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From state integrals to q-series

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It is well-known to the experts that multi-dimensional state integrals of products of Faddeev's quantum dilogarithm which arise in Quantum Topology can be written as finite sums of products of basic hypergeometric series in $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$. We illustrate this fact by giving a detailed proof for a family of one-dimensional integrals which includes state-integral invariants of 4_1 and 5_2 knots.

1. Introduction

1.1. State-integrals and their q-series

Multi-dimensional state integrals of products of Faddeev's quantum dilogarithm appear in abundance in Quantum Topology, and were studied among others by Hikami [Hik01], Dimofte–Gukov–Lennels–Zagier [DGLZ09], Andersen–Kashaev [AK], and Kashaev–Luo–Vartanov [KLV16]. It is well-known to the experts that such state-integrals can be written as finite sums of products of pairs of q-series and $\tilde{q}$-series. The reason for this is a factorized structure of Faddeev's quantum dilogarithm, the structure of the set of its poles, and the specific form of exponential factors of the integrand of the state-integrals, while its derivation is based on an application of the residue theorem. Instead of formulating a general theorem for multi-dimensional integrals which obscures the principle, we will give a detailed proof for the case of a family of 1-dimensional integrals and illustrate it with some concrete examples taken from [AK, KLV16]. Similar computations appear in mathematical physics [BDP14].

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To state our results, recall that *Faddeev's quantum dilogarithm function* $\Phi_b(x)$ is given by [Fad95]

$$(1)\qquad \Phi_b(x) = \frac{(e^{2\pi b(x+c_b)};q)_\infty}{(e^{2\pi b^{-1}(x-c_b)};\tilde{q})_\infty},$$

where

$$q = e^{2\pi i b^2}, \qquad \tilde{q} = e^{-2\pi i b^{-2}}, \qquad c_b = \frac{i}{2}(b+b^{-1}), \qquad \Im(b^2) > 0.$$

Remarkably, this function admits an extension to all values of b with $b^2 \notin \mathbb{R}_{\leq 0}$. $\Phi_b(x)$ is a meromorphic function of x with

$$\text{poles: } c_b + i\mathbb{N}b + i\mathbb{N}b^{-1}, \qquad \text{zeros: } -c_b - i\mathbb{N}b - i\mathbb{N}b^{-1}.$$

The functional equation

$$\Phi_b(x)\Phi_b(-x) = e^{\pi i x^2}\Phi_b(0)^2, \qquad \Phi_b(0) = q^{\frac{1}{24}}\tilde{q}^{-\frac{1}{24}}$$

allows us to move $\Phi_b(x)$ from the denominator to the numerator of the integrand of a state-integral.

For natural numbers A, B with $B > A > 0$, we consider the absolutely convergent integral

$$\mathcal{I}_{A,B}(b) = \int_{\mathbb{R}+i\epsilon} \Phi_b(x)^B e^{-A\pi i x^2} dx$$

with small positive ϵ. The condition $B > A > 0$ ensures not only the convergence of $\mathcal{I}_{A,B}(b)$ for $\Im(b^2) > 0$, but also the convergence of the q-series and the $\tilde{q}$-series (for $|q|, |\tilde{q}| < 1$) that appear in Theorem 1.1 below.

To express the above state-integral in terms of series, consider the generating series

$$(2)\qquad F_{A,B}(q,x) = \sum_{m=0}^\infty \frac{(-1)^{Am} q^{A\frac{m(m+1)}{2}}}{(q)_m^B} x^m, \qquad \widetilde{F}_{A,B}(q,x) = F_{B-A,B}(q,x).$$

Consider the operators δ and δ_k (for k a positive natural number) which act on the space of functions of x as follows

$$(3)\qquad (\delta F)(x) = x\partial_x F(x), \qquad (\delta_k F)(x) = \sum_{s=1}^\infty \frac{s^{k-1}q^s}{1-q^s} F(q^s x).$$

Likewise, there are operators $\widetilde{\delta}$ and $\widetilde{\delta}_k$ which act on the space of functions of $\widetilde{x}$ and with q replaced by $\tilde{q}$. It is easy to see that any two of the operators δ, δ_k, $\widetilde{\delta}$, $\widetilde{\delta}_k$ commute and they freely generate over $\mathbb{Q}$ a commutative ring $\mathcal{D} \otimes \widetilde{\mathcal{D}}$, where

$$\mathcal{D} = \mathbb{Q}[\delta, \delta_1, \delta_2, \dots], \qquad \widetilde{\mathcal{D}} = \mathbb{Q}[\widetilde{\delta}, \widetilde{\delta}_1, \widetilde{\delta}_2, \dots] .$$

Let

$$\begin{aligned} \mathcal{D}_b &= \mathcal{D}[(2\pi i)^{-1}, b^{\pm 1}, e_2, e_4, e_6, \dots], \\ \widetilde{\mathcal{D}}_b &= \widetilde{\mathcal{D}}[(2\pi i)^{-1}, b^{\pm 1}, e_2, e_4, e_6, \dots] , \end{aligned}$$

where $e_l = e_l(\tilde{q}) = \widetilde{\delta}_l(1) \in \mathbb{Z}[[\tilde{q}]]$. Consider the following *operator valued polynomial*:

$$\begin{aligned} P_{A,B} = \mathrm{Res}_{w=0} & \left(e^{\frac{1}{4\pi i} w^2 + A w (b(\delta + \frac{1}{2}) + b^{-1}(\widetilde{\delta} + \frac{1}{2}))} \right)^A \\ & \times \left(\frac{\phi(bw, \delta_\bullet) \widetilde{\phi}(b^{-1} w, \widetilde{\delta}_\bullet)}{b(1 - e^{b^{-1} w})} \right)^B \in \mathcal{D}_b \otimes \widetilde{\mathcal{D}}_b , \end{aligned} \tag{4}$$

where

$$\phi(w, \delta_\bullet) = \exp\left(- \sum_{l=1}^{\infty} \frac{\delta_l}{l!} w^l \right) \tag{5a}$$

$$\begin{aligned} \widetilde{\phi}(w, \widetilde{\delta}_\bullet) = \exp(-\widetilde{\delta} w) & \exp\left(2 \sum_{l=\text{even}>0} e_l(\tilde{q}) \frac{w^l}{l!} \right) \\ & \times \exp\left(- \sum_{l=1}^{\infty} \frac{\widetilde{\delta}_l}{l!} (-w)^l \right) . \end{aligned} \tag{5b}$$

For a series $F(x, \widetilde{x})$, we define:

$$\langle F(x, \widetilde{x}) \rangle = F(1, 1) . \tag{6}$$

Theorem 1.1. *We have:*

$$\mathcal{I}_{A,B}(b) = \left(\frac{\tilde{q}}{q} \right)^{\frac{B-3A}{24}} e^{\pi i \frac{B+2(A+1)}{4}} \left\langle P_{A,B} \left(F_{A,B}(q, x) \widetilde{F}_{A,B}(\tilde{q}, \widetilde{x}) \right) \right\rangle . \tag{7}$$

Corollary 1.2. Writing $P_{A,B} = \sum_k p_k P_k$ (a finite sum), for $p_k \in \mathcal{D}_b$ and $P_k \in \widetilde{\mathcal{D}}_b$, it follows that

$$\mathcal{I}_{A,B}(b) = \left(\frac{\tilde{q}}{q}\right)^{\frac{B-3A}{24}} e^{\pi i \frac{B+2(A+1)}{4}} \sum_k g_k(q) G_k(\tilde{q}) \tag{8}$$

where

$$g_k(q) = \langle p_k F_{A,B} \rangle, \qquad G_k(\tilde{q}) = \langle P_k \widetilde{F}_{A,B} \rangle \,. \tag{9}$$

Remark 1.3. The left hand side of Equation (8) has analytic continuation to the cut plane $\mathbb{C} \setminus \{b^2 \,|\, b^2 < 0\}$ whereas each of the series g_k and G_k is only well-defined in the upper-half plane $\{b^2 \,|\Im(b^2) > 0\}$.

Remark 1.4. $P_{A,B}$, as a polynomial in the variables $e_2, e_4, \dots$ has degree $B-1$, where the degree of e_l is l. $P_{A,B}$ as a Laurent polynomial in b has b-monomials of degrees in $\{-B+1, -B+3, \dots, B-3, B-1\}$.

1.2. *q*-difference equations

Next we describe a linear q-difference equation of $F_{A,B}(q,x)$. Consider the operators $\hat{x}$ and $\hat{E}$ which act on $f(x) \in \mathbb{Q}(q)[[x]]$ by:

$$(\hat{E}f)(x) = f(qx), \qquad (\hat{x}f)(x) = xf(x) \,.$$

Observe that $\hat{E}\hat{x} = q\hat{x}\hat{E}$.

Lemma 1.5. (a) We have:

$$F_{A,B}(q^{-1}, x) = \widetilde{F}_{A,B}(q, x) \,. \tag{10}$$

(b) $F_{A,B}$ satisfies the linear q-difference equation

$$\left((1-\hat{E})^B - (-1)^A q^A x \hat{E}^A\right) F_{A,B}(q,x) = 0 \,. \tag{11}$$

Corollary 1.6. (a) If we define $\omega(q,x) = F_{A,B}(q,qx)/F_{A,B}(q,x)$ and $\omega(q,x)_n = \prod_{j=1}^{n} \omega(q,q^j x)$, then ω satisfies the nonlinear equation

$$\sum_{j=0}^{B}(-1)^j \binom{B}{j} \omega(q,x)_j - (-1)^A q^A x \omega(q,x)_A = 0 \,.$$

(b) F is an admissible power series in the sense of Kontsevich-Soibelman [KS11, Sec.6], the limit $\lim_{q\to 1} \omega(q,x) = \omega(x) \in \overline{\mathbb{Q}}[[x]]$ exists and satisfies the algebraic equation (also known as the Nahm equation or the gluing equation)

$$(1-\omega(x))^B = (-1)^A x \omega(x)^A \,. \tag{12}$$

The Nahm equation has been studied by several authors including [Zag07, Sec.3], [Vla, VZ11], [RV, Sec.4].

1.3. The case of the 4_1 knot

We now specialize Corollary 1.2 to the invariant of the 4_1 and 5_2 knots is given by [KLV16, AK]

$$\mathcal{I}_{1,2} = \mathcal{I}_{4_1} \qquad \mathcal{I}_{2,3} = \mathcal{I}_{5_2} \,.$$

In this section, let

$$F(q,x) = F_{1,2}(q,x) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2} x^n \,. \tag{13}$$

Corollary 1.7. (a) We have:

$$\mathcal{I}_{4_1}(b) = -\frac{i}{2}\left(\frac{q}{\tilde{q}}\right)^{\frac{1}{24}} \big(b\, G(q) g(\tilde{q}) - b^{-1} G(\tilde{q}) g(q)\big) \tag{14}$$

where

$$g(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2} \tag{15a}$$

$$G(q) = \sum_{m=0}^{\infty} \left(1 + 2m - 4\sum_{s=1}^{\infty} \frac{q^{s(m+1)}}{1-q^s}\right) (-1)^m \frac{q^{\frac{1}{2}m(m+1)}}{(q)_m^2} \tag{15b}$$

(b) The series $g(q)$ and $G(q)$ are given in terms of $F(q,x)$ by:

$$g(q) = \langle F \rangle \tag{16a}$$

$$G(q) = \langle (2 + 2\delta - 4\delta_1) F \rangle \tag{16b}$$

(c) F satisfies the linear q-difference equation

$$F(q, q^{-1}x) + F(q, qx) = (2 - x)F(q, x) \tag{17}$$

The series $g(q)$ that appears in Theorem 1.7 was known to the first author and Zagier to be closely related to the 4_1 knot. For a detailed discussion of experimental facts below, see [GZ]. Empirically, it appears that

- the pair $(g(q), G(q))$ is related to the 3D index of the 4_1 knot,
- the radial asymptotics of the pair $(g(q), G(q))$ are related to the asymptotics of the Kashaev invariant of the 4_1 knot,
- the above observations for 4_1 also hold for the case of 5_2 knot discussed below.

Recall that the index of an ideal triangulation was introduced in [DGG14, DGG13], necessary and sufficient conditions for its convergence was established in [Gar16] and its topological invariance was proven in [GHRS15]. For a detailed discussion of the above experimental facts, see [GZ].

1.4. The case of the 5_2 knot

In this section, let

$$F(q,x) = F_{2,3}(q,x) = \sum_{m=0}^{\infty} t_m(q) x^m, \quad \widetilde{F}(q,\widetilde{x}) = F_{1,3}(q,\widetilde{x}) = \sum_{m=0}^{\infty} T_m(q)\widetilde{x}^m$$

where

$$t_m(q) = \frac{q^{m(m+1)}}{(q)_m^3}, \qquad T_n(q) = (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^3} = t_n(q^{-1}) \,.$$

Let

$$\begin{aligned} R_{m,n}(q,\tilde{q}) = &-\frac{b^2}{2}\left(1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_1^{(m)2}(q) - 3E_2^{(m)}(q)\right) \\ &- \frac{1}{2\pi i} + \frac{1}{2}\left(1 + 2m - 3E_1^{(m)}(q)\right)\left(1 + 2n - 6E_1^{(n)}(\tilde{q})\right) \\ &+ \frac{b^{-2}}{2}\left(-n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) - 9E_1^{(n)2}(\tilde{q}) + 3E_2^{(n)}(\tilde{q})\right), \end{aligned}$$

where $E_l^{(m)}(q)$ are defined in Equation (29a). For $k = 1, \dots, 4$ let

$$g_k(q) = \sum_{m=0}^{\infty} p_k(m) t_m(q), \qquad G_k(\tilde{q}) = \sum_{n=0}^{\infty} P_k(n) T_n(\tilde{q}) \,, \tag{18}$$

where

$$p_{1,m}(q) = 1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_1^{(m)2}(q) - 3E_2^{(m)}(q) \tag{19a}$$

$$p_{2,m}(q) = 1 + 2m - 3E_1^{(m)}(q) \tag{19b}$$

$$p_{3,m}(q) = 1 \tag{19c}$$

and

$$P_{1,m}(q) = 1 \tag{20a}$$

$$P_{2,m}(q) = 1 + 2n - 6E_1^{(n)}(\tilde{q}) \tag{20b}$$

$$P_{3,m}(q) = -n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) - 9E_1^{(n)2}(\tilde{q}) + 3E_2^{(n)}(\tilde{q}) \,. \tag{20c}$$

Corollary 1.8. (a) We have:

$$\begin{aligned} \mathcal{I}_{2,3}(q) &= -e^{\frac{3\pi i}{4}} \left(\frac{q}{\tilde{q}}\right)^{\frac{1}{8}} \sum_{m,n=0}^{\infty} R_{m,n}(q,\tilde{q}) t_m(q) T_n(\tilde{q}) \\ &= -e^{\frac{3\pi i}{4}} \left(\frac{q}{\tilde{q}}\right)^{\frac{1}{8}} \Big(-\frac{b^2}{2} g_1(q) G_1(\tilde{q}) - \frac{1}{2\pi i} g_3(q) G_1(\tilde{q}) \\ &\qquad + \frac{1}{2} g_2(q) G_2(\tilde{q}) + \frac{b^{-2}}{2} g_3(q) G_3(\tilde{q}) \Big) \end{aligned} \tag{21}$$

(b) F and $\widetilde{F}$ satisfy the linear q-difference equations

$$F(q, q^3 x) - (3 - q^2 x) F(q, q^2 x) + 3F(q, qx) - F(q, x) = 0$$

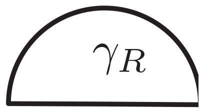
$$\widetilde{F}(q, q^3 x) - 3\widetilde{F}(q, q^2 x) + (3 - q^2 x)\widetilde{F}(q, qx) - \widetilde{F}(q, x) = 0 \,.$$

Remark 1.9. A computation gives that $P(A, B) = P(B - A, B)$ for $(A, B) = (1, 2)$ and $(A, B) = (2, 3)$ corresponding to the invariants of the 4_1 and 5_2 knots. In all other cases that we tried, we found that $P(A, B)$ is not equal to $P(B - A, B)$.

2. Proofs

2.1. A residue computation

To relate the state-integral $\mathcal{I}_{A,B}$ to a sum, we will apply the residue theorem on a semicircle γ_R with center 0 and radius R, oriented counterclockwise in the upper half-plane:



Then, we will take the limit $R \to \infty$. To compute the residue of the integrand, we need to expand $\Phi_b(x)$ near the pole

$$x_{m,n} = c_b + ibm + ib^{-1}n$$

for natural numbers m and n. Let

$$\phi_m(x) = \frac{(q^{m+1}e^x; q)_\infty}{(q^{m+1}; q)_\infty} \tag{23}$$

$$\widetilde{\phi}_n(x) = \frac{(\tilde{q}; \tilde{q})_\infty}{(\tilde{q}e^x; \tilde{q})_\infty} \frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^x; \tilde{q}^{-1})_n} \tag{24}$$

Lemma 2.1. We have:

$$\Phi_b(x + x_{m,n}) = \frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty} \frac{1}{(q;q)_m} \frac{1}{(\tilde{q}^{-1};\tilde{q}^{-1})_n} \frac{\phi_m(2\pi bx)\, \widetilde{\phi}_n(2\pi b^{-1}x)}{1 - e^{2\pi b^{-1}x}}. \tag{25}$$

Proof. Equation (1) implies the functional equations

$$\frac{\Phi_b(x + c_b + ib)}{\Phi_b(x + c_b)} = \frac{1}{1 - qe^{2\pi bx}}$$
$$\frac{\Phi_b(x + c_b + ib^{-1})}{\Phi_b(x + c_b)} = \frac{1}{1 - \tilde{q}^{-1}e^{2\pi b^{-1}x}}$$

which give

$$\Phi_b(x + x_{m,n}) = \Phi_b(x + c_b) \frac{1}{(qe^{2\pi bx}; q)_m} \frac{1}{(\tilde{q}^{-1}e^{2\pi b^{-1}x}; \tilde{q}^{-1})_n}$$
$$\Phi_b(x + c_b) = \frac{1}{1 - e^{2\pi b^{-1}x}} \frac{(qe^{2\pi bx}; q)_\infty}{(\tilde{q}e^{2\pi b^{-1}x}; \tilde{q})_\infty}.$$

Thus,

$$\begin{aligned}\Phi_b(x+x_{m,n}) &= \frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\frac{1}{(q;q)_m}\frac{1}{(\tilde{q}^{-1};\tilde{q}^{-1})_n}\\ &\quad\times\frac{1}{1-e^{2\pi b^{-1}x}}\frac{(qe^{2\pi bx};q)_\infty}{(q;q)_\infty}\frac{(\tilde{q};\tilde{q})_\infty}{(\tilde{q}e^{2\pi b^{-1}x};\tilde{q})_\infty}\frac{(q;q)_m}{(qe^{2\pi bx};q)_m}\frac{(\tilde{q}^{-1};\tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^{2\pi b^{-1}x};\tilde{q}^{-1})_n}\\ &= \frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\frac{1}{(q;q)_m}\frac{1}{(\tilde{q}^{-1};\tilde{q}^{-1})_n}\\ &\quad\times\frac{1}{1-e^{2\pi b^{-1}x}}\frac{(q^{m+1}e^{2\pi bx};q)_\infty}{(q^{m+1};q)_\infty}\frac{(\tilde{q};\tilde{q})_\infty}{(\tilde{q}e^{2\pi b^{-1}x};\tilde{q})_\infty}\frac{(\tilde{q}^{-1};\tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^{2\pi b^{-1}x};\tilde{q}^{-1})_n}\end{aligned}$$

The result follows. □

The decoupling of (m,n) in the quadratic form comes as follows: since A,m,n are integers, $e^{A\pi imn}=1$ and a computation gives

$$e^{-A\pi i(x+x_{n,m})^2} = i^A\left(\frac{q}{\tilde{q}}\right)^{\frac{A}{8}} t_m^A(q)\,\widetilde{t}_n^A(\tilde{q})e^{-A\pi ix^2+2A\pi x\left(b(m+\frac{1}{2})+b^{-1}(n+\frac{1}{2})\right)}$$

where

$$t_m^A(q)=(-1)^{Am}q^{A\frac{m(m+1)}{2}}, \qquad \widetilde{t}_n^A(\tilde{q})=(-1)^{An}\tilde{q}^{-A\frac{n(n+1)}{2}}.$$

The Dedekind function $\eta(\tau)=q^{1/24}(q;q)_\infty$ (with $q=e^{2\pi i\tau}$) satisfies the modular equation $\eta(-\tau^{-1})=\sqrt{-i\tau}\eta(\tau)$ [And76]. It follows that

$$\frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}=e^{\frac{\pi i}{4}}\left(\frac{\tilde{q}}{q}\right)^{\frac{1}{24}}b^{-1}. \tag{26}$$

After we set $w=x/(2\pi)$, the above discussion implies that

$$\begin{aligned}\mathcal{I}_{A,B}(b) &= \left(\frac{\tilde{q}}{q}\right)^{\frac{B-3A}{24}} e^{\pi i\frac{B+2(A+1)}{4}}\\ &\quad\times\sum_{m,n=0}^{\infty}\left(\operatorname{Res}_{w=0}F_{A,B,m,n}(w)\right)\frac{t_m^A(q)}{(q;q)_m^B}\frac{\widetilde{t}_n^A(\tilde{q})}{(\tilde{q}^{-1};\tilde{q}^{-1})_n^B},\end{aligned} \tag{27}$$

where

$$F_{A,B,m,n}(w)=e^{\frac{A}{4\pi i}w^2+Aw\left(b(m+\frac{1}{2})+b^{-1}(n+\frac{1}{2})\right)}\left(\frac{\phi_m(bw)\,\widetilde{\phi}_n(b^{-1}w)}{b(1-e^{b^{-1}w})}\right)^B. \tag{28}$$

2.2. The Taylor series of $\phi_m(x)$ and $\widetilde{\phi}_n(x)$

In this section we express the Taylor series of $\phi_m(x)$ and $\widetilde{\phi}_n(x)$ in terms of the q-series $E_l^{(m)}(q)$ and $\widetilde{E}_l^{(m)}(\tilde{q})$ defined by:

$$E_l^{(m)}(q) = \sum_{s=1}^{\infty} \frac{s^{l-1} q^{s(m+1)}}{1-q^s} = \langle \delta_l(x^m) \rangle \tag{29a}$$

$$\widetilde{E}_l^{(n)}(\tilde{q}) = \begin{cases} -n + E_1^{(n)}(\tilde{q}) & \text{if } l = 1 \\ E_l^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is odd} \\ 2E_l^{(0)}(\tilde{q}) - E_l^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is even} \end{cases} \tag{29b}$$

Proposition 2.2. We have:

$$\phi_m(x) = \exp\left(- \sum_{l=1}^{\infty} \frac{1}{l!} E_l^{(m)}(q) x^l \right) \tag{30a}$$

$$\widetilde{\phi}_n(x) = \exp\left(\sum_{l=1}^{\infty} \frac{1}{l!} \widetilde{E}_l^{(m)}(\tilde{q}) x^l \right) . \tag{30b}$$

The proof of this proposition is given in Section 2.6. The first few terms in Equations (30a)–(30a) are given by:

(31a)

$$\begin{aligned} \phi_m(x) &= \exp\left(-E_1^{(m)} x - \frac{1}{2} E_2^{(m)} x^2 - \frac{1}{6} E_3^{(m)} x^3 - \frac{1}{24} E_4^{(m)} x^4 - \cdots \right) \\ &= 1 - E_1^{(m)} x + \frac{1}{2}(E_1^{(m)2} - E_2^{(m)}) x^2 \\ &\quad + \frac{1}{6}(-E_1^{(m)3} + 3E_1^{(m)} E_2^{(m)} - E_3^{(m)}) x^3 \\ &\quad + \frac{1}{24}(E_1^{(m)4} - 6E_1^{(m)2} E_2^{(m)} + 3E_2^{(m)2} + 4E_1^{(m)} E_3^{(m)} - E_4^{(m)}) x^4 + \cdots \end{aligned}$$

(31b)

$$\begin{aligned} \widetilde{\phi}_n(x) &= \exp\left(\widetilde{E}_1^{(n)} x + \frac{1}{2} \widetilde{E}_2^{(n)} x^2 + \frac{1}{6} \widetilde{E}_3^{(n)} x^3 + \frac{1}{24} \widetilde{E}_4^{(n)} x^4 - \cdots \right) \\ &= 1 + \widetilde{E}_1^{(n)} x + \frac{1}{2}(\widetilde{E}_1^{(n)2} + \widetilde{E}_2^{(n)}) x^2 \\ &\quad + \frac{1}{6}(\widetilde{E}_1^{(n)3} + 3\widetilde{E}_1^{(n)} \widetilde{E}_2^{(n)} + \widetilde{E}_3^{(n)}) x^3 \\ &\quad + \frac{1}{24}(\widetilde{E}_1^{(n)4} + 6\widetilde{E}_1^{(n)2} \widetilde{E}_2^{(n)} + 3\widetilde{E}_2^{(n)2} + 4\widetilde{E}_1^{(n)} \widetilde{E}_3^{(n)} + \widetilde{E}_4^{(n)}) x^4 + \cdots \end{aligned}$$

where $E_l^{(m)} = E_l^{(m)}(q)$ and $\widetilde{E}_l^{(m)} = \widetilde{E}_l^{(m)}(\tilde{q})$.

2.3. The connection with the differential operators δ_l and $\widetilde{\delta}_l$

In this section we connect the series $E_l^{(m)}(q)$ and $\widetilde{E}_l^{(m)}(\tilde{q})$ with the action of the differential operators δ_l and $\widetilde{\delta}_l$ on a series $F(x)$ and $\widetilde{F}(\widetilde{x})$ respectively. Consider formal power series

$$F(x) = \sum_{m=0}^{\infty} t(m)x^m \qquad \widetilde{F}(\widetilde{x}) = \sum_{m=0}^{\infty} \widetilde{t}(m)\widetilde{x}^m \,.$$

Lemma 2.3. We have:

$$\sum_{m=0}^{\infty} \left(\prod_{j=1}^{r} E_{l_j}^{(m)}(q) \right) t(m) = \left\langle \prod_{j=1}^{r} \delta_{l_j} F \right\rangle \tag{32}$$

$$\sum_{m=0}^{\infty} m^r t(m) = \langle \delta^r F \rangle \tag{33}$$

and

$$\sum_{n=0}^{\infty} \left(\prod_{j=1}^{r} \widetilde{E}_{l_j}^{(n)}(\tilde{q}) \right) \widetilde{t}(n) = \left\langle \prod_{j=1}^{r} \widetilde{\delta}_{l_j} \widetilde{F} \right\rangle \tag{34}$$

$$\sum_{n=0}^{\infty} n^r \widetilde{t}(n) = \langle \widetilde{d}^r \widetilde{F} \rangle \,. \tag{35}$$

Proof. For a positive natural number l we have:

$$\sum_{m=0}^{\infty} E_l^{(m)}(q)t(m) = \sum_{m=0}^{\infty} \langle \delta_l(x^m) \rangle t(m) = \left\langle \delta_l(\sum_{m=0}^{\infty} t(m)x^m) \right\rangle = \langle \delta_l F \rangle \,.$$

Moreover, for positive natural numbers l, l' we have:

$$\begin{aligned} \sum_{m=0}^{\infty} E_l^{(m)}(q)E_{l'}^{(m)}(q)t(m) &= \sum_{m=0}^{\infty} \langle \delta_l(x^m) \rangle \langle \delta_{l'}(x^m) \rangle t(m) \\ &= \left\langle \delta_l \left(\sum_{m=0}^{\infty} \langle \delta_{l'}(x^m) \rangle t(m) x^m \right) \right\rangle . \end{aligned}$$

Now,

$$\langle \delta_{l'}(x^m) \rangle t(m) x^m = \sum_{s=1}^{\infty} \frac{s^{l'-1} q^s}{1-q^s} q^{sm} t(m) x^m = \delta_{l'}(x^m) t(m)$$

and summing up over m, we obtain that

$$\sum_{m=0}^{\infty} \langle \delta_{l'}(x^m) \rangle t(m) x^m = \delta_{l'} F(q,x) \,.$$

It follows that

$$\sum_{m=0}^{\infty} E_l^{(m)}(q) E_{l'}^{(m)}(q) t(m) = \langle \delta_l \delta_{l'} F \rangle \,.$$

The general case of Equation (32) follows by induction on r. Equation (33) is obvious. □

2.4. Proof of Theorem 1.1

Fix natural numbers A and B with $B > A \geq 1$, and let

$$t(m) = \frac{(-1)^{Am} q^{A\frac{m(m+1)}{2}}}{(q)_m^B}, \qquad F(q,x) = \sum_{m=0}^{\infty} t(m) x^m$$

and

$$\widetilde{t}(n) = \frac{(-1)^{(B-A)n} \tilde{q}^{(B-A)\frac{n(n+1)}{2}}}{(\tilde{q})_n^B}, \qquad \widetilde{F}(\tilde{q},\widetilde{x}) = \sum_{n=0}^{\infty} \widetilde{t}(n) x^n \,.$$

Use Equations (27) and (28) and Proposition 2.2 to expand $F_{A,B,m,n}(w)$ as a power series with coefficients polynomials in the variables $m, E_l^{(m)}(q)$ and $n, \widetilde{E}_l^{(n)}(\tilde{q})$ and $b^{\pm 1}$ and $(2\pi i)^{-1}$. Now apply Lemma 2.3 to convert the variables $m, E_l^{(m)}(q), n, \widetilde{E}_l^{(n)}(\tilde{q})$ in terms of the action of the operators $\delta, \delta_l, \widetilde{\delta}, \widetilde{\delta}_l$ respectively. This concludes the proof of Theorem 1.1. □

2.5. Some auxiliary power series

Consider the auxiliary series

$$\frac{1}{ae^x - 1} = \sum_{n=0}^{\infty} p_n(a) x^n \tag{36}$$

where

$$p_n(a) = -\frac{a}{n!(1-a)^{n+1}} \sum_{m=0}^{n-1} A_{n,m} a^m \qquad p_0(a) = -\frac{1}{1-a}$$

and $A_{n,m}$ are *Euler triangular numbers* (sequence `A008292` in the online encyclopedia of integer sequences [Slo]) that satisfy the recursion

$$A_{n,m} = (n-m)A_{n-1,m-1} + (m+1)A_{n-1,m}$$

and also given by the sum

$$A_{n,m} = \sum_{k=0}^{m} (-1)^k \binom{n+1}{k} (m+1-k)^n \,.$$

For a detailed discussion on this subject, see [FS70]. A table of the first few numbers $A_{n,m}$ is given by

$n \setminus m$	0	1	2	3	4	5	6	7	8
1	1								
2	1	1							
3	1	4	1						
4	1	11	11	1					
5	1	26	66	26	1				
6	1	57	302	302	57	1			
7	1	120	1191	2416	1191	120	1		
8	1	247	4293	15619	15619	4293	247	1	
9	1	502	14608	88234	156190	88234	14608	502	1

Lemma 2.4. For $l \geq 1$, we have:

$$\frac{d^l}{dx^l} \log(1 - q^k e^{bx})|_{x=0} = b^l p_{l-1}(q^k) + b\, \delta_{l,1} \tag{37}$$

Proof. It follows from

$$\frac{d}{dx}\log(1-q^k e^{bx}) = b\left(1+\frac{1}{q^k e^{bx}-1}\right)$$

and Equation (36). □

For positive natural numbers l, r with $l \geq r$ and m consider the q-series $E^{(m)}_{l,r}(q)$ defined by

$$E^{(m)}_{l,r}(q) = \sum_{k=m+1}^{\infty} \frac{q^{kr}}{(1-q^k)^l} \tag{38}$$

Lemma 2.5. (a) We have

$$E^{(m)}_{l,r}(q) = \sum_{s=r}^{\infty} a_{l,s}\frac{q^{s(m+1)}}{1-q^s} \tag{39}$$

where

$$\frac{x^r}{(1-x)^l} = \sum_{s=r}^{\infty} a_{l,s}x^s$$

(b) It follows that

$$\sum_{r=0}^{l-1} A_{l-1,r}E^{(m)}_{l,r+1}(q) = E^{(m)}_{l}(q) \tag{40}$$

Proof. For (a), interchange k and s summation:

$$\begin{aligned} E^{(m)}_{l,r}(q) &= \sum_{k=m+1}^{\infty}\sum_{s=r}^{\infty} a_{l,s}q^{sk} = \sum_{s=r}^{\infty}\sum_{k=m+1}^{\infty} a_{l,s}q^{sk} \\ &= \sum_{s=r}^{\infty} q^{(m+1)s}\sum_{k=0}^{\infty} a_{l,s}q^{sk} = \sum_{s=r}^{\infty} a_{l,s}\frac{q^{(m+1)s}}{1-q^s} \end{aligned}$$

(b) follows from (a) and the fact that

$$\frac{\sum_{r=0}^{l-1} A_{l-1,r}x^r}{(1-x)^l} = \sum_{s=1}^{\infty} s^{l-1}x^s\,.$$

□

Lemma 2.6. We have:

$$\phi_m(x) = \exp\left(-\sum_{l=1}^{\infty}\frac{1}{l!}\sum_{r=0}^{l-1}A_{l-1,r}E^{(m)}_{l,r+1}(q)x^l\right) \tag{41}$$

Proof. It follows from Lemma 2.4 combined with

$$\begin{aligned}\log(\phi_m(x)) &= \log\left(\frac{(q^{m+1}e^x;q)_\infty}{(q^{m+1};q)_\infty}\right)\\ &= \sum_{l=m+1}^{\infty}\Big(\log(1-q^le^x)-\log(1-q^l)\Big)\end{aligned}$$

□

2.6. Proof of Proposition 2.2

Part (a) of Proposition 2.2 follows from Lemma 2.5 and Lemma 2.6. For part (b), we will use the series

$$E^{[m]}_l(q) = \sum_{s=1}^{\infty}\frac{s^{k-1}q^{s(m+1)}}{1-q^s}$$

Using

$$\log(\widetilde{\phi}_n(x)) = \log\left(\frac{(\tilde q;\tilde q)_\infty}{(\tilde q e^x;\tilde q)_\infty}\right) + \log\left(\frac{(\tilde q^{-1};\tilde q^{-1})_n}{(\tilde q^{-1}e^x;\tilde q^{-1})_n}\right)$$

and the proof of part (a) of Proposition 2.2, it follows that

$$\begin{aligned}\log(\widetilde{\phi}_n(x)) &= \log\left(\frac{(\tilde q;\tilde q)_\infty}{(\tilde q e^x;\tilde q)_\infty}\right) + \log\left(\frac{(\tilde q^{-1};\tilde q^{-1})_n}{(\tilde q^{-1}e^x;\tilde q^{-1})_n}\right)\\ &= \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{r=0}^{l-1}A_{l-1,r}E^{(0)}_{l,r+1}(\tilde q)x^l + \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{r=0}^{l-1}A_{l-1,r}E^{[n]}_{l,r+1}(\tilde q^{-1})x^l\\ &= \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{r=0}^{l-1}A_{l-1,r}\left(E^{(0)}_{l,r+1}(\tilde q)+E^{[n]}_{l,r+1}(\tilde q^{-1})\right)x^l\end{aligned}$$

where

$$E^{[n]}_{l,r}(q) = \sum_{k=1}^{n}\frac{q^{kr}}{(1-q^k)^l}\,. \tag{42}$$

Let

$$\widetilde{E}^{(n)}_{l,r}(\tilde{q}) = \begin{cases} -n + E^{(n)}_{1,1}(\tilde{q}) & \text{if } l = r = 1 \\ E^{(n)}_{l,r}(\tilde{q}) & \text{if } l > 1 \text{ is odd} \\ 2E^{(0)}_{l,r}(\tilde{q}) - E^{(n)}_{l,r}(\tilde{q}) & \text{if } l > 1 \text{ is even} \end{cases} \tag{43}$$

We claim that

$$E^{(0)}_{l,r}(\tilde{q}) + E^{[n]}_{l,l-r}(\tilde{q}^{-1}) = \widetilde{E}^{(n)}_{l,r}(\tilde{q}) \tag{44}$$

for $l > r \geq 1$ and

$$E^{(0)}_{1,1}(\tilde{q}) + E^{[n]}_{1,1}(\tilde{q}^{-1}) = \widetilde{E}^{(n)}_{1,1}(\tilde{q}) \tag{45}$$

Assuming Equations (44) and (45), it follows that

$$\begin{aligned} \log(\widetilde{\phi}_n(x)) &= \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} \widetilde{E}^{(n)}_{l,r+1}(\tilde{q}) x^l \\ &= \sum_{l=1}^{\infty} \frac{1}{l!} \widetilde{E}^{(n)}_{l}(\tilde{q}) x^l \end{aligned}$$

where the last step follows from part (b) of Lemma 2.5.

It remains to prove Equations (44) and (45). Equation (44) follows from the definition of $\widetilde{E}^{(n)}_{1,1}(\tilde{q})$ and

$$\begin{aligned} E^{(0)}_{l,r}(\tilde{q}) + E^{[n]}_{l,l-r}(\tilde{q}^{-1}) &= \sum_{k=1}^{\infty} \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} + \sum_{k=1}^{n} \frac{\tilde{q}^{-k(l-r)}}{(1-\tilde{q}^{-k})^l} \\ &= \sum_{k=1}^{\infty} \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} + (-1)^l \sum_{k=1}^{n} \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} \\ &= (1+(-1)^l) \sum_{k=1}^{n} \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} + \sum_{k=n+1}^{\infty} \frac{\tilde{q}^{kr}}{(1-\tilde{q}^k)^l} \end{aligned}$$

Equation (45) follows from

$$\begin{aligned} E^{(0)}_{1,1}(\tilde{q}) + E^{[n]}_{1,1}(\tilde{q}^{-1}) &= \sum_{k=1}^{\infty} \frac{\tilde{q}^k}{1-\tilde{q}^k} + \sum_{k=1}^{n} \frac{\tilde{q}^{-k}}{1-\tilde{q}^{-k}} \\ &= \sum_{k=1}^{\infty} \frac{1-1+\tilde{q}^k}{1-\tilde{q}^k} - \sum_{k=1}^{n} \frac{1}{1-\tilde{q}^k} = -n + \sum_{k=n+1}^{\infty} \frac{\tilde{q}^k}{1-\tilde{q}^k} \end{aligned}$$

This completes the proof of Proposition 2.2. □

2.7. Proof of Lemma 1.5

Part (a) of Lemma 1.5 follows from the definition of $F_{A,B}$ and $\widetilde{F}_{A,B}$.

Part (b) follows from an application of Zeilberger's creative telescoping [Zei91]. To apply the method, define

$$t(m,x) = \frac{(-1)^{Am} q^{A\frac{m(m+1)}{2}}}{(q)_m^B} x^m$$

Then, observe that t satisfies the recursions with respect to m and x:

$$(1-q^{m+1})^B t(m+1,x) = (-1)^A q^{A(m+1)} t(m,x) \qquad t(m,qx) = q^m t(m,x)\,.$$

Now, we eliminate q^m from the above equations as follows. The second equation implies that $t(m+1,q^j x) = q^{j(m+1)} t(m+1,x)$. Expanding the first equation, it follows that

$$\sum_{j=0}^{B} (-1)^j \binom{B}{j} t(m+1,q^j x) = (-1)^A q^A x t(m,q^A x)$$

Summing for $m \geq 0$ implies (b). □

Proof. (of Corollary 1.6) The admissibility of F in the sense of Kontsevich-Soibelman, follows from [KS11, Sec.6.1] and [KS11, Thm.9]. Given this, the Nahm Equation (12) for ω follows easily from part (b) of Lemma 1.5. □

3. An application: state-integrals of the 4_1 and 5_2 knots

3.1. Proof of Corollary 1.7

Assume now that $(A,B) = (1,2)$. Then,

$$\begin{aligned}
\frac{1}{(b(1-e^{b^{-1}w}))^2} &= \frac{1}{w^2} - \frac{b^{-1}}{w} + O(1) \\
(\phi_m(bw))^2 &= 1 - 2E_1^{(m)}(q) bw + O(w^2) \\
(\widetilde{\phi}_n(b^{-1}w))^2 &= 1 + 2\widetilde{E}_1^{(n)}(\tilde{q}) b^{-1} w + O(w^2) \\
e^{\frac{1}{4\pi i}w^2 + w(b(m+1/2)+b^{-1}(n+1/2))} &= 1 + \left(\frac{1}{2}+m\right) bw + \left(\frac{1}{2}+n\right) b^{-1} w + O(w^2)
\end{aligned}$$

Combined with $\widetilde{E}_1^{(n)}(\tilde{q}) = -n + E_1^{(n)}(\tilde{q})$, it follows that the residue $R = \mathrm{Res}_{w=0}(F_{1,2,m,n}(w))$ is given by

$$R = \left(b\left(\frac{1}{2} + m - 2E_1^{(m)}(q) \right) - b^{-1}\left(\frac{1}{2} + n - 2E_1^{(n)}(\tilde{q}) \right) \right)$$

The above, together with the fact that $t_n(q) = (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2}$ satisfies $t_n(q^{-1}) = t_n(q)$ implies Equation (14). Equation (17) follows from Equation (11) for $(A, B) = (1, 2)$.

3.2. Proof of Corollary 1.8

Assume now that $(A, B) = (2, 3)$. Then,

$$\begin{aligned}
\frac{1}{(b(1 - e^{b^{-1}w}))^3} &= -\frac{1}{w^3} + \frac{3b^{-1}}{2w^2} - \frac{b^{-2}}{w} + O(1), \\
(\phi_m(bw))^3 &= 1 - 3E_1^{(m)}(q)\, bw + \frac{3}{2}\left(3E_1^{(m)2}(q) - E_2^{(m)}(q)\right) b^2 w^2 \\
&\quad + O(w^3), \\
(\widetilde{\phi}_n(b^{-1}w))^3 &= 1 + 3\widetilde{E}_1^{(n)}(\tilde{q})\, b^{-1} w + \frac{3}{2}\left(3\widetilde{E}_1^{(n)2}(\tilde{q}) + \widetilde{E}_2^{(n)}(\tilde{q})\right) b^{-2} w^2 \\
&\quad + O(w^3), \\
e^{\frac{2}{4\pi i}w^2 + 2w(b(m+1/2) + b^{-1}(n+1/2))} & \\
&= 1 + \left((1+2m)b + (1+2n)b^{-1}\right) w \\
&\quad + \left(1 + \frac{b^2 + b^{-2}}{2} + \frac{1}{2\pi i} + 2b^2 m^2 + 2b^{-2} n^2 + 4mn \right. \\
&\qquad \left. + 2(1+b^2)m + 2(1+b^{-2})n \right) w^2 + O(w^3).
\end{aligned}$$

If $R = \mathrm{Res}_{w=0}(F_{2,3,m,n}(w))$, then

$$\begin{aligned}
R_{m,n} = &-\frac{b^2}{2}\left(1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_1^{(m)2}(q) - 3E_2^{(m)}(q)\right) \\
&- \frac{1}{2\pi i} + \frac{1}{2}\left(1 + 2m - 3E_1^{(m)}(q)\right)\left(1 + 2n - 6E_1^{(n)}(\tilde{q})\right) \\
&+ \frac{b^{-2}}{2}\left(-n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) - 9E_1^{(n)2}(\tilde{q}) + 3E_2^{(n)}(\tilde{q})\right),
\end{aligned}$$

This proves part (a) of Corollary 1.8. Part (b) follows from Equation (11) for $(A, B) = (2, 3)$ and $(A, B) = (1, 3)$. Note that Theorem 1.1 states that

$$\mathcal{I}_{2,3}(q) = -e^{\frac{3\pi i}{4}} \langle P_{2,3}(F\widetilde{F}) \rangle \tag{46}$$

where

$$\begin{aligned} P_{2,3} = & -\frac{b^2}{2}\left(1 + 4\delta + 4\delta^2 - 6\delta_1 - 12\delta\delta_1 + 9\delta_1^2 - 3\widetilde{\delta_2}\right) \\ & + \frac{1}{2}\left(1 + 2\delta + \frac{i}{\pi} + 2\widetilde{\delta} + 4\delta\widetilde{\delta} - 3\delta_1 - 6\widetilde{\delta}\delta_1 - 6e_2(\tilde{q}) - 6\widetilde{\delta_1} - 12\delta\widetilde{\delta_1} + 18\delta_1\widetilde{\delta_1}\right) \\ & + \frac{b^{-2}}{2}\left(-\widetilde{\delta} - \widetilde{\delta}^2 + 3\widetilde{\delta_1} + 6\widetilde{\widetilde{\delta}\delta_1} - 9\widetilde{\delta}_1^2 + 3\widetilde{\delta_2}\right). \end{aligned}$$

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