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A characterization of Clifford hypersurfaces among embedded constant mean curvature hypersurfaces in a unit sphere

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Let Σ be an $n(\geq 3)$-dimensional compact embedded hypersurface in a unit sphere with constant mean curvature $H \geq 0$ and with two distinct principal curvatures λ and μ of multiplicity $n-1$ and 1, respectively. It is known that if $\lambda > \mu$, there exist many compact embedded constant mean curvature hypersurfaces [26]. In this paper, we prove that if $\mu > \lambda$, then Σ is congruent to a Clifford hypersurface. The proof is based on the arguments used by Brendle [10].

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1. Introduction

Let Σ be an n-dimensional compact embedded hypersurface in an $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$ with constant mean curvature H. In case of minimal surfaces in $\mathbb{S}^3$ (i.e., $n=2$ and $H=0$), Brendle [10] ingeniously proved the famous Lawson conjecture which states that the only embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus from a sharp estimate for a two-point function by using the maximum principle. It was observed that the embeddedness condition can be replaced by the weaker assumption that the minimal torus is Alexandrov-immersed in $\mathbb{S}^3$ [9]. The technique using the maximum principle for a two-point function was also used by Andrews-Li [5], who gave a complete classification of embedded constant mean curvature tori in $\mathbb{S}^3$. More generally, the proof of Lawson conjecture was extended to a class of embedded Weingarten tori in $\mathbb{S}^3$ [11]. Hauswirth-Kilian-Schmidt [15] obtained that every mean-convex Alexandrov embedded constant mean curvature torus in $\mathbb{S}^3$ is rotationally symmetric by using integrable systems.

It is interesting to find the higher-dimensional analogues of these results. One possible approach to the higher-dimensional problem is to characterize a Clifford hypersurface among embedded constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$. Unfortunately, even when $H=0$, it is well-known that there exist infinitely many mutually noncongruent embedded minimal hypersurfaces in $\mathbb{S}^{n+1}$ which are homeomorphic to the Clifford hypersurface [17]. Recall that an n-dimensional *Clifford hypersurface* in $\mathbb{S}^{n+1}$ with constant mean curvature H has two distinct principal curvatures λ and μ of multiplicity $n-k$ and k, respectively. Moreover it is given by

$$\mathbb{S}^{n-k}\left(\frac{1}{\sqrt{1+\lambda^2}}\right)\times\mathbb{S}^{k}\left(\frac{1}{\sqrt{1+\mu^2}}\right),$$

where λ and μ satisfy $nH=(n-k)\lambda+k\mu$ and $\lambda\mu+1=0$.

In view of this observation, we restrict ourselves to consider compact embedded constant mean curvature hypersurfaces in a unit sphere with two distinct principal curvatures. Otsuki [22] proved that if the multiplicities of two distinct principal curvatures are greater than 1, then the minimal hypersurface is locally congruent to a Clifford minimal hypersurface. Later, by studying an ordinary differential equation derived from the two distinct principal curvature condition, Otsuki [23, 24] also proved that a compact embedded minimal hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures of multiplicity $n-1$ and 1, respectively, is congruent to a Clifford minimal hypersurface (see also [12]). Therefore he gave the following characterization of Clifford minimal hypersurfaces in $\mathbb{S}^{n+1}$.

Theorem 1.1 ([22–24]). *Let Σ be an $n(\geq 3)$-dimensional compact embedded minimal hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures of multiplicity $n-k$ and k for $1 \leq k \leq n-1$. Then Σ is congruent to a Clifford minimal hypersurface $\mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right) \times \mathbb{S}^{k}\left(\sqrt{\frac{k}{n}}\right)$.*

In case of constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, Wei [31] obtained the analogue of Otsuki's result, provided the multiplicities of two principal curvatures are at least 2, applying a similar argument as in [22].

Theorem 1.2 ([31]). *Let Σ be an $n(\geq 3)$-dimensional hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature H and with two distinct principal curvatures λ and μ of multiplicities $n-k$ and k, respectively, for $2 \leq k \leq n-2$. Then Σ is isometric to a Clifford hypersurface $\mathbb{S}^{n-k}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^{k}\left(\frac{1}{\sqrt{1+\mu^2}}\right)$, where λ and μ satisfy $nH = (n-k)\lambda + k\mu$ and $\lambda\mu + 1 = 0$.*

Therefore it suffices to consider constant mean curvature hypersurfaces with two distinct principal curvatures λ and μ, μ being simple (i.e., multiplicity 1). Perdomo [26] obtained the existence of compact embedded constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ other than the totally geodesic n-spheres and Clifford hypersurfaces (see also [12, 27, 33]). Indeed, he constructed such examples by analyzing an ordinary differential equation arising from the two distinct principal curvatures λ and μ satisfying that $\lambda > \mu$. More precisely, he proved

Theorem 1.3 ([26]). *For any integer $m \geq 2$ and H between $\cot\frac{\pi}{m}$ and $\frac{(m^2-2)\sqrt{n-1}}{n\sqrt{m^2-1}}$, there exists a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature H other than the totally geodesic n-spheres and Clifford hypersurfaces.*

On the other hand, in the study of n-dimensional constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures of multiplicity $n-1$ and 1, it mostly requires an additional assumption to obtain a characterization of Clifford hypersurfaces. For instance, Perdomo [25] and Wang [30] independently obtained a curvature integral inequality for minimal hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, which characterizes a Clifford minimal hypersurface. Later, Wei [32] showed that the similar curvature integral inequality holds for hypersurfaces with the vanishing m-th order mean curvature (i.e., $H_m \equiv 0$). More precisely, they proved

Theorem ([25, 30, 32]). *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple. Then*

$$\int_M |A|^2 \leq \frac{n(m^2 - 2m + n)}{m(n-m)} \mathrm{Vol}(M),$$

where equality holds if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

In [3], Andrews-Huang-Li obtained a uniqueness of Clifford hypersurface among compact embedded Weingarten hypersurfaces in the unit sphere with two distinct principal curvatures satisfying a linear relation between them. Very recently, the authors [21] obtained a more general sharp curvature integral inequality for hypersurfaces in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature and with two distinct principal curvatures, which generalizes Simons' integral inequality and gives a characterization of Clifford hypersurfaces in $\mathbb{S}^{n+1}$.

In contrast to the 2-dimensional problem for embedded constant mean curvature tori, we consider embedded constant mean curvature hypersurfaces with two distinct principal curvatures of multiplicity $n-1$ and 1 without assuming any topological restriction. In this paper, we give the following characterization theorem (Theorem 5.3) of Clifford hypersurfaces:

Theorem. *Let Σ be an $n(\geq 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature $H \geq 0$ and with two distinct principal curvatures λ and μ, μ being simple. If $\mu > \lambda$, then Σ is congruent to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where $\lambda = \frac{nH - \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)}$.*

The key ingredients in the proof of our theorem are the following: We first define a suitable two-point function on an embedded constant mean curvature hypersurface based on the non-collapsing argument and compute the first and second order derivatives of the two-point function. This technique was pioneered by Huisken [18] and was developed by Andrews [2].

Secondly, we obtain a Simons-type identity for constant mean curvature hypersurfaces with two distinct principal curvatures. Indeed, this provides a sufficient condition for constant mean curvature hypersurfaces to attain the equality in Kato's inequality. Combining with Simons-type identity and adapting the arguments by Brendle [10] with a slight modification finally gives a characterization of Clifford hypersurfaces.

We remark that every constant mean curvature torus in $\mathbb{S}^3$ has two distinct principal curvatures which implies that there is no umbilic point. (See [20] for minimal tori and [14, 16] for constant mean curvature tori in $\mathbb{S}^3$.) Moreover, constant mean curvature tori in $\mathbb{S}^3$ automatically satisfy the condition that $\mu > \lambda$. Hence our main theorem can be regarded as an extension of the results by Brendle [10] and Andrews-Li [5] to higher-dimensional cases.

2. Preliminaries

Let $F : \Sigma^n \to \mathbb{S}^{n+1} (\subset \mathbb{R}^{n+2})$ be a compact embedded constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, one of them being simple. Let $\nu(x)$ be the unit normal vector at $x \in \Sigma$ in $\mathbb{S}^{n+1}$. Let h and A be the second fundamental form and the shape operator of Σ, respectively. Note that A is a self-adjoint endomorphism of the tangent space at each point x in Σ such that $\langle A(X), Y\rangle = h(X, Y)$ for all $X, Y \in T_x\Sigma$. Since Σ has two distinct principal curvatures and one of them is simple, we may assume that $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$, where each λ_i denotes the principal curvature on Σ for $1 \le i \le n$. The *normalized mean curvature* H is defined by

$$H = \frac{1}{n}\mathrm{tr}(h) = \frac{1}{n}\sum_{i=1}^{n}\lambda_i = \frac{n-1}{n}\lambda + \frac{1}{n}\mu.$$

Since Σ is a compact embedded hypersurface, Σ divides $\mathbb{S}^{n+1}$ into two connected components. Because the mean curvature of $F(\Sigma)$ in $\mathbb{S}^{n+1}$ is constant, we may assume that $H \ge 0$ by choosing the suitable orientation of Σ. Let R be the region satisfying that ν points out of R. The *mean curvature vector* $\vec{H}$ satisfies that $\vec{H} = -nH\nu(x)$.

For a positive function Ψ on Σ, we denote by $B_T(x, \frac{1}{\Psi(x)})$ a ball with radius $\frac{1}{\Psi(x)}$ which touches Σ at $F(x)$ inside the region R in $\mathbb{S}^{n+1}$. Note that our notation $B_T(x, r)$ is different from a ball $B_r(x)$ centered at x with radius $r > 0$. Then $B_T(x, \frac{1}{\Psi(x)})$ is a ball of radius $\frac{1}{\Psi(x)}$ centered at $p(x) =$

$F(x) - \frac{1}{\Psi(x)}\nu(x)$ in $\mathbb{R}^{n+2}$. Define the two-point function $Z : \Sigma \times \Sigma \to \mathbb{R}$ by

$$Z(x, y) := \Psi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \,. \tag{1}$$

It is easy to check that for any $y \in \Sigma$,

$$\begin{cases} Z(x,y) > 0 & \text{if } F(y) \in \operatorname{int} B_T(x, \frac{1}{\Psi(x)}), \\ Z(x,y) = 0 & \text{if } F(y) \in \partial B_T(x, \frac{1}{\Psi(x)}), \\ Z(x,y) < 0 & \text{if } F(y) \notin B_T(x, \frac{1}{\Psi(x)}), \end{cases}$$

since

$$\frac{2}{\Psi(x)} Z(x, y) = |F(y) - p(x)|^2 - \left(\frac{1}{\Psi(x)}\right)^2 .$$

We recall the definition of the interior ball curvature at $x \in \Sigma$, which was originally given by Andrews-Langford-McCoy [4] (see also [5]).

Definition 2.1. The *interior ball curvature* k is a positive function on Σ defined by

$$k(x) := \inf\left\{\frac{1}{r} : B_T(x, r) \cap \Sigma = \{x\},\ r > 0\right\}.$$

Because Σ is compact and embedded in $\mathbb{S}^{n+1}$, one can see that the function k is a well-defined positive function on Σ. From the definition of $k(x)$ for every point $x \in \Sigma$, it follows that

$$k(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geq 0$$

for all $y \in \Sigma$.

Let $\Phi(x) := \max\{\lambda(x), \mu(x)\}$ be the maximum value of the principal curvatures of Σ in $\mathbb{S}^{n+1}$ at $F(x)$. Note that the two distinct principal curvature condition guarantees that Σ has no umbilic point and hence $\Phi(x) - H > 0$.

Motivated by the works of Brendle [10] and Andrews-Li [5], we introduce the constant κ as follows:

$$\kappa := \sup_{x \in \Sigma} \frac{k(x) - H}{\Phi(x) - H}.$$

For convenience, we will write $\varphi(x) := \Phi(x) - H$.

Proposition 2.2 (Uniform boundedness of κ). *Let Σ be a compact embedded constant mean curvature hypersurface with two distinct principal curvatures in $\mathbb{S}^{n+1}$. Then there exists a constant $K > 0$ satisfying*

$$1 \leq \kappa < K.$$

Proof. By definition, one sees that $\varphi > 0$. Because Σ is compact, φ is uniformly bounded and k is uniformly bounded above. From the definition of k, it immediately follows that $k(x) \geq \Phi(x)$ for all $x \in \Sigma$, which gives the conclusion. □

Define a positive function $\Psi(x) := \kappa\varphi(x) + H = \kappa(\Phi(x) - H) + H$ on Σ. Then $\Psi(x) \geq k(x)$. It follows that

$$Z(x, y) = \Psi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geq 0 \tag{2}$$

for all $(x, y) \in \Sigma \times \Sigma$. Therefore if there exists a point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ satisfying that $Z(\overline{x}, \overline{y}) = 0$, then

$$\frac{\partial Z}{\partial x_i}(\overline{x}, \overline{y}) = \frac{\partial Z}{\partial y_i}(\overline{x}, \overline{y}) = 0,$$

since the function Z attains its global minimum at $(\overline{x}, \overline{y})$. Note that the global minimum of the function Z is attained at $(x, x) \in \Sigma \times \Sigma$ for all $x \in \Sigma$. Furthermore, one can see that there exists a point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ satisfying that $Z(\overline{x}, \overline{y}) = 0$ and $\overline{x} \neq \overline{y}$ by making use of the compactness of Σ and the property of interior ball curvature derived from the embeddedness of Σ (see Lemma 5.1).

3. Simons-type identity for constant mean curvature hypersurfaces

Let Σ be an $n(\geq 3)$-dimensional compact embedded constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. The traceless part of the second fundamental form h is defined to be a differential 2-form η on Σ with the coefficient function η_{ij} in local coordinates as follows:

$$\eta_{ij} := h_{ij} - \delta_{ij} H,$$

where δ_{ij} is the Kronecker delta. The corresponding traceless shape operator $\mathring{A}$ is defined by

$$\left\langle \mathring{A}(X), Y \right\rangle = \eta(X, Y)$$

for all $X, Y \in T_x\Sigma$, where $T_x\Sigma$ denotes the tangent space of Σ at $x \in \Sigma$.

In 1970, Otsuki [22] observed the following interesting property of the eigenspace of principal curvatures.

Theorem 3.1 ([22]). *Let Σ be a hypersurface immersed in an $(n+1)$-dimensional Riemannian manifold of constant curvature such that the multiplicities of principal curvatures are all constant. Then we have the following:*

- *The distribution of the space of principal vectors corresponding to each principal curvature is completely integrable.*
- *If the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

Let $(x_1, \ldots, x_n)$ be the geodesic normal coordinates at $x \in \Sigma$ (i.e. the metric tensor is given by $g_{ij} = \delta_{ij}$ and the Christoffel symbol $\Gamma_{ij}^k(x)$ at x vanishes). We may assume that $h_{ij} = \lambda_i \delta_{ij}$ with $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$. We will denote the coefficient function of the covariant derivative $\nabla^\Sigma h$ by h_{ijk}. Then

$$h_{ijk}(x) = \frac{\partial h_{ij}}{\partial x_k}(x)$$

at $x \in \Sigma$. As a consequence of Theorem 3.1, one can compute η_{ijk} for $1 \leq i, j, k \leq n$.

Lemma 3.2. Let Σ be a constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ, μ being simple: $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$. Then for all $1 \leq i, j, k \leq n$, we have

$$\eta_{ijk} = h_{ijk}$$

and

$$\begin{cases} \eta_{ijk} = 0 & \text{if } i, j, k \text{ are all distinct,} \\ \eta_{iik} = 0 & \text{if } k \neq n, \\ \eta_{nnn} = -(n-1)\eta_{iin} & \text{for } i = 1, \ldots, n-1. \end{cases}$$

Proof. One can easily see that $\eta_{ijk} = h_{ijk}$ for $1 \leq i, j, k \leq n$ on Σ. If i, j, k are all distinct, then at least two of them are contained in the set $\{1, \ldots, n-1\}$. Using the Codazzi equations, we may assume that i and j are in the set

$\{1, \ldots, n-1\}$. Since $h_{ijk} = \frac{\partial h_{ij}}{\partial x_k}$ at x, the first part of Theorem 3.1 implies

$$\eta_{ijk} = \frac{\partial h_{ij}}{\partial x_k} = \frac{\partial \lambda_i}{\partial x_k}\delta_{ij} = 0$$

at $x \in \Sigma$. To check the last two equalities, we let $i, j \in \{1, \ldots, n-1\}$. Then $h_{iik} = h_{jjk}$ for any $k \in \{1, \ldots, n\}$, and $h_{iij} = 0$ are direct consequences of the second part of Theorem 3.1. The constant mean curvature assumption implies that $h_{nnk} = -\sum_{i=1}^{n-1} h_{iik}$. Hence the conclusion immediately follows. □

When a constant mean curvature hypersurface has two distinct principal curvatures, we first prove the following useful identity.

Proposition 3.3. *Let Σ be a constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ, μ being simple. Then $|\mathring{A}|$ is strictly positive and*

$$|\nabla^\Sigma \mathring{A}|^2 = \frac{n+2}{n}|\nabla^\Sigma |\mathring{A}||^2. \tag{3}$$

Remark 3.4. It is well-known that a constant mean curvature hypersurface Σ in space forms satisfies

$$|\nabla^\Sigma \mathring{A}|^2 - |\nabla^\Sigma |\mathring{A}||^2 \geq \frac{2}{n}|\nabla^\Sigma |\mathring{A}||^2, \tag{4}$$

which is so-called Kato's inequality [6, 19, 28, 34]. It would be interesting to characterize the equality case. Proposition 3.3 gives a sufficient condition for Kato's inequality (4) to attain the equality.

Proof of Proposition 3.3. Since Σ has two distinct principal curvatures λ and μ, the functions $\lambda - H$ and $\mu - H$ never vanish. Thus the function $|\mathring{A}|$ is strictly positive. For $x \in \Sigma$, we choose the geodesic normal coordinates at x as above. Then we have

$$\begin{aligned} |\nabla^\Sigma \mathring{A}|^2 = \sum_{i,j,k=1}^n \eta_{ijk}^2 &= \sum_{i=1}^n \eta_{iii}^2 + 3\sum_{\substack{i,k=1 \\ i\neq k}}^n \eta_{iik}^2 + \sum_{\substack{i,j,k=1 \\ i,j,k \text{ distinct}}}^n \eta_{ijk}^2 \\ &= \eta_{nnn}^2 + 3(n-1)\eta_{11n}^2 \\ &= (n-1)^2\eta_{11n}^2 + 3(n-1)\eta_{11n}^2 \\ &= (n-1)(n+2)\eta_{11n}^2, \end{aligned} \tag{5}$$

where we used the relations $\eta_{nnn} = -\sum_{i=1}^{n-1} \eta_{iin} = -(n-1)\eta_{11n}$ in the second and third equality. Since $2|\mathring{A}|\ \nabla^\Sigma|\mathring{A}| = \nabla^\Sigma|\mathring{A}|^2$,

$$
\begin{aligned}
(6) \qquad |\nabla^\Sigma|\mathring{A}||^2 &= \frac{1}{4|\mathring{A}|^2}|\nabla^\Sigma|\mathring{A}|^2|^2 \\
&= \frac{1}{|\mathring{A}|^2}\sum_{i,j,k=1}^{n} \eta_{ii}\eta_{iik}\eta_{jj}\eta_{jjk} \\
&= \frac{1}{|\mathring{A}|^2}\sum_{\substack{i,j,k=1\\ k\neq n}}^{n} \eta_{ii}\eta_{iik}\eta_{jj}\eta_{jjk} + \frac{1}{|\mathring{A}|^2}\sum_{i,j=1}^{n} \eta_{ii}\eta_{iin}\eta_{jj}\eta_{jjn}.
\end{aligned}
$$

Using Lemma 3.2, one sees that the first term of the right hand side of the identity (6) vanishes. Moreover

$$\sum_{i=1}^{n}\eta_{ii} = (n-1)\eta_{11} + \eta_{nn} = 0.$$

Therefore the second term of the right hand side of the identity (6) can be written as

$$
\begin{aligned}
&\frac{1}{|\mathring{A}|^2}\sum_{i,j=1}^{n} \eta_{ii}\eta_{iin}\eta_{jj}\eta_{jjn} \\
&= \frac{1}{|\mathring{A}|^2}\sum_{i,j=1}^{n-1} \eta_{ii}\eta_{iin}\eta_{jj}\eta_{jjn} + \frac{2}{|\mathring{A}|^2}\sum_{i=1}^{n-1}\eta_{ii}\eta_{iin}\eta_{nn}\eta_{nnn} + \frac{1}{|\mathring{A}|^2}\eta_{nn}\eta_{nnn}\eta_{nn}\eta_{nnn} \\
&= \frac{1}{|\mathring{A}|^2}\left((n-1)^2\eta_{11}^2\eta_{11n}^2 + 2(n-1)^3\eta_{11}^2\eta_{11n}^2 + (n-1)^4\eta_{11}^2\eta_{11n}^2\right) \\
&= \frac{n^2(n-1)^2}{|\mathring{A}|^2}\eta_{11}^2\eta_{11n}^2.
\end{aligned}
$$

From the fact that

$$|\mathring{A}|^2 = (n-1)\eta_{11}^2 + (n-1)^2\eta_{11}^2 = n(n-1)\eta_{11}^2 > 0,$$

we finally obtain

$$(7) \qquad |\nabla^\Sigma|\mathring{A}||^2 = n(n-1)\eta_{11n}^2.$$

Hence combining the equations (5) and (7),

$$|\nabla^\Sigma \mathring{A}|^2 = (n-1)(n+2)\eta_{11n}^2 = \frac{n+2}{n}|\nabla^\Sigma|\mathring{A}||^2,$$

which completes the proof. □

The following second order partial differential equation of the second fundamental form of a minimal hypersurface in $\mathbb{S}^{n+1}$ was established by Simons [29].

$$\Delta_\Sigma |A|^2 - 2|\nabla^\Sigma A|^2 + 2(|A|^2 - n)|A|^2 = 0.$$

More generally one can obtain the analogue of the above equation by Simons for a constant mean curvature hypersurface Σ in a Riemannian manifold. The Gauss equations and the Ricci formulas state that

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$
$$h_{ijkl} = h_{ijlk} + \sum_{r=1}^{n} h_{rj}R_{rikl} + \sum_{r=1}^{n} h_{ir}R_{rjkl},$$

where R_{ijkl} denotes the components of the Riemann curvature tensor of Σ. The Laplacian of h can be computed by making use of the Codazzi equations as follows:

$$\begin{aligned}
\Delta_\Sigma h_{ij} &= \sum_{k=1}^{n} h_{ijkk} = \sum_{k=1}^{n} h_{kijk} \\
&= \sum_{k=1}^{n} h_{kikj} + \sum_{k,r=1}^{n} h_{ri}R_{rkjk} + \sum_{k,r=1}^{n} h_{kr}R_{rijk} \\
&= \sum_{k=1}^{n} h_{kikj} + \sum_{k,r=1}^{n} h_{ri}(\delta_{rj}\delta_{kk} - \delta_{rk}\delta_{kj} + h_{rj}h_{kk} - h_{rk}h_{kj}) \\
&\quad + \sum_{k,r=1}^{n} h_{kr}(\delta_{rj}\delta_{ik} - \delta_{rk}\delta_{ij} + h_{rj}h_{ik} - h_{rk}h_{ij}) \\
&= \sum_{k=1}^{n} h_{kkij} + nh_{ij} - h_{ij} + nH\sum_{r=1}^{n} h_{ri}h_{rj} \\
&\quad - \sum_{k,r=1}^{n} h_{ir}h_{rk}h_{kj} + h_{ij} - nH\delta_{ij} + \sum_{k,r=1}^{n} h_{ik}h_{kr}h_{rj} - |A|^2 h_{ij} \\
&= \sum_{k=1}^{n} h_{kkij} + (n - |A|^2)h_{ij} + nH\sum_{r=1}^{n} h_{ri}h_{rj} - nH\delta_{ij}.
\end{aligned}$$

Therefore we have

$$\Delta_\Sigma h_{ij} = (n - |A|^2)h_{ij} + nH_{ij} - nH\delta_{ij} + nH\sum_{k=1}^{n} h_{ik}h_{kj}. \tag{8}$$

Note that the the above equation (8) holds for any hypersurface Σ in $\mathbb{S}^{n+1}$.

In the following we have second-order elliptic partial differential equation on the trace-less second fundamental form, which was obtained by Alías-de Almeida-Brasil [1]. For completeness we give the proof which is slightly different from their proof.

Proposition 3.5 ([1]). *Let Σ be a constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ, μ being simple. Then*

$$\Delta_\Sigma|\mathring{A}| - \frac{2}{n}\frac{|\nabla^\Sigma|\mathring{A}||^2}{|\mathring{A}|} + (|A|^2 - n)|\mathring{A}| \\ - 2nH^2|\mathring{A}| + \operatorname{sgn}(\lambda - \mu)\frac{n(n-2)}{\sqrt{n(n-1)}}H|\mathring{A}|^2 = 0.$$

Proof. Using the equation (8), we get

$$\begin{aligned} \sum_{i,j=1}^{n} \eta_{ij}\Delta_\Sigma\eta_{ij} &= \sum_{i,j=1}^{n} \eta_{ij}\Delta_\Sigma h_{ij} \\ &= \sum_{i,j=1}^{n} (n - |A|^2)\eta_{ij}^2 + nH\sum_{i,j,k=1}^{n} \eta_{ij}\left(\eta_{ik}\eta_{kj} + H\eta_{ik}\delta_{kj} + H\delta_{ik}\eta_{kj}\right) \\ &= (n - |A|^2)\sum_{i=1}^{n}\eta_{ii}^2 + nH\sum_{i=1}^{n}\eta_{ii}^3 + 2nH^2\sum_{i=1}^{n}\eta_{ii}^2. \end{aligned}$$

The left hand side is equal to $\frac{1}{2}\Delta_\Sigma|\mathring{A}|^2 - |\nabla^\Sigma\mathring{A}|^2$, and the right hand side is equal to $(n - |A|^2)|\mathring{A}|^2 + 2nH^2|\mathring{A}|^2 + nH\sum_{i=1}^{n}\eta_{ii}^3$. We also see that

$$\begin{aligned} \sum_{i=1}^{n}\eta_{ii}^3 &= (n-1)\eta_{11}^3 - (n-1)^3\eta_{11}^3 \\ &= -n(n-1)(n-2)\eta_{11}^3 = -\operatorname{sgn}(\lambda - \mu)\frac{n-2}{\sqrt{n(n-1)}}|\mathring{A}|^3, \end{aligned}$$

since $|\mathring{A}|^3 = (|\mathring{A}|^2)^{\frac{3}{2}} = \operatorname{sgn}(\lambda-\mu)\sqrt{n(n-1)}^3\eta_{11}^3$. Therefore we have the following Simons-type identity:

$$\Delta_\Sigma|\mathring{A}|^2 - 2|\nabla^\Sigma\mathring{A}|^2 + 2(|A|^2-n)|\mathring{A}|^2 - 4nH^2|\mathring{A}|^2 + \operatorname{sgn}(\lambda-\mu)\frac{2n(n-2)}{\sqrt{n(n-1)}}H|\mathring{A}|^3 = 0.$$

Since $\Delta_\Sigma|\mathring{A}|^2 = 2|\mathring{A}|\Delta_\Sigma|\mathring{A}| + 2|\nabla^\Sigma|\mathring{A}||^2$,

$$\Delta_\Sigma|\mathring{A}| + \frac{|\nabla^\Sigma|\mathring{A}||^2}{|\mathring{A}|} - \frac{|\nabla^\Sigma\mathring{A}|^2}{|\mathring{A}|} + (|A|^2-n)|\mathring{A}| - 2nH^2|\mathring{A}| + \operatorname{sgn}(\lambda-\mu)\frac{n(n-2)}{\sqrt{n(n-1)}}H|\mathring{A}|^2 = 0.$$

Therefore applying the equation (3) gives the conclusion. □

Applying Proposition 3.5 to the function $\varphi = \Phi - H$, where Φ is the maximum value of the principal curvatures, we get the following:

Corollary 3.6. *Let Σ be a constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ, μ being simple. Then*

$$\Delta_\Sigma\varphi - \frac{2}{n}\frac{|\nabla^\Sigma\varphi|^2}{\varphi} + (|A|^2-n)\varphi - 2nH^2\varphi + \operatorname{sgn}(\lambda-\mu)nf(n)H\varphi^2 = 0,$$

where the function $f(n)$ is defined by

$$f(n) := \begin{cases} \frac{n-2}{n-1} & \text{if } \Phi = \mu, \\ n-2 & \text{if } \Phi = \lambda. \end{cases}$$

Proof. Note that if $\Phi = \mu$, then $|\mathring{A}|^2 = \frac{n}{n-1}\varphi^2$ and if $\Phi = \lambda$, then $|\mathring{A}|^2 = n(n-1)\varphi^2$. The conclusion follows from Proposition 3.5 and the linearity of Δ_Σ and ∇^Σ. □

For later use, we define a constant $g(n)$ depending on the dimension n as follows:

$$g(n) = \begin{cases} \frac{1}{n-1} & \text{if } \Phi = \mu, \\ n-1 & \text{if } \Phi = \lambda. \end{cases}$$

Then one can write $|\mathring{A}|^2 = ng(n)\varphi^2$.

4. First and second order derivatives of the two-point function

Let Σ be an $n(\geq 3)$-dimensional compact embedded constant mean curvature hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ of the multiplicity $n-1$ and 1, respectively. Consider a pair of points $(\overline{x},\overline{y}) \in \Sigma \times \Sigma$ such that $Z(\overline{x},\overline{y}) = 0$. Then by the equation (2)

$$\frac{\partial Z}{\partial x_i}(\overline{x},\overline{y}) = \frac{\partial Z}{\partial y_i}(\overline{x},\overline{y}) = 0.$$

Let us choose geodesic normal coordinates $(x_1,\dots,x_n)$ at $\overline{x}$ in Σ satisfying that

$$h_{ij} = \lambda_i \delta_{ij}$$

with $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$ and geodesic normal coordinates $(y_1,\dots,y_n)$ at $\overline{y}$ in Σ. Therefore the first order derivatives of the function $Z(\overline{x},\overline{y})$ are given by

$$\begin{aligned}(9)\qquad 0 = \frac{\partial Z}{\partial x_i}(\overline{x},\overline{y}) = \frac{\partial \Psi(\overline{x})}{\partial x_i}&(1 - \langle F(\overline{x}), F(\overline{y})\rangle - \Psi(\overline{x}) \left\langle \frac{\partial F(\overline{x})}{\partial x_i}, F(\overline{y}) \right\rangle \\ &+ \sum_{k=1}^{n} {h_i}^k(\overline{x}) \left\langle \frac{\partial F(\overline{x})}{\partial x_k}, F(\overline{y}) \right\rangle,\end{aligned}$$

and

$$(10)\qquad 0 = \frac{\partial Z}{\partial y_i}(\overline{x},\overline{y}) = -\Psi(\overline{x}) \left\langle F(\overline{x}), \frac{\partial F(\overline{y})}{\partial x_i} \right\rangle + \left\langle \nu(\overline{x}), \frac{\partial F(\overline{y})}{\partial x_i} \right\rangle.$$

In this section, using these relations in geodesic normal coordinates as above, we are able to compute the second order derivatives of the function Z.

Proposition 4.1. *At the point $(\overline{x},\overline{y}) \in \Sigma \times \Sigma$ satisfying that $Z(\overline{x},\overline{y}) = 0$ and $\overline{x} \neq \overline{y}$, we have*

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2}(\overline{x},\overline{y}) = &\left(\Delta_\Sigma \Psi(\overline{x}) - 2\sum_{i=1}^{n} \frac{\left|\frac{\partial \Psi(\overline{x})}{\partial x_i}\right|^2}{\Psi(\overline{x}) - \lambda_i(\overline{x})} \right. \\ &\left. + \left(|A(\overline{x})|^2 - n\right)\Psi(\overline{x}) \right) (1 - \langle F(\overline{x}), F(\overline{y})\rangle) \\ &+ n\Psi(\overline{x}) + nH\Psi(\overline{x}) \langle \nu(\overline{x}), F(\overline{y})\rangle - nH \langle F(\overline{x}), F(\overline{y})\rangle.\end{aligned}$$

Proof. Differentiating the equation (9) in the direction $\frac{\partial}{\partial x_i}$ gives

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2 Z}{\partial {x_i}^2}(\overline{x},\overline{y}) &= \Delta_\Sigma\Psi(\overline{x})\left(1-\langle F(\overline{x}),F(\overline{y})\rangle\right) - 2\sum_{i=1}^{n}\frac{\partial\Psi(\overline{x})}{\partial x_i}\left\langle\frac{\partial F(\overline{x})}{\partial x_i},F(\overline{y})\right\rangle\\ &\quad -\Psi(\overline{x})\left\langle\Delta_\Sigma F(\overline{x}),F(\overline{y})\right\rangle + \sum_{k=1}^{n}\sum_{i=1}^{n}\frac{\partial {h_i}^k(\overline{x})}{\partial x_i}\left\langle\frac{\partial F(\overline{x})}{\partial x_k},F(\overline{y})\right\rangle\\ &\quad +\sum_{i,k=1}^{n}{h_i}^k(\overline{x})\left\langle -h_{ik}(\overline{x})\nu(\overline{x})-\delta_{ik}F(\overline{x}),F(\overline{y})\right\rangle.\end{aligned}$$

Since $F:\Sigma\to\mathbb{S}^{n+1}\subset\mathbb{R}^{n+2}$ is a constant mean curvature hypersurface,

$$\Delta_\Sigma F(x)+nF(x)=-nH\nu(x).$$

By using the Codazzi equations,

$$\sum_{i=1}^{n}\frac{\partial {h_i}^k(\overline{x})}{\partial x_i}=\sum_{i=1}^{n}\nabla^\Sigma_{\frac{\partial}{\partial x_i}}{h_i}^k(\overline{x})=\sum_{i=1}^{n}h_{iki}(\overline{x})=\sum_{i=1}^{n}h_{iik}(\overline{x})=0$$

at $\overline{x}$. Thus

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2 Z}{\partial {x_i}^2}(\overline{x},\overline{y}) &= \Delta_\Sigma\Psi(\overline{x})\left(1-\langle F(\overline{x}),F(\overline{y})\rangle\right) - 2\sum_{i=1}^{n}\frac{\partial\Psi(\overline{x})}{\partial x_i}\left\langle\frac{\partial F(\overline{x})}{\partial x_i},F(\overline{y})\right\rangle\\ &\quad + n\Psi(\overline{x})\left\langle F(\overline{x}),F(\overline{y})\right\rangle + n\Psi(\overline{x})H\left\langle\nu(\overline{x}),F(\overline{y})\right\rangle\\ &\quad -|A(\overline{x})|^2\left\langle\nu(\overline{x}),F(\overline{y})\right\rangle - nH\left\langle F(\overline{x}),F(\overline{y})\right\rangle.\end{aligned}$$

Rearranging the above formula by using the equation (1) with $Z(\overline{x},\overline{y})=0$ yields

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2 Z}{\partial {x_i}^2}(\overline{x},\overline{y}) &= \left(\Delta_\Sigma\Psi(\overline{x})+\left(|A(\overline{x})|^2-n\right)\Psi(\overline{x})\right)\left(1-\langle F(\overline{x}),F(\overline{y})\rangle\right)\\ &\quad + n\Psi(\overline{x}) - 2\sum_{i=1}^{n}\frac{\partial\Psi(\overline{x})}{\partial x_i}\left\langle\frac{\partial F(\overline{x})}{\partial x_i},F(\overline{y})\right\rangle\\ &\quad + nH\Psi(\overline{x})\left\langle\nu(\overline{x}),F(\overline{y})\right\rangle - nH\left\langle F(\overline{x}),F(\overline{y})\right\rangle.\end{aligned}\tag{11}$$

Using the formula (9) with $\frac{\partial Z}{\partial x_i}(\overline{x},\overline{y})=0$, we have

$$\left\langle\frac{\partial F(\overline{x})}{\partial x_i},F(\overline{y})\right\rangle=\frac{\left(1-\langle F(\overline{x}),F(\overline{y})\rangle\right)}{\Psi(\overline{x})-\lambda_i(\overline{x})}\,\frac{\partial\Psi(\overline{x})}{\partial x_i},\tag{12}$$

Putting the equation (12) in the equation (11), we get the conclusion. □

Let $w_i(\overline{x},\overline{y})$ be the reflection of the vector $\frac{\partial F(\overline{x})}{\partial x_i}$ in $\mathbb{R}^{n+2}$ with respect to the hyperplane orthogonal to $F(\overline{x})-F(\overline{y})$ and passing through the origin. The vector $w_i(\overline{x},\overline{y})$ is given by

$$w_i(\overline{x},\overline{y}) = \frac{\partial F(\overline{x})}{\partial x_i} - 2\left\langle \frac{\partial F(\overline{x})}{\partial x_i}, \frac{F(\overline{x})-F(\overline{y})}{|F(\overline{x})-F(\overline{y})|}\right\rangle \frac{F(\overline{x})-F(\overline{y})}{|F(\overline{x})-F(\overline{y})|}.$$

We remark that $\{w_1(\overline{x},\overline{y}),\ldots,w_n(\overline{x},\overline{y})\}$ is the set of mutually orthogonal unit tangent vectors in $T_{F(\overline{y})}\mathbb{S}^{n+1}$. On the other hand, the following three properties hold at $(\overline{x},\overline{y})$ for $1\le i\le n$.

- $\left\langle \frac{\partial F(\overline{y})}{\partial y_i}, \Psi(\overline{x})F(\overline{x})-\nu(\overline{x})\right\rangle = -\frac{\partial Z}{\partial y_i}(\overline{x},\overline{y}) = 0,$
- $\langle w_i(\overline{x},\overline{y}), \Psi(\overline{x})F(\overline{x})-\nu(\overline{x})\rangle = \frac{\left\langle \frac{\partial F(\overline{x})}{\partial x_i}, F(\overline{y})\right\rangle}{1-\langle F(\overline{x}),F(\overline{y})\rangle} Z(\overline{x},\overline{y}) = 0,$
- $|F(\overline{y})|^2\,|\Psi(\overline{x})F(\overline{x})-\nu(\overline{x})|^2 - \langle F(\overline{y}), \Psi(\overline{x})F(\overline{x})-\nu(\overline{x})\rangle^2$
  $= (1+\Psi(\overline{x})^2) - \Psi(\overline{x})^2 = 1 \neq 0.$

Thus one sees that $\mathrm{Span}\left(\frac{\partial F(\overline{y})}{\partial y_1},\ldots,\frac{\partial F(\overline{y})}{\partial y_n}\right) = \mathrm{Span}\left(w_1(\overline{x},\overline{y}),\ldots,w_n(\overline{x},\overline{y})\right)$. Moreover, if we choose the coordinates at $\overline{y}$ satisfying that for $1\le i\ne j\le n$

$$\left\langle w_i(\overline{x},\overline{y}), \frac{\partial F(\overline{y})}{\partial y_i}\right\rangle \ge 0 \quad \text{and} \quad \left\langle w_i(\overline{x},\overline{y}), \frac{\partial F(\overline{y})}{\partial y_j}\right\rangle = 0,$$

then the above three properties implies that

$$w_i(\overline{x},\overline{y}) = \frac{\partial F(\overline{y})}{\partial y_i}. \tag{13}$$

Equipped with the local coordinates chosen as above, we are able to get the following second order derivatives at the global minimum points of the two-point function Z.

Proposition 4.2. *At the point $(\overline{x},\overline{y})\in\Sigma\times\Sigma$ satisfying that $Z(\overline{x},\overline{y})=0$ and $\overline{x}\ne\overline{y}$, we have*

$$\frac{\partial^2 Z}{\partial x_i\partial y_i}(\overline{x},\overline{y}) = \lambda_i(\overline{x}) - \Psi(\overline{x}).$$

Proof. Differentiating the equation (9) in the direction $\frac{\partial}{\partial y_i}$ gives

$$\begin{aligned}\frac{\partial^2 Z}{\partial x_i \partial y_i}(\overline{x},\overline{y}) &= -\frac{\partial \Psi(\overline{x})}{\partial x_i}\left\langle F(\overline{x}), \frac{\partial F(\overline{y})}{\partial y_i}\right\rangle - \Psi(\overline{x})\left\langle \frac{\partial F(\overline{x})}{\partial x_i}, \frac{\partial F(y)}{\partial y_i}\right\rangle \\ &\quad + \sum_{k=1}^{n} {h_i}^k(\overline{x})\left\langle \frac{\partial F(\overline{x})}{\partial x_k}, \frac{\partial F(\overline{y})}{\partial y_i}\right\rangle \\ &= \frac{1}{1-\langle F(\overline{x}), F(\overline{y})\rangle} \\ &\quad \times \left((\lambda_i(\overline{x}) - \Psi(\overline{x}))\left\langle \frac{\partial F(\overline{x})}{\partial x_i}, F(\overline{y})\right\rangle\right)\left\langle F(\overline{x}), \frac{\partial F(\overline{y})}{\partial y_i}\right\rangle \\ &\quad + (\lambda_i(\overline{x}) - \Psi(\overline{x}))\left\langle \frac{\partial F(\overline{x})}{\partial x_i}, \frac{\partial F(\overline{y})}{\partial y_i}\right\rangle .\end{aligned}$$

Here the second equality follows from the equation (9) with $\frac{\partial Z}{\partial x_i}(\overline{x},\overline{y}) = 0$. Moreover it can be expressed in terms of $w_i(\overline{x},\overline{y})$ as follows:

$$\begin{aligned}\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) &= -2\left(\lambda_i(\overline{x}) - \Psi(\overline{x})\right)\left\langle \frac{\partial F(\overline{x})}{\partial x_i}, \frac{F(\overline{x})-F(\overline{y})}{|F(\overline{x})-F(\overline{y})|}\right\rangle \\ &\quad \times \left\langle \frac{F(\overline{x})-F(\overline{y})}{|F(\overline{x})-F(\overline{y})|}, \frac{\partial F(\overline{y})}{\partial y_i}\right\rangle \\ &\quad + \left(\lambda_i(\overline{x}) - \Psi(\overline{x})\right)\left\langle \frac{\partial F(\overline{x})}{\partial x_i}, \frac{\partial F(\overline{y})}{\partial y_i}\right\rangle \\ &= \left(\lambda_i(\overline{x}) - \Psi(\overline{x})\right)\left\langle w_i(\overline{x},\overline{y}), \frac{\partial F(\overline{y})}{\partial y_i}\right\rangle .\end{aligned}$$

Plugging the equation (13) into the above identity gives the conclusion. □

Proposition 4.3. *At the point* $(\overline{x},\overline{y}) \in \Sigma\times\Sigma$ *satisfying that* $Z(\overline{x},\overline{y}) = 0$ *and* $\overline{x}\neq\overline{y}$*, we have*

$$\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2}(\overline{x},\overline{y}) = n\Psi(\overline{x}) + nH\Psi(\overline{x})\left\langle F(\overline{x}), \nu(\overline{y})\right\rangle - nH\left\langle \nu(\overline{x}), \nu(\overline{y})\right\rangle .$$

Proof. Differentiating the equation (10) in the direction $\frac{\partial}{\partial y_i}$ gives

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2}(\overline{x},\overline{y}) &= -\Psi(\overline{x})\left\langle F(\overline{x}), \Delta_\Sigma F(\overline{y})\right\rangle + \left\langle \nu(\overline{x}), \Delta_\Sigma F(\overline{y})\right\rangle \\ &= n\Psi(\overline{x})\left\langle F(\overline{x}), F(\overline{y})\right\rangle + nH\Psi(\overline{x})\left\langle F(\overline{x}), \nu(\overline{y})\right\rangle \\ &\quad - n\left\langle \nu(\overline{x}), F(\overline{y})\right\rangle - nH\left\langle \nu(\overline{x}), \nu(\overline{y})\right\rangle \\ &= n\Psi(\overline{x}) + nH\Psi(\overline{x})\left\langle F(\overline{x}), \nu(\overline{y})\right\rangle - nH\left\langle \nu(\overline{x}), \nu(\overline{y})\right\rangle .\end{aligned}$$

□

Proposition 4.4. *For* $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ *satisfying that* $Z(\overline{x}, \overline{y}) = 0$ *and* $\overline{x} \neq \overline{y}$,

$$\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2}$$
$$= (1 - \langle F(\overline{x}), F(\overline{y})\rangle)$$
$$\times \left(\Delta_\Sigma \Psi(\overline{x}) - 2\sum_{i=1}^{n} \frac{|\frac{\partial \Psi(\overline{x})}{\partial x_i}|^2}{\Psi(\overline{x}) - \lambda_i(\overline{x})} + \left(|A(\overline{x})|^2 - n\right)\Psi(\overline{x}) - nH\Psi(\overline{x})^2 + nH \right).$$

Proof. Applying Proposition 4.1, Proposition 4.2, and Proposition 4.3, we have

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2} \\ &= (1 - \langle F(\overline{x}), F(\overline{y})\rangle) \\ &\quad \times \left(\Delta_\Sigma \Psi(\overline{x}) - 2\sum_{i=1}^{n} \frac{|\frac{\partial \Psi(\overline{x})}{\partial x_i}|^2}{\Psi(\overline{x}) - \lambda_i(\overline{x})} + \left(|A(\overline{x})|^2 - n\right)\Psi(\overline{x}) \right) \\ &\quad + 2nH + nH\Psi(\overline{x})\langle \nu(\overline{x}), F(\overline{y})\rangle - nH\langle F(\overline{x}), F(\overline{y})\rangle \\ &\quad + nH\Psi(\overline{x})\langle F(\overline{x}), \nu(\overline{y})\rangle - nH\langle \nu(\overline{x}), \nu(\overline{y})\rangle . \end{aligned} \tag{14}$$

In order to get the conclusion, we need the following computations:

- $\langle \nu(\overline{x}), F(\overline{y})\rangle = -\Psi(\overline{x})(1 - \langle F(\overline{x}), F(\overline{y})\rangle)$,

Since $B_T(\overline{x}, \frac{1}{\Psi(\overline{x})})$ touches Σ at $F(\overline{x})$ and $F(\overline{y})$ simultaneously, the center $p(\overline{x})$ of the geodesic ball $B_T(\overline{x}, \frac{1}{\Psi(\overline{x})})$ is given by

$$p(\overline{x}) = F(\overline{x}) - \frac{1}{\Psi(\overline{x})}\nu(\overline{x}) = F(\overline{y}) - \frac{1}{\Psi(\overline{x})}\nu(\overline{y}),$$

which gives $\nu(\overline{y}) = \nu(\overline{x}) + \Psi(\overline{x})(F(\overline{y}) - F(\overline{x}))$. Thus

- $\begin{aligned}\langle F(\overline{x}), \nu(\overline{y})\rangle &= \langle F(\overline{x}), \nu(\overline{x}) + \Psi(\overline{x})(F(\overline{y}) - F(\overline{x}))\rangle \\ &= -\Psi(\overline{x})(1 - \langle F(\overline{x}), F(\overline{y})\rangle).\end{aligned}$

Moreover

- $\begin{aligned}\langle \nu(\overline{x}), \nu(\overline{y})\rangle &= \langle \nu(\overline{x}), \nu(\overline{x}) + \Psi(\overline{x})(F(\overline{y}) - F(\overline{x}))\rangle \\ &= 1 + \Psi(\overline{x})\langle \nu(\overline{x}), F(\overline{y})\rangle = 1 - \Psi(\overline{x})^2(1 - \langle F(\overline{x}), F(\overline{y})\rangle),\end{aligned}$
- $\langle F(\overline{x}), F(\overline{y})\rangle = 1 - (1 - \langle F(\overline{x}), F(\overline{y})\rangle)$.

Combining these computations with the equation (14), we get the conclusion. □

Since $\Phi(x) \leq k(x) \leq \Psi(x)$, one sees that for $1 \leq i \leq n$

$$\begin{aligned}\Psi(x) - \lambda_i &= \Psi(x) - \left(nH - \sum_{j \neq i} \lambda_j \right) \\ &= \Psi(x) + \sum_{j \neq i} \lambda_j - nH \leq n(\Psi(x) - H).\end{aligned}$$

We remark that $\Psi(x) - \lambda_j < n(\Psi(x) - H)$ for some $1 \leq j \leq n$ because $F(\Sigma)$ has two distinct principal curvatures. As a consequence of Proposition 4.4, we have the following:

Corollary 4.5. *For $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ satisfying that $Z(\overline{x}, \overline{y}) = 0$ and $\overline{x} \neq \overline{y}$,*

$$\begin{aligned}&\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2 \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2} \\ &\leq (1 - \langle F(\overline{x}), F(\overline{y}) \rangle) \\ &\quad \times \left(\Delta_\Sigma \Psi(\overline{x}) - \frac{|\nabla^\Sigma \Psi(\overline{x})|^2}{\Psi(\overline{x}) - H} + \left(|A(\overline{x})|^2 - n \right) \Psi(\overline{x}) - nH\Psi(\overline{x})^2 + nH \right).\end{aligned}$$

Moreover, if $\kappa > 1$, equality holds only when $\nabla^\Sigma \Psi(\overline{x}) = 0$.

5. Proof of Main Theorem

We begin with showing the existence of a global minimum point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ of the function Z which is not contained in the diagonal $D = \{(x, x) : x \in \Sigma\} \subset \Sigma \times \Sigma$ when $\kappa > 1$.

Lemma 5.1. *If $\kappa > 1$, then there exists a point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma \setminus D$ such that $Z(\overline{x}, \overline{y}) = 0$, where D is the diagonal.*

Proof. Since Σ is compact, κ is attained at some point $\overline{x} \in \Sigma$. Thus

$$\Psi(\overline{x}) = \kappa\varphi(\overline{x}) + H = k(\overline{x}). \tag{15}$$

By the definition of the interior ball curvature $k(\overline{x})$ at $\overline{x} \in \Sigma$, there exists a point $\overline{y} \in \Sigma$ satisfying that $\overline{y} \in B_T(\overline{x}, \frac{1}{k(\overline{x})}) \cap \Sigma \setminus \{\overline{x}\}$. This is equivalent to that there exists a point $\overline{y} \in \Sigma \setminus \{\overline{x}\}$ such that $Z(\overline{x}, \overline{y}) = 0$, which follows from the definition of the function Z and the equation (15). □

By definition of the interior ball curvature $k(x)$, it holds $\Phi(x) \leq k(x)$ for every $x \in \Sigma$, in general. However the following proposition shows that if Σ has a constant mean curvature $H > 0$ and two distinct principal curvatures of multiplicity $n-1$ and 1, then $k(x) = \Phi(x)$ for every $x \in \Sigma$.

Proposition 5.2. *Let Σ be an $n(\geq 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature H with two distinct principal curvatures, one of them being simple. If $H > 0$. Then the interior ball curvature $k(x)$ is the same as the maximum principal curvature $\Phi(x)$ for all $x \in \Sigma$.*

Proof. Suppose that $\kappa > 1$. By Lemma 5.1, there exists a point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma$ with $\overline{x} \neq \overline{y}$ satisfying that $Z(\overline{x}, \overline{y}) = 0$. Using Corollary 3.6 and Corollary 4.5 together with $\Psi(\overline{x}) = \kappa\varphi(\overline{x}) + H$, we get

$$\begin{aligned}
(16)\qquad &\frac{1}{(1 - \langle F(\overline{x}), F(\overline{y})\rangle)} \left(\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2} \right) \\
&\leq \kappa \Delta_\Sigma \varphi(\overline{x}) - \frac{2\kappa}{n} \frac{|\nabla^\Sigma \varphi(\overline{x})|^2}{\varphi(\overline{x})} + \left(|A(\overline{x})|^2 - n\right)(\kappa\varphi(\overline{x}) + H) \\
&\quad - nH(\kappa\varphi(\overline{x}) + H)^2 + nH \\
&= H|A(\overline{x})|^2 - \kappa^2 nH\varphi(\overline{x})^2 - nH^3 - \operatorname{sgn}(\lambda - \mu)\kappa n f(n) H\varphi(\overline{x})^2,
\end{aligned}$$

where $f(n) = \frac{n-2}{n-1}$ if $\Phi = \mu$, and $f(n) = n-2$ if $\Phi = \lambda$. Using the relation

$$|\mathring{A}|^2 = |A|^2 - nH^2 = ng(n)\varphi^2,$$

we get

$$\begin{aligned}
&\frac{1}{(1 - \langle F(\overline{x}), F(\overline{y})\rangle)} \left(\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2} \right) \\
&\leq -nH\varphi(\overline{x})^2(\kappa^2 + \operatorname{sgn}(\lambda - \mu) f(n)\kappa - g(n)) \\
&< -nH\varphi(\overline{x})^2(1 + \operatorname{sgn}(\lambda - \mu) f(n) - g(n)) \\
&\leq 0
\end{aligned}$$

where we used the identity $1 + \operatorname{sgn}(\lambda - \mu) f(n) - g(n) = 0$.

However, since the point $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma \setminus D$ is a global minimum point of the function Z, we see

$$0 \leq \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2},$$

which is a contradiction. From Proposition 2.2, it follows that

$$k(x) = \Phi(x) = \Psi(x)$$

for all $x \in \Sigma$. □

We are now ready to prove our main theorem.

Theorem 5.3. *Let Σ be an $n(\geq 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature $H \geq 0$ and with two distinct principal curvatures λ and μ, μ being simple. If $\mu > \lambda$, then Σ is congruent to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where $\lambda = \frac{nH-\sqrt{n^2H^2+4(n-1)}}{2(n-1)}$.*

Proof. If $H = 0$, then Σ is congruent to a Clifford minimal hypersurfaces from Theorem 1.1 by Otsuki. Thus it suffices to consider the case of $H > 0$. Since $\mu > \lambda$, we have $\Phi = \mu$. From Proposition 5.2, we have

$$\Phi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geq 0,$$

for all $x, y \in \Sigma$.

Fix $x \in \Sigma$ and choose an orthonormal frame $\{e_1, \ldots, e_n\}$ in a neighborhood of x such that $h(e_n, e_n) = \Phi$. Let $\gamma(t)$ be a geodesic on Σ such that $\gamma(0) = F(x)$ and $\gamma'(0) = e_n$. For simplicity, let us identify the hypersurface Σ with its image under the embedding F, so that $F(x) = x$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) := Z(F(x), \gamma(t)) = \Phi(x)(1 - \langle F(x), \gamma(t)\rangle) + \langle \nu(x), \gamma(t)\rangle .$$

Then, by definition, $f(t) \geq 0$ and $f(0) = 0$. A simple computation shows

$$\begin{aligned} f'(t) &= -\left\langle \Phi(x)F(x) - \nu(x), \gamma'(t)\right\rangle, \\ f''(t) &= \left\langle \Phi(x)F(x) - \nu(x), \gamma(t) + h(\gamma'(t), \gamma'(t))\nu(\gamma(t))\right\rangle, \\ f'''(t) &= \Big\langle \Phi(x)F(x) - \nu(x), \gamma'(t) + (\nabla^{\Sigma}_{\gamma'(t)}h)(\gamma'(t), \gamma'(t))\nu(\gamma(t)) \\ &\qquad + h(\gamma'(t), \gamma'(t))\nabla_{\gamma'(t)}\nu(\gamma(t))\Big\rangle, \end{aligned}$$

where ∇ is the covariant derivative of $\mathbb{R}^{n+2}$. In particular, it follows that

$$f(0) = f'(0) = 0,$$

$$f''(0) = \langle \Phi(x)F(x) - \nu(x), F(x) + \Phi(x)\nu(x)\rangle = 0.$$

Moreover the fact that $f \geq 0$ implies that $f'''(0) = 0$. Hence

$$0 = f'''(0) = \langle \Phi(x)F(x) - \nu(x), e_n + h_{nnn}(x)\nu(x)\rangle = -h_{nnn}(x),$$

since $\nabla_{\gamma'(t)}\nu(\gamma(t))$ is tangent to Σ. Therefore we get $e_n\lambda = h_{11n} = -\frac{1}{n-1}h_{nnn} = 0$. Combining this with Lemma 3.2, one sees that λ and μ are constant on Σ, which implies that Σ is an isoparametric hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. From the classification of isoparametric hypersurfaces with two principal curvatures due to Cartan [13], it follows that Σ is congruent to the Riemannian product $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{1}{\sqrt{1+\mu^2}}\right)$, where λ and μ satisfy $nH = (n-1)\lambda + \mu$.

We now claim that $\lambda\mu + 1 = 0$ on Σ. To see this, let $\{e_1, \ldots, e_n, e_{n+1}\}$ be a local orthonormal frame around $p \in \Sigma$ such that e_{n+1} is normal to Σ and $h_{ij} = \lambda_i\delta_{ij}$ at p. Let $\{\omega_1, \ldots, \omega_n, \omega_{n+1}\}$ be a dual coframe. We use the following convention on the ranges of indices:

$$1 \leq A, B, C, \ldots \leq n+1 \quad \text{and} \quad 1 \leq i, j, k, \ldots \leq n.$$

Then the structure equations of a unit sphere $\mathbb{S}^{n+1}$ are given by

$$\begin{aligned} d\omega_A &= -\sum_{B=1}^{n+1} \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= -\sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \\ \Omega_{AB} &= \frac{1}{2}\sum_{C,D=1}^{n+1} K_{ABCD}\, \omega_C \wedge \omega_D, \\ K_{ABCD} &= \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}. \end{aligned} \tag{17}$$

We restrict these forms to Σ. Then we have $\omega_{n+1} = 0$ on Σ. Moreover

$$0 = d\omega_{n+1} = -\sum_{i=1}^{n} \omega_{n+1,i} \wedge \omega_i \quad \text{and} \quad \omega_{n+1,i} = \sum_{j=1}^{n} h_{ij}\omega_j = \sum_{i=1}^{n} \lambda_i\omega_i.$$

Recall that h_{ijk} is defined by

$$\sum_{k=1}^{n} h_{ijk}\omega_k = dh_{ij} - \sum_{k=1}^{n} h_{ik}\omega_{kj} - \sum_{k=1}^{n} h_{kj}\omega_{ki}.$$

Let $\theta_{ij} := (\lambda_i - \lambda_j)\omega_{ij} = \theta_{ji}$. Then we have

$$\sum_{k=1}^{n} h_{ijk}\omega_k = \delta_{ij} d\lambda_j - (\lambda_i - \lambda_j)\omega_{ij} = \delta_{ij} d\lambda_j - \theta_{ij}.$$

Since each λ_i is constant on Σ, Lemma 3.2 shows that

$$\theta_{in} = \delta_{in} d\lambda_n - \sum_{k=1}^{n} h_{ink}\omega_k = -\sum_{k=1}^{n-1} h_{ink}\omega_k - h_{inn}\omega_n = 0$$

for $1 \le i \le n-1$. This implies that $\omega_{in} = \frac{\theta_{in}}{\lambda - \mu} = 0$. Therefore, for $1 \le i \le n-1$, using the equation (17) gives

$$0 = d\omega_{in} = -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kn} - \omega_{i,n+1} \wedge \omega_{n+1,n} + \omega_i \wedge \omega_n = (\lambda\mu + 1)\, \omega_i \wedge \omega_n,$$

which shows that $\lambda\mu + 1 = 0$ on Σ. Therefore Σ is congruent to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^1\left(\frac{|\lambda|}{\sqrt{1+\lambda^2}}\right)$, where $\lambda = \frac{nH - \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)}$ since $\mu > \lambda$. □

6. Appendix: The case of $H = 0$

Our proof of Theorem 5.3 still works for the case of $H = 0$. Although Otsuki gave a classification theorem for embedded minimal hypersurfaces with two distinct principal curvatures in Theorem 1.1, we here give another proof of Theorem 1.1. If $H = 0$, then $\mu = -(n-1)\lambda$. Therefore, by choosing a suitable orientation, we have $\mu(x) > \lambda(x)$ for every $x \in \Sigma$. The proof is divided into two cases: $\kappa = 1$ and $\kappa > 1$. The proof in case of $\kappa = 1$ is similar to that of Theorem 5.3 with $\Phi = \mu$. For this reason, it suffices to consider the case of $\kappa > 1$. The proof uses basically Brendle's argument [10].

Proposition 6.1. *Let* $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma \setminus D$ *such that* $Z(\overline{x}, \overline{y}) = 0$. *Then* $\nabla^{\Sigma}\Phi(\overline{x}) = 0$.

Proof. Since the function Z attains its global minimum at $(\overline{x}, \overline{y})$, it follows from the inequality (16) and $H = 0$ that

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2} \\ &= H|A(\overline{x})|^2 - \kappa^2 nH\varphi(\overline{x})^2 - nH^3 - \mathrm{sgn}(\lambda - \mu)\kappa n f(n) H\varphi(\overline{x})^2 \\ &= 0. \end{aligned}$$

Therefore equality holds in Corollary 4.5, which implies that $\nabla^{\Sigma}\Psi(\overline{x}) = 0$. Hence we obtain that $\nabla^{\Sigma}\Phi(\overline{x}) = 0$. □

For a point $(\overline{x}, \overline{y})$ such that $Z(\overline{x}, \overline{y}) = 0$, we choose open neighborhoods U_1 and U_2 of $\overline{x}$ and $\overline{y}$, respectively, such that $\overline{U_1 \times U_2} \cap D = \emptyset$. Then there exist a constant $\Lambda_1 > 0$ depending on $\overline{x}$ and $\overline{y}$ such that

$$\sup_{U_1 \times U_2} \{|\nabla^{\Sigma}\Psi|, |\nabla^{\Sigma}F|, |A|^2\} < \Lambda_1, \tag{18}$$

and

$$\inf_{U_1 \times U_2} \{\Psi - \lambda, \Psi - \mu, 1 - \langle F(x), F(y)\rangle\} > \frac{1}{\Lambda_1}. \tag{19}$$

For a sufficiently small $\varepsilon > 0$, there exist open neighborhoods $N_1 \subset U_1$ and $N_2 \subset U_2$ of $\overline{x}$ and $\overline{y}$ such that

$$|Z(x, y)| < \varepsilon \text{ and } |dZ(x, y)| < \varepsilon$$

for $(x, y) \in N_1 \times N_2$. Obviously, the neighborhood $N_1 \times N_2$ is disjoint from D. In order to compute the second order derivatives for an arbitrary point $(x, y) \in N_1 \times N_2$, let us choose geodesic normal coordinates $(x_1, \ldots, x_n)$ at x satisfying that $h_{ij} = \lambda_i \delta_{ij}$ at x.

We recall that the vector $w_i(x, y)$ is defined by the reflection of $\frac{\partial F(x)}{\partial x_i}$ in $\mathbb{R}^{n+2}$ with respect to the hyperplane orthogonal to $F(x) - F(y)$ and passing through the origin. Thus the vector $w_i(x, y)$ is given by

$$w_i(x, y) = \frac{\partial F(x)}{\partial x_i} - 2\left\langle \frac{\partial F(x)}{\partial x_i}, \frac{F(x) - F(y)}{|F(x) - F(y)|} \right\rangle \frac{F(x) - F(y)}{|F(x) - F(y)|}.$$

For any point $(x, y) \in N_1 \times N_2$, we obtain the following estimates:

$$\left|\left\langle \frac{\partial F}{\partial y_i}(y), \Psi(x)F(x) - \nu(x) \right\rangle\right| = \left|-\frac{\partial Z}{\partial y_i}(x, y)\right| < \varepsilon, \tag{20}$$

$$|\langle w_i(x,y), \Psi(x)F(x) - \nu(x)\rangle| = \left| \frac{\left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle}{1 - \langle F(x), F(y)\rangle} \right| Z(x,y) < \Lambda_1^2 \varepsilon, \tag{21}$$

and

$$\begin{aligned} &|F(y)|^2 \, |\Psi(x)F(x) - \nu(x)|^2 - \langle F(y), \Psi(x)F(x) - \nu(x)\rangle^2 \\ &= (1 + \Psi(x)^2) - (\Psi(x) - Z(x,y))^2 \\ &= 1 + 2\Psi(x)Z(x,y) - Z(x,y)^2 \\ &> 1 - \varepsilon^2 \neq 0. \end{aligned} \tag{22}$$

Let $V(x,y)$ be the orthogonal projection of $\Psi(x)F(x) - \nu(x)$ onto $T_{F(y)}\mathbb{S}^{n+1}$, which is spanned by $\{\frac{\partial F}{\partial y_i}(y)\}_{i=1}^n$ and $\nu(y)$. For a suitably chosen small ε, we can conclude that

- $F(y)$ and $\Psi(x)F(x) - \nu(x)$ are linearly independent,
- $|V(x,y)|^2 = 1 + 2\Psi(x)Z(x,y) - Z(x,y)^2$

from the formula (22). The inequality (20) implies that

- The set $\{\frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial y_n}, V\}$ is a basis of $T_{F(y)}\mathbb{S}^{n+1}$. Moreover, we can make the angle between V and $\nu(y)$ arbitrarily close to 0 in $N_1 \times N_2$ for a sufficiently small $\varepsilon > 0$.

Note that the vectors $w_i(x,y)$ and $w_j(x,y)$ for $1 \leq i, j \leq n$ and $i \neq j$ are mutually orthogonal unit vectors in $T_{F(y)}\mathbb{S}^{n+1}$. Finally the inequality (21) implies that

- The set $\{w_1, \ldots, w_n, V\}$ is also a basis of $T_{F(y)}\mathbb{S}^{n+1}$. Moreover, we can make the angle between w_i $(1 \leq i \leq n)$ and V arbitrarily close to $\frac{\pi}{2}$ in $N_1 \times N_2$ for a sufficiently small $\varepsilon > 0$.

Therefore we choose geodesic normal coordinates at y satisfying that for $1 \leq i < j \leq n$

- the angle between $\frac{\partial F}{\partial y_i}(y)$ and $w_i(x,y)$ is sufficiently small, which depends on $\varepsilon > 0$,
- $\left\langle w_i(x,y), \frac{\partial F}{\partial y_i}(y) \right\rangle \geq 0$ and $\left\langle w_i(x,y), \frac{\partial F}{\partial y_j}(y) \right\rangle = 0$.

Moreover, the magnitude of the difference vector between $w_i(x,y)$ and $\frac{\partial F}{\partial y_i}(y)$ can be controlled by $Z(x,y)$ and $|dZ(x,y)|$ as follows:

Lemma 6.2. *Let $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma \setminus D$ such that $Z(\overline{x}, \overline{y}) = 0$. Then there exist open neighborhoods N_1 of $\overline{x}$ and N_2 of $\overline{y}$ in Σ satisfying that $(N_1 \times N_2) \cap D = \emptyset$ and there exists a constant $\Lambda_2 > 0$ depending only on $\overline{x}$, $\overline{y}$ such that*

$$\left| w_i(x,y) - \frac{\partial F}{\partial y_i}(y) \right|^2 \leq \Lambda_2 \Big(Z(x,y) + |dZ(x,y)| \Big)$$

for any point $(x,y) \in N_1 \times N_2$.

Proof. Let us choose the open neighborhoods N_1 and N_2 of $\overline{x}$ and $\overline{y}$ as above. Furthermore, for an arbitrary point $(x,y) \in N_1 \times N_2$, we choose the geodesic normal coordinates at x and y as above. If $w_i(x,y) = \frac{\partial F}{\partial y_i}(y)$, then it is trivial. Thus we may assume that $w_i(x,y) \neq \frac{\partial F}{\partial y_i}(y)$. Since we can make the angle between $\frac{\partial F}{\partial y_i}(y)$ and $w_i(x,y)$ sufficiently small, we may assume that the angle between the vectors $w_i(x,y) - \frac{\partial F}{\partial y_i}(y)$ and V is less than $\frac{\pi}{4}$. It is easy to see that

$$\left| w_i(x,y) - \frac{\partial F}{\partial y_i}(y) \right|^2 < 2 \left| \left\langle w_i(x,y) - \frac{\partial F}{\partial y_i}(y), \frac{V}{|V|} \right\rangle \right|^2.$$

Therefore

$$\begin{aligned}
\left| w_i(x,y) - \frac{\partial F}{\partial y_i}(y) \right|^2 &< \frac{2}{|V|^2} \left| \left\langle w_i(x,y) - \frac{\partial F}{\partial y_i}(y), V \right\rangle \right|^2 \\
&= \frac{2}{|V|^2} \left| \left\langle w_i(x,y) - \frac{\partial F}{\partial y_i}(y), \Psi(x)F(x) - \nu(x) \right\rangle \right|^2 \\
&= 4 \left| \frac{\left\langle \frac{\partial F}{\partial x_i}(x), F(y) \right\rangle}{1 - \langle F(x), F(y) \rangle} Z(x,y) + \frac{\partial Z}{\partial y_i}(x,y) \right|^2 \\
&\leq 4\Lambda^4 \varepsilon Z(x,y) + 4\varepsilon(2\Lambda_1^2 + 1)\, |dZ(x,y)| \\
&\leq \Lambda_2 \Big(Z(x,y) + |dZ(x,y)| \Big),
\end{aligned}$$

for any point $(x,y) \in N_1 \times N_2$. □

In the proof of Proposition 4.1, Proposition 4.2, and Proposition 4.3, we used the property

$$Z(\overline{x}, \overline{y}) = \frac{\partial Z}{\partial x_i}(\overline{x}, \overline{y}) = \frac{\partial Z}{\partial y_i}(\overline{x}, \overline{y}) = 0$$

at a global minimum point $(\overline{x}, \overline{y})$. However it is no longer valid in general. Using the geodesic normal coordinates we picked in the above, we have the following second order derivatives in general. For any point $(x, y) \in N_1 \times N_2$,

$$\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2}(x,y) = \left(\Delta_\Sigma \Psi(x) - 2\sum_{i=1}^{n} \frac{|\frac{\partial \Psi(x)}{\partial x_i}|^2}{\Psi(x) - \lambda_i(x)} + \big(|A(x)|^2 - n\big)\,\Psi(x) \right)(1 - \langle F(x), F(y)\rangle) + n\Psi(x) + 2\sum_{i=1}^{n} \frac{\frac{\partial \Psi(x)}{\partial x_i}}{\Psi(x) - \lambda_i(x)} \frac{\partial Z}{\partial x_i} - |A(x)|^2 Z(x,y), \tag{23}$$

$$\frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) = \lambda_i(x) - \Psi(x) - \frac{1}{2}\left| w_i(x,y) - \frac{\partial F(y)}{\partial y_i}\right|^2 - \frac{1}{1 - \langle F(x), F(y)\rangle} \left\langle F(x), \frac{\partial F(y)}{\partial y_i} \right\rangle \frac{\partial Z}{\partial x_i}(x,y), \tag{24}$$

$$\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2}(x,y) = n\Psi(x) - nZ(x,y). \tag{25}$$

Lemma 6.3. *Let $(\overline{x}, \overline{y}) \in \Sigma \times \Sigma \setminus D$ such that $Z(\overline{x}, \overline{y}) = 0$. Then there exist open neighborhoods N_1 of $\overline{x}$ and N_2 of $\overline{y}$ in Σ satisfying that $(N_1 \times N_2) \cap D = \emptyset$ and there exists a constant $\Lambda > 0$ depending only on $\overline{x}$, $\overline{y}$ such that*

$$\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2}(x,y) + 2\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i}(x,y) + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2}(x,y) \le \Lambda\big(Z(x,y) + |dZ(x,y)|\big) \tag{26}$$

for all $(x, y) \in N_1 \times N_2$.

Proof. Applying the estimates (18), (19), the equalities (23), (24), (25), and Lemma 6.2, we have

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {x_i}^2} + 2\sum_{i=1}^{n} \frac{\partial^2 Z}{\partial x_i \partial y_i} + \sum_{i=1}^{n} \frac{\partial^2 Z}{\partial {y_i}^2} \\ &\le (1 - \langle F(\overline{x}), F(\overline{y})\rangle) \left(\Delta_\Sigma \Psi(\overline{x}) - \frac{2}{n}\frac{|\nabla^\Sigma \Psi(\overline{x})|^2}{\Psi(\overline{x})} + \big(|A(\overline{x})|^2 - n\big)\,\Psi(\overline{x}) \right) \\ &\quad + 2n\Lambda_1^2|dZ| + \Lambda_1 Z + n(\Lambda_2 Z + \Lambda_2|dZ|) + 2n\Lambda_1^2|dZ| + nZ \\ &\le \Lambda\big(Z + |dZ|\big). \end{aligned}$$

□

We define the set Ω by

$$\Omega := \{\overline{x} \in \Sigma : \text{there exists a point } \overline{y} \in \Sigma \setminus \{\overline{x}\} \text{ such that } Z(\overline{x}, \overline{y}) = 0\}.$$

Lemma 5.1 shows that the set Ω is nonempty. Moreover we can prove the following:

Theorem 6.4. *The set Ω is an open subset of Σ.*

We need the following strict maximum principle for a degenerate second order elliptic partial differential equation.

Theorem 6.5 ([7, 8]). *Let Ω be an open subset of an n-dimensional Riemannian manifold, and let $X_1, \ldots, X_m$ be smooth vector fields on Ω. Assume that $\varphi : \Omega \to \mathbb{R}$ is a nonnegative smooth function satisfying*

$$\sum_{j=1}^{m} (D^2\varphi)(X_j, X_j) \leq -L \inf_{|\xi| \leq 1} (D^2\varphi)(\xi, \xi) + L|d\varphi| + L\varphi,$$

where L is a positive constant. Let $F = \{x \in \Omega : \varphi(x) = 0\}$ be the zero set of the function φ. Moreover, suppose that $\gamma : [0,1] \to \Omega$ is a smooth path such that $\gamma(0) \in F$ and $\gamma'(s) = \sum_{j=1}^{m} f_j(s) X_j(\gamma(s))$ for suitable smooth functions $f_1, \ldots, f_m : [0,1] \to \mathbb{R}$. Then $\gamma(s) \in F$ for all $s \in [0,1]$.

Proof of Theorem 6.4. Take an open subset Ω of $2n$-dimensional manifold $\Sigma \times \Sigma$ as an open neighborhood $N_1 \times N_2$ of $(\overline{x}, \overline{y})$ in Lemma 6.3 with the usual product topology. Let $X_i = \frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i}$ for $i = 1, \ldots, n$. Then Lemma 6.3 shows that the condition of Theorem 6.5 is satisfied. Applying Theorem 6.5 with $L = \Lambda$, we have the conclusion. □

We now complete our proof of Theorem 1.1. By Proposition 6.1 and Theorem 6.4, we see that $\Delta_\Sigma \Psi(\overline{x}) = 0$ for all $\overline{x} \in \Omega$. Thus Corollary 3.6 implies that $|A(\overline{x})|^2 = n$ for all $\overline{x} \in \Omega$. Furthermore, since $|A|^2 = n(n-1)\lambda^2$ by minimality and two distinct principal curvatures condition, we conclude that λ and μ are constant on Ω, which implies that Ψ is constant on Ω. By analytic continuation for solutions of elliptic partial differential equations, we see that Ψ is constant on Σ. Hence λ and μ are constant on Σ, which shows that Σ is an isoparametric minimal hypersurface with two distinct principal curvatures. From the Cartan's classification of isoparametric hypersurfaces and analysis of the structure equations as before, it follows that

Σ is congruent to a Clifford minimal hypersurface. However, this is a contradiction to the fact that $\kappa = 1$ on any Clifford minimal hypersurface, which completes the proof.

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