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Some characters that depend only on length

Alexander R. Miller

The author recently introduced Foulkes characters for a wide variety of reflection groups, and the hyperoctahedral ones have attracted some special attention. Diaconis and Fulman connected them to adding random numbers in balanced ternary and other number systems that minimize carries, and Goldstein, Guralnick, and Rains observed experimentally that they also play the role of irreducibles among the hyperoctahedral characters that depend only on length. We prove this conjecture and show that the same is true for a more general family of reflection groups.

1. Introduction

1.1

For type B our main theorem says that the characters $\chi(x)$ of the hyperoctahedral group B_n that depend only on length

$$\ell(x) = \min\{k : x = r_1 r_2 \cdots r_k,\ r_i \text{ a reflection}\}$$

are the $\mathbf{N}$-linear combinations

$$a_0\phi_0 + a_1\phi_1 + \cdots + a_n\phi_n \quad (a_i = 0, 1, 2, \ldots)$$

where

$$\phi_i(x) = \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} (2j+1)^{n-\ell(x)}.$$

We introduced the ϕ_i's in [5]. They are defined in terms of some homology groups and are called *hyperoctahedral Foulkes characters* because when the construction is applied to the symmetric group it gives an old set of characters for S_n that Foulkes got by summing Specht modules of certain ribbon shapes according to height.

Classic (type A) Foulkes characters have been the subject of many investigations, most recently because of connections with random numbers, shuffling cards, the Veronese embedding, and combinatorial Hopf algebras; see [1]. Hyperoctahedral Foulkes characters have remarkable properties as well [5], and Diaconis and Fulman recently connected them to adding random numbers in balanced ternary and other number systems that minimize carries [2]. Our result says that the hyperoctahedral Foulkes characters are the minimal characters that depend only on length, in the same way that irreducible characters are minimal. In fact we prove this for all wreath products $Z_r \wr S_n$ with $r > 1$, not just $r = 2$.

1.2

The group $Z_r \wr S_n$ can be thought of as the group of n-by-n monomial matrices (one nonzero entry in each row and column) with r-th roots of unity for nonzero entries. Reflections are elements x whose fixed space $\ker(x - I)$ is a hyperplane, and they generate the whole group. Foulkes characters $\phi_0, \phi_1, \ldots, \phi_l$ of $Z_r \wr S_n$ are given by [5, Thm. 5]

$$\phi_i(x) = \sum_{j=0}^{l} (-1)^{i-j} \binom{n+1}{i-j} (rj+1)^{n-\ell(x)} \tag{1}$$

where

$$\ell(x) = \min\{k : x = r_1 r_2 \cdots r_k,\ r_i \text{ a reflection}\} \tag{2}$$

and the number l (called the rank) equals $n - 1$ if $r = 1$ and equals n if $r > 1$.

Foulkes characters of $Z_r \wr S_n$ come from [5] some homology representations, and they are a basis for the class functions that depend only on length. In fact the following is true.[1]

Proposition 1. *If $\chi(x)$ is a character of $Z_r \wr S_n$ that depends only on $\ell(x)$ then*

$$\chi = a_0\phi_0 + a_1\phi_1 + \cdots + a_l\phi_l \tag{3}$$

with coefficients a_i that are unique, rational, and nonnegative.

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[1] The $r = 1$ cases of Proposition 1 and Eq. (1) go back to Kerber–Thürlings and Diaconis–Fulman, respectively; see [3, Thm. 8.5.8] and [1, Eq. 2.7 and Cor. 2.2].

Goldstein, Guralnick, and Rains together [6] observed experimentally that for $r = 2$ (the hyperoctahedral case) the coefficients in (3) are in fact integers. We prove this conjecture and show that the same is true for all $r > 1$. The claim does not extend to S_n for $n > 2$.

Theorem A. *If $n > 2$ then there exists a character of S_n that both depends only on length and is not a* $\mathbf{Z}$*-linear combination of Foulkes characters.*

Theorem B. *If $r > 1$ then the characters of $Z_r \wr S_n$ that depend only on length are the* $\mathbf{N}$*-linear combinations of Foulkes characters.*

Preliminaries span the next two sections: the first recalls wreath products and their conjugacy classes and then translates reflection length into partition length; the second highlights some facts about complete symmetric functions and the characteristic map. §4 applies the map to another basis (Eq. (1) summands stripped of binomials and signs) and then expands in complete symmetric functions to end with an integrality result about differences of characters. §5 adds the integrality to the positivity of Proposition 1 to conclude Theorem B. Theorem A follows from another result plus Chebyshev's theorem about primes.

2. Wreath products and length

This section gives the dictionary between $Z_r \wr S_n$ and the group of n-by-n monomial matrices whose nonzero entries are r-th roots of unity. When $r = 1$ this is the usual correspondence between S_n and n-by-n permutation matrices, so that reflections correspond to transpositions and reflection length corresponds to n minus the number of cycles. The general version (in Proposition 2 below) is similar. Let G be a finite group.

2.1

The symmetric group S_n acts on $G^n = G \times \cdots \times G$ by permuting coordinates according to $s(g_1, \ldots, g_n) = (g_{s^{-1}(1)}, \ldots, g_{s^{-1}(n)})$ and the wreath product $G \wr S_n$ is the semidirect product $G^n \rtimes S_n$. Write $G_n = G \wr S_n$ and let $(g; s)$ be short for $(g_1, g_2, \ldots, g_n; s) \in G_n$. Conjugacy classes of G_n are indexed by partition-valued functions in a natural way.

2.2

A partition λ of n (denoted by $\lambda \vdash n$ or $|\lambda| = n$) is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ whose parts λ_i sum to n. The length $\ell(\lambda) = k$ is the number of parts. The number of i's in λ is denoted by $m_i(\lambda)$, so that $\sum m_i(\lambda) = \ell(\lambda)$, and we abbreviate λ to $1^{m_1(\lambda)}2^{m_2(\lambda)} \ldots n^{m_n(\lambda)}$. The set of all partitions is denoted by $\mathrm{Par} = \{\lambda : |\lambda| \geq 0\}$.

2.3

Let G_* be the set of conjugacy classes of G. The *type* of an element $(g; s)$ in G_n is the partition-valued function $\rho : G_* \to \mathrm{Par}$ whose value at a conjugacy class C of G is the partition $\rho(C)$ whose parts are the periods of the cycles $(i_1 \ldots i_k)$ of s whose *cycle product* $g_{i_k} g_{i_{k-1}} \cdots g_{i_1}$ belongs to C. Two elements are conjugate if and only if they have the same type [5, p. 170] and we index each conjugacy class C_ρ of G_n by the type ρ of $x \in C_\rho$, so that the conjugacy classes of G_n are indexed by the partition-valued functions $\rho : G_* \to \mathrm{Par}$ that satisfy $\sum_C |\rho(C)| = n$. For conjugacy classes with only one member g we abbreviate $\rho(\{g\})$ to $\rho(g)$, so at the class of the identity element we write $\rho(1)$.

2.4

When G is the cyclic group Z_r of r-th roots of unity, let $\varphi : Z_r \wr S_n \to \mathrm{GL}(n, \mathbf{C})$ be the map that sends an element $(z; s)$ to the n-by-n matrix whose i-th column is $z_{s(i)} e_{s(i)}$ where e_i is the standard column vector in $\mathbf{C}^n$ with 1 in the i-th spot and zero elsewhere, so that $\varphi((z; s))e_i = z_{s(i)} e_{s(i)}$. This is a faithful representation mapping $Z_r \wr S_n$ onto the group of all n-by-n monomial matrices whose nonzero entries are r-th roots of unity.

Call $x \in Z_r \wr S_n$ a reflection if $\varphi(x)$ is a reflection, so that $\ker(\varphi(x) - I)$ is a hyperplane. If we write the type ρ of an element $x \in Z_r \wr S_n$ as an r-tuple $(\rho(1), \rho(\zeta), \ldots, \rho(\zeta^{r-1}))$ for some primitive r-th root of unity ζ then the reflections in $Z_r \wr S_n$ are the elements whose type ρ is either $(1^{n-2}2, \emptyset, \ldots, \emptyset)$ or $(1^{n-1}, \emptyset, \ldots, \emptyset, 1, \emptyset, \ldots, \emptyset)$.

2.5

Reflections in S_n are transpositions $(i\, j)$. The k-cycle $(1\,2\, \ldots\, k)$ factors into $k - 1$ transpositions and no fewer, so its length equals $k - 1$, and in general the length of a permutation $x \in S_n$ equals n minus the number of cycles of x.

The number of cycles can in turn be read off from the dimension of the fixed-space $\ker(\varphi(x) - I)$. The cycle $(1\,2\,\ldots\,k)$ acting on $\mathbf{C}^k$ has characteristic polynomial $X^k - 1$, so in this case the fixed-space dimension equals 1, and in general the fixed-space dimension of $\varphi(x)$ equals the number of cycles of x. Here is how the story goes in general.

Proposition 2. *Let $x \in Z_r \wr S_n$ and let ρ be the type of x. Then the following are equal:*

(i) $n - \ell(x)$,

(ii) $\ell(\rho(1))$,

(iii) $\dim \ker_{\mathbf{C}^n}(\varphi(x) - I)$.

Proof. (iii)=(ii). First consider an element $c = (z; s)$ in $Z_r \wr S_k$ where s is a k-cycle. Write $\omega = z_1 z_2 \cdots z_k$. Then c is conjugate to $(\omega, 1, \ldots, 1; (1 \ldots k))$, so the characteristic polynomial of $\varphi(c)$ equals $X^k - \omega$. It follows that characteristic polynomial of $\varphi(x)$ equals

$$\prod_{\omega \in Z_r} \prod_k (X^k - \omega)^{m_k(\rho(\omega))} \tag{4}$$

and so the dimension of the 1-eigenspace equals $\sum m_k(\rho(1)) = \ell(\rho(1))$.

(ii)=(i). Write $x = (z; s)$. The set of reflections is stable under conjugation, so $\ell(x)$ depends only on ρ and we can conjugate x to assume that the places where $z_i \neq 1$ are in one-to-one correspondence with the parts of the partitions $\rho(\omega)$, $\omega \neq 1$. For each $z_i \neq 1$ the element $(1, \ldots, 1, z_i, 1, \ldots, 1; 1)$ in $Z_r \wr S_n$ is a reflection, and together they multiply to $(z; 1)$. Hence $\ell((z; 1))$ is at most $\sum_{\omega \neq 1} \ell(\rho(\omega))$. Elements of the form $(1; (i\,j)) \in Z_r \wr S_n$ are also reflections, and since a k-cycle is a certain product of $k - 1$ transpositions it follows that $\ell((1; s)) \leq n - \sum_\omega \ell(\rho(\omega))$. Write $x = (z; 1)(1; s)$ to conclude that $n - \ell(x) \geq \ell(\rho(1))$.

If $x = r_1 \cdots r_k$ is a product of k reflections then $\varphi(x)$ fixes the intersection of the k reflecting hyperplanes, so that $\dim \ker(\varphi(x) - I) \geq n - k$. Now we have that

$$\dim \ker(\varphi(x) - I) \geq n - \ell(x) \geq \ell(\rho(1))$$

and the equality (iii)=(ii) from before says that the left-hand side equals the right-hand side. $\square$

Remark 1. The equality of (i) and (iii) in Proposition 2 is well known. We have included it here to reconcile (1) with the fixed-space version that we wrote down in [5, Thm. 5].

2.6

For the purposes of this paper it is best to work in the language of partitions, so rewrite the Foulkes characters $\phi_0, \phi_1, \dots, \phi_l$ of $Z_r \wr S_n$ using the equality $n - \ell(x) = \ell(\rho(1))$ to get

$$\phi_i(x) = \sum_{j=0}^{l} (-1)^{i-j} \binom{n+1}{i-j} (rj+1)^{\ell(\rho(1))}. \tag{5}$$

Remark 2. The ϕ_i's are a basis for the space of functions $f(x)$ on $Z_r \wr S_n$ that depend only on the length $\ell(\rho(1))$, which can be $1, 2, \dots, n$ if $r = 1$, and can be $0, 1, \dots, n$ if $r > 1$. This is why $l = n - 1$ if $r = 1$ and why $l = n$ if $r > 1$.

3. The characteristic map

We require some facts about the characteristic map for wreath products [4, App. 8]. In particular we require one (Theorem 4) that does not appear in [4].

Let G be a finite group. Write $\mathrm{Par}_n(X)$ for the set of functions $\rho : X \to \mathrm{Par}$ that satisfy $\sum_{x\in X} |\rho(x)| = n$, so that conjugacy classes C_ρ of G_n are indexed by the functions $\rho \in \mathrm{Par}_n(G_*)$ where G_* is the set of conjugacy classes of G, and the irreducible characters of G_n can be indexed by the functions $\lambda \in \mathrm{Par}_n(G^*)$ where G^* is the set of irreducible characters of G. For $\gamma \in G^*$ and $C \in G_*$ define $\gamma(C) = \gamma(g)$, $g \in C$.

3.1

The characteristic map

$$\mathrm{ch} : R(G) \to \Lambda(G)$$

is a certain linear isomorphism from the graded vector space $R(G) = \bigoplus_{n\geq 0} R(G_n)$ where $R(G_n)$ is the space of complex-valued class functions of G_n, onto the polynomial ring

$$\Lambda(G) = \mathbf{C}[p_k(C) : C \in G_*,\ k \geq 1]$$

where the $p_k(C)$'s are independent indeterminates over $\mathbf{C}$ and $p_k(C)$ is assigned degree k. It is the $\mathbf{C}$-linear map from $R(G)$ to $\Lambda(G)$ such that for $f \in R(G_n)$

$$\mathrm{ch}(f) = \frac{1}{|G_n|} \sum_{\rho \in \mathrm{Par}_n(G_*)} |C_\rho| f(C_\rho) \prod_{C \in G_*} p_{\rho(C)}(C)$$

where $f(C_\rho)$ means the value $f(x)$ at any $x \in C_\rho$.

3.1.1. Change of basis. We require more flexibility than allowed for in [4, App. 8], which restricts to irreducible $\gamma \in G^*$ only. The following setup allows for all characters. One benefit of the relaxed setup is Theorem 4.

For each character γ of G and each integer $k \geq 1$ define

$$p_k(\gamma) = \frac{1}{|G|} \sum_{C \in G_*} |C| \gamma(C) p_k(C) \tag{6}$$

and define $h_n(\gamma)$ according to the formal power series

$$\sum_{n \geq 0} h_n(\gamma) X^n = \exp\left(\sum_{k \geq 1} \frac{1}{k} p_k(\gamma) X^k \right) \tag{7}$$

where X is an indeterminate. Then for each partition μ define

$$s_\mu(\gamma) = \det(h_{\mu_i - i + j}(\gamma)), \qquad h_\mu(\gamma) = h_{\mu_1}(\gamma) h_{\mu_2}(\gamma) \cdots \tag{8}$$

and for each $\lambda \in \mathrm{Par}(G^*)$ define

$$S_\lambda = \prod_{\gamma \in G^*} s_{\lambda(\gamma)}(\gamma), \qquad H_\lambda = \prod_{\gamma \in G^*} h_{\lambda(\gamma)}(\gamma). \tag{9}$$

The set $\{p_k(\gamma) : \gamma \in G^*,\ k \geq 1\}$ is an algebraically independent set of generators for $\Lambda(G)$, so we may regard $p_k(\gamma)$ (for $\gamma \in G^*$) as the k-th power sum $y_{1\gamma}^k + y_{2\gamma}^k + \cdots$ in a new sequence of variables $y_\gamma = (y_{1\gamma}, y_{2\gamma}, \ldots)$. Then for any partition μ the polynomials $h_\mu(\gamma)$ and $s_\mu(\gamma)$ have their usual meaning as complete symmetric and Schur functions.

3.1.2. We require four properties of the linear isomorphism $\mathrm{ch}: R(G) \to \Lambda(G)$. The first two are (I) and (II) below; they extend a couple of basic facts about the characteristic map for S_n. The third is Proposition 3 and the fourth is Theorem 4.

- **(I)** The irreducible characters of G_n are $\chi_\lambda = \mathrm{ch}^{-1}(S_\lambda)$ for $\lambda \in \mathrm{Par}_n(G^*)$.
- **(II)** $\{H_\lambda : \lambda \in \mathrm{Par}_n(G^*)\}$ and $\{S_\lambda : \lambda \in \mathrm{Par}_n(G^*)\}$ generate the same free $\mathbf{Z}$-module.

Remark 3. [4, App. B §9] details (I). (II) follows from (9) plus the fact that for $\gamma \in G^*$ $\{h_\mu(\gamma) : \mu \vdash k\}$ and $\{s_\mu(\gamma) : \mu \vdash k\}$ generate the same free $\mathbf{Z}$-module [4, p. 101].

For each character γ of G define the formal power series

$$H(\gamma) = \sum_{n \geq 0} h_n(\gamma) X^n \tag{10}$$

and let $\eta_n(\gamma) \in R(G_n)$ be the function whose value at an element $x \in G_n$ of type ρ is

$$\eta_n(\gamma)(x) = \prod_{C \in G_*} \gamma(C)^{\ell(\rho(C))}. \tag{11}$$

Macdonald proves the next result for irreducible characters $\gamma \in G^*$. The same calculations [4, App. B §8] work for characters in general (assuming the definitions in §3.1.1).

Proposition 3. *If γ is a character of a finite group G then:*

- (i) *$\eta_n(\gamma)$ is a character of G_n.*
- (ii) *$H(\gamma) = \sum_{n \geq 0} \mathrm{ch}(\eta_n(\gamma)) X^n$.* □

Remark 4. If G-module V affords γ then the G_n-module on the n-fold tensor product $V \otimes \cdots \otimes V$ given by $(g; s)(v_1 \otimes \cdots \otimes v_n) = g_1 v_{s^{-1}(1)} \otimes \cdots \otimes g_n v_{s^{-1}(n)}$ affords $\eta_n(\gamma)$.

Here is a useful benefit of the relaxed setup.

Theorem 4. *$H(\alpha + \beta) = H(\alpha) H(\beta)$ for any characters α, β of G.*

Proof. $H(\alpha + \beta) = \exp \sum \frac{1}{k} p_k(\alpha + \beta) X^k$ and $p_k(\gamma)$ is linear in γ. □

4. An integrality result

Fix a finite group G and integer $n \geq 0$. Let $l = n - 1$ if $|G| = 1$ and let $l = n$ if $|G| > 1$.

Definition 1. For $0 \leq k \leq l$ and $x \in G_n$ define

$$\chi_k(x) = (k|G| + 1)^{\ell(\rho(1))} \tag{12}$$

where ρ is the type of x.

Proposition 5. *If $G = Z_r$ and $\phi_0, \phi_1, \ldots, \phi_l$ are the Foulkes characters of $Z_r \wr S_n$ then:*

(i) $\phi_i = \sum_{j=0}^{l} (-1)^{i-j} \binom{n+1}{i-j} \chi_j \qquad (i = 0, 1, \ldots, l).$

(ii) $\{\chi_0, \chi_1, \ldots, \chi_l\}$ *and* $\{\phi_0, \phi_1, \ldots, \phi_l\}$ *generate the same free* $\mathbf{Z}$*-module.*

Proof. (i) restates (5), our earlier translation of ϕ_i into partition length. (ii) is true because the l-by-l matrix $[(-1)^{i-j}\binom{n+1}{i-j}]$ is lower triangular with 1's on the diagonal. □

The χ_i's are related to the function η_n. Let **reg** and **1** be the regular and principal characters of G, and define $\gamma_k = k \cdot \mathbf{reg} + \mathbf{1}$ for $k = 0, 1, 2, \ldots$.

Proposition 6. $\chi_k = \eta_n(\gamma_k)$. *In particular χ_k is a character of G_n.*

Proof. Since

$$\gamma_k(g) = \begin{cases} k|G| + 1 & \text{if } g = 1 \\ 1 & \text{otherwise} \end{cases} \tag{13}$$

we have that for $\rho \in \mathrm{Par}_n(G_*)$

$$\prod_{C \in G_*} \gamma_k(C)^{\ell(\rho(C))} = (k|G| + 1)^{\ell(\rho(1))}.$$

The left-hand side is $\eta_n(\gamma_k)(x)$ and the right-hand side is $\chi_k(x)$. Hence $\eta_n(\gamma_k) = \chi_k$. Proposition 3(i) says that $\eta_n(\gamma_k)$ is a character of G_n because γ_k is a character of G. □

Proposition 7. $\sum_{n \geq 0} \mathrm{ch}(\eta_n(\gamma_k)) X^n = H(\mathbf{1}) \prod_{\gamma \in G^*} H(\gamma)^{k\gamma(1)}$.

Proof. Use Proposition 3(ii) to write

$$\sum_{n\geq 0} \mathrm{ch}(\eta_n(\gamma_k))X^n = H(\gamma_k),$$

and then decompose γ_k into irreducibles

$$\gamma_k = \mathbf{1} + \sum_{\gamma\in G^*} k\gamma(1)\gamma$$

and use Theorem 4 to write $H(\gamma_k) = H(\mathbf{1})\prod_{\gamma\in G^*} H(\gamma)^{k\gamma(1)}$. □

Corollary 8. *Write* $\mathrm{ch}(\chi_k) = \sum_{\lambda\in \mathrm{Par}_n(G^*)} a_\lambda H_\lambda$. *Then the coefficient* a_λ *of* H_λ *equals*

$$\binom{k+1}{\ell(\lambda(\mathbf{1}))}\binom{\ell(\lambda(\mathbf{1}))}{m_1(\lambda(\mathbf{1})),\, m_2(\lambda(\mathbf{1})),\ldots} \times \prod_{\gamma\neq \mathbf{1}} \binom{k\gamma(1)}{\ell(\lambda(\gamma))}\binom{\ell(\lambda(\gamma))}{m_1(\lambda(\gamma)),\, m_2(\lambda(\gamma)),\ldots}$$

(a product of multinomial coefficients) where $m_i(\mu)$ *is the number of* i*'s in* μ. □

Let $G' \leq G$ be the subgroup generated by the commutators $xyx^{-1}y^{-1}$. Then $|G/G'|$ is the number of linear characters of G, so there is a nontrivial one if and only if $G \neq G'$.

Theorem 9. *If* $G\neq G'$ *and* $a_0\chi_0 + a_1\chi_1 + \cdots + a_n\chi_n$ *is a difference of characters of* G_n *then the coefficients* a_i *are integers.*

Proof. Suppose that $G \neq G'$, so that $l = n$ and G has a nontrivial linear character γ. Fix an index m and suppose that $a_0\chi_0 + \cdots + a_m\chi_m$ is a difference of characters, so that

$$a_0\mathrm{ch}(\chi_0) + a_1\mathrm{ch}(\chi_1) + \cdots + a_m\mathrm{ch}(\chi_m) \in \mathcal{H} \tag{14}$$

where $\mathcal{H}$ is the free $\mathbf{Z}$-module generated by the H_λ's. The object is to show that $a_0,\ldots,a_m$ are integers. By induction it is enough to show that the last coefficient a_m is an integer because then $a_m\mathrm{ch}(\chi_m)\in\mathcal{H}$ and hence $a_0\mathrm{ch}(\chi_0) + \cdots + a_{m-1}\mathrm{ch}(\chi_{m-1}) \in \mathcal{H}$.

Corollary 8 tells us that the coefficient of $h_1(\gamma)^n$ in $\mathrm{ch}(\chi_k)$ is 1 if $k = n$ and is 0 if $k < n$. Comparing coefficients in (14) we conclude that if $m = n$ then a_m is an integer. If $m < n$ then

$$\text{coeff. of } h_{n-m}(\mathbf{1})h_1(\gamma)^m \text{ in } \mathrm{ch}(\chi_k) = \begin{cases} m+1 & \text{if } k = m \\ 0 & \text{if } k < m \end{cases}$$

and

$$\text{coeff. of } h_1(\mathbf{1})^{m+1}h_{n-m-1}(\gamma) \text{ in } \mathrm{ch}(\chi_k) = \begin{cases} m & \text{if } k = m \\ 0 & \text{if } k < m, \end{cases}$$

so that $(m+1)a_m$ and ma_m are integers. Hence a_m is an integer in this case as well. □

5. Proofs of Theorems A and B

Proof of Theorem A. For the symmetric group S_n (so that $G = Z_1$) Corollary 8 says

$$\mathrm{ch}(\chi_k) = \sum_{\lambda \vdash n} \binom{k+1}{\ell(\lambda)} \binom{\ell(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots} h_\lambda(\mathbf{1}). \tag{15}$$

Suppose that $n > 2$. Then Chebyshev tells us that $n > p > n/2$ for some prime p. Fix such a p. Then if a partition λ of n has length p there must be an $i > 0$ such that

$$p > m_i(\lambda) > 0, \tag{16}$$

and hence all partitions λ of n satisfy the equation

$$\binom{p}{\ell(\lambda)} \binom{\ell(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots} \equiv 0 \pmod{p}. \tag{17}$$

Together (15) and (17) tell us that χ_{p-1}/p is a character. The χ_i's are linearly independent, so χ_{p-1}/p is not in their $\mathbf{Z}$-span, which by Proposition 5 is the same as saying that the character χ_{p-1}/p is not in the $\mathbf{Z}$-span of the Foulkes characters of S_n. □

Proof of Theorem B. Let $r > 1$ and let $\chi(x)$ be a character of $Z_r \wr S_n$ that depends only on $\ell(x)$. Theorem 9 tells us that χ is an integral linear combination of the χ_i's, which by Proposition 5 is the same as saying that χ is an integral linear combination of the Foulkes characters ϕ_i of $Z_r \wr S_n$. Proposition 1 adds that the coefficients must be nonnegative. Hence χ is an $\mathbf{N}$-linear combination of Foulkes characters. □

6. Remarks

Proposition 1 extends to all wreath products $G \wr S_n$ that satisfy $G \neq G'$, where now the length of $x \in G_n$ is defined to be $\ell(\rho(1))$ and ϕ_i is defined according to Proposition 5(i): the hook-content formula (or more directly, [4, Ex. 1, pp. 65–66] and Proposition 7) expresses χ_j as a sum of irreducible characters $\chi_\lambda = \mathrm{ch}^{-1}(S_\lambda)$, and if we define a sequence of partition-valued functions $\lambda^0, \lambda^1, \ldots, \lambda^n \in \mathrm{Par}_n(G^*)$ according to $\lambda^k(\mathbf{1}) = n - k$ and $\lambda^k(\gamma) = 1^k$ $(k = 0, 1, \ldots, n)$ for some nontrivial linear character γ of G, then the coefficient of χ_{λ^k} in χ_j equals $\binom{n}{k}\binom{n+j-k}{n}$ for $j = 0, 1, \ldots, n$, so that in terms of the class functions $\theta_k = \chi_{\lambda^k}/\binom{n}{k}$ and the usual inner product on the space of class functions we have that

$$\langle \theta_k, \phi_i \rangle = \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} \binom{n+j-k}{n} = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

The second equality is a basic identity that relates to Eulerian idempotents [5, Eqs. 11,9]. It follows that if $\chi(x)$ is a class function that depends only on $\ell(\rho(1))$ then

$$\chi = \sum_{i=0}^{n} \langle \chi, \theta_i \rangle \phi_i, \tag{19}$$

and if χ is also a character of G_n then the coefficients $\langle \chi, \theta_i \rangle$ are rational and nonnegative. The upshot is that (by Theorem 9) we have the following extension of Theorem B.

Theorem 10. *If $G \neq G'$ then the characters $\chi(x)$ of $G \wr S_n$ that depend only on $\ell(\rho(1))$ are precisely the $\mathbf{N}$-linear combinations of the class functions $\phi_0, \phi_1, \ldots, \phi_n$ given by*

$$\phi_i(x) = \sum_{j=0}^{n} (-1)^{i-j} \binom{n+1}{i-j} (rj+1)^{\ell(\rho(1))} \tag{20}$$

where ρ is the type of x. □

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Fakultät für Mathematik, Universität Wien
Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria

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