Box^{(1)}_{b,k}$ is injective, we conclude that $v = \widetilde{v}$. Thus, $\overline{\partial}^*_{b,k}N^{(1)}_k g = \overline{\partial}^*_{b,k}v \in H^m(X, L^k)$ and $\|\overline{\partial}^*_{b,k}N^{(1)}_k g\|_{m,k} \leq k^{M(m)}\|g\|_{m,k}$. The theorem follows. □

Theorem 5.5. *With the notations above, for every $m \in \mathbb{N}$, $m \geq 2$, there is a $N_m > 0$ such that for every $k \geq N_m$,*

$$\Pi_k = I - \overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k} \text{ on } \mathscr{C}^\infty(X, L^k), \tag{5.16}$$

$$\Pi_k : \mathscr{C}^\infty(X, L^k) \to H^m(X, L^k) \tag{5.17}$$

and

$$\|(I - \Pi_k)u\|_{m,k} \leq k^{M(m)}\left\|\overline{\partial}_{b,k}u\right\|_{m,k}, \quad \forall u \in \mathscr{C}^\infty(X, L^k), \tag{5.18}$$

where $M(m) > 0$ is a constant independent of k and u.

Proof. Fix $m \in \mathbb{N}$, $m \geq 2$ and let $N_m > 0$ be as in Theorem 5.5. We assume that $k \geq N_m$. Let $g \in \mathscr{C}^\infty(X, L^k)$. From Theorem 5.4, we know that $\overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \in H^m(X, L^k)$. Since $m \geq 2$, it is clearly that $\overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \in \operatorname{Dom}\Box^{(0)}_{b,k}$. Moreover, it is easy to check that

$$\overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \perp \operatorname{Ker}\overline{\partial}_{b,k} = \operatorname{Ker}\Box^{(0)}_{b,k}. \tag{5.19}$$

We claim that

$$g - \overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \in \operatorname{Ker}\Box^{(0)}_{b,k}. \tag{5.20}$$

Let $f \in \mathscr{C}^\infty(X, L^k)$. We have

$$\begin{aligned}(g - \overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \,|\, \Box^{(0)}_{b,k}f)_k &= (\Box^{(0)}_{b,k}g \,|\, f)_k - (\overline{\partial}^*_{b,k}N^{(1)}_k\overline{\partial}_{b,k}g \,|\, \Box^{(0)}_{b,k}f)_k \\ &= (\Box^{(0)}_{b,k}g \,|\, f)_k - (\overline{\partial}_{b,k}g \,|\, N^{(1)}_k\Box^{(1)}_{b,k}\overline{\partial}_{b,k}f)_k \\ &= (\Box^{(0)}_{b,k}g \,|\, f)_k - (\overline{\partial}_{b,k}g \,|\, \overline{\partial}_{b,k}f)_k = 0.\end{aligned}$$

The claim (5.20) follows. From (5.19) and (5.20), we get (5.16). Theorem 5.4 and (5.16) yield (5.17) and (5.18). □

From Theorem 5.5 and the Sobolev embedding theorem, we get Theorem 1.1.

6. Asymptotic expansion of the Szegő kernel

In this section, we will prove Theorem 1.2 and Theorem 1.3. Let s be a local trivializing section of L on an open set $D \subset X$ and let $\Pi_{k,s}$ be the localized operator of Π_k (see (1.6)). Let $\mathcal{S}_k$ and $\mathcal{G}_k$ be as in Theorem 4.13. From the constructions of $\mathcal{G}_k$ and $\mathcal{S}_k$, it is straightforward to see that we can find $\widetilde{\mathcal{G}_k} : H^s_{\rm comp}(D) \to H^{s+1}_{\rm loc}(D)$, $\widetilde{\mathcal{S}_k} : H^s_{\rm comp}(D) \to H^s_{\rm loc}(D)$, for every $s \in \mathbb{Z}$, such that $\widetilde{\mathcal{G}_k}$ and $\widetilde{\mathcal{S}_k}$ are properly supported on D,

$$\begin{aligned} &\widetilde{\mathcal{S}_k} - \mathcal{S}_k = O(k^{-\infty}) : H^s_{\rm comp}(D) \to H^s_{\rm loc}(D), \quad \forall s \in \mathbb{Z}, \\ &\widetilde{\mathcal{G}_k} - \mathcal{G}_k = O(k^{-\infty}) : H^s_{\rm comp}(D) \to H^{s+1}_{\rm loc}(D), \quad \forall s \in \mathbb{Z}, \end{aligned} \tag{6.1}$$

and

$$\widetilde{\chi}\, \widetilde{\mathcal{S}_k}\, \chi = O(k^{-\infty}) : H^s_{\rm comp}(D) \to H^s_{\rm loc}(D), \quad \forall s \in \mathbb{Z}, \tag{6.2}$$

for every $\widetilde{\chi}, \chi \in C^\infty_0(D)$ with $\operatorname{supp}\widetilde{\chi} \cap \operatorname{supp}\chi = \emptyset$, and

$$\Box^{(0)}_{s,k}\, \widetilde{\mathcal{G}_k} + \widetilde{\mathcal{S}_k} = I + R_k \text{ on } D, \tag{6.3}$$

where R_k is properly supported on D and

$$R_k = O(k^{-\infty}) : H^s_{\rm loc}(D) \to H^{s-1}_{\rm loc}(D), \quad \forall s \in \mathbb{Z}. \tag{6.4}$$

From (6.3), it is easy to see that

$$\Pi_{k,s} + \Pi_{k,s} R_k = \Pi_{k,s} \widetilde{\mathcal{S}_k} \ \text{ on } D. \tag{6.5}$$

Theorem 6.1. *With the notations above, for every $\ell \in \mathbb{N}_0$, there is a $N_\ell > 0$ such that for every $k \geq N_\ell$, $\widetilde{\chi}\,\Pi_k \chi = O(k^{-\infty}) : \mathscr{C}^\infty(X, L^k) \to \mathscr{C}^\ell(X, L^k)$, for every $\chi \in \mathscr{C}^\infty_0(D)$, $\widetilde{\chi} \in \mathscr{C}^\infty(X)$ with* $\operatorname{supp}\widetilde{\chi} \cap \operatorname{supp}\chi = \emptyset$, *and*

$$\Pi_{k,s} - \mathcal{S}_k = O(k^{-\infty}) : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\ell(D). \tag{6.6}$$

Proof. Fix $\ell \in \mathbb{N}_0$. From Theorem 5.5, there exists $N_\ell > 0$ such that for every $k \geq N_\ell$,

$$\begin{aligned} &\Pi_k = I - \overline{\partial}^*_{b,k} N^{(1)}_k \overline{\partial}_{b,k} \text{ on } \mathscr{C}^\infty(X, L^k), \\ &\Pi_k : \mathscr{C}^\infty(X, L^k) \to H^{\ell+n}(X, L^k), \\ &\|(I - \Pi_k)u\|_{n+\ell,k} \leq k^{M(\ell)} \left\|\overline{\partial}_{b,k} u\right\|_{n+\ell,k}, \quad \forall u \in \mathscr{C}^\infty(X, L^k), \end{aligned} \tag{6.7}$$

where $M(\ell) > 0$ is a constant independent of k and u. Now, we assume that $k \geq N_\ell$. By the Sobolev embedding theorem we have $H^{\ell+n}(X, L^k) \subset \mathscr{C}^\ell(X, L^k)$.

Fix $N_1 > 0$ and let $u \in \mathscr{C}^\infty_0(D)$. Consider

$$v = U_{k,s}\widetilde{\mathcal{S}_k}u - \Pi_k(U_{k,s}\widetilde{\mathcal{S}_k}u) = (I - \Pi_k)(U_{k,s}\widetilde{\mathcal{S}_k}u). \tag{6.8}$$

From (6.5), we have

$$\begin{aligned} &v = U_{k,s}(\widetilde{\mathcal{S}_k} - \Pi_{k,s}\widetilde{\mathcal{S}_k})u \text{ on } D, \\ &v = U_{k,s}(\widetilde{\mathcal{S}_k}u) - \Pi_k(U_{k,s}(I + R_k)u) \text{ on } X. \end{aligned} \tag{6.9}$$

From (6.7) and (6.8), we obtain

$$\left\|(I - \Pi_k)(U_{k,s}\widetilde{\mathcal{S}_k}u)\right\|_{n+\ell,k} \leq k^{M(\ell)} \left\|\overline{\partial}_{b,k}(U_{k,s}\widetilde{\mathcal{S}_k}u)\right\|_{n+\ell,k}. \tag{6.10}$$

Note that $\overline{\partial}_{s,k}\widetilde{\mathcal{S}_k} = O(k^{-\infty}) : H^s_{\rm comp}(D) \to H^{s-1}_{\rm loc}(D)$ for all $s \in \mathbb{Z}$. From this observation, (6.10) and the second formula of (6.9) we conclude that

$$U_{k,s}\widetilde{\mathcal{S}_k} - \Pi_k U_{k,s} - \Pi_k U_{k,s} R_k = O(k^{-\infty}) : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\ell(X, L^k). \tag{6.11}$$

From (6.4) and (6.7), it is easy to see that

$$\Pi_k U_{k,s} R_k = O(k^{-\infty}) : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\ell(X, L^k). \tag{6.12}$$

From (6.11) and (6.12), we conclude that

$$U_{k,s}\widetilde{\mathcal{S}_k} - \Pi_k U_{k,s} = O(k^{-\infty}) : \mathscr{C}^\infty_0(D) \to \mathscr{C}^\ell(X, L^k). \tag{6.13}$$

From (6.13) and (6.1), (6.6) follows.

Finally, from (6.13), (6.2) and noting that $\widetilde{\mathcal{S}_k}$ is properly supported on D, we deduce that $\widetilde{\chi}\Pi_k\chi = O(k^{-\infty}) : \mathscr{C}^\infty(X, L^k) \to \mathscr{C}^\ell(X, L^k)$, for every $\chi \in \mathscr{C}^\infty_0(D)$, $\widetilde{\chi} \in \mathscr{C}^\infty(X)$ with $\operatorname{supp}\widetilde{\chi} \cap \operatorname{supp}\chi = \emptyset$. ☐

Proof of Theorem 1.2. This follows immediately from Theorems 4.14 and 6.1. □

Proof of Theorem 1.3. Let $\mathcal{A}_k$ be as in Theorem 1.3. It is not difficult to see that for every $s \in \mathbb{Z}$ and $N \in \mathbb{N}$, there exists $n(N, s) > 0$ independent of k, such that

$$\mathcal{A}_k = O(k^{n(N,s)}) : H^s_{\mathrm{comp}}(D) \to \mathscr{C}^N_0(D). \tag{6.14}$$

From (6.14), (6.6) and since $\mathcal{A}_k : H^s_{\mathrm{comp}}(D) \to \mathscr{C}^\infty_0(D)$ for every $s \in \mathbb{Z}$, we conclude that

$$\Pi_{k,s}\mathcal{A}_k \equiv \mathcal{S}_k\mathcal{A}_k \mod O(k^{-\infty}). \tag{6.15}$$

From (6.15) and Theorem 4.15, Theorem 1.3 follows. □

7. Kodaira Embedding theorem for Levi-flat CR manifolds

In this section, we will prove Theorem 1.4. Let s be a local trivializing section of L on an open set $D \subset X$. Fix $p \in D$ and let $x = (x_1, \ldots, x_{2n-1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, be local coordinates of X defined in some small neighbourhood of p such that (4.33) hold. We may assume that the local coordinates x defined on D. We write $x' = (x_1, \ldots, x_{2n-2})$. Let $M > 1$ be a large constant so that

$$\left|-2\mathrm{Im}\,\overline{\partial}_b\phi(x) + u\omega_0(x)\right|^2 \leq \frac{M^2}{8}, \quad \forall x \in D, |u| \leq 1. \tag{7.1}$$

Consider

$$\tau \in \mathscr{C}^\infty_0(\mathbb{R}, [0,1]),\ \tau = 1 \text{ on } \left[\tfrac{1}{4}, \tfrac{1}{2}\right], \operatorname{supp}\tau \subset [0,1],$$

$\chi \in \mathscr{C}^\infty_0(\mathbb{R}, [0,1])$, $\chi = 1$ on $\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]$, $\operatorname{supp}\chi \subset [-1,1]$, $\chi(t) = \chi(-t)$, $t \in \mathbb{R}$. Fix $0 < \delta < 1$. Put

$$\alpha_\delta(x, \eta, k) := \tau\left(\frac{\langle\, \eta \,|\, \omega_0(x)\,\rangle}{\delta}\right)\chi\Big(\frac{4\,|\eta|^2}{M^2}\Big) \in S^0_{\mathrm{cl}}(1, T^*D) \tag{7.2}$$

and let $\mathcal{A}_{k,\delta}$ be a properly supported classical semi-classical pseudodifferential operator on D with

$$\mathcal{A}_{k,\delta}(x,y) \equiv \frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik\langle x-y,\eta\rangle}\alpha_\delta(x,\eta,k)d\eta \mod O(k^{-\infty}).$$

Fix $\ell \in \mathbb{N}$, $\ell \geq 2$. In view of Theorem 1.3, we see that there is a $N_\ell > 0$ such that for every $k \geq N_\ell$, $\Pi_{k,s}\mathcal{A}_{k,\delta}(x,y) \in \mathscr{C}^\ell(D \times D)$ and

$$(\Pi_{k,s}\mathcal{A}_{k,\delta})(x,y) \equiv \int e^{ik\psi(x,y,u)} a_\delta(x,y,u,k)du \quad \text{mod } O(k^{-\infty}) \text{ in } \mathscr{C}^\ell(D \times D), \tag{7.3}$$

where

$$\begin{aligned} &a_\delta(x,y,u,k) \in \mathscr{C}_0^\infty(D \times D \times (-M,M)) \cap S^n_{\mathrm{loc,cl}}(1; D \times D \times (-M,M)),\\ &a_\delta(x,y,u,k) \sim \sum_{j=0}^{\infty} a_{j,\delta}(x,y,u)k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1; D \times D \times (-M,M)). \end{aligned} \tag{7.4}$$

From (1.14), (7.1) and (7.3), we get

$$a_{0,\delta}(x,x,u) = \frac{1}{2}\pi^{-n}\left|\det R^L_x\right| \tau\Big(\frac{u}{\delta}\Big), \ \ \forall (x,x,u) \in D \times D \times (-M,M). \tag{7.5}$$

From now on, we assume that $k \geq N_\ell$.

We will use the following rescaling of the coordinates:

$$F^*_k : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, \quad F^*_k y := \left(\frac{y_1}{\sqrt{k}}, \frac{y_2}{\sqrt{k}}, \ldots, \frac{y_{2n-2}}{\sqrt{k}}, \frac{y_{2n-1}}{k}\right).$$

We introduce the shorthand notations

$$\begin{aligned} \boldsymbol{\chi}(y) &:= \chi(y_1)\cdots\chi(y_{2n-2})\chi(y_{2n-1}),\\ \boldsymbol{\chi}(k,y) &:= \chi(\sqrt{k}y_1)\cdots\chi(\sqrt{k}y_{2n-2})\chi(ky_{2n-1}). \end{aligned}$$

hence $\boldsymbol{\chi}(y) = \boldsymbol{\chi}(k, F^*_k y)$.

For $j = 1, \ldots, n-1$, let $\lambda_j = \lambda_j(p)$ are the eigenvalues of R^L_p with respect to $\langle \cdot | \cdot \rangle$ and let $\alpha_j \in \mathbb{C}$ be as in (4.33). Set

$$R(w) = \sum_{l=1}^{n-1}(\alpha_l w_l - \overline{\alpha}_l \overline{w}_l) + iuy_{2n-1} + \frac{1}{2}\sum_{j=1}^{n-1}\lambda_j \left|w_j\right|^2$$

where $w_j = y_{2j-1} + iy_{2j}$. Let

$$u_{k,\delta,p} := \Pi_k U_{k,s}\mathcal{A}_{k,\delta}\Big(e^{kR(w)}\boldsymbol{\chi}(k,y)\Big), \tag{7.6}$$

so $u_{k,\delta,p}$ is a global $\mathscr{C}^\ell$ CR section. We write $u_{k,\delta,p} = U_{k,s}\widetilde{u}_{k,\delta,p}$ on D, with $\widetilde{u}_{k,\delta,p} \in C^\ell(D)$. Then, $|u_{k,\delta,p}(x)|_{h^k} = |\widetilde{u}_{k,\delta,p}(x)|$, $x \in D$. Put

$$\begin{aligned}\psi_0(x,y,u) &:= \psi(x,y,u) - i\sum_{j=1}^{n-1}(\alpha_j w_j - \overline{\alpha}_j\overline{w}_j) + uy_{2n-1} - \frac{i}{2}\sum_{j=1}^{n-1}\lambda_j\,|w_j|^2 \\ &= \psi(x,y,u) - iR(w).\end{aligned}$$

From (7.3), we can check that we have $\bmod O(k^{-\infty})$ in $\mathscr{C}^\ell(D)$,

$$\begin{aligned}\widetilde{u}_{k,\delta,p}(x) &\equiv \int e^{ik\psi_0(x,y,u)}a_\delta(x,y,u,k)\boldsymbol{\chi}(k,y) \\ &\equiv \int e^{ik\psi_0(x,F_k^*y,u)}k^{-n}a_\delta(x,F_k^*y,u,k)\boldsymbol{\chi}(y)dudy.\end{aligned} \tag{7.7}$$

Put

$$\widehat{u}_{k,\delta,p} := \exp\left(-k\sum_{j=1}^{n-1}(\alpha_j z_j - \overline{\alpha}_j\overline{z}_j)\right)\widetilde{u}_{k,\delta,p} \in \mathscr{C}^\ell(D). \tag{7.8}$$

Lemma 7.1. *With the notations above, there is a $k_0 > 0$ such for all $k \geq k_0$ and $p \in X$,*

$$\begin{aligned}&\frac{1}{8}\delta c_p \leq |\widehat{u}_{k,\delta,p}(p)| \leq 2\delta c_p, \quad \frac{1}{32}\delta^2 c_p \leq \left|\frac{1}{k}\frac{\partial \widehat{u}_{k,\delta,p}}{\partial x_{2n-1}}(p)\right| \leq 2\delta^2 c_p, \\ &\left|\frac{1}{k}\frac{\partial \widehat{u}_{k,\delta,p}}{\partial x_j}(p)\right| \leq \delta^4,\end{aligned} \tag{7.9}$$

where $j = 1, 2, \ldots, 2n-2$, and $c_p = \frac{1}{2}\pi^{-n}\left|\det R_p^L\right|\int \boldsymbol{\chi}(y)dy$.

Proof. From (7.7), (7.5), (4.36) and note that $\psi_0(0,0,u) = 0$, $\forall u \in \mathbb{R}$, we can check that

$$\begin{aligned}&\lim_{k\to\infty}|\widehat{u}_{k,\delta,p}(p)| = \frac{1}{2}\pi^{-n}\left|\det R_p^L\right|\int \tau\Big(\frac{u}{\delta}\Big)\boldsymbol{\chi}(y)dydu, \\ &\lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial \widehat{u}_{k,\delta,p}}{\partial x_{2n-1}}(p)\right| = \frac{1}{2}\pi^{-n}\left|\det R_p^L\right|\int u\tau\Big(\frac{u}{\delta}\Big)\boldsymbol{\chi}(y)dydu, \\ &\lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial \widehat{u}_{k,\delta,p}}{\partial x_j}(p)\right| = 0, \quad j = 1, 2, \ldots, 2n-2.\end{aligned}$$

Since $\frac{\delta}{4} \leq \int \tau\big(\frac{u}{\delta}\big)du \leq \delta$ and $\frac{\delta^2}{16} \leq \int u\tau\big(\frac{u}{\delta}\big)du \leq \delta^2$, there is $k_0 > 0$ such that for every $k \geq k_0$, (7.9) hold. Since X is compact, k_0 can be taken to be independent of the point p. □

For every $j=1,2,\ldots,n-1$, let

$$u^j_{k,\delta,p}:=\Pi_k U_{k,s}\mathcal{A}_{k,\delta}\Big(e^{kR(w)}\sqrt{k}(y_{2j-1}+iy_{2j})\boldsymbol{\chi}(k,y)\Big). \tag{7.10}$$

Then, $u^j_{k,\delta,p}$ is a global $\mathscr{C}^\ell$ CR section. On D, we write $u^j_{k,\delta,p}=U_{k,s}\widetilde{u}^j_{k,\delta,p}$, with $\widetilde{u}^j_{k,\delta,p}\in\mathscr{C}^\ell(D)$. From (7.3), we can check that

$$\widetilde{u}^j_{k,\delta,p}(x)\equiv\int e^{ik\psi_0(x,F^*_k y,u)}k^{-n}a_\delta(x,F^*_k y,u,k)(y_{2j-1}+iy_{2j})\boldsymbol{\chi}(y)dudy, \tag{7.11}$$

$\bmod O(k^{-\infty})$ in $\mathscr{C}^\ell(D)$. Put

$$\begin{aligned}\widehat{u}^j_{k,\delta,p}&:=\exp\left(-k\sum_{l=1}^{n-1}(\alpha_l z_l-\overline{\alpha}_l\overline{z}_l)\right)\widetilde{u}^j_{k,\delta,p}\in\mathscr{C}^\ell(D),\\ j&=1,2,\ldots,n-1.\end{aligned} \tag{7.12}$$

Lemma 7.2. *With the notations above, there exists $k_0>0$ such that for all $p\in X$ and $k\geq k_0$,*

$$\begin{aligned}&\left|\widehat{u}^j_{k,\delta,p}(p)\right|\leq\delta^4,\quad\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial x_{2n-1}}(p)\right|\leq\delta^4,\\ &\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial z_j}(p)\right|\geq\frac{1}{8}\delta\lambda_j d_p,\quad j=1,2,\ldots,n-1,\\ &\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial\overline{z}_s}(p)\right|\leq\delta^4,\quad j,s=1,2,\ldots,n-1,\\ &\left|\frac{1}{k}\frac{\partial\widehat{u}^j_{k,\delta,p}}{\partial z_s}(p)\right|\leq\delta^4,\quad j,s=1,2,\ldots,n-1,\quad j\neq s,\end{aligned} \tag{7.13}$$

where $\{\lambda_j\}_{j=1}^{n-1}$ are the eigenvalues of R^L_p with respect to $\langle\cdot|\cdot\rangle$ and

$$d_p=\frac{1}{2\pi^n}\left|\det R^L_p\right|\int|y_1+iy_2|^2\boldsymbol{\chi}(y)dy.$$

Proof. From (7.11), (7.5), (4.36) and observing that $\psi_0(0,0,u)=0$ for all $u\in\mathbb{R}$, it is straightforward to check that for every $j,s,t=1,\ldots,n-1$,

$s \neq j$,

$$\lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial \widehat{u}^j_{k,\delta,p}}{\partial z_j}(p)\right| = \frac{\lambda_j}{2\pi^n}\left|\det R^L_p\right|\int \tau\Big(\frac{u}{\delta}\Big)\left|y_{2j-1}+iy_{2j}\right|^2 \boldsymbol{\chi}(y)dydu,$$

$$\begin{aligned}\lim_{k\to\infty}\left|\widehat{u}^j_{k,\delta,p}(p)\right| &= \lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial \widehat{u}^j_{k,\delta}}{\partial x_{2n-1}}(p)\right| \\ &= \lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial \widehat{u}^j_{k,\delta,p}}{\partial z_s}(p)\right| = \lim_{k\to\infty}\left|\frac{1}{k}\frac{\partial \widehat{u}^j_{k,\delta}}{\partial \overline{z}_t}(p)\right| = 0.\end{aligned}$$

Since $\frac{\delta}{4} \leq \int \tau\Big(\frac{u}{\delta}\Big)du \leq \delta$, there is a constant $k_0 > 0$ such that (7.13) holds for every $k \geq k_0$. Since X is compact, k_0 can be taken to be independent of the point p. The lemma follows. ☐

Consider the $\mathscr{C}^\ell$ map

$$\text{(7.14)}\qquad \Phi_{k,\delta,p} : D \to \mathbb{C}^n,\ \ x \longmapsto \left(\frac{\widetilde{u}_{k,\delta,p}}{\widetilde{u}_{k,\delta^2,p}}(x), \frac{\widetilde{u}^1_{k,\delta,p}}{\widetilde{u}_{k,\delta^2,p}}(x), \ldots, \frac{\widetilde{u}^{n-1}_{k,\delta,p}}{\widetilde{u}_{k,\delta^2,p}}(x)\right).$$

The following Lemma is a consequence of (7.13) and (7.9) together with a straightforward computation and therefore we omit the details.

Lemma 7.3. *With the notations above, there are $k_0 > 0$ and $0 < \delta_0 < 1$ such that for all $k \geq k_0$, $0 < \delta \leq \delta_0$ and $p \in X$, the differential of $\Phi_{k,\delta,p}$ is injective at p.*

Let $\operatorname{dist}(\cdot,\cdot)$ denote the Riemannian distance on X and for $x \in X$ and $r > 0$, put $B(x,r) := \{y \in X; \operatorname{dist}(x,y) < r\}$. From now on, we fix $k > k_0$ and $0 < \delta < \delta_0$, where $k_0 > 0$ and $0 < \delta_0 < 1$ are as in Lemma 7.3. Since X is compact there exists $r_k > 0$ such that for every $x_0 \in X$, $\widetilde{u}_{k,\delta^2,x_0}(x) \neq 0$ for every $x \in B(x_0, 2r_k)$ and the maps Φ_{k,δ,x_0} and $d\Phi_{k,\delta,x_0}$ are injective on $B(x_0, 2r_k)$. We can find $x_1, x_2, \ldots, x_{d_k} \in X$ such that

$$\text{(7.15)}\qquad X = B(x_1, r_k) \cup B(x_2, r_k) \cup \cdots \cup B(x_{d_k}, r_k).$$

For every $j = 1, 2, \ldots, d_k$, let $u_{k,\delta^2,x_j}, u_{k,\delta,x_j}, u^1_{k,\delta,x_j}, \ldots, u^{n-1}_{k,\delta,x_j} \in \mathscr{C}^\ell(X, L^k)$ be as in (7.6) and (7.10). Consider the map $\Phi_{k,\delta} : X \to \mathbb{CP}^{(n+1)d_k-1}$,

$$\begin{aligned}\text{(7.16)}\qquad \Phi_{k,\delta} = \big[&u_{k,\delta^2,x_1}, u_{k,\delta,x_1}, u^1_{k,\delta,x_1}, \ldots, u^{n-1}_{k,\delta,x_1}, \\ &\ldots, u_{k,\delta^2,x_{d_k}}, u_{k,\delta,x_{d_k}}, u^1_{k,\delta,x_{d_k}}, \ldots, u^{n-1}_{k,\delta,x_{d_k}}\big].\end{aligned}$$

Let $q \in X$. Then, $q \in B(x_j, r_k)$ for some $j = 1, 2, \ldots, d_k$. From the discussion before (7.15), we see that $u_{k,\delta^2,x_j}(q) \neq 0$. Thus, $\Phi_{k,\delta}$ is well-defined as a $\mathscr{C}^\ell$ map.

Theorem 7.4. *With the notations above, the differential of $\Phi_{k,\delta}$ is injective at every $x \in X$ and for every $x_0, y_0 \in X$ with $\operatorname{dist}(x_0, y_0) \leq \frac{r_k}{2}$, we have $\Phi_{k,\delta}(x_0) \neq \Phi_{k,\delta}(y_0)$.*

Proof. Let $q \in X$. Assume that $q \in B(x_1, r_k)$. Then, $u_{k,\delta^2,x_1}(q) \neq 0$. On $B(x_1, r_k)$, consider the map $\Psi : B(x_1, r_k) \to \mathbb{C}^{(n+1)d_k-1}$,

$$\begin{aligned}(7.17)\qquad \Psi = \Bigg(&\frac{u_{k,\delta,x_1}}{u_{k,\delta^2,x_1}}, \frac{u^1_{k,\delta,x_1}}{u_{k,\delta^2,x_1}}, \ldots, \frac{u^{n-1}_{k,\delta,x_1}}{u_{k,\delta^2,x_1}}, \ldots,\\ &\frac{u_{k,\delta^2,x_{d_k}}}{u_{k,\delta^2,x_1}}, \frac{u_{k,\delta,x_{d_k}}}{u_{k,\delta^2,x_1}}, \frac{u^1_{k,\delta,x_{d_k}}}{u_{k,\delta^2,x_1}}, \ldots, \frac{u^{n-1}_{k,\delta,x_{d_k}}}{u_{k,\delta^2,x_1}}\Bigg).\end{aligned}$$

From the discussion before (7.15), we see that $d\Phi_{k,\delta,x_1}$ is injective on $B(x_1, 2r_k)$. Thus, $d\Psi$ is injective at q and hence $d\Phi_{k,\delta}$ is injective at q.

Let $x_0, y_0 \in X$ with $\operatorname{dist}(x_0, y_0) \leq \frac{r_k}{2}$. We may assume that $x_0 \in B(x_1, r_k)$. Thus, $x_0, y_0 \in B(x_1, 2r_k)$. From the discussion before (7.15), we see that Φ_{k,δ,x_1} is injective on $B(x_1, 2r_k)$. Hence,

$$(7.18)\qquad \Phi_{k,\delta,x_1}(x_0) \neq \Phi_{k,\delta,x_1}(y_0).$$

By the definition (7.14) of Φ_{k,δ,x_1}, relation (7.18) implies that $\Phi_{k,\delta}(x_0) \neq \Phi_{k,\delta}(y_0)$. The lemma follows. □

Let s be a local trivializing section of L on an open set $D \subset X$. As before, we fix $p \in D$ and let $x = (x_1, \ldots, x_{2n-1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, be local coordinates of X defined in some small neighbourhood of p such that (4.33) hold. We may assume that the local coordinates x defined on D. Take $m > N_\ell$ be a large constant and let $u_{m,\delta,p}$ be as in (7.6). On D, we write $u_{m,\delta,p} = U_{k,s}\widetilde{u}_{m,\delta,p}$, $\widetilde{u}_{m,\delta,p} \in \mathscr{C}^\ell(D)$. Put

$$D_{p,m} := \left\{x = (x_1, \ldots, x_{2n-1});\ |x| < \frac{1}{m\log m}\right\}.$$

We need the following.

Lemma 7.5. *With the notations above, there exists* $m_0 > 0$ *such that* $r_k m_0^{1/3} > 4$ *and for all* $m \geq m_0$ *and* $p \in X$,

$$\inf \left\{ |u_{m,\delta,p}(x)|_{h^m} \, ; \, x \in D_{p,m} \right\} \geq \frac{1}{8} \delta c_p, \tag{7.19}$$

where $c_p = \frac{1}{2}\pi^{-n} \left|\det R_p^L\right| \int \boldsymbol{\chi}(y)dy$*, and for every* $q \in X$ *with* $\operatorname{dist}(q, x) \geq \frac{r_k}{4}$*, for all* $x \in D_{p,m}$*, we have*

$$|u_{m,\delta,p}(q)|_{h^m} \leq \frac{1}{2} \inf \left\{ |u_{m,\delta,p}(x)|_{h^m} \, ; \, x \in D_{p,m} \right\}, \tag{7.20}$$

where $r_k > 0$ *is as in Theorem 7.4.*

Proof. Let $m > N_\ell$ be large enough so that

$$r_k m^{1/3} > 4. \tag{7.21}$$

As in (7.7), we have $\operatorname{mod} O(m^{-\infty})$ in $\mathscr{C}^\ell(D)$

$$\widetilde{u}_{m,\delta,p}(x) \equiv \int e^{im\psi_0(x, F_m^* y, u)} m^{-n} a_\delta(x, F_m^* y, u, m) \boldsymbol{\chi}(y) du dy. \tag{7.22}$$

From (7.22), we can repeat the proof of the first formula of (7.9) with minor changes and get (7.19). We only need to prove (7.20). Let $q \in X$ with $\operatorname{dist}(q, x) \geq \frac{r_k}{4}$, for all $x \in D_{p,m}$. If $q \notin D$, from (i) in Theorem 1.2, we can check that $|u_{m,\delta,p}(q)|_{h^m} = O(m^{-\infty})$.

We may thus assume that $q \in D$. For simplicity, we may suppose that $\operatorname{dist}(x_1, x_2) = |x_1 - x_2|$ on D. We write $q = (q_1, \ldots, q_{2n-1})$. Since $\operatorname{dist}(q, x) \geq \frac{r_k}{4}$, for all $x \in D_{p,m}$, from (7.21), we have $|q| \geq \frac{1}{4m^{1/3}}$ for m large. Thus, $|q'| \geq \frac{1}{8m^{1/3}\log m}$ or $|q_{2n-1}| \geq \frac{1}{8m^{1/3}}$, where $q' = (q_1, \ldots, q_{2n-2})$. If $|q'| \geq \frac{1}{8m^{1/3}\log m}$, by using the fact that $m \operatorname{Im} \psi_0(q, F_m^* y, u) \geq c m^{1/3} \frac{1}{(\log m)^2}$, $\forall y \in \operatorname{supp} \boldsymbol{\chi}(y)$, where $c > 0$ is a constant independent of m, we conclude that

$$|\widetilde{u}_{m,\delta,p}(q)| = O(m^{-\infty}), \text{ if } \left|q'\right| \geq \frac{1}{8m^{1/3}\log m}. \tag{7.23}$$

If $|q_{2n-1}| \geq \frac{1}{8m^{1/3}}$ and $|q'| < \frac{1}{8m^{1/3}\log m}$, from (4.36), we can integrate by parts with respect to u several times and conclude that

$$|\widetilde{u}_{m,\delta,p}(q)| = O(m^{-\infty}), \text{ if } |q_{2n-1}| \geq \frac{1}{8m^{1/3}\log m} \text{ and } \left|q'\right| < \frac{1}{8m^{1/3}\log m}. \tag{7.24}$$

From (7.23) and (7.24), (7.20) follows. □

Now, we fix $m \geq N_\ell + m_0$, where m_0 is as Lemma 7.5. From Lemma 7.5, we see that we can find $x_1 \in X, x_2 \in X, \ldots, x_{d_m} \in X$ such that $X = \bigcup_{j=1}^{d_m} U_{x_j,m}$, where for each j, $U_{x_j,m}$ is an open neighbourhood of x_j with $\sup\{\mathrm{dist}(q_1,q_2);\, q_1,q_2 \in U_{x_j,m}\} < \frac{r_k}{4}$, and for each j, we can find a $\mathscr{C}^\ell$ global CR section u_{m,δ,x_j} such that

$$\inf\left\{\left|u_{m,\delta,x_j}(x)\right|_{h^m};\, x \in U_{x_j,m}\right\} > 0, \tag{7.25}$$

and for every $q \in X$ with $\mathrm{dist}(q,x) \geq \frac{r_k}{4}$, for all $x \in U_{x_j,m}$, we have

$$\left|u_{m,\delta,x_j}(q)\right|_{h^m} \leq \frac{1}{2}\inf\left\{\left|u_{m,\delta,x_j}(x)\right|_{h^m};\, x \in U_{x_j,m}\right\}, \tag{7.26}$$

where $r_k > 0$ is as in Theorem 7.4. Consider the map:

$$\Psi_{m,\delta} : X \to \mathbb{CP}^{d_m-1},\ \ x \longmapsto [u_{m,\delta,x_1}, u_{m,\delta,x_2}, \ldots, u_{m,\delta,x_{d_m}}](x). \tag{7.27}$$

Let $q \in X$. Then, $q \in U_{x_j,m}$ for some $j = 1,2,\ldots,d_m$. In view of (7.25), we see that $u_{m,\delta,x_j}(q) \neq 0$. Thus, $\Psi_{m,\delta}$ is well-defined as a smooth map.

Theorem 7.6. *The map* $(\Phi_{k,\delta}, \Psi_{m,\delta}) : X \to \mathbb{CP}^{(n+1)d_k-1} \times \mathbb{CP}^{d_m-1}$ *is a* $\mathscr{C}^\ell$ *CR embedding, where* $\Phi_{k,\delta}$ *is given by* (7.16)

Proof. In view of Theorem 7.4, we only need to show that $(\Phi_{k,\delta}, \Psi_{m,\delta})$ is injective. Let $q_1, q_2 \in X$, $q_1 \neq q_2$. Assume first that $\mathrm{dist}(q_1,q_2) \leq \frac{r_k}{4}$. From Theorem 7.4, we know that $\Phi_{k,\delta}(q_1) \neq \Phi_{k,\delta}(q_2)$ and hence $(\Phi_{k,\delta}(q_1), \Psi_{m,\delta}(q_1)) \neq (\Phi_{k,\delta}(q_2), \Psi_{m,\delta}(q_2))$. We assume that $\mathrm{dist}(q_1,q_2) > \frac{r_k}{4}$. From (7.26), it is straightforward to check that $\Psi_{m,\delta}(q_1) \neq \Psi_{m,\delta}(q_2)$ and thus

$$(\Phi_{k,\delta}(q_1), \Psi_{m,\delta}(q_1)) \neq (\Phi_{k,\delta}(q_2), \Psi_{m,\delta}(q_2)).$$

The theorem follows. $\square$

Note that $\Phi_{k,\delta}$ are defined by collecting many local embedding CR maps and it is difficult to show that $\Phi_{k,\delta}$ is injective on X.

Proof of Theorem 1.4. With the notations above, consider the Segre map

$$\begin{aligned} &\Upsilon : \mathbb{CP}^{(n+1)d_k-1} \times \mathbb{CP}^{d_m-1} \to \mathbb{CP}^{(n+1)d_k d_m-1}, \\ &([z_1,\ldots,z_{(n+1)d_k}], [w_1,\ldots,w_{d_m}]) \\ &\quad \to [z_1w_1, z_1w_2, \ldots, z_1w_{d_m}, z_2w_1, \ldots, z_{(n+1)d_k}w_{d_m}], \end{aligned} \tag{7.28}$$

which is a holomorphic embedding. By Theorem 7.6, we deduce that

$$\Upsilon \circ (\Phi_{k,\delta}, \Psi_{m,\delta}) : X \to \mathbb{CP}^{(n+1)d_k d_m - 1},$$

is a $\mathscr{C}^\ell$ CR embedding. We have proved that for every $M \geq k + N_\ell + m_0$, we can find CR sections $s_0, s_1, \ldots, s_{d_M} \in \mathscr{C}^\ell(X, L^M)$, such that the map $x \in X \to [s_0(x), s_1(x), \ldots, s_{d_M}(x)] \in \mathbb{CP}^{d_M}$ is an embedding. Theorem 1.4 follows. □

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