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The strong topological monodromy conjecture for Weyl hyperplane arrangements

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The Bernstein–Sato polynomial, or the b-function, is an important invariant of hypersurface singularities. The local topological zeta function is also an invariant of hypersurface singularities that has a combinatorial description in terms of a resolution of singularities. The Strong Topological Monodromy Conjecture of Denef and Loeser states that poles of the local topological zeta function are also roots of the b-function.

We use a result of Opdam to produce a lower bound for the b-function of hyperplane arrangements of Weyl type. This bound proves the "n/d conjecture", by Budur, Mustaţă, and Teitler for this class of arrangements, which implies the Strong Monodromy Conjecture for this class of arrangements.

1. Introduction

The goal of this short paper is to prove the Strong Topological Monodromy Conjecture for hyperplane arrangements of Weyl type, i.e., Coxeter arrangements arising from a finite Weyl group. This conjecture links two invariants of hypersurface singularities: the local topological zeta function, and the Bernstein–Sato polynomial (or b-function).

The *Bernstein–Sato polynomial*, also called the *b-function*, is a relatively fine invariant of singularities of hypersurfaces. Let f be a polynomial function on an affine space X, and let $\mathcal{D}_X$ be the ring of differential operators on X. Then the b-function of f can be defined as the minimal polynomial $b_f(s)$ for the operator of multiplication by s on the holonomic $\mathcal{D}_X[s]$-module $\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}$ [8].

The *local topological zeta function* associated to a hypersurface $V(f)$ is a function $Z_{\mathrm{top},f}(s)$ on $\mathbb{C}$. Defined by Denef and Loeser [4], it is computed in terms of the Euler-Poincaré characteristic of the irreducible components of an embedded resolution of singularities of the hypersurface $V(f)$. Thus it forms a topological analog to the more analytic *local Igusa zeta function* [7].

In the case of f a relative invariant on a prehomogenous vector space, poles of the Igusa zeta function correspond to roots of the b-function [7]. Consequently, by work of Malgrange [10, 11] and Kashiwara [8], the poles also give the eigenvalues of the monodromy operator on the cohomology of the Milnor fiber. The Topological Monodromy Conjecture of Denef and Loeser [4] is an analog of this work for topological zeta functions. The weak form states that exponentiating the poles of $Z_{\mathrm{top},f}$ gives eigenvalues of the monodromy operator. The strong form states that the poles of $Z_{\mathrm{top},f}$ are roots of b_f, which, by Malgrange and Kashiwara, implies the weak version.

We will consider the case of f a hyperplane arrangement. This case has proved particularly tractable for computation, especially to compute and relate singularity invariants such as b-functions, zeta functions, Milnor monodromy, and jumping coefficients [1–3, 14, 15, 17]. In particular, Budur, Mustaţă, and Teitler have proved the weak version of the Topological Monodromy Conjecture for hyperplane arrangements [2, Theorem 1.3(a)]. We will prove the strong version for a particular class of arrangements.

Theorem 1.1. *Let $\mathfrak{h}$ be a Cartan subalgebra of a simple complex Lie algebra $\mathfrak{g}$. Let $\xi \in \mathbb{C}[\mathfrak{h}]$ be the product of the positive roots. If c is a pole of $Z_{\mathrm{top},\xi}(s)$, then $b_\xi(c) = 0$.*

2. Hyperplane arrangements of Weyl type

Let G be a complex connected reductive Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let $R \subset \mathfrak{h}^*$ be the associated root system with Weyl group W. Define ξ to be the product of the positive roots:

$$\xi = \prod_{\alpha \in R^+} \alpha.$$

The zero locus $V(\xi)$ is a union of hyperplanes. This is the hyperplane arrangement we wish to study.

The function ξ is anti-symmetric with respect to the W-action on $\mathfrak{h}$, and is the Jacobian determinant of the quotient map $\mathfrak{h} \to \mathfrak{h}/W$. The set $V(\xi)$ consists of points fixed by at least one non-trivial element of W. Thus $V(\xi)$ is the complement of $\mathfrak{h}^{\mathrm{reg}}$. The W-invariant function ξ^2 is called the *discriminant* of the root system R. Let Δ denote the pullback of ξ^2 under the Chevalley isomorphism $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\cong} \mathbb{C}[\mathfrak{h}]^W$.

When the root system R is of type A_{n-1}, this polynomial is recognized as the *Vandermonde determinant*:

$$\xi_n = \prod_{1 \le i < j \le n} (x_i - x_j).$$

In this case, Δ sends a matrix in $\mathfrak{g}$ to the discriminant of its characteristic polynomial.

Since ξ^2 is in $\mathbb{C}[\mathfrak{h}]^W$, we can consider its image in $\mathbb{C}[\mathfrak{h}/W]$. Specifically, by the Chevalley-Shephard-Todd theorem, $\mathfrak{h}/W$ is an n-dimensional affine space, where $n = \mathrm{rk}(G) = \dim(\mathfrak{h})$. Hence $\mathbb{C}[\mathfrak{h}/W]$ is a polynomial ring in n variables. Fix a homogeneous free set of generators for this polynomial ring, so that $\mathbb{C}[\mathfrak{h}/W] = \mathbb{C}[e_1, \ldots, e_n]$. We write $\mathbb{C}[\mathfrak{h}/W]$ to mean polynomials in the generators $\{e_1, \ldots, e_n\}$, and $\mathbb{C}[\mathfrak{h}]^W$ to mean polynomials in the generators $\{x_1, \ldots, x_n\}$ of $\mathbb{C}[\mathfrak{h}]$. Let g be the polynomial corresponding to ξ^2 in $\mathbb{C}[\mathfrak{h}/W]$, that is, $g(e_1, \ldots, e_n) = \xi^2(x_1, \ldots, x_n)$.

In [12], Eric Opdam found the b-function for g. We show in the next section that $b_g(s)$ divides $b_{\xi^2}(s)$, but evidence suggests that it falls far short of equality. Moreover, for a general f, it is always true that $b_{f^2}(s) \mid b_f(2s+1)b_f(2s)$, but equality does not always hold.

3. Proofs

In [2, Theorem 1.3(b)], Budur, Mustaţă, and Teitler reduce the Strong Monodromy Conjecture to the so-called *n/d conjecture* [2, Conjecture 1.2]. We prove Theorem 3.3, which is the n/d conjecture in the case of Weyl arrangements. As a corollary, we deduce Theorem 1.1, which is the Strong Monodromy Conjecture in this case.

We begin by proving the following relationship between the b-functions of g and ξ.

Theorem 3.1. *The function $b_g(s)$ divides the function $b_\xi(2s+1)$.*

Proof. The inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$ induces a restriction map $\rho\colon \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$, which is an isomorphism by the Chevalley restriction theorem. Let $\Delta = \rho^*(\xi^2)$, which is an element of $\mathbb{C}[\mathfrak{g}]^G$.

Let $L_{\xi^2}(s) \in D(\mathfrak{h})[s]$ be an operator that satisfies

$$L_{\xi^2}(s)(\xi^{2(s+1)}) = b_{\xi^2}(s) \cdot (\xi^2)^s.$$

Since ξ^2 is W-invariant, we may assume (by averaging) that $L_{\xi^2}(s) \in D(\mathfrak{h})^W[s]$.

The space $D(\mathfrak{h})^W$ of W-invariant operators acts on $\mathbb{C}[\mathfrak{h}/W]$, by pulling back via the isomorphism $\mathbb{C}[\mathfrak{h}/W] \cong \mathbb{C}[\mathfrak{h}]^W$. For any $L \in D(\mathfrak{h})^W$, let $\varphi(L)$ be the corresponding differential operator in $D(\mathfrak{h}/W)$. Clearly, φ extends to a map $\varphi\colon D(\mathfrak{h})^W[s] \to D(\mathfrak{h}/W)[s]$. Applying φ to $L_{\xi^2}(s)$, we see that

$$\varphi(L_{\xi^2}(s))(g^{s+1}) = b_{\xi^2}(s) \cdot g^s.$$

This equation shows that the b-function of g divides $b_{\xi^2}(s)$, that is,

$$b_g(s) \mid b_{\xi^2}(s). \tag{1}$$

Similarly, we have a map $D(\mathfrak{g})^G[s] \to D(\mathfrak{g}/\!\!/G)[s]$. Let $L_\Delta(s)$ be an operator that satisfies

$$L_\Delta(s)(\Delta^{s+1}) = b_\Delta(s) \cdot \Delta^s. \tag{2}$$

The action of G on $D(\mathfrak{g})[s]$ is locally finite because it is compatible with the Bernstein filtration on $D(\mathfrak{g})[s]$. By complete reducibility, we can then decompose (2) into isotypic components, which shows that only the G-invariant part of $L_\Delta(s)$ contributes to the right-hand side. We may thus assume that $L_\Delta(s) \in D(\mathfrak{g})^G[s]$. By a similar argument as above for the quotient $\mathfrak{g} \to \mathfrak{g}/\!\!/G$ instead of $\mathfrak{h} \to \mathfrak{h}/W$, we see that

$$b_g(s) \mid b_\Delta(s). \tag{3}$$

Let $L_\xi(s) \in D(\mathfrak{h})$ such that $L_\xi(s)(\xi^{s+1}) = b_\xi(s) \cdot \xi^s$. Observe that

$$L_\xi(2s)L_\xi(2s+1)(\xi^{2(s+1)}) = b_\xi(2s)b_\xi(2s+1) \cdot (\xi^2)^s.$$

Therefore the b-function of ξ^2 divides $b_\xi(2s)b_\xi(2s+1)$, that is,

$$b_{\xi^2}(s) \mid b_\xi(2s)b_\xi(2s+1). \tag{4}$$

From (1) and (4), we see that

$$b_g(s) \mid b_\xi(2s)b_\xi(2s+1). \tag{5}$$

We use the following theorem. The existence is due to Harish-Chandra [5], and the surjectivity is due to Wallach [16], and Levasseur–Stafford [9].

Proposition 3.2. *Conjugating the radial part map* Rad *by ξ yields a surjective homomorphism of algebras* HC: $D(\mathfrak{g})^G \to D(\mathfrak{h})^W$, *called the* Harish-Chandra homomorphism.

Clearly, HC extends to a map HC: $D(\mathfrak{g})^G[s] \to D(\mathfrak{h})^W[s]$. Recall that $L_\Delta(s)$ is in $D(\mathfrak{g})^G[s]$ and was chosen such that $L_\Delta(s)(\Delta^{s+1}) = b_\Delta(s) \cdot \Delta^s$. Since Δ corresponds to the function ξ^2 under the Chevalley restriction map, we have

$$\begin{aligned} \mathrm{HC}(L_\Delta(s-1/2)) \cdot (\xi^2)^{s+1} &= \xi \circ \mathrm{Rad}(L_\Delta(s-1/2)) \circ \xi^{-1}(\xi^2)^{s+1} \\ &= \xi \circ \mathrm{Rad}(L_\Delta(s-1/2))(\xi^2)^{(2s+1)/2} \\ &= \xi \cdot b_\Delta(s-1/2) \cdot (\xi^2)^{(2s-1)/2} \\ &= b_\Delta(s-1/2) \cdot \xi^{2s}, \end{aligned}$$

which shows that $b_{\xi^2}(s) \mid b_\Delta(s-1/2)$.

Since HC is surjective, $L_{\xi^2}(s) \in D(\mathfrak{h})^W$ can be lifted to an operator in $D(\mathfrak{g})^G$. By running the previous argument in reverse, we can see that $b_\Delta(s-1/2) \mid b_{\xi^2}(s)$. We conclude that $b_{\xi^2}(s) = b_\Delta(s-1/2)$, and by changing variables that

$$b_{\xi^2}(s+1/2) = b_\Delta(s). \tag{6}$$

From (3), (4), and (6), we see that

$$b_g(s) \mid b_\xi(2s+1) b_\xi(2s+2). \tag{7}$$

Suppose that $b_g(s) \nmid b_\xi(2s+1)$. This means that there is some c that is a root of $b_g(s)$ of some multiplicity m, but is a root of $b_\xi(2s+1)$ of multiplicity $k < m$ (where k may be zero). By (5), c must be a root of $b_\xi(2s)$, and by (7), c must be a root of $b_\xi(2s+2)$.

By [15, Theorem 1], the difference between any two roots of the b-function of ξ, a hyperplane arrangement, is less than 2. So c cannot be a root of both $b_\xi(2s)$ and $b_\xi(2s+2)$, and we have a contradiction. This argument proves that $b_g(s) \mid b_\xi(2s+1)$. □

The proof of the n/d conjecture for Weyl arrangements now follows quite easily, which also proves Theorem 1.1.

Theorem 3.3. *Let $\mathfrak{h}$ be a Cartan subalgebra of a simple complex Lie algebra $\mathfrak{g}$. Let $\xi \in \mathbb{C}[\mathfrak{h}]$ be the product of the positive roots as defined earlier. Let $d = \deg(\xi)$ and let $n = \dim(\mathfrak{h})$. Then $-n/d$ is always a root of the b-function of ξ.*

Proof. Let $d_1 \leq \cdots \leq d_n$ be a list of the degrees of the fundamental invariants of the Lie group G. The degree of the highest fundamental invariant is equal to the Coxeter number. Recall that n is the rank of the root system, and the total number of roots equals $2d$. It is known (see, e.g., [6, Section 3.18]) that $d_n \cdot n = 2d$.

From [12], we know that

$$b_g(s) = \prod_{i=1}^{n} \prod_{j=1}^{d_i - 1} \left(s + \frac{1}{2} + \frac{j}{d_i} \right).$$

Notice that one of the factors above is

$$\left(s + \frac{1}{2} + \frac{1}{d_n} \right) = \left(s + \frac{1}{2} + \frac{n}{2d} \right).$$

So $-(1/2 + n/(2d))$ is a root of $b_g(s)$ and hence of $b_\xi(2s+1)$, which precisely means that $b_\xi(-n/d) = 0$. □

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