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On uniformly bounded orthonormal Sidon systems

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In answer to a question raised recently by Bourgain and Lewko, we show that any uniformly bounded subGaussian orthonormal system is $\otimes^2$-Sidon. This sharpens their result that it is "5-fold tensor Sidon", or $\otimes^5$-Sidon in our terminology. The proof is somewhat reminiscent of the author's original one for (Abelian) group characters, based on ideas due to Drury and Rider. However, we use Talagrand's majorizing measure theorem in place of Fernique's metric entropy lower bound. We also show that a uniformly bounded orthonormal system is randomly Sidon if it is $\otimes^4$-tensor Sidon, or equivalently $\otimes^k$-Sidon for some (or all) $k \geq 4$. Various generalizations are presented, including the case of random matrices, for systems analogous to the Peter-Weyl decomposition for compact non-Abelian groups. In the latter setting we also include a new proof of Rider's unpublished result that randomly Sidon sets are Sidon, which implies that the union of two Sidon sets is Sidon.

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The study of "thin sets" and in particular Sidon sets in discrete Abelian groups was actively developed in the 1970's and 1980's. Among the early fundamental results, Drury's proof of the stability of Sidon sets under finite unions stands out (see [14]). Rider's work [26] connected Sidon sets to random Fourier series. This led the author to a characterization of Sidon sets in terms of Rudin's $\Lambda(p)$-sets (see [15, 17]) and eventually in [20] to an arithmetic characterization of Sidon sets. Bourgain [2] gave a different proof, as well as a host of other results on related questions. The 2013 book [7] by Graham and Hare gives an account of this subject, updating the 1975 one [14] by Lopez and Ross. Concerning $\Lambda(p)$-sets see Bourgain's survey [4]. See also [12] for connections with Banach space theory.

Most of the results on lacunary sets crucially use the group structure ([3] is a notable exception). However, quite recently Bourgain and Lewko [5] were able to obtain several analogues for uniformly bounded orthonormal systems. We pursue the same theme in this paper.

Let $\psi_2(x) = \exp x^2 - 1$. Let C be a constant. We will say that an orthonormal system (φ_n) in $L_2(T, m)$ (here (T, m) is any probability space) is a *C-subGaussian system* if for any sequence $y = (y_n)$ in l_2 we have

$$\left\| \sum y_n \varphi_n \right\|_{L_{\psi_2}} \leq C \left(\sum |y_n|^2 \right)^{\frac{1}{2}}, \tag{0.1}$$

where L_{ψ_2} is the Orlicz space on (T, m) associated to ψ_2, with norm defined by

$$\|f\|_{\psi_2} = \inf\{t > 0 \mid \mathbb{E} \exp |f/t|^2 \leq e\}. \tag{0.2}$$

In [5] this is called a $\psi_2(C)$-system but we prefer to use a different term. This is a variant of the notion of subGaussian random process, as considered in [9] or [15, p. 24]. It also appears under the name "σ-generalized Gaussian" in e.g. [28, p. 236]. Indeed, assuming without loss of generality that the φ_n's all have vanishing mean, then (0.1) holds for some C iff for some $\sigma > 0$ we have for all $y \in \ell_2$

$$\mathbb{E} \exp \Re \left(\sum y_n \varphi_n \right) \leq \exp \sigma^2 \sum |y_n|^2/2. \tag{0.3}$$

Clearly the latter forces $\mathbb{E}\varphi_n = 0$ for all n while (0.1) does not. But this is the only significative difference. Indeed, assuming $\mathbb{E}\varphi_n = 0$ for all n, it is not hard to show that (0.1) implies (0.3) for some $0 \le \sigma < \infty$ depending only on C. Conversely (0.3) for some σ implies (0.1) for some $C < \infty$ depending only on σ (and $C \simeq \sigma$ for the best possible values).

Classical examples of such systems include sequences of independent identically distributed (i.i.d.) Gaussian random variables, or independent mean zero uniformly bounded ones. They also include Hadamard lacunary sequences of the form $\varphi_n(t) = \exp(iN(n)t)$ where $\{N(n)\}$ is an increasing sequence such that

$$\liminf N(n+1)/N(n) > 1.$$

Roughly, one may interpret (0.1) as expressing a certain form of independence of the system (φ_n). When the system is bounded in $L_\infty(T, m)$, the notion of Sidon system constitutes another form of independence: we say that (φ_n) is Sidon if there is a constant α such that for any finitely supported scalar sequence

$$\sum |a_n| \le \alpha \left\| \sum a_n \varphi_n \right\|_\infty . \tag{0.4}$$

Assume that (φ_n) are characters on a compact Abelian group. Then (0.4) $\Rightarrow$ (0.1) by a classical result due to Rudin [27]. Conversely, in [17] the author combined harmonic analysis results (due to Drury and Rider, see [26]) with probabilistic results on stationary Gaussian processes (due mainly to Fernique) to show that (0.1) $\Rightarrow$ (0.4). Bourgain gave an alternate proof in [2].

Recently, Bourgain and Lewko [5] considered the question whether the preceding implications still held for more general orthonormal systems bounded in $L_\infty(T, m)$. In such generality it is easy to see that (0.4) $\not\Rightarrow$ (0.1), because the direct sum of a Sidon system with an *arbitrary* system satisfies (0.4). Conversely Bourgain and Lewko construct an example showing that also (0.1) $\not\Rightarrow$ (0.4), but that (0.1) nevertheless implies a weak form of (0.4), namely they show in [5] that (0.1) implies that the system $\{\varphi_n(t_1) \cdots \varphi_n(t_5)\}$ defined on the 5-fold product $T \times \cdots \times T$ satisfies (0.4). Since the latter clearly implies (0.4) when the φ_n's are group (or semi-group) morphisms, this provides one more proof of (0.1) $\Rightarrow$ (0.4) for characters. Naturally they raised the question whether 5-fold can be replaced by 2-fold, which would then be optimal. Our main result Theorem 1.1 gives a positive answer. We give a more general version in §1 which leads to several possibly interesting variants. The proof makes crucial use of a consequence (see Lemma 1.4) of Talagrand's majorizing measure Theorem from [29].

In §2 we consider the analogue of random Fourier series for uniformly bounded orthonormal systems. We call randomly Sidon the systems that satisfy the analogue of Rider's condition (i.e. that satisfy (0.4) with the right hand side replaced by the average over all signs of $\|\sum \pm a_n\varphi_n\|_\infty$), and we prove that 4-fold tensor Sidon is equivalent to randomly Sidon. Thus the k-fold variant of the Sidon property (we use for this the term $\otimes^k$-Sidon) is the same notion for all $k \geq 4$.

In §3 we apply a similar generalization to the natural "non-commutative" analogue of Sidon sets on non-Abelian compact groups. Here orthonormal functions are replaced by matrix valued functions (generalizing irreducible representations), for which the entries suitably renormalized form an orthonormal system. We obtain an analogue of subGaussian $\Rightarrow \otimes^2$-Sidon (see Corollary 3.11).

The simplest case of interest is provided by a random $d \times d$-matrix

$$t \mapsto [\varphi_{ij}(t)]$$

with orthonormal entries satisfying (0.1), and such that for some C' we have a uniform bound

$$\sup_{t\in T} \|d^{-1/2}\varphi(t)\|_{M_d} \leq C'. \tag{0.5}$$

Then Corollary 3.11 (see also Remark 3.14) shows that there is a constant $\alpha = \alpha(C, C') > 0$ such that for any matrix $a \in M_d$ we have

$$\alpha \mathrm{tr}|a| \leq \sup_{t_1,t_2\in T} |\mathrm{tr}(d^{-1}\varphi(t_1)\varphi(t_2)a)|. \tag{0.6}$$

The prototypical example of φ satisfying (0.6) and (0.5) (with $C' = \alpha = 1$) with orthonormal entries satisfying (0.1) for some numerical C (independent of d) is the case when $d^{-1/2}\varphi$ is a random unitary $d \times d$-matrix uniformly distributed over the unitary group. In §4 we illustrate by an example the possible applications to matrices of our generalized setting.

In §5 we consider the notion of "randomly Sidon" for matrix valued functions. We obtain an analogue of randomly Sidon $\Rightarrow \otimes^4$-Sidon (see Theorem 5.7).

In §6 we briefly discuss a reinforcement of the implication [$\{\varphi_{ij}\}$ C-subGaussian] $\Rightarrow$ (0.6) valid when $d^{-1/2}\varphi$ is a representation π on a compact group. In that case it suffices to assume that the character of π (namely $t \mapsto \mathrm{tr}(\pi(t)) = d^{-1/2}\sum \varphi_{ii}$) is C-subGaussian.

1. Sidon systems

Theorem 1.1. *Let* (φ_n) *be an orthonormal system satisfying* (0.1) *and moreover such that*

$$\|\varphi_n\|_\infty \le C' \tag{1.1}$$

for any $n \ge 1$*. Then there is a constant* $\alpha = \alpha(C, C')$ *such that* $\forall a \in \ell_1$

$$\sum |a_n| \le \alpha \sup_{(t_1,t_2)\in T\times T} \left|\sum a_n \varphi_n(t_1)\varphi_n(t_2)\right|. \tag{1.2}$$

Proof. This will be deduced from the more general Corollary 1.12 below. □

Remark 1.2. The same proof also shows that

$$\sum |a_n| \le \alpha \sup_{(t_1,t_2)\in T\times T} \left|\sum a_n \varphi_n(t_1)\overline{\varphi_n(t_2)}\right|.$$

Actually, assuming that (φ^1_n) and (φ^2_n) are two uniformly bounded orthonormal systems satisfying (0.1) with respective constants C_1, C_2 and bounds C'_1, C'_2. we will show there is $\alpha = \alpha(C_1, C_2, C'_1, C'_2)$ such that $\forall a \in l_1$

$$\sum |a_n| \le \alpha \sup_{(t_1,t_2)\in T\times T} \left|\sum a_n \varphi^1_n(t_1)\varphi^2_n(t_2)\right|. \tag{1.3}$$

In [5] the system is called $\otimes^2$-Sidon if (1.2) holds, and $\otimes^k$-Sidon if (1.2) holds with k-factors $(\varphi_n(t_1)\varphi_n(t_2)\cdots\varphi_n(t_k))$ in place of 2. Theorem 1.1 answers the question raised in [5], whether (0.1) implies $\otimes^2$-Sidon. According to [5], (0.1) implies $\otimes^5$-Sidon but not $\otimes^1$-Sidon, so “2” is optimal.

When the φ_n's are Abelian group characters Theorem 1.1 was established in [17]. Our method closely follows our original approach in [17], modulo the later progress allowed by Talagrand's majorizing measure theorem from [29]. One could also use the subsequent proof of Talagrand's Bernoulli conjecture by Bednorz and Latała [1], and use Bernoulli random variables in a similar fashion (as we did in our initial draft), but we will content ourselves with the Gaussian case.

Let (g_n) (resp. $(g^{\mathbb{R}}_n)$) denote an i.i.d. sequence of complex (resp. real) valued standard Gaussian random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. In the complex case $g_n = 2^{-1/2}(g'_n + ig''_n)$ with (g'_n), (g''_n) mutually independent and each having the same distribution as $(g^{\mathbb{R}}_n)$.

It is worthwhile to record here the following two easy and well known observations: for any Banach space B and any $x_1, \ldots, x_N \in B$

$$(1.4)\qquad 2^{-1/2}\mathbb{E}\left\|\sum x_n g_n^{\mathbb{R}}\right\| \le \mathbb{E}\left\|\sum x_n g_n\right\| \le 2^{1/2}\mathbb{E}\left\|\sum x_n g_n^{\mathbb{R}}\right\|.$$

For any matrix $a \in M_N$ with $\|a\|_{M_N} \le 1$, we have

$$(1.5)\qquad \mathbb{E}\left\|\sum\nolimits_{i=1}^{N} x_i \sum\nolimits_j a_{ij} g_j\right\| \le \mathbb{E}\left\|\sum\nolimits_1^N x_j g_j\right\|.$$

Indeed, since equality holds when a is unitary, (1.5) follows by an extreme point argument.

In the sequel, we mostly use the complex Gaussians (g_n) but one could use the real ones $(g_n^{\mathbb{R}})$, except that it introduces irrelevant factors equal to 2 at various places.

In addition to Talagrand's work, the crucial ingredient in the proof is an "interpolation property" which uses the following observation (for convenience we state this only for $N < \infty$ but this restriction is not necessary).

Lemma 1.3. *Let $0 < \delta < 1$ and $N \ge 1$. Let P_1 be the orthogonal projection onto the span of $[g_1, \ldots, g_N]$. There is an operator $T_\delta : L_1(\mathbb{P}) \to L_1(\mathbb{P})$ such that for some $w_0(\delta)$ (independent of N) we have*

$$(1.6)\qquad \begin{array}{l} T_\delta(g_n) = g_n \ \ \forall n = 1, \ldots, N, \quad \|T_\delta\| \le w_0(\delta) \\ \textit{and} \quad \|T_\delta(1 - P_1) : L_2(\mathbb{P}) \to L_2(\mathbb{P})\| \le \delta. \end{array}$$

Moreover the same result holds for $(g_n^{\mathbb{R}})$.

Proof. First consider the real case when (g_n) are real valued Gaussians. We have a classical (hyper) contractive semigroup (sometimes called the Mehler semigroup) $t \mapsto T(e^{-t})$ multiplying the Hermite polynomials of (multivariate) degree d by e^{-dt}. We prefer to replace e^{-t} by δ. More explicitly, if we work on $\mathbb{R}^N$ equipped with the standard Gaussian measure γ, then

$$(1.7)\qquad T(\delta)(f) = \int f(\delta x + (1 - \delta^2)^{1/2} y)\, d\gamma(y).$$

Let P_0 be the projection onto the constant function 1. We can then take $T_\delta = \delta^{-1}(T(\delta) - P_0)$. Then (1.6) holds with $w_0(\delta) = 2/\delta$. The complex case requires a small adjustment: We write $g_n = 2^{-1/2}(g_n' + i g_n'')$ and we apply the preceding argument in $\mathbb{R}^{2N}$. This provides us with S_δ such that $S_\delta(g_n) = g_n$ but also $S_\delta(\overline{g_n}) = \overline{g_n}$ and $\|S_\delta(1 - Q) : L_2(\mathbb{P}) \to L_2(\mathbb{P})\| \le \delta$, where Q is the orthogonal projection onto the orthogonal of span$[g_n, \overline{g_n}]$. Consider now for any $z \in \mathbb{T}$ the measure preserving mapping $V_z : L_1 \to L_1$ taking g_n to zg_n, and let $W = \int \bar{z} V_z$. Note $W(g_n) = g_n$ and $W(\overline{g_n}) = 0$. Clearly $\|W : L_p \to L_p\| \le 1$ for any $1 \le p \le \infty$ and in particular for $p = 1, 2$. Then the operator $T_\delta = W S_\delta$ satisfies (1.6). □

We will use Talagrand's work in the form of the following result from [29] (see also [30, §2.4]).

Lemma 1.4 ([29]). *Assume* (0.1)*. There is a numerical constant K such that for any N and any $x_1, \ldots, x_N$ in l_∞ we have*

$$\int \left\| \sum \varphi_n(t) x_n \right\|_\infty dm(t) \le KC \int \left\| \sum g_n x_n \right\|_\infty d\mathbb{P}. \tag{1.8}$$

Proof. By (1.4) we may assume (g_n) are real Gaussians if we wish. In the tradition originating in Slepian's comparison Lemma, (1.8) is immediate from Theorem 15 in [29]. This is a simple Corollary of the Fernique majorizing measure conjecture, proved by Talagrand in [29]. □

The next result, originally due to Mireille Lévy [11] in the case $p = 1$ is a consequence of the Hahn-Banach theorem. We use the following notation: Let $u : L_p(\mathbb{P}) \to L_p(m)$ be a linear operator. We say that u is regular if $\exists C$ such that for any N and any $x_1, \ldots, x_N \in L_p(\mathbb{P})$ we have

$$\left\| \sup_n |u(x_n)| \right\|_p \le C \left\| \sup_n |x_n| \right\|_p. \tag{1.9}$$

More generally, we extend this definition to any linear operator $u : E \to L_p(m)$ defined only on a subspace $E \subset L_p(\mathbb{P})$. We denote by $\|u\|_{reg}$ the smallest constant C for which (1.9) holds for any finite set $x_1, \ldots, x_N \in E$. By a well known property of L_1-spaces with respect to the projective tensor product when $p = 1$ and E is the full space we have $\|u\|_{reg} = \|u\|$.

Proposition 1.5 ([11, 22]). *Let $(\Omega, \mathbb{P})$ and (T, m) be arbitrary measure spaces. Let $1 \le p \le \infty$.*
Let $\{g_n\} \subset L_p(\Omega, \mathbb{P})$, $\{\varphi_n\} \subset L_p(T, m)$ be arbitrary sets. The following are equivalent, for a fixed constant C.

(i) *For any N and any $x_1, \ldots, x_N \in \ell_\infty$*

$$\int \left\| \sum \varphi_n x_n \right\|_\infty^p dm \le C^p \int \left\| \sum g_n x_n \right\|_\infty^p d\mathbb{P}. \tag{1.10}$$

(i)' *Same as (i) for any Banach space B any N and any $x_1, \ldots, x_N \in B$.*

(ii) *There is a regular operator $u\ : L_p(\mathbb{P}) \to L_p(m)$ with $\|u\|_{reg} \le C$ such that $u(g_n) = \varphi_n$.*

Note that (i) $\Leftrightarrow$ (i)' holds because any separable Banach space embeds isometrically in ℓ_∞.

It is worthwhile to observe that the assumption (0.1) can be replaced in Theorem 1.1 by the following:

$$\text{There are } C \ge 0 \text{ and } u:\ L_1(\mathbb{P}) \to L_1(m) \text{ with } \|u\| \le C \text{ such that } \forall n\ \ u(g_n) = \varphi_n. \tag{1.11}$$

Definition 1.6. When (1.11) holds we will say that (φ_n) is C-dominated by (g_n).

Combining Proposition 1.5 with Talagrand's result, we find

Theorem 1.7. *There is a numerical constant τ_0 such that any C-subGaussian sequence $\{\varphi_n\} \subset L_1(T, m)$ is $\tau_0 C$-dominated by (g_n).*

Remark 1.8. It would be interesting to find the best constant τ_0. We suspect there may be an explicit formula for the kernel of the operator u by which a subGaussian system (say assumed satisfying (0.3) with $\sigma = 1$) is dominated by (g_n).

Remark 1.9 (Comparing L_{ψ_2} and L_p). Let f be a random variable on (T, m). Recall that $f \in L_{\psi_2}$ (i.e. $\|f\|_{\psi_2} < \infty$) iff

$$f \in \cap_{p<\infty} L_p \quad \text{and} \quad \sup_{p<\infty} p^{-1/2}\|f\|_p < \infty.$$

Moreover, $f \mapsto \sup_{2\le p<\infty} p^{-1/2}\|f\|_p$ is a norm equivalent to the norm (0.2) on L_{ψ_2}. These elementary and well known facts are proved using Stirling's formula and the Taylor expansion of the exponential function.
Therefore, a system (φ_n) is subGaussian (i.e. satisfies (0.1)) iff there is a constant L such that for any $y \in \ell_2$ and any $2 \le p < \infty$

$$\left\|\sum y_n \varphi_n\right\|_p \le L p^{1/2} \left(\sum |y_n|^2\right)^{1/2}. \tag{1.12}$$

Moreover the smallest C in (0.1) and the smallest L in (1.12) are equivalent quantities, up to numerical factors.

In particular, a sequence of characters on a compact Abelian group is sub-Gaussian iff it is a $\Lambda(p)$-set (in Rudin's sense [27]) for all $2 < p < \infty$ with $\Lambda(p)$-constant $O(p^{1/2})$. See Bourgain's survey for more information on $\Lambda(p)$-sets.

We will prove a more "abstract" form of Theorem 1.1. Note that when dealing with a tensor $T \in L_1(m_1) \otimes L_1(m_2)$ say $T = \sum x_j \otimes y_j$ then the projective and injective tensor product norm denoted respectively by $\|\cdot\|_\wedge$ and $\|\cdot\|_\vee$ are very simply explicitly described by

$$\|T\|_\wedge = \int \left|\sum x_j(t_1)y_j(t_2)\right| dm_1(t_1)dm_2(t_2)$$
$$\|T\|_\vee = \sup\left\{\left|\sum \langle x_j, \psi_1\rangle\langle y_j, \psi_2\rangle\right| \;\middle|\; \|\psi_1\|_\infty \le 1, \|\psi_2\|_\infty \le 1\right\}.$$

Theorem 1.10. *Let $(T_1, m_1), (T_2, m_2)$ be two probability spaces. Let (g_n) be an i.i.d. sequence of complex Gaussian random variables as above. For any $0 < \delta < 1$ there is $w(\delta) > 0$ for which the following property holds. Let (φ_n^1) and (φ_n^2) ($1 \le n \le N$) be functions respectively in $L_1(m_1)$ and $L_1(m_2)$. Assume that (φ_n^1) and (φ_n^2) are both 1-dominated by (g_n) (i.e. they satisfy (1.11) or equivalently (1.10) with $p = C = 1$). Then there is a decomposition in $L_1(m_1) \otimes L_1(m_2)$ of the form*

$$\sum\nolimits_1^N \varphi_n^1 \otimes \varphi_n^2 = t + r \tag{1.13}$$

satisfying

$$\|t\|_\wedge \le w(\delta) \tag{1.14}$$
$$\|r\|_\vee \le \delta. \tag{1.15}$$

Moreover the same result holds with $(g_n^{\mathbb{R}})$ in place of (g_n).

Corollary 1.11. *In the situation of the theorem, for any matrix $a \in M_N$ with $\|a\|_{M_N} \le 1$, there is a decomposition*

$$\sum\nolimits_{1 \le i,j \le N} a_{ij}\varphi_i^1 \otimes \varphi_j^2 = t + r$$

such that (1.14) and (1.15) hold.

Proof. We simply observe that by (1.5) we can replace $(\varphi_i^2)_{1 \le i \le N}$ by the "rotated" sequence $(\sum_j a_{ij}\varphi_j^2)_{1 \le i \le N}$, which still satisfies (1.11). □

The assumption that (φ_n) is orthonormal in Theorem 1.1 will now be weakened. It suffices to assume given a system (ψ_n) in L_∞ that is biorthogonal to (φ_n), in the sense that $\int \varphi_n \psi_k dm = \delta_{nk}$. The advantage of this formulation is that it includes the Gaussian case, e.g. with $\varphi_n = g_n^{\mathbb{R}} \|g_n^{\mathbb{R}}\|_1^{-1}$ and $\psi_n = \mathrm{sign}(g_n^{\mathbb{R}})$ (or the complex analogue).

Corollary 1.12. *In the situation of Theorem 1.10, let $(\psi_n^1), (\psi_n^2)$ be systems biorthogonal respectively to $(\varphi_n^1), (\varphi_n^2)$ and uniformly bounded respectively by C_1', C_2', then*

$$\sum |a_n| \le \alpha \sup_{(t_1,t_2)\in T\times T} \left|\sum a_n \psi_n^1(t_1)\psi_n^2(t_2)\right|, \tag{1.16}$$

where α is a constant depending only on C_1' and C_2'.
In particular, for any uniformly bounded orthonormal system (φ_n), subGaussian implies $\otimes^2$-Sidon.

Proof. Let $\varepsilon_n \in \mathbb{T}$ be such that $|a_n| = \varepsilon_n a_n$. By Corollary 1.11 we have a decomposition $\sum \varepsilon_n \varphi_n^1 \otimes \varphi_n^2 = t + r$. Let $f(t_1, t_2) = \sum a_n \psi_n^1(t_1)\psi_n^2(t_2)$. We have

$$\langle t + r, f\rangle = \int \left(\sum \varepsilon_n \varphi_n^1 \otimes \varphi_n^2\right)(f) = \sum \varepsilon_n a_n = \sum |a_n|.$$

Therefore

$$\begin{aligned}\sum |a_n| &\le |\langle t, f\rangle| + |\langle r, f\rangle| \le w(\delta)\|f\|_\infty + \sum |a_n||\langle r, \psi_n^1 \otimes \psi_n^2\rangle| \\ &\le w(\delta)\|f\|_\infty + \delta C_1' C_2' \sum |a_n|.\end{aligned}$$

Choosing δ such that $\delta C_1' C_2' = 1/2$ we obtain the conclusion with $\alpha = 2w(\delta)$. For the last part of Corollary 1.12 recall that by Theorem 1.7 subGaussian implies domination by (g_n). Thus the last assertion is obtained by taking (up to a renormalization, as in Remark 1.18) $\varphi_n^1 = \varphi_n^2 = \varphi_n$ and $\psi_n^1 = \psi_n^2 = \overline{\varphi_n}$. □

Remark 1.13. Actually the preceding proof requires only that $(\psi_n^1 \otimes \psi_n^2)$ be biorthogonal to $(\varphi_n^1 \otimes \varphi_n^2)$. For instance, it suffices to have (ψ_n^1) biorthogonal to (φ_n^1) and to have $\int \varphi_n^2 \psi_n^2 = 1$ for all n.

Remark 1.14. It is worthwhile to observe that Theorem 1.10 actually reduces to the case when $\varphi_n^1 = \varphi_n^2 = g_n$. Indeed, once we have obtained a decomposition $\sum_1^N g_n \otimes g_n = t + r$, we simply let

$$\sum\nolimits_1^N \varphi_n^1 \otimes \varphi_n^2 = (u_1 \otimes u_2)(t) + (u_1 \otimes u_2)(r).$$

However, it turns out that the proof below is essentially the same in the case $\varphi_n^1 = \varphi_n^2 = g_n$ as in the general case, so we proceed without using the present observation.

Proof of Theorem 1.10. By (1.4) it suffices to prove this when the (g_n)'s are real valued. This is arguably irrelevant but it simplifies notation, allowing us to avoid complex conjugation. Let $u_j : \ L_1(\mathbb{P}) \to L_1(m_j)$ with $\|u_j\| \leq 1$ such that $u_j(g_n^{\mathbb{R}}) = \varphi_n^j$ for $1 \leq n \leq N$. By classical results on L_1-spaces, for any $\varepsilon > 0$ there is a finite rank projection Q_j on $L_1(m_j)$ with $\|Q_j\| < 1 + \varepsilon$ that is the identity on span$[\varphi_1^j, \ldots, \varphi_N^j]$. Thus replacing u_j by $Q_j u_j$ we may clearly assume that u_j has finite rank. Thus each u_j can be identified with an element $\Phi^j \in L_\infty(\mathbb{P}) \otimes L_1(m_j)$.

It is easy and well known that, since u_j has finite rank

$$\|\Phi^j\|_{L_\infty(\mathbb{P};L_1(m_j))} = \|u_j\|.$$

Indeed, if k is the rank of u_j, this is clear when $\Phi^j = \sum_1^k x_q \otimes y_q$ with x_q,y_q measurable with respect to a finite σ-subalgebra. The general case can then be checked by a simple approximation argument by step functions.

A fortiori $\Phi^j \in L_2(\mathbb{P}) \otimes L_1(m_j)$.

The fact that Φ^j represents u_j is expressed for all $\psi^j \in L_\infty(m_j)$ by

$$u_j^*(\psi^j) = \int \Phi^j \psi^j dm_j. \tag{1.17}$$

Let

$$T = \int \Phi^1(\omega) \otimes \Phi^2(\omega) d\mathbb{P}(\omega). \tag{1.18}$$

Note that

$$\begin{aligned} \|T\|_\wedge &= \left\| \int \Phi^1(\omega) \otimes \Phi^2(\omega) d\mathbb{P}(\omega) \right\|_\wedge \\ &\leq \|\Phi^1\|_{L_\infty(\mathbb{P};L_1(m_1))} \|\Phi^2\|_{L_\infty(\mathbb{P};L_1(m_2))} \leq \|u_1\|\|u_2\|. \end{aligned} \tag{1.19}$$

We claim that

$$\|T\|_\vee \leq \|u_1 : \ L_2 \to L_1\|\|u_2 : \ L_2 \to L_1\| \leq \|u_1\|\|u_2\|. \tag{1.20}$$

Indeed, by (1.17) we have

$$\langle T, \psi^1 \otimes \psi^2 \rangle = \int u_1^*(\psi^1)(\omega) u_2^*(\psi^2)(\omega) d\mathbb{P}(\omega) \tag{1.21}$$

and hence

$$\begin{aligned}|\langle T, \psi^1 \otimes \psi^2 \rangle| &\le \|u_1^*(\psi^1)\|_2 \|u_2^*(\psi^2)\|_2 \\ &\le \|u_1^* :\ L_\infty \to L_2\| \|\psi^1\|_{L_\infty(m_1)} \|u_2^* :\ L_\infty \to L_2\| \|\psi^2\|_{L_\infty(m_2)},\end{aligned}$$

from which (1.20) follows. We will now use the orthogonal projection $P_1 : L_2(\mathbb{P}) \to \mathrm{span}[g_n^{\mathbb{R}}]$ from Lemma 1.3.

Let $S = \sum_1^N \varphi_n^1 \otimes \varphi_n^2$. We first claim that we have a decomposition (recall (1.18))

$$S = T + R$$

such that

$$\|R\|_\vee \le \|u_1(I - P_1) :\ L_2 \to L_1\| \|u_2\|. \tag{1.22}$$

Note that since $S = (u_1 \otimes u_2)(\sum g_n^{\mathbb{R}} \otimes g_n^{\mathbb{R}})$ and $\sum g_n^{\mathbb{R}} \otimes g_n^{\mathbb{R}}$ represents P_1 we have

$$\langle S, \psi^1 \otimes \psi^2 \rangle = \int \langle g_n^{\mathbb{R}}, u_1^*(\psi^1) \rangle \langle g_n^{\mathbb{R}}, u_2^*(\psi^2) \rangle d\mathbb{P} = \int P_1 u_1^*(\psi^1) u_2^*(\psi^2) d\mathbb{P},$$

and hence by (1.21)

$$S = T + R$$

where

$$\langle R, \psi^1 \otimes \psi^2 \rangle = -\int (I - P_1) u_1^*(\psi^1) u_2^*(\psi^2).$$

Applying (1.20) to R we find (1.22) which proves our claim.

We will now use the "interpolation property" from Lemma 1.3.

Fix $0 < \delta < 1$. We will now replace u_1 by $u_1 T_\delta$. Note that we still have $u_1 T_\delta(g_n^{\mathbb{R}}) = u_1(g_n^{\mathbb{R}}) = \varphi_n^1$ but in addition we now have

$$\|u_1 T_\delta\| \|u_2\| \le w_0(\delta) \|u_1\| \|u_2\| \text{ and } \|u_1 T_\delta (I - P_1) :\ L_2 \to L_1\| \le \delta \|u_1\|.$$

Therefore, the preceding decomposition becomes $S = t + r$ with $\|t\|_\wedge \le w_0(\delta)\|u_1\|\|u_2\| \le w_0(\delta)$ and $\|r\|_\vee \le \delta \|u_1\|\|u_2\| \le w_0(\delta)$. The minor correction to pass from the real case to the complex one leads to doubled constants $w_0(\delta), \delta$. □

Remark 1.15. Let $J: L_\infty(\mathbb{P}) \to L_1(\mathbb{P})$, $J_2: L_\infty(\mathbb{P}) \to L_2(\mathbb{P})$ and $J_1: L_2(\mathbb{P}) \to L_1(\mathbb{P})$ be the natural inclusions, so that $J = J_1 J_2$. A more abstract way to run the previous proof is to observe that the tensor S corresponds to the operator $u_1 J_1 P_1 J_2 u_2^*: L_\infty(m_2) \to L_1(m_1)$, which can be decomposed as

$$\begin{aligned} u_1 J_1 P_1 J_2 u_2^* &= u_1 J_1 T_\delta J_2 u_2^* - u_1 J_1 T_\delta (1 - P_1) J_2 u_2^* \\ &= u_1 T_\delta J u_2^* - u_1 J_1 T_\delta (1 - P_1) J_2 u_2^*. \end{aligned}$$

This is the operator version of the decomposition $S = t + r$. Then, $\|t\|_\wedge$ is equal to the integral norm of $u_1 T_\delta J u_2^*$ which is clearly (since J appears inside) $\leq \|u_1 T_\delta\| \|u_2^*\| \leq w_0(\delta) \|u_1\| \|u_2\|$ and $\|r\|_\vee$ is $\leq \|u_1 J_1 T_\delta (1 - P_1)\| \|J_2 u_2^*\| \leq \delta \|u_1 J_1\| \|J_2 u_2^*\| \leq \delta \|u_1\| \|u_2\|$.

Remark 1.16. It is known that the best estimate for $w_0(\delta)$ with properties (1.6) is $w_0(\delta) = O(\log(1/\delta))$. This follows easily from a result proved already in the Sidon set context by J.F. Méla, namely Lemma 3 in [16]. The latter says that for any $0 < \delta < 1$ there is a measure σ on $[0,1]$ such that $\int s d\sigma(s) = 1$, $|\int s^n d\sigma(s)| \leq \delta$ for all odd n and $|\sigma|([0,1]) \leq C |\log \delta|$ with C independent of δ. Now let $T(s)$ be the operator defined in (1.7). We have $T(s) = \sum_{n \geq 0} s^n P_n$ where the P_n's are the orthogonal projections onto the span (or the "chaos") of Hermite polynomials of degree n. Consider then

$$T_\delta = \int (T(s) - T(-s))/2 \, d\sigma(s) = P_1 + \sum_{n \text{ odd}>1} P_n \int s^n d\sigma(s).$$

It is easy to check (1.6) with $w_0(\delta) \leq C |\log \delta|$. Of course this implies the same growth for $w(\delta)$.

An alternate proof can be given using complex interpolation by the same idea as in [18, p. 11].

Remark 1.17 ($\otimes^k$ implies $\otimes^{k+1}$). Let (ψ_n^1) be a Sidon sequence. Then for any sequence (ψ_n^2) such that $\delta = \inf \|\psi_n^2\|_1 > 0$, the sequence $(\psi_n^1 \otimes \psi_n^2)$ is Sidon. Indeed, for any fixed s we have $\sum |a_n \psi_n^2(s)| \leq \alpha \| \sum a_n \psi_n^1 \psi_n^2(s) \|_\infty$, which, after integration over s, implies $\delta \sum |a_n| \leq \alpha \| \sum a_n \psi_n^1 \otimes \psi_n^2 \|_\infty$. In particular, for a uniformly bounded othonormal system, $\otimes^k$-Sidon implies $\otimes^{k+1}$-Sidon.

Remark 1.18 (On homogeneity). Assume that (φ_n) is biorthogonal to (ψ_n) and $\|\psi_n\|_\infty \leq C'$. Let $c > 0$. If (φ_n) is c-dominated by (g_n), then

$(c^{-1}\varphi_n)$ is 1-dominated by (g_n), is biorthogonal to $(c\psi_n)$, and the latter has norm $\le cC'$ in L_∞. Thus Corollary 1.12 applies in this case too.

Corollary 1.19. *Let $\{\psi_n \mid 1 \le n \le N\} \subset L_\infty(m)$ and $\{\varphi_n \mid 1 \le n \le N\} \subset L_1(m)$. Let $a_{ij} = \langle \varphi_i, \psi_j\rangle$. Assume that $a = [a_{ij}]$ is invertible and $\|a^{-1}\|_{M_N} \le c$. Then if (φ_n) is 1-dominated by (g_n), and if $\sup_n \|\psi_n\|_\infty \le C'$ there is a number $\alpha = \alpha(c, C')$ (depending only on c and C') such that (ψ_n) is $\otimes^2$-Sidon with constant α.*

Proof. There is a system $\{\varphi'_n \mid 1 \le n \le N\} \subset L_1(m)$ that is biorthogonal to (ψ_n) and c-dominated by (g_n). Indeed, setting $b = a^{-1}$, and $\varphi'_i = \sum_k b_{ik}\varphi_k$ we have $\langle \varphi'_i, \psi_j\rangle = \sum_k b_{ik}a_{kj} = \delta_{ij}$. Clearly, (φ'_n) is 1-dominated by $(\sum_k b_{nk}g_k)$, and by the rotational invariance of Gaussian measure, the latter is c-dominated by (g_n). Therefore the present statement follows from Corollary 1.12 (and Remark 1.18). □

Remark 1.20 (On almost biorthogonal systems). In the situation of the preceding Corollary, let $\theta = \|a - I\|$. If $\theta < 1$ then $\|b\| \le (1-\theta)^{-1}$.

For convenience, we record here the following elementary fact.

Lemma 1.21. *Let $\{g_n \mid 1 \le n \le N\} \subset L_1(\mathbb{P})$ and $\{\varphi_n \mid 1 \le n \le N\} \subset L_\infty(m)$. Assume there is $T:\ L_1(\mathbb{P}) \to L_1(m)^{**}$ of norm 1 such that*

$$\forall n, k \le N \quad \langle T(g_n), \varphi_k\rangle = \delta_{nk}. \tag{1.23}$$

Then for any $\varepsilon > 0$ there is $T^\varepsilon:\ L_1(\mathbb{P}) \to L_1(m)$ with norm $\le 1 + \varepsilon$ satisfying (1.23).

Proof. Fix $0 < \varepsilon < 1$. Let $E = \mathrm{span}\{T(g_n) \mid 1 \le n \le N\} \subset L_1(m)^{**}$. The space $L_1(m)^{**}$ is an abstract L_1-space and more generally a $\mathcal{L}_1$-space in the sense of [13]. In particular, there is a finite rank operator $S:\ L_1(m)^{**} \to L_1(m)^{**}$ with norm $< 1 + \varepsilon/4$ that is the identity on E. Let F be the range of S. Note $E \subset F$. By the local reflexivity principle, the inclusion $F \subset L_1(m)^{**}$ is the weak* limit of a net $J_i : F \to L_1(m)$ with $\|J_i\| < 1 + \varepsilon$. By a simple perturbation argument (see [13] for details) we may adjust J_i so that $\langle J_i(e), \varphi_k\rangle = \langle e, \varphi_k\rangle$ for any $e \in F$ (and hence for any $e \in E$). Then $T^\varepsilon = J_iST :\ L_1(\mathbb{P}) \to L_1(m)$ satisfies (1.23) and $\|T^\varepsilon\| \le 1 + \varepsilon$. □

2. Randomly Sidon systems

In this section, we denote simply by (g_n) the sequence denoted previously by $(g_n^{\mathbb{C}})$. In connection with Rider's paper [26], let us say that a sequence (ψ_n) is randomly Sidon if there is a constant C such that for any finite scalar sequence (a_n) we have

$$\sum |a_n| \le C\mathbb{E}\left\|\sum g_n a_n \psi_n\right\|_\infty .$$

Clearly, Sidon implies randomly Sidon.
Assuming (ψ_n) bounded in L_∞, it is easy to see by a truncation argument (as in [17] or in Lemma 3.2 below) that this is equivalent to the same property with a Bernoulli sequence (ε_n) (i.e. independent uniformly distributed choices of signs) in place of (g_n). The latter case was considered by Rider [26] when the φ_n's are distinct characters (γ_n) on a compact Abelian group and he proved that randomly Sidon sets of characters are Sidon.

Assume (ψ_n) bounded in L_∞. Bourgain and Lewko [5] observed using Slepian's lemma (see Remark 2.6) that, for any fixed k, if the k-fold tensor product $(\psi_n \otimes \cdots \otimes \psi_n)$ is randomly Sidon, then (ψ_n) is randomly Sidon. Thus every uniformly bounded $\otimes^k$-Sidon sequence is randomly Sidon. In particular, they proved that every uniformly bounded orthonormal system satisfying (0.1) is randomly Sidon.

In the remarks that follow we try to clarify the relationship between this notion and the notion of sequence dominated by (g_n) introduced in Definition 1.6.

Proposition 2.1. *Consider a sequence $(\varphi_n)_{1\le n\le N}$ in $L_1(m)$, The following properties are equivalent.*

(i) *For any $(f_1,\dots,f_N)$ in $L_\infty(m)$ we have $\sum |\langle f_n,\varphi_n\rangle| \le \mathbb{E}\|\sum g_n f_n\|_\infty$.*

(i)' *For any $(f_1,\dots,f_N)$ in $L_\infty(m)$ we have $|\sum\langle f_n,\varphi_n\rangle| \le \mathbb{E}\|\sum g_n f_n\|_\infty$.*

(ii) *There is an operator $u:\ L_1 \to L_1$ with $\|u\|\le 1$ such that $u(g_n)=\varphi_n$.*

Proof. The equivalence $(i) \Leftrightarrow$ (i)' is obvious because for any $z_n \in \mathbb{T}$ the right hand side is unchanged when we replace (f_n) by $(z_n f_n)$.
Assume (i). Consider the linear form

$$\sum g_n f_n \mapsto \sum \langle f_n,\varphi_n\rangle$$

and extend it by Hahn-Banach to $\xi \in L^1(\mathbb{P}; L_\infty(m))^*$ of norm ≤ 1 such that $\xi(g_n \otimes f_n) = \langle f_n, \varphi_n \rangle$. This linear form $\xi \in L^1(\mathbb{P}; L_\infty(m))^*$ defines an operator $u: \ L^1(\mathbb{P}) \to L_1(m)^{**}$ with norm ≤ 1 such that $\langle u(g_n), f_n \rangle = \xi(g_n \otimes f_n) = \langle f_n, \varphi_n \rangle$. Therefore $u(g_n) = \varphi_n$. Composing u with the norm 1 projection from $L_1(m)^{**}$ to $L_1(m)$ (associated to the Hahn decomposition) we obtain (ii). Conversely, if (ii) holds we have

$$\int \left\| \sum u(g_n)(t) \otimes f_n \right\|_\infty dm(t) \leq \mathbb{E} \left\| \sum g_n f_n \right\|_\infty$$

and since

$$\left| \int \sum u(g_n)(t) f_n(t) dm(t) \right| \leq \int ess \sup_s \left| \sum u(g_n)(t) f_n(s) \right| dm(t)$$

and $u(g_n) = \varphi_n$ we obtain (ii) $\Rightarrow$ (i)'. $\square$

When the functions $(\varphi_n)_{1 \leq n \leq N}$ are of the form $\varphi_n = \overline{\gamma_n}/C$, where (γ_n) are distinct characters on a compact Abelian group, and C a fixed constant, (i) implies

(iii) For any $a_n \in \mathbb{C}$

$$\left| \sum a_n \right| \leq C \mathbb{E} \left\| \sum a_n g_n \gamma_n \right\|_\infty .$$

By the "sign invariance" (complex sense) of (g_n) the latter is equivalent to

(iii)' $\sum |a_n| \leq C \mathbb{E} \| \sum a_n g_n \gamma_n \|_\infty$.

Moreover, a simple averaging shows that (iii) is equivalent to

(iv) For any $f_n \in L_\infty$

$$\sum |\hat{f}_n(\gamma_n)| \leq C \mathbb{E} \| \sum g_n f_n \|_\infty.$$

Indeed, this follows from

$$\begin{aligned} \mathbb{E} \left\| \sum g_n f_n \right\|_\infty &\geq \sup_s \mathbb{E} \sup_t \left| \sum g_n \gamma_n(s) f_n(t-s) \right| \\ &\geq \mathbb{E} \left\| \sum g_n \gamma_n * f_n \right\|_\infty = \mathbb{E} \left\| \sum g_n \hat{f}_n(\gamma_n) \gamma_n \right\|_\infty . \end{aligned} \tag{2.1}$$

Thus we conclude:

Remark 2.2. The set (γ_n) is randomly Sidon with constant C iff (γ_n) (or equivalently $(\overline{\gamma_n})$) is C-dominated by (g_n).

We now turn to the analogous questions for more general function systems.

Proposition 2.3. *Consider a finite sequence $(\psi_n)_{1\le n\le N}$ in $L_\infty(m)$. The following properties are equivalent.*

(i) *For any complex $N\times N$-matrix $[a_{nk}]$ we have*

$$\left|\sum\nolimits_1^N a_{nn}\right| \le \mathbb{E}\left\|\sum\nolimits_n g_n\left(\sum\nolimits_k a_{nk}\psi_k\right)\right\|_\infty \tag{2.2}$$

(ii) *For any $\varepsilon>0$, there is an operator $u\colon\ L_1(\mathbb{P})\to L_1(m)$ with $\|u\|\le 1+\varepsilon$ such that $(u(g_n))$ is biorthogonal to (ψ_n).*

Proof. Let $E=\mathrm{span}[\psi_n]$. (i) $\Rightarrow$ (ii) is proved using Hahn-Banach as for Proposition 2.1, but a priori this leads to an operator $u\colon\ L_1\to E^*$ with $\|u\|\le 1$ and $(u(g_n))$ biorthogonal to (ψ_n). But since E (being finite dimensional) is weak*-closed, we may identify E^* to L_1/N where N is the preannihilator of E. Applying the lifting property of L_1-spaces, we obtain (ii). More precisely, for any $\varepsilon>0$ there is a subspace $G\subset L_1(\mathbb{P})$ containing $\{g_n\}$ that is $(1+\varepsilon)$-isomorphic to a finite dimensional ℓ_1-space and $(1+\varepsilon)$-complemented in $L_1(\mathbb{P})$. Then it suffices to lift $u_{|G}$ and that is immediate. The proof of (ii) $\Rightarrow$ (i) is similar to the one for (ii) $\Rightarrow$ (i)' in Proposition 2.1. $\square$

Remark 2.4. By Theorem 1.10, if a sequence (ψ_n) assumed bounded in L_∞ satisfies (2.2) for all N then $(\psi_n\otimes\psi_n)$ is Sidon (in other words (ψ_n) is $\otimes^2$-Sidon).

Remark 2.5. In the converse direction, let (φ_n) and (ψ_n) be mutually biorthogonal sequences in L_∞. Assume $\|\varphi_n\|_\infty\le 1$ for all n. We claim that if (ψ_n) is randomly Sidon with constant α, then $(\psi_n\otimes\psi_n)$ satisfies (2.2) with the same constant α. Indeed, we have

$$\begin{aligned}&\mathbb{E}\left\|\sum\nolimits_n g_n\left(\sum\nolimits_k a_{nk}\psi_k\otimes\psi_k\right)\right\|_\infty\\ &\ge \sup_s \mathbb{E}\left\|\sum\nolimits_n g_n\left(\sum\nolimits_k a_{nk}\psi_k(s)\otimes\psi_k(\cdot)\right)\right\|_\infty\end{aligned}$$

and since $(g_n\varphi_n(s))$ is 1-dominated by (g_n)

$$\ge \sup_s \mathbb{E}\left\|\sum\nolimits_n g_n\varphi_n(s)\left(\sum\nolimits_k a_{nk}\psi_k(s)\otimes\psi_k(\cdot)\right)\right\|_\infty.$$

Therefore, integrating in s and using Jensen again we obtain

$$\mathbb{E}\left\|\sum\nolimits_n g_n\left(\sum\nolimits_k a_{nk}\psi_k\otimes\psi_k\right)\right\|_\infty \geq \mathbb{E}\left\|\sum\nolimits_n a_{nn}g_n\psi_n(\cdot)\right\|_\infty \geq \alpha^{-1}\sum|a_{nn}|.$$

This proves our claim.

Remark 2.6 (randomly $\otimes^k$-Sidon implies randomly Sidon). As observed by Bourgain and Lewko [5], a well known variant (due to Sudakov) of Slepian's comparison Lemma shows that if a uniformly bounded system is randomly $\otimes^k$-Sidon for some $k\geq 2$ then it is already randomly Sidon. This can be seen by an idea due to Simone Chevet [6]: let $K_1,\ldots,K_k$ be compact sets, $x_j^1\in C(K_1),\ldots,x_j^k\in C(K_k)$ finitely supported families, then,

$$\begin{aligned}&\mathbb{E}\left\|\sum g_j x_j^1\otimes\cdots\otimes x_j^k\right\|_{C(K_1\times\cdots\times K_k)}\\ &\leq\sqrt{k}\sum_{m=1}^k\left(\sup_j\prod_{q\neq m}\|x_j^q\|\right)\mathbb{E}\left\|\sum g_j x_j^m\right\|.\end{aligned}$$

Chevet's idea can be applied in a somewhat more general context (see [6]), it gives a similar bound for

$$\mathbb{E}\left\|\sum_{i(1),\ldots,i(k)} g_{i(1),\ldots,i(k)}x_{i(1)}^1\otimes\cdots\otimes x_{i(k)}^k\right\|_{C(K_1\times\cdots\times K_k)},$$

with $(g_{i(1),\ldots,i(k)})$ i.i.d. real valued Gaussian.
In any case, if we apply this to $x_j^1=\cdots=x_j^k=\psi_k,\ B_1=\cdots=B_k=L_\infty$ (recall L_∞ is isometric to $C(K)$ for some K), we find that, for (ψ_j) uniformly bounded in L_∞, randomly $\otimes^k$-Sidon implies randomly Sidon.

Theorem 2.7. *Let (ψ_n) be bounded in $L_\infty(T,m)$, and such that there is another system (φ_n) bounded in $L_\infty(T,m)$ such that $\int\psi_n\overline{\varphi_k}dm=\delta_{n,k}$. Then (ψ_n) is randomly Sidon iff it is $\otimes^4$-Sidon and this holds iff it is $\otimes^k$-Sidon for some (or all) $k\geq 4$. In particular, this is valid when (ψ_n) is a uniformly bounded orthonormal system.*

Proof. If (ψ_n) is randomly Sidon, by Remarks 2.5 and 2.4 $(\psi_n\otimes\psi_n)$ is $\otimes^2$-Sidon, which means (ψ_n) is $\otimes^4$-Sidon. Conversely, if (ψ_n) is $\otimes^k$-Sidon for some $k\geq 4$, then it is randomly $\otimes^k$-Sidon, and by Remark 2.6 it is randomly Sidon. $\square$

We can now state the generalization of Drury's union Theorem in our context.

Corollary 2.8. *In the situation of Theorem 2.7, if $\{\psi_n\}$ is the union of two Sidon systems, then it is $\otimes^4$-Sidon.*

Remark 2.9. We do not know whether, in the situation of Corollary 2.8, the assumption already implies $\otimes^2$-Sidon. By [25], it does not imply Sidon.

The next statement aims to clarify the connection between the subGaussian property of a system and its domination by a Gaussian sequence.

We will need the following

Notation. Let $(T;m)$ be a probability space. Let Z be a scalar valued random variable on $(T;m)$. we denote by $(Z^{[k]})$ an i.i.d. sequence of copies of Z on $(T;m)^{\mathbb{N}}$ so that

$$\forall t \in T^{\mathbb{N}} \quad Z^{[k]}(t) = Z(t_k).$$

We will use the following well known elementary fact.

There is an absolute constant $\theta > 0$ such that for any Z with $\mathbb{E}Z = 0$

$$\theta^{-1}\|Z\|_{\psi_2} \leq \mathbb{E}\sup|Z^{[k]}|(\log k)^{-1/2} \leq \theta\|Z\|_{\psi_2}. \tag{2.3}$$

This is easily proved by relating the growth of the function $t \mapsto \mathbb{P}\{|Z| > t\}$ to the infinite product appearing in $\mathbb{P}\{\sup|Z^{[k])}|(\log k)^{-1/2} > 2\theta\} = 1 - \prod_k(1 - \mathbb{P}\{|Z| > 2\theta(\log k)^{-1/2}\}$.

Proposition 2.10. *Let (φ_n) $(1 \leq n \leq N)$ be a system in $L_1(T;m)$. Consider the following assertions, where C and C' are positive constants.*

(i) *The system (φ_n) $(1 \leq n \leq N)$ satisfies* (0.1) *(i.e. it is C-subGaussian) and is such that $\mathbb{E}\varphi_n = 0$ for all n.*

(ii) *The system $(\varphi_n^{[k])})$ $(1 \leq n \leq N, k \in \mathbb{N})$ is C''-dominated by the (Gaussian i.i.d.) sequence $(g_n^{[k])})$ $(1 \leq n \leq N, k \in \mathbb{N})$.*

Then we have (i) $\Rightarrow$ (ii) *(resp.* (ii) $\Rightarrow$ (i)*) for some constant C'' (resp. C) depending only on C (resp. C'').*

Proof. Using the equivalence with (0.3), one checks easily that (i) is essentially equivalent to:

(i)' The system $(\varphi_n^{[k]})$ $(1 \leq n \leq N, k \in \mathbb{N})$ satisfies (0.1) for some possibly different constant C' depending only on C.

Then (i)$'$ $\Rightarrow$ (ii) by Talagrand's (1.8) and Proposition 1.5. The converse (ii) $\Rightarrow$ (i) follows from (2.3) and Proposition 1.5. Indeed, (1.10) with $p=1$ applied to $(\varphi_n^{[k]})$ (with a suitable choice of $x_n \in \ell_\infty$) yields for $Z = \sum a_n \varphi_n$ and $S = \sum a_n g_n$

$$\mathbb{E}\sup_k |Z^{[k]}|(\log k)^{-1/2} \le C\mathbb{E}\sup_k |S^{[k]}|(\log k)^{-1/2}.$$

By (2.3) this implies (i). □

3. Systems of random matrices

Assume given a sequence of finite dimensions d_n.

From now on g_n will be an independent sequence of random $d_n \times d_n$-matrices, such that $\{{d_n}^{1/2} g_n(i,j) \mid 1 \le i,j \le d_n\}$ are i.i.d. normalized $\mathbb{C}$-valued Gaussian random variables. Note $\|g_n(i,j)\|_2 = {d_n}^{-1/2}$.

For each n let (φ_n) be a random matrix of size $d_n \times d_n$ on (T,m). We call this a "matricial system". We will compare (φ_n) with the sequence (u_n) that is an independent sequence where each u_n is uniformly distributed over the unitary group $U(d_n)$. The subGaussian condition becomes: for any N and $y_n \in M_{d_n}$ $(n \le N)$ we have

$$\left\|\sum d_n \mathrm{tr}(y_n\varphi_n)\right\|_{\psi_2} \le C\left(\sum d_n \mathrm{tr}|y_n|^2\right)^{1/2} = \left\|\sum d_n \mathrm{tr}(y_n g_n)\right\|_2. \tag{3.1}$$

In other words, $\{d_n^{1/2}\varphi_n(i,j) \mid n \ge 1, 1 \le i,j \le d_n\}$ is a C-subGaussian system of functions. The uniform boundedness assumption becomes

$$\exists C' \ \forall n \quad \|\varphi_n\|_{L_\infty(M_{d_n})} \le C'. \tag{3.2}$$

As for the orthonormality condition it becomes

$$\int \varphi_n(i,j)\overline{\varphi_{n'}(k,\ell)} = d_n^{-1}\delta_{n,n'}\delta_{i,k}\delta_{j,\ell}. \tag{3.3}$$

In other words, $\{d_n^{1/2}\varphi_n(i,j) \mid n \ge 1, 1 \le i,j \le d_n\}$ is an orthonormal system.

This is modeled on the case when (φ_n) is a sequence of distinct irreducible representations on a compact group.

Actually, we will consider a slightly more general situation. We assume that there are complex-valued $\{\psi_n(i,j) \mid n \ge 1, 1 \le i,j \le d_n\}$ in $L_\infty(m)$ such that

$$\sup_n \|\psi_n\|_{L_\infty(m;M_{d_n})} \le C' \tag{3.4}$$

and

$$\int \varphi_n(i,j)\overline{\psi_{n'}(k,\ell)}\, dm = d_n^{-1}\delta_{n,n'}\delta_{i,k}\delta_{j,\ell}. \tag{3.5}$$

Equivalently

$$\begin{aligned} &\int \varphi_n \otimes \overline{\psi_{n'}}\, dm = 0 \text{ if } n \neq n' \\ &\text{and } \int \varphi_n \otimes \overline{\psi_n}\, dm = d_n^{-1} \sum_{i,j\le d_n} e_{ij}\otimes e_{ij}. \end{aligned} \tag{3.6}$$

Applying transposition on the second factor this is also equivalent to

$$\begin{aligned} &\int \varphi_n \otimes \psi_{n'}^*\, dm = 0 \text{ if } n \neq n' \\ &\text{and } \int \varphi_n \otimes \psi_n^*\, dm = d_n^{-1} \sum_{i,j\le d_n} e_{ij}\otimes e_{ji}. \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned} &\int \varphi_n \otimes \psi_n^*\, dm = d_n^{-1} \sum_{i,j\le d_n} e_{ij}\otimes e_{ji} \\ &\Leftrightarrow \forall a \in M_{d_n} \quad \int \varphi_n a \psi_n^*\, dm = d_n^{-1}\mathrm{tr}(a) I, \end{aligned} \tag{3.8}$$

and also

$$\Leftrightarrow \forall a \in M_{d_n} \quad \int \psi_n a \varphi_n^*\, dm = d_n^{-1}\mathrm{tr}(a) I. \tag{3.9}$$

Thus, if both (3.2) and the orthonormality (3.3) hold, then (3.4) and (3.5) hold for the choice $\psi_n = \varphi_n$. In any case, we will conclude from this (see Corollary 3.11) that $\exists \alpha$ such that for any (a_n) with $a_n \in M_{d_n}$

$$\sum d_n \mathrm{tr}|a_n| \le \alpha \sup_{(t_1,t_2)\in T\times T} \left| \sum d_n \mathrm{tr}(a_n \psi_n(t_1)\psi_n(t_2)) \right|. \tag{3.10}$$

Let $U(d)$ denote the (compact) group of unitary $d\times d$ matrices. In the next Lemma, we give a simple argument from [15] showing that the family

$\{u_k(i,j)\}$ $(k \geq 1,\ 1 \leq i,j \leq d_k)$ is dominated by $\{g_k(i,j)\}$ $(k \geq 1,\ 1 \leq i,j \leq d_k)$, using an explicit positive operator T, bounded on L_p for all $1 \leq p \leq \infty$. By work due to Figà-Talamanca and Rider, this family has long been known to be subGaussian, see [8, §36, p. 390]. The idea in Lemma 3.1 was used in [15] to give a simpler proof of the latter fact.

Lemma 3.1. *Let $(d_k)_{k\in I}$ be an arbitrary collection of integers. Let $G = \prod_{k\in I} U(d_k)$. Let $u \mapsto u_k$ denote the coordinates on G, and $u_k(i,j)$ $(1 \leq i,j \leq d_k)$ the entries of u_k. Let $\{g_k(i,j)\}$ $(1 \leq i,j \leq d_k)$ be a collection of independent complex valued Gaussian random variables such that $\mathbb{E}(g_k(i,j)) = 0$ and $\mathbb{E}|g_k(i,j)|^2 = 1/d_k$, on a probability space $(\Omega,\mathbb{P})$. For some $C_0 > 0$ there is an operator $T:\ L_1(\Omega,\mathbb{P}) \to L_1(G,m_G)$ with $\|T:\ L_p(\Omega,\mathbb{P}) \to L_p(G,m_G)\| \leq C_0$ for all $1 \leq p \leq \infty$ such that*

$$\forall k \forall i,j \leq d_k \quad T(g_k(i,j)) = u_k(i,j).$$

Proof. Let $g_k = v_k|g_k|$ be the polar decomposition of g_k. The key observation is that (v_k) and $(|g_k|)$ are independent random variables, and that (v_k) and (u_k) have the same distribution. Also for any fixed $v \in U(d_k)$, g_k has the same distribution as (vg_kv^{-1}), and hence $(|g_k|)$ has the same distribution as $(v|g_k|v^{-1})$. Let V denote the conditional expectation with respect to (v_k) on $(\Omega,\mathbb{P})$. Then $V(g_k) = v_k\mathbb{E}|g_k|$. Since $\mathbb{E}|g_k|$ commutes with any $v \in U(d_k)$, we have $\mathbb{E}|g_k| = \delta_k I$ for some $\delta_k > 0$. We claim that $\delta = \inf_k \delta_k > 0$. Indeed, let $c_1 = d_k^{1/2}\mathbb{E}|g_k(i,j)|$. Note that c_1 is independent of (k,i,j). We have $(\sum_{i,j}(\mathbb{E}|g_k(i,j)|)^2)^{1/2} \leq \mathbb{E}(\sum_{i,j}|g_k(i,j)|^2)^{1/2} = \mathbb{E}(\mathrm{tr}|g_k|^2)^{1/2}$ and hence

$$\begin{aligned} d_k^{1/2}c_1 &\leq \mathbb{E}(\mathrm{tr}|g_k|^2)^{1/2} \leq \mathbb{E}(\mathrm{tr}|g_k|\|g_k\|)^{1/2} \\ &\leq (\mathbb{E}\mathrm{tr}|g_k|)^{1/2}(\mathbb{E}\|g_k\|)^{1/2} = (\delta_k d_k)^{1/2}(\mathbb{E}\|g_k\|)^{1/2} \end{aligned}$$

and since, as is well known $\sup_k \mathbb{E}\|g_k\| < \infty$, the claim $\delta > 0$ follows. Since (v_k) and (u_k) have the same distribution we can identify V to an operator $V_1:\ L_1(\Omega,\mathbb{P}) \to L_1(G,m_G)$ such that $V_1(g_k(i,j)) = \delta_k u_k(i,j)$. We will now modify V_1 to replace δ_k by δ. Let $\mathcal{E}_k:\ L_1(G,m_G) \to L_1(G,m_G)$ denote the conditional expectation with respect to the σ-algebra generated by the coordinates $\{u_j \mid j \neq k\}$ on G, so that $\mathcal{E}_k(u_j) = u_j$ if $j \neq k$, and $\mathcal{E}_k(u_k) = 0$. Let Id denote the identity on $L_1(G,m_G)$. Recall $0 < \delta/\delta_k \leq 1$. Then let

$$W = \prod((1-\delta/\delta_k)\mathcal{E}_k + (\delta/\delta_k)Id).$$

By a simple limiting argument, this infinite product makes sense and defines an operator $W:\ L_p(G,m_G) \to L_p(G,m_G)$ with $\|W\| \leq 1$ for any $1 \leq p \leq \infty$, such that $W(u_k(i,j)) = (\delta/\delta_k)u_k(i,j)$. Thus, setting $T = \delta^{-1}WV_1$ we have

$$T(g_k(i,j)) = \delta^{-1}W(V_1(g_k(i,j))) = u_k(i,j),$$

and $\|T : \ L_p(G,m_G) \to L_p(G,m_G)\| \leq C_0 = \delta^{-1}$. In addition, note that T is actually positive. $\square$

The following basic fact compares the notions of randomly Sidon for (g_k) and (u_k). It is proved by the same truncation trick that was used in [17]. See [15, Chap.V and VI] for further details and more general facts.

Lemma 3.2. *Let $\psi_n \in L_\infty(m; M_{d_n})$ $(n \geq 1)$ be an arbitrary matricial system satisfying* (3.4). *The following are equivalent:*

(i) *There is a constant α_1 such that for any n and any $x_k \in M_{d_k}$*

$$\sum d_k \mathrm{tr}|x_k| \leq \alpha_1 \mathbb{E} \left\| \sum d_k \mathrm{tr}(x_k g_k \psi_k) \right\|_\infty .$$

(ii) *There is a constant α_2 such that for any n and any $x_k \in M_{d_k}$*

$$\sum d_k \mathrm{tr}|x_k| \leq \alpha_2 \int \left\| \sum d_k \mathrm{tr}(x_k u_k \psi_k) \right\|_\infty m_G(du).$$

where $u = (u_n)$ denotes (as before) a random sequence of unitaries uniformly distributed in $G = \prod_{n\geq 1} U(d_n)$.

Sketch. From Lemma 3.1 it is easy to deduce that

$$\int \left\| \sum d_k \mathrm{tr}(x_k u_k \psi_k) \right\|_\infty m_G(du) \leq C_0 \mathbb{E} \left\| \sum d_k \mathrm{tr}(x_k g_k \psi_k) \right\|_\infty ,$$

and hence (ii) $\Rightarrow$ (i). To check the converse, recall the well known fact that $c_4 = \sup \mathbb{E}\|g_n\|^2 < \infty$, from which it is easy to deduce by Chebyshev's inequality that there exists $c_5 > 0$ such that

$$\sup \mathbb{E}(\|g_n\| 1_{\{\|g_n\| > c_5\}}) \leq (2\alpha_1 C')^{-1}.$$

We may assume that the sequences (u_n) and (g_n) are mutually independent. Then the sequences (g_n) and $(u_n g_n)$ have the same distribution. Then by the triangle inequality and by Remark 5.1

$$\begin{aligned}
&\mathbb{E}\left\|\sum d_k \mathrm{tr}(x_k g_k \psi_k)\right\|_\infty = \mathbb{E}\left\|\sum d_k \mathrm{tr}(x_k u_k g_k \psi_k)\right\|_\infty \\
&\le \mathbb{E}\left\|\sum d_k \mathrm{tr}(x_k u_k g_k 1_{\{\|g_k\|\le c_5\}}\psi_k)\right\|_\infty \\
&\quad + \mathbb{E}\left\|\sum d_k \mathrm{tr}(x_k u_k g_k 1_{\{\|g_k\|> c_5\}}\psi_k)\right\|_\infty \\
&\le c_5\mathbb{E}\left\|\sum d_k \mathrm{tr}(x_k u_k \psi_k)\right\|_\infty + (2\alpha_1 C')^{-1}\sum d_k \mathrm{tr}|x_k|\|\psi_k\|_\infty \\
&\le c_5\mathbb{E}\left\|\sum d_k \mathrm{tr}(x_k u_k \psi_k)\right\|_\infty + (2\alpha_1)^{-1}\sum d_k \mathrm{tr}|x_k|.
\end{aligned}$$

Using this we see that (i) implies

$$\sum d_k \mathrm{tr}|x_k| \le \alpha_1 c_5 \mathbb{E}\left\|\sum d_k \mathrm{tr}(x_k u_k \psi_k)\right\|_\infty + (1/2)\sum d_k \mathrm{tr}|x_k|,$$

and hence (i) $\Rightarrow$ (ii) with $\alpha_2 \le 2\alpha_1 c_5$. □

Definition 3.3. Let (φ_n) be a sequence with $\varphi_n \in L_\infty(T, m; M_{d_n})$ for all n and let $C > 0$.

(i) We say that (φ_n) is Sidon with constant C if for any n and any sequence (x_k) with $x_k \in M_{d_k}$ we have

$$\sum_1^n d_k \mathrm{tr}|x_k| \le C\left\|\sum_1^n d_k \mathrm{tr}(x_k \varphi_k)\right\|_\infty.$$

(ii) We say that (φ_n) is randomly Sidon with constant C if for any n and any $x_k \in M_{d_k}$ we have

$$\sum_1^n d_k \mathrm{tr}|x_k| \le C\mathbb{E}\left\|\sum_1^n d_k \mathrm{tr}(x_k g_k \varphi_k)\right\|_\infty.$$

By Lemma 3.2 this is equivalent to the previous definition with random unitaries (u_k) in place of (g_k).
If this holds only for scalar matrices (i.e. for $x_k \in \mathbb{C}I_{d_k}$) we say that (φ_n) is randomly central Sidon with constant C.

(iii) Let $k \ge 1$. We say that (φ_n) is $\dot{\otimes}^k$-Sidon with constant C if the system $\{\varphi_n(t_1)\cdots\varphi_n(t_k)\}$ is Sidon with constant C.
We say that (φ_n) is randomly $\dot{\otimes}^k$-Sidon with constant C if $\{\varphi_n(t_1)\cdots\varphi_n(t_k)\}$ is randomly Sidon with constant C.

Now assume merely that $\{\varphi_n\} \subset L_2(T,m)$.

(iv) We say that (φ_n) is subGaussian with constant C (or C-subGaussian) if for any n and any complex sequence (x_k) we have

$$\left\|\sum_1^n d_k \operatorname{tr}(x_k\varphi_k)\right\|_{\psi_2(m)} \leq C\left(\sum_1^n d_k \operatorname{tr}|x_k|^2\right)^{1/2}.$$

Remark 3.4. Using (u_k) for the randomization it is clear that Sidon implies randomly Sidon (with at most the same constant). A fortiori, $\dot{\otimes}^k$-Sidon implies randomly $\dot{\otimes}^k$-Sidon.

Remark 3.5. We should emphasize that central Sidon does not imply randomly central Sidon, in contrast with the preceding remark.

As earlier, we will consider the following more general form of the assumption (3.1):

$$\begin{aligned}&\exists C \geq 0\ \exists u:\ \ L_1(\mathbb{P}) \to L_1(m)\\ &\text{such that } \|u\| \leq C \text{ and } \forall n,i,j\ \ u(g_n(i,j)) = \varphi_n(i,j).\end{aligned} \tag{3.11}$$

In other words, the φ_n's are entrywise C-dominated by the g_n's.

Here again, Talagrand's inequality (1.8) is crucial. Restated in the present context:

Theorem 3.6. *For any matricial system* (φ_n)*,* (3.1) $\Rightarrow$ (3.11) *(possibly with a different* C*).*

Notation. Let (T_1, m_1) and (T_2, m_2) be probability spaces.
Let $\psi^1 \in L_\infty(T_1, m_1; M_d)$, $\psi^2 \in L_\infty(T_2, m_2; M_d)$. We denote by

$$\psi^1\dot{\otimes}\psi^2 \in L_\infty(T_1 \times T_2, m_1 \times m_2; M_d)$$

the function defined on $T_1 \times T_2$ by

$$\psi^1\dot{\otimes}\psi^2(t_1,t_2) = \psi^1(t_1)\psi^2(t_2).$$

We now state the matricial generalization of Corollary 1.12.

Theorem 3.7. *Assuming* (3.4) *and* (3.5)*, we have* (3.1) $\Rightarrow$ (3.10)*. More generally, given two systems* $(\varphi_n^1), (\varphi_n^2)$ *satisfying* (3.11) *with respective constants* C_1, C_2*, and two systems* $(\psi_n^1), (\psi_n^2)$ *satisfying* (3.4) *with respective* *constants C_1', C_2' and such that the pairs $(\varphi_n^1), (\psi_n^1)$ and $(\varphi_n^2), (\psi_n^2)$ satisfy (3.5), the system $(\psi_n^1 \dot{\otimes} \psi_n^2)$ is Sidon with a constant depending only on C_1, C_2, C_1', C_2'.*

Let $v \in L_1 \otimes L_1$. We denote

$$\gamma_2^*(v) = \sup\left\{\left|\sum\nolimits_1^N \langle v, x_j \otimes y_j\rangle\right|\right\}$$

where the sup runs over all $\sum_1^N x_j \otimes y_j \in L_\infty \otimes L_\infty$ such that

$$\left\|\left(\sum\nolimits_1^N |x_j|^2\right)^{1/2}\right\|_\infty \left\|\left(\sum\nolimits_1^N |y_j|^2\right)^{1/2}\right\|_\infty \le 1\ .$$

Theorem 3.7 will be deduced rather easily from Theorem 1.10 using the following simple fact.

Lemma 3.8. *Let $v \in L_1(m_1) \otimes L_1(m_2)$ be a tensor such that $\gamma_2^*(v) \le 1$. Let $x \in M_d$ and let $\psi^j \in L_\infty(m_j; M_d)$. Let $f(t_1, t_2) = \mathrm{tr}(a\psi^1\psi^2) \in L_\infty(m_1 \times m_2)$. Then*

$$|\langle v, f\rangle| \le \mathrm{tr}|a| \|\psi^1\|_{L_\infty(m_1;M_d)} \|\psi^2\|_{L_\infty(m_2;M_d)}.$$

Proof. We may assume (by polar decomposition) $a = a_2a_1$ with $\mathrm{tr}|a_1|^2 = \mathrm{tr}|a_2|^2 = \mathrm{tr}|a|$. Then $f = \mathrm{tr}([a_1\psi^1][\psi^2 a_2]) = \sum_{k,\ell}(a_1\psi^1)(\ell, k) \otimes (\psi^2 a_2)(k, \ell)$. Then

$$\langle v, f\rangle = \sum\nolimits_{k,\ell} \langle v, (a_1\psi^1)(\ell, k) \otimes (\psi^2 a_2)(k, \ell)\rangle$$

and hence since $\gamma_2^*(v) \le 1$

$$|\langle v, f\rangle| \le \left\|\left(\sum\nolimits_{k,\ell} |(a_1\psi^1)(\ell, k)|^2\right)^{1/2}\right\|_\infty \left\|\left(\sum\nolimits_{k,\ell} |(\psi^2 a_2)(k, \ell)|^2\right)^{1/2}\right\|_\infty$$

but we have

$$\left(\sum\nolimits_{k,\ell} |(a_1\psi^1)(\ell, k)|^2\right)^{1/2} = (\mathrm{tr}|a_1\psi^1|^2)^{1/2} \le (\mathrm{tr}|a_1|^2)^{1/2} \|\psi^1\|_{M_d}$$

and similarly for $\psi^2 a_2$. Thus we obtain

$$\begin{aligned} |\langle v, f\rangle| &\le (\mathrm{tr}|a_1|^2)^{1/2} (\mathrm{tr}|a_2|^2)^{1/2} \|\psi^1\|_{L_\infty(M_d)} \|\psi^2\|_{L_\infty(M_d)} \\ &= (\mathrm{tr}|a|) \|\psi^1\|_{L_\infty(M_d)} \|\psi^2\|_{L_\infty(M_d)} \end{aligned}$$

proving the Lemma. $\square$

Remark 3.9. By Grothendieck's well known inequality (see e.g. [23, Theorem 2.1]) we have $\gamma_2^*(v) \le K_G\|v\|_\vee$ for any $v \in L_1(m_1) \otimes L_1(m_2)$. Thus in Theorem 1.10 we have $\gamma_2^*(r) \le K_G\|r\|_\vee \le K_G\delta$. But actually a close examination (see Remark 1.15) shows that we directly obtain a bound for $\gamma_2^*(r)$ without recourse to Grothendieck's theorem. Indeed, with the notation of the proof of Theorem 1.10, one has $\gamma_2^*(R) \le \|T_\delta(I-P_1) : \ L_2 \to L_2\|\|u_1\|\|u_2\|$.

Remark 3.10. In case the reader is wondering about that, the general definition of the γ_2^*-norm for an element r in the algebraic tensor product $X \otimes Y$ of two Banach spaces is

$$\gamma_2^* = \inf\left\{\|a\|_{M_n}\left(\sum\nolimits_1^n \|x_j\|^2\right)^{1/2}\left(\sum\nolimits_1^N \|y_j\|^2\right)^{1/2}\right\},$$

where the infimum runs over all possible ways to write r as

$$r = \sum\nolimits_{1\le i,j\le n} a_{ij} x_i \otimes y_j, \quad n \ge 1,\ x_i \in X,\ y_j \in Y,\ a_{ij} \in \mathbb{C}.$$

When $X = Y = L_1$ this is identical to the preceding definition (see [23]).

Proof of Theorem 3.7. We will apply Theorem 1.10. By homogeneity, we may assume that (φ_n^1) and (φ_n^2) satisfy (3.11) with $C_1 = C_2 = 1$ (then C_j' is replaced by C_jC_j' and ψ_n^j by $C_j\psi_n^j$). Let V_n be arbitrary in the unit ball of M_{d_n}. Consider the tensor

$$S = \sum\nolimits_n d_n \sum\nolimits_{i,k,\ell} V_n(i,k)\varphi_n^1(k,\ell) \otimes \varphi_n^2(\ell,i),$$

which roughly could be written as $\sum_n d_n \mathrm{tr}(V_n\varphi_n^1\varphi_n^2)$ using tensor product to form the products of matrix coefficients in φ_n^1 and φ_n^2. Let $\varphi_n'^1 = V_n\varphi_n^1$ (this denotes product of the scalar matrix V_n by the L_1-valued matrix φ_n^1). Note that, by (1.5) (and Proposition 1.5) applied to the standard normal family $\{d_n^{1/2}\varphi_n^1(i,j) \mid 1 \le n \le N, i,j \le d_n\}$, if we replace (φ_n^1) by $(\varphi_n'^1)$, then (3.11) still holds. This gives us

$$S = \sum\nolimits_n \sum\nolimits_{i,\ell} [d_n^{1/2}\varphi_n'^1(i,\ell)] \otimes [d_n^{1/2}\varphi_n^2(\ell,i)].$$

By Theorem 1.10 (actually we could invoke Corollary 1.11), and using Remark 3.9, this shows that we have a decomposition $S = t + r$ with $\|t\|_\wedge \le w(\delta)$ and $\gamma_2^*(r) \le \delta$. Now let $f = \sum d_n \mathrm{tr}(a_n\psi_n^1\psi_n^2) \in L_\infty(m_1) \otimes L_\infty(m_2)$, or more explicitly

$$f = \sum\nolimits_n d_n \sum\nolimits_{i,k,\ell} a_n(i,k)\psi_n^1(k,\ell) \otimes \psi_n^2(\ell,i).$$

Recalling (3.5) we find $\langle S, f\rangle = \sum d_n \mathrm{tr}({}^tV_n a_n)$. Then (denoting simply $\|f\|_\infty = \|f\|_{L_\infty(m_1 \times m_2)}$)

$$|\langle t, f\rangle| \le \|t\|_\wedge \|f\|_\infty \le w(\delta)\|f\|_\infty$$

and by Lemma 3.8 and (3.4)

$$\begin{aligned}|\langle r, f\rangle| &\le \delta \sum d_n \mathrm{tr}|a_n| \|\psi_n^1\|_{L_\infty(m_1;M_{d_n})} \|\psi_n^2\|_{L_\infty(m_2;M_{d_n})}\\ &\le \delta \sum d_n \mathrm{tr}|a_n| C_1' C_2'.\end{aligned}$$

Therefore

$$\left|\sum d_n \mathrm{tr}({}^tV_n a_n)\right| = |\langle S, f\rangle| \le w(\delta)\|f\|_\infty + \delta \sum d_n \mathrm{tr}|a_n| C_1' C_2',$$

and taking the sup over all V_n's we find

$$\sum d_n \mathrm{tr}|a_n| \le w(\delta)\|f\|_\infty + \delta \sum d_n \mathrm{tr}|a_n| C_1' C_2',$$

and we conclude choosing δ small enough so that $\delta C_1' C_2' < 1$ that we have

$$\sum d_n \mathrm{tr}|a_n| \le w(\delta)(1 - \delta C_1' C_2')^{-1} \|f\|_\infty.$$

Taking $\delta C_1' C_2' = 1/2$, this completes the proof with $\alpha = 2w((2C_1' C_2')^{-1})$, and using Remark 1.16 we obtain the announced bound on α. $\square$

In particular, we have

Corollary 3.11. *Let $\{\psi_n \mid 1 \le n \le N\} \subset L_\infty(m; M_{d_n})$ satisfying (3.4). Assume that the system $\{d_n^{1/2}\psi_n(i,j) \mid 1 \le n \le N, 1 \le i,j \le d_n\}$ admits a biorthogonal system that is 1-dominated by $\{d_n^{1/2} g_n(i,j) \mid 1 \le n \le N, 1 \le i,j \le d_n\}$. Then there is a number $\alpha = \alpha(C')$ (depending only on C') such that (ψ_n) is $\dot\otimes^2$-Sidon with constant α.*

Remark 3.12 (Returning to group representations). Let G be a compact group. Let $\Lambda = \{\pi_n\} \subset \hat G$ be a sequence of distinct unitary representations on G. Let $d_n = \dim(\pi_n)$. Then (Peter-Weyl) $\{d_n^{1/2}\pi_n(i,j)\}$ is an orthonormal system in $L_2(G)$. Thus we may apply Corollary 3.11 with

$\psi_n = \varphi_n = \pi_n$ on (G, m_G). Recalling Theorem 3.6, we find that if $\Lambda = \{\pi_n\}$ satisfies (3.1), then Λ is a Sidon set. Indeed, for representations, $\dot{\otimes}^2$-Sidon (or $\dot{\otimes}^k$-Sidon) obviously implies Sidon. This was first proved in [17, 19].

Remark 3.13 (On almost biorthogonal systems). In the situation of the preceding Corollary, just like in Remark 1.20 it suffices to have a system almost biorthogonal to $\{d_n^{1/2}\psi_n(i,j) \mid 1 \le n \le N, 1 \le i,j \le d_n\}$. More precisely, let $I = \{(n,i,j) \mid 1 \le n \le N, 1 \le i,j \le d_n\}$. Let $p = (n,i,j) \in I$ and $p' = (n',i',j') \in I$. Let $a = [a(p,p')]$ be the matrix defined by $a(p,p') = \langle d_n^{1/2}\varphi_n(i,j), d_{n'}^{1/2}\psi_{n'}(i',j')\rangle$. Assume a invertible with inverse b such that $\|b\| \le c$. If $\{d_n^{1/2}\varphi_n(i,j) \mid 1 \le n \le N, 1 \le i,j \le d_n\}$ is 1-dominated by $\{d_n^{1/2}g_n(i,j) \mid 1 \le n \le N, 1 \le i,j \le d_n\}$, then there is a number $\alpha = \alpha(c, C')$ (depending only on c, C') such that (ψ_n) is $\dot{\otimes}^2$-Sidon with constant α.

Remark 3.14 (On almost biorthogonal single systems). The preceding Remark is already significant for a single random $d \times d$-matrix $\psi \in L_\infty(m; M_d)$ with $\|\psi\|_{L_\infty(m;M_d)} \le C'$. Indeed, let $\varphi \in L_1(m; M_d)$. Let $\{d^{1/2}g_{ij} \mid 1 \le i,j \le d\}$ be a family of Gaussian complex variables with $\mathbb{E}(g_{ij}) = 0$ and $\mathbb{E}|g_{ij}|^2 = 1$. We again replace (3.5) by almost orthogonality. Let a be the $d^2 \times d^2$-matrix defined by $a(i,j;i',j') = d\int \psi_{ij}^*\varphi_{i'j'}\,dm$. If a is invertible and $\|a^{-1}\|_{M_{d^2}} \le c$, and if $(d^{1/2}\varphi_{ij})$ is 1-dominated by $(d^{1/2}g_{ij})$, there is $\alpha(c, C')$ *(independent of d)* such that

$$\forall x \in M_d \quad \mathrm{tr}|x| \le \alpha \|tr(x(\psi\dot{\otimes}\psi))\|_\infty.$$

In other words, the singleton $\{\psi\}$ is $\dot{\otimes}^2$-Sidon with constant α.

4. An example

The following example provides us with an illustration of the possible use of Corollary 3.11 and Remark 3.14. Although there may well be an alternate argument, we do not see a direct proof of the phenomenon appearing in Corollary 4.1.

Let $\chi \ge 1$ be a constant (to be specified later). Let T_n be the set of $n \times n$-matrices $a = [a_{ij}]$ with $a_{ij} = \pm 1/\sqrt{n}$. Let

$$A_n^\chi = \{a \in T_n \mid \|a\| \le \chi\}.$$

This set includes the famous Hadamard matrices. We have then

Corollary 4.1. *There is a numerical $\chi \geq 1$ such that for some C we have*

$$\forall n \geq 1\ \forall x \in M_n \quad \mathrm{tr}|x| \leq C \sup_{a', a'' \in A_n^\chi} |\mathrm{tr}(x a' a'')|.$$

Equivalently, denoting the set $\{a'a'' \mid a', a'' \in A_n^\chi\}$ by $A_n^\chi A_n^\chi$, its absolutely convex hull satisfies

$$(\chi)^2 \, absconv[A_n^\chi A_n^\chi] \subset B_{M_n} \subset C \, absconv[A_n^\chi A_n^\chi]$$

Proof. Let $(\Omega, \mathbb{P})$ be a probability space. Let $g_n^{\mathbb{R}}$ (resp. $g_n^{\mathbb{C}}$) be a random $n \times n$-matrix on $(\Omega, \mathbb{P})$ with i.i.d. real-valued (resp. complex-valued) normalized Gaussian entries of mean 0 and L_2-norm $= (1/n)^{1/2}$ as usual. Let

$$\varepsilon_n(i,j)(\omega) = n^{-1/2} \mathrm{sign}(g_n^{\mathbb{R}}(i,j)(\omega)).$$

Let $\gamma(1)$ be the L_1-norm of a normal Gaussian variable (i.e. $\gamma(1) = (2\pi)^{-1/2} \int |x| \exp -(x^2/2) dx$). Clearly, for any i, j, i', j'

$$\mathbb{E}(\varepsilon_n(ij) g_n^{\mathbb{R}}(i'j')) = \delta_{ii'} \delta_{jj'} n^{-1/2} \mathbb{E}|g_n^{\mathbb{R}}(i'j')| = \delta_{ii'} \delta_{jj'} n^{-1} \gamma(1).$$

We define $\psi_n \in L_\infty(\mathbb{P}; M_n)$ by

$$\psi_n = (2^{1/2}/\gamma(1)) \varepsilon_n 1_{\varepsilon_n \in A_n^\chi}.$$

Note $\|\psi_n\|_{M_n} \leq 2^{1/2} \chi/\gamma(1)$. We will use Corollary 3.11. Clearly $\mathbb{E}(\psi_n^* \otimes g_{n'}) = 0$ whenever $n \neq n'$. To handle the case $n = n'$, it is well known that there is $c_0 > 0$ such that

$$\forall n \forall \chi \geq c_0 \quad \mathbb{P}(\varepsilon_n \notin A_n^\chi) = \mathbb{P}\{\|\varepsilon_n\| > \chi\} \leq \exp -(c_0 n \chi^2).$$

Therefore, if $\chi \geq c_0$ for any i, j, i', j' we have

$$\begin{aligned} |\mathbb{E}(\psi_n(ij) 2^{-1/2} g_n^{\mathbb{R}}(i'j')) - \delta_{ii'} \delta_{jj'} n^{-1}| &\leq \gamma(1)^{-1} \mathbb{E}(|g_n^{\mathbb{R}}(i'j')| 1_{\{\varepsilon_n \notin A_n^\chi\}}) \\ &\leq \gamma(1)^{-1} n^{-1/2} \exp -(c_0 n \chi^2/2). \end{aligned}$$

Fix $\chi \geq c_0$. This shows that the matrix

$$a(i,j;i',j') = n \int \psi_n(ij) 2^{-1/2} g_n^{\mathbb{R}}(i'j') \, dm$$

is a perturbation of the identity when n is large enough so that (say) when $n \geq n_0(\chi)$ it is invertible with inverse of norm ≤ 2. Let $\varphi_n(ij) =$

$(2^{-1/2}g_n^{\mathbb{R}}(ij))$. Note that $(\varphi_n(ij))$ is obviously 1-dominated by $(g_n^{\mathbb{C}}(ij))$. The conclusion follows from Remark 3.14 (applied here with $d = n$ and $C' = 2^{1/2}\chi/\gamma(1))$ for all $n \geq n_0(\chi)$. But the case $n < n_0(\chi)$ can be handled trivially by adjusting α. $\square$

5. Randomly Sidon matricial systems

We first recall a useful basic fact (see [15] for variations on this theme).

Remark 5.1 (Contraction principle). Let (u_k) and G be as in Lemma 3.1. Let $\{x_k(i,j) \mid k \geq 1, 1 \leq i,j \leq d_k\}$ be a finitely supported family in an arbitrary Banach space B. For any matrix $a \in M_{d_k}$ with complex entries, we denote by ax and xa the matrix products (with entries in B). By convention, we write $\mathrm{tr}(u_k x_k) = \sum_{ij} u_k(i,j)x_k(j,i)$. With this notation, the following"contraction principle" holds

$$\int \left\| \sum d_k \mathrm{tr}(a_k u_k b_k x_k) \right\| dm_G$$
$$\leq \sup_k \|a_k\|_{M_{d_k}} \sup_k \|b_k\|_{M_{d_k}} \int \left\| \sum d_k \mathrm{tr}(u_k x_k) \right\| dm_G.$$

Indeed, this is obvious by the translation invariance of m_G if a_k, b_k are all unitary. Then the result follows by an extreme point argument, since the unit ball of M_{d_k} is the closed convex hull of its unitary elements. The same inequality (same proof) holds with (g_k) (as in Lemma 3.2) in place of (u_k).

Proposition 5.2. *Let (ψ_n^1) be a randomly central Sidon system with constant C_1. Let (φ_n^2, ψ_n^2) be a system satisfying (3.4) with constant C_2' and (3.8). Then the system $(\psi_n^1 \dot{\otimes} \varphi_n^2)$ is randomly Sidon with constant $C_1 C_2'$.*

Proof. Let $x_k \in M_{d_k}$. We have (for simplicity in the sequel we always abusively write sup for essential suprema)

$$\mathbb{E} \sup_{t_1,t_2} \left| \sum d_k \mathrm{tr}(x_k g_k \psi_k^1(t_1) \varphi_k^2(t_2) \right| \geq \sup_{t_2} \mathbb{E} \sup_{t_1} \left| \sum d_k \mathrm{tr}(x_k g_k \psi_k^1(t_1) \varphi_k^2(t_2) \right|.$$

Assume (ψ_k^2) satisfies (3.4) with constant C_2'. Then, by Remark 5.1, we have for a.a. fixed t_2

$$\mathbb{E} \sup_{t_1} \left| \sum d_k \mathrm{tr}(x_k g_k \psi_k^1(t_1) \varphi_k^2(t_2) \right|$$
$$\geq \frac{1}{C_2'} \mathbb{E} \sup_{t_1} \left| \sum d_k \mathrm{tr}(|x_k^*| \psi_k^2(t_2)^* g_k \psi_k^1(t_1) \varphi_k^2(t_2) \right|$$

and by the trace identity, this is

$$= \frac{1}{C_2'} \mathbb{E} \sup_{t_1} \left| \sum d_k \mathrm{tr}(g_k \psi_k^1(t_1) \varphi_k^2(t_2) |x_k^*| \psi_k^2(t_2)^* \right|$$

and hence

$$\begin{aligned} &\sup_{t_2} \mathbb{E} \sup_{t_1} \left| \sum d_k \mathrm{tr}(x_k g_k \psi_k^1(t_1) \varphi_k^2(t_2) \right| \\ &\geq \frac{1}{C_2'} \mathbb{E} \sup_{t_1} \left| \sum d_k \mathrm{tr}(g_k \psi_k^1(t_1) \left[\int \varphi_k^2 |x_k^*| \psi_k^{2^*} dm(t_2) \right] \right| \end{aligned}$$

and by (3.8) and the randomly central Sidon assumption on (ψ_k^1) the last term is

$$= (C_2')^{-1} \mathbb{E} \sup_{t_1} \left| \sum \mathrm{tr}(g_k \psi_k^1(t_1) [\mathrm{tr}|x_k^*|] \right| \geq (C_2')^{-1} C_1^{-1} \sum d_k \mathrm{tr}|x_k^*|.$$

Since $\mathrm{tr}|x_k^*| = \mathrm{tr}|x_k|$ this proves the announced result. □

Remark 5.3. For irreducible representations on a compact group Proposition 5.2 shows that randomly central Sidon implies randomly Sidon with identical constants.

Proposition 5.4. *Let (ψ_n^2) be a randomly Sidon system on (T_2, m_2). Let (φ_n^1, ψ_n^1) be a system satisfying* (3.7)*, or equivalently* (3.5)*, on (T_1, m_1). We also assume that (ψ_n^1), (ψ_n^2) and (φ_n^1) are all uniformly bounded, i.e. satisfy* (3.4)*. Then the system $(\psi_n^1 \dot{\otimes} \psi_n^2)$ is $\dot{\otimes}^2$-Sidon.*

Proof. Let $f_k(t_1, t_2) = \sum_\ell a_{k\ell}(\psi_\ell^1(t_1)\psi_\ell^2(t_2)) b_{k\ell}$ where $a_{k\ell}$ (resp. $b_{k\ell}$) is a matrix of size $d_k \times d_\ell$ (resp. $d_\ell \times d_k$). Assume (ψ_n^2) randomly Sidon with constant C_2, and $\|\varphi_n^1\|_{L_\infty(M_{d_n})} \leq C_1'$ for all n. We claim that

$$\sum d_k |\mathrm{tr}(a_{kk}) \mathrm{tr}(b_{kk})| \leq C_2 C_1' \mathbb{E} \sup_{t_1, t_2} \left| \sum d_k \mathrm{tr}(g_k f_k) \right|.$$

By (3.7), (3.8) and (3.9) we have

$$d_k \int \varphi_k^1(t_1)^* f_k dm(t_1) = \mathrm{tr}(a_{kk}) \psi_k^2(t_2) b_{kk}.$$

Therefore since (ψ_n^2) is randomly Sidon with constant C_2 we have

$$\begin{aligned}\sum |\mathrm{tr}(a_{kk})\mathrm{tr}(b_{kk})| &\le C_2\mathbb{E}\sup_{t_2}\left|\int \sum d_k \mathrm{tr}(g_k \varphi^1_k(t_1)^* f_k) dm(t_1)\right| \\ &\le C_2 \int \mathbb{E}\sup_{t_2}\left|\sum d_k \mathrm{tr}(g_k \varphi^1_k(t_1)^* f_k)\right| dm(t_1) \\ &\le C_2 \sup_{t_1}\mathbb{E}\sup_{t_2}\left|\sum d_k \mathrm{tr}(g_k \varphi^1_k(t_1)^* f_k)\right|\end{aligned}$$

and by the contraction principle in Remark 5.1

$$\le C_2 C_1' \sup_{t_1}\mathbb{E}\sup_{t_2}\left|\sum d_k \mathrm{tr}(g_k f_k)\right|$$

and a fortiori

$$\le C_2 C_1' \mathbb{E}\sup_{t_1,t_2}\left|\sum d_k \mathrm{tr}(g_k f_k)\right|.$$

Thus we obtain the claim.

Let $m = m_1 \times m_2$. Let $E \subset L_1(\mathbb{P}; L_\infty(m))$ be the subspace formed of all the functions of the form $f = \sum d_k \mathrm{tr}(g_k f_k)$. Let $\xi :\ E \to \mathbb{C}$ be the linear form defined by

$$\xi(f) = \sum \mathrm{tr}(a_{kk})\mathrm{tr}(b_{kk}).$$

By Hahn-Banach there is an extension $\xi' :\ L_1(\mathbb{P}; L_\infty(m)) \to \mathbb{C}$ with norm $\le C_2 C_1'$. Since $L_1(\mathbb{P}; L_\infty(m))$ can be identified to the projective tensor product of $L_1(\mathbb{P})$ and $L_\infty(m)$, ξ defines a bounded linear map $T :\ L_1(\mathbb{P}) \to L_\infty(m)^*$ with $\|T\| \le C_2 C_1'$ such that

$$\forall k \forall f_k \quad d_k \left\langle \sum_{i,j} T(g_k(i,j)), f_k(j,i) \right\rangle = \mathrm{tr}(a_{kk})\mathrm{tr}(b_{kk}). \tag{5.1}$$

Let $\theta_k(i,j) = T(g_k(i,j))$ and $\psi_k = \psi^1_k \dot{\otimes} \psi^2_k$. Then (5.1) implies

$$d_k \int \theta_k(j',i)\psi_\ell(j,i') = \delta_{k\ell}\delta_{ij}\delta_{i'j'}.$$

By Lemma 1.21 we may assume that $\theta_k(i,j) \in L_1(m)$ and $\|T\| \le (1+\varepsilon) C_2 C_1'$. Then we have $\int \theta_k \otimes \psi_\ell \ dm = 0$ if $\ell \ne k$ and

$$d_k \int \theta_k \otimes \psi_k \ dm = \sum_{i,j \le d_k} e_{ij} \otimes e_{ij}.$$

Thus if we let $\varphi_k = \theta_k^*$, then (φ_k, ψ_k) satisfies (3.7) and (φ_k) (as well as (θ_k)) is $\|T\|$-dominated by (g_k). Therefore, by Theorem 3.7 we conclude that (ψ_k) is $\dot{\otimes}^2$-Sidon. $\square$

The next statement records a simple observation.

Proposition 5.5. *Let (φ_n, ψ_n) be systems satisfying* (3.4) *and* (3.5). *In addition assume that (φ_n) satisfies* (3.2). *The following are equivalent:*

(i) *(ψ_n) is randomly Sidon.*

(ii) *(ψ_n) is randomly $\dot{\otimes}^k$-Sidon for all $k \geq 1$.*

Proof. Assume (i). Let (ψ_n^1) be any randomly Sidon system. We will show that $(\psi_n^1 \dot{\otimes} \psi_n)$ is randomly Sidon. Fix t_2. Then

$$\sum d_k \mathrm{tr}|\psi_k(t_2)x_k| \leq C\mathbb{E}\operatorname{ess\,sup}_{t_1}\Big|\sum d_k \mathrm{tr}(\psi_k(t_2)x_k g_k \psi_k^1(t_1))\Big|$$

and hence

$$\int \sum d_k \mathrm{tr}|\psi_k x_k| dm \leq C\mathbb{E}\operatorname{ess\,sup}_{t_1,t_2}\Big|\sum d_k \mathrm{tr}(\psi_k(t_2)x_k g_k \psi_k^1(t_1))\Big|$$

$$= C\mathbb{E}\|\sum d_k \mathrm{tr}(x_k g_k \psi_k^1 \dot{\otimes} \psi_k)\|.$$

Now we claim that (3.2) and (3.5) imply

$$\sum d_k \mathrm{tr}|x_k| \leq C' \int \sum d_k \mathrm{tr}|\psi_k(t)x_k| dm(t).$$

Indeed, let $x_k = v_k|x_k|$ be the polar decomposition. Then

$$\begin{aligned}\mathrm{tr}|x_k| &= \int \mathrm{tr}(v_k^*\varphi_k(t)^*\psi_k(t)v_k|x_k|)dm(t)\\ &= \int \mathrm{tr}(v_k^*\varphi_k(t)^*\psi_k(t)x_k)dm(t)\\ &\leq C' \int \mathrm{tr}|\psi_k(t)x_k|dm(t),\end{aligned}$$

from which the claim follows. This shows that (i) implies that $(\psi_n^1 \dot{\otimes} \psi_n)$ is randomly Sidon. In particular taking $\psi_n^1 = \psi_n$ we find that $(\psi_n \dot{\otimes} \psi_n)$ is randomly Sidon. Iterating this argument we obtain (ii). (ii) $\Rightarrow$ (i) is trivial. $\square$

Remark 5.6. By the same reasoning, assuming that (φ_n, ψ_n) satisfy (3.4) and (3.5), and that (φ_n) satisfies (3.2), one shows that if (ψ_n) is $\dot{\otimes}^k$-Sidon then it is $\dot{\otimes}^{k+1}$-Sidon.

We now come to the main point: the comparison between Sidon and randomly Sidon. The generalization of Rider's result from [26] in our new framework is:

Theorem 5.7. *Let (φ_n, ψ_n) be systems satisfying (3.2), (3.4) and (3.5). If (ψ_n) is randomly Sidon (in particular if it is the union of two Sidon systems), then it is $\dot{\otimes}^k$-Sidon for all $k \geq 4$.*

Proof. Proposition 5.4 with $\varphi^1 = \varphi^2$ and $\psi^1 = \psi^2 = \psi$ shows that $(\psi_n \dot{\otimes}^2 \psi_n)$ is $\dot{\otimes}^2$-Sidon. Equivalently (ψ_n) is $\dot{\otimes}^4$-Sidon. By Remark 5.6, it is $\dot{\otimes}^k$-Sidon for all $k \geq 4$. $\square$

Corollary 5.8 (Rider, circa 1975, unpublished). *A sequence (φ_n) of distinct irreducible representations on a compact group is Sidon iff it is randomly Sidon.*

Remark 5.9. In the case of function systems, randomly $\dot{\otimes}^k$-Sidon for some k implies randomly Sidon (see Remark 2.6), and hence the converse to Theorem 5.7 holds, but it seems unclear for general matricial systems. However, the argument in Remark 2.6 based on Slepian's Lemma does work for randomly *central* $\dot{\otimes}^k$-Sidon.

Remark 5.10 (Rider's unpublished results). For subsets of duals of compact non-Abelian groups, Rider [26] announced in 1975 that he had solved the (then still open) "union problem" by proving that the union of two Sidon sets is Sidon. He also extended to the non-Abelian case that randomly Sidon implies Sidon. However, he never published the proof. Around 1979, since I needed to use his result, I wrote to him and he kindly sent me a long detailed handwritten letter describing his proof, based on a delicate estimate of the ratio of determinants appearing in Weyl's famous character formulae [31] for representations of the unitary groups. Unfortunately that letter was lost since then and Rider passed away in 2008.

In Corollary 5.8 we have obtained a new proof of Rider's unpublished result that a randomly Sidon set Λ is Sidon when Λ is a set of irreducible representations on a compact group. In particular, since randomly Sidon is obviously stable by finite unions, this is the first *published* proof of the stability of Sidon sets under finite unions. (See however [32] for *connected* compact groups, using the structure theory of Lie groups).

In a sequel to the present paper [24] we present what is most likely but a reconstruction of Rider's original proof. The heart of that proof is a uniform spectral gap estimate for the sequence of the unitary groups $U(n)$ $(n \geq 1)$, which may be of independent interest for random matrix or free probability theory.

6. Sidon sets of characters on a non-Abelian compact group

Although we have nothing new to add to this, we would like to emphasize here a curious phenomenon already observed in [19], concerning Sidon sets that are singletons, i.e. simply formed of a single irreducible representation. Though simple, this is a nontrivial example, because the dimension d is allowed to tend to ∞, while the constants remain fixed.

Theorem 6.1 ([19]). *Let π be an irreducible representation of dimension d on a compact group G equipped with its Haar probability m_G. Let $\chi(x) = \mathrm{tr}(\pi(x))$ be its character. Assume that for some constant C*

$$\|\chi\|_{\psi_2} \leq C. \tag{6.1}$$

Then there is α depending only on C (and not *on $d = \dim(\pi)$) such that for any $a \in M_d$ we have*

$$\mathrm{tr}|a| \leq \alpha \sup_{t \in G} |\mathrm{tr}(a\pi(t))|. \tag{6.2}$$

Conversely, (6.2) *for some α implies* (6.1) *with a constant C depending only on α.*

It seems curious that the subGaussian nature of the character of an irreducible representation π expressed by (6.1) suffices by itself to imply the strong property of the whole range of π expressed by (6.2).

Proof. Assume (6.1). Let g be a random $d \times d$ Gaussian matrix as in §3. Let δ be the metric defined on G by $\delta(x, y) = (d\ \mathrm{tr}|\pi(x) - \pi(y)|^2)^{1/2}$. Let $N(\varepsilon)$ be the smallest number of a covering of G by open balls of radius ε for the metric δ. Since this metric is translation invariant, we have

$$m_G(\{x \mid \delta(x,1) < \varepsilon\})^{-1} \leq N(\varepsilon)$$

(and in fact $N(\varepsilon)$ is essentially equivalent to $m_G(\{x \mid \delta(x,1) < \varepsilon\})^{-1}$). Since $\mathrm{tr}|\pi(t) - I|^2 = 2(d - \Re(\chi(t)))$, we have for any $0 < \varepsilon < \sqrt{2}$

$$\begin{aligned} m(\{t \mid \delta(t,1) < \varepsilon d\}) &= m(\{\Re(\chi) > d(1-\varepsilon^2/2)\}) \\ &\leq e \exp -(d^2(1-\varepsilon^2/2)^2 C^{-2}). \end{aligned} \tag{6.3}$$

Therefore Sudakov's minoration (see e.g. [21, p.69] or [30]) tells us that that there is a numerical constant c' such that

$$\sup_{\varepsilon > 0} \varepsilon d \left(\log \frac{1}{m_G(\{t \mid \delta(t,1) < \varepsilon d\}} \right)^{1/2} \leq c' \mathbb{E} \sup_{t \in G} |d \ \mathrm{tr}(g\pi(t)|. \tag{6.4}$$

Choosing e.g. $\varepsilon = 1$, (6.3) gives us $d^2(2C)^{-1} - 1 \leq c' \mathbb{E} \sup_{t\in G} |d \ \mathrm{tr}(g\pi(t)|$, and hence for all large enough d we have $d^2(2C)^{-1}/2 \leq c' \mathbb{E} \sup_{t\in G} |d \, \mathrm{tr}(g\pi(t)|$. The latter means that the singleton $\{\pi\}$ is randomly *central* Sidon with constant $c'4C^2/2$. By Proposition 5.2, $\{\pi \dot{\otimes} \pi\}$, and hence $\{\pi\}$ itself, is Sidon with the same constant. The converse implication follows from the non-Abelian analogue (due to Figà-Talamanca and Rider) of Rudin's classical result that Sidon sets have $\Lambda(p)$-constants growing like $\sqrt{p}$, and hence (recall (1.12)) are subGaussian. For details see [8, (37.25) p. 437] or [15]. □

In our follow-up paper [24] we review (and partly amend) the results of [19]. We refer the reader to [24] for more on the themes of the present paper.

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