

Group actions on spheres with rank one prime power isotropy

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We show that a rank two finite group G admits a finite G -CW-complex $X \simeq S^n$ with rank one prime power isotropy if and only if G does not p' -involve $\mathrm{Qd}(p)$ for any odd prime p . This follows from a more general theorem which allows us to construct a finite G -CW-complex by gluing together a given G -invariant family of representations defined on the Sylow subgroups of G .

1. Introduction

Actions of finite groups on spheres can be studied in various different geometrical settings. The fundamental examples come from the unit spheres $S(V)$ in a real or complex G -representation V , and already natural questions arise for these examples about the dimensions of the non-empty fixed sets $S(V)^H$, $H \leq G$, and the structure of the isotropy subgroups.

A useful way to measure the complexity of the isotropy is the *rank*. We say that G has *rank* k if it contains a subgroup isomorphic to $(\mathbb{Z}/p)^k$, for some prime p , but no subgroup $(\mathbb{Z}/p)^{k+1}$, for any prime p . In this paper we answer the following question:

Question. For which finite groups G , does there exist a finite G -CW-complex $X \simeq S^n$ with all isotropy subgroups of rank one?

By P. A. Smith theory, the rank one assumption on the isotropy subgroups implies that G must have $\mathrm{rank}(G) \leq 2$ (see [6, Corollary 6.3]). Since every rank one finite group can act freely on a finite complex homotopy equivalent to a sphere (Swan [17]), we can restrict our attention to rank two groups. Here are three natural settings for the study of finite group actions on spheres:

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- (A) smooth G -actions on closed manifolds homotopy equivalent to spheres;
- (B) finite G -homotopy representations (see tom Dieck [20, Definition 10.1]);
- (C) finite G -CW-complexes $X \simeq S^n$.

In contrast to G -representation spheres $S(V)$, the non-linear smooth G -actions on a smooth manifold $M \simeq S^n$ exhibit more flexibility. For example, in the linear case, the fixed sets $S(V)^H$ are always linear subspheres. For smooth actions, the fixed sets are smoothly embedded submanifolds but may not even be integral homology spheres.

Well-known general constraints on smooth actions arise from P. A. Smith theory: if H is a subgroup of p -power order, for some prime p , then M^H is a $\mathbb{Z}_{(p)}$ -homology sphere. In addition, even if the fixed sets are diffeomorphic to spheres, they may be knotted or linked as embedded subspheres in M (see [21], [3]). One can also consider topological G -actions, usually with the assumption of local linearity, otherwise the fixed sets may not be locally flat submanifolds.

In the setting (B) of G -homotopy representations, the objects of study are finite (or more generally finite-dimensional) G -CW-complexes X satisfying the property that for each $H \leq G$, the fixed point set X^H is homotopy equivalent to a sphere $S^{n(H)}$ where $n(H) = \dim X^H$. We could also consider a version of this setting where $\dim X^H$ is the same as its homological dimension, and X^H is a $\mathbb{Z}_{(p)}$ -homology $n(H)$ -sphere, for H of p -power order.

The third setting (C) is the most flexible of all. Here we suppose that $X \simeq S^n$ is a finite G -CW-complex homotopy equivalent to a sphere, but do not require that $\dim X = n$. Moreover, we make no initial assumptions about the homology of the fixed sets X^H , although the conditions imposed by P. A. Smith theory with \mathbb{F}_p -coefficients still hold. In the setting (C), we will see that $\dim X^H$ must be (much) higher in general than its homological dimension, and this provides new obstructions to understanding our motivating question in setting (A) or (B).

In this paper we provide a complete answer for the existence question in setting (C). Our construction produces G -CW-complexes with prime power isotropy.

Theorem A. *Let G be a finite group of rank two. If G admits a finite G -CW-complex $X \simeq S^n$ with rank one isotropy then G is $\text{Qd}(p)$ -free. Conversely, if G is $\text{Qd}(p)$ -free, then there exists a finite G -CW-complex $X \simeq S^n$ with rank one prime power isotropy.*

The group $\text{Qd}(p)$ is defined as the semidirect product

$$\text{Qd}(p) = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes SL_2(p)$$

with the obvious action of $SL_2(p)$ on $\mathbb{Z}/p \times \mathbb{Z}/p$. We say $\text{Qd}(p)$ is p' -involved in G if there exists a subgroup $K \leq G$, of order prime to p , such that $N_G(K)/K$ contains a subgroup isomorphic to $\text{Qd}(p)$. If a group G does not p' -involve $\text{Qd}(p)$ for any odd prime p , then we say that G is $\text{Qd}(p)$ -free.

In our earlier work [5] and [6], we studied this problem in the setting (B) of G -homotopy representations, introduced by tom Dieck (see [20, Definition 10.1]). We found a list of conditions on a rank two finite group G that guarantees the existence of a finite G -homotopy representation with rank one prime power isotropy. Identifying the full list of necessary and sufficient conditions is still an open problem, but we did provide a complete answer [6, Theorem C] for rank two finite simple groups.

The necessity of the $\text{Qd}(p)$ -free condition was established in [23, Theorem 3.3] and [6, Proposition 5.4]. In the other direction, if G is a rank two finite group which is $\text{Qd}(p)$ -free then G has a p -effective representation $V_p: G_p \rightarrow U(n)$ (see Definition 5.6) which can be used to construct finite G -CW-complexes $X \simeq S^n$ with rank one isotropy. The existence of these p -effective representations was proved by Jackson [8, Theorem 47] and they were also one of the main ingredients for the constructions in Hambleton-Yalçın [6].

To do the construction in Theorem A, we prove a more technical theorem. We now introduce more terminology to state this theorem. For each prime p dividing the order of G , let G_p denote a fixed Sylow p -subgroup of G .

Definition 1.1. Suppose that we are given a family of Sylow representations $\{V_p\}$ defined on Sylow p -subgroups G_p , over all primes p . We say the family $\{V_p\}$ is G -invariant if

- (i) V_p respect fusion in G , i.e., the character χ_p of V_p satisfies $\chi_p(gxg^{-1}) = \chi_p(x)$ whenever $gxg^{-1} \in G_p$ for some $g \in G$ and $x \in G_p$; and
- (ii) for all p , $\dim V_p$ is equal to a fixed positive integer n .

Given a G -invariant family of Sylow representations $\{V_p\}$, we construct a G -equivariant spherical fibration $q: E \rightarrow B$ over a contractible G -space B with isotropy in \mathcal{P} such that for every $x \in \text{Fix}(B, G_p) = B^{G_p}$, the fiber $q^{-1}(x)$ is G_p -homotopy equivalent to $S(V_p^{\oplus k})$ for some $k \geq 1$ (see Theorem 3.4). The total space of this G -fibration has many interesting properties:

in particular, it admits a G -map

$$f_0: \coprod_p G \times_{G_p} S(V_p^{\oplus k}) \rightarrow E.$$

By adapting the G -CW-surgery techniques introduced by Oliver-Petrie [12] to this G -map, we obtain a finite G -CW-complex $X \simeq S^{2kn-1}$ whose restriction to Sylow p -subgroups resembles the linear spheres $S(V_p^{\oplus k})$. In particular, we prove the following theorem (see Definition 3.6 for the definition of p -local G -equivalence).

Theorem B. *Let G be a finite group. Suppose that $\{V_p: G_p \rightarrow U(n)\}$ is a G -invariant family of Sylow representations. Then there exists a positive integer $k \geq 1$ and a finite G -CW-complex $X \simeq S^{2kn-1}$ with prime power isotropy, such that the G_p -CW-complex $\text{Res}_{G_p}^G X$ is p -locally G_p -equivalent to $S(V_p^{\oplus k})$, for every prime $p \mid |G|$.*

This theorem was stated by Petrie [13, Theorem C] in a slightly different form and a sketched proof was provided. Related results were proved by tom Dieck (see [18, Satz 2.5], [19, Theorem 1.7]). Although we use some of the steps of these arguments, we believe that a proof of Theorem B does not exist in the literature. All the previous constructions seem to aim towards obtaining a finite G -CW-complex $X \simeq S^m$ with $\dim X = m$. However, we showed in [5] and [6] that there are additional necessary conditions for obtaining such a complex with prime power isotropy. Here is a specific example.

Example 1.2. Let G denote the dihedral group of order $2q$, with q an odd prime. Let V_2 be a trivial representation of $G_2 = \mathbb{Z}/2$, and let V_q be a free unitary representation of $G_q = \mathbb{Z}/q$, such that $\dim V_2 = \dim V_q$. Then Theorem B shows that there exists a finite G -CW-complex $X \simeq S^m$, with $\text{Fix}(X, G_2) \simeq S^m$ (2-locally), and $\text{Fix}(X, G_q) = \emptyset$, for some integer $m = 2kn - 1$. However, these conditions imply that $\dim X > m$ by [5, Proposition 2.10] (compare [15, Theorem 4.2]).

The paper is organized as follows: In Section 2, we show that for every finite group G , there is a finite-dimensional contractible G -space B with prime power isotropy, such that for every p -subgroup H , the fixed point set X^H is $\mathbb{Z}_{(p)}$ -acyclic. This might be of independent interest, since P. A. Smith theory only guarantees that the fixed sets are \mathbb{F}_p -acyclic. In Section 3, using this space as base space, we construct a G -equivariant fibration $q: E \rightarrow B$

with fiber type $S(V_H^{\oplus k})$, for a given compatible family $\{V_H\}$ of representations. The total space E has only prime power isotropy and its restriction to G_p is p -locally G_p -equivalent to $S(V_p^{\oplus k})$ for some $k \geq 1$. However, E is not a finite G -CW complex, and this means that the methods of [12] must be applied with care.

In Section 4, we prove Proposition 4.1 which allows us to kill homology groups to reach to a p -local homotopy equivalence on fixed points of p -subgroups. In Section 5, we prove our main theorems (Theorem A and Theorem B). Theorem A essentially follows from Theorem B once we apply a theorem of Jackson [8] on the existence of p -effective characters for rank two finite groups which are $\text{Qd}(p)$ -free.

Finally, we remark that Theorem A was also stated in Jackson [8, Proposition 48], but the indication of proof appears to confuse homotopy actions with finite G -CW-complexes. The motivation for Theorem A comes from the work of Adem and Smith [1] on the existence of free actions of finite groups on a product of two spheres. There is an interesting set of conditions related to this problem which we discussed in detail in [6, Section 1]. We refer the reader to this discussion for further details on the history of this problem.

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2. Acyclic complexes with prime power isotropy

The main purpose of this section is to prove the following theorem.

Theorem 2.1. *Let G be a finite group and \mathcal{P} denote the family of all subgroups of G with prime power order. Then there exists a finite-dimensional contractible G -CW-complex X , with isotropy in \mathcal{P} , such that for every p -subgroup $P \leq G$, the fixed point subspace X^P is $\mathbb{Z}_{(p)}$ -acyclic.*

There is a similar theorem by Leary and Nucinkis [10, Proposition 3.1] for infinite groups acting on contractible complexes, which implies in particular that for a finite group G , there is a finite-dimensional contractible G -CW-complex X with isotropy in \mathcal{P} . But this contractible complex is constructed

using a mapping telescope, and the fixed point subspaces are \mathbb{F}_p -acyclic but do not have finitely generated $\mathbb{Z}_{(p)}$ -homology.

Let \mathcal{F}_p denote the family of all p -subgroups of G . The family \mathcal{P} is the union of families \mathcal{F}_p over all primes p dividing the order of G . To prove Theorem 2.1, we first prove the following result.

Proposition 2.2. *Let G be a finite group and p be a prime such that $p \mid |G|$. Then, there exists a finite-dimensional G -CW-complex X , with isotropy in \mathcal{F}_p , such that for every p -subgroup $P \leq G$, the fixed point subspace X^P is $\mathbb{Z}_{(p)}$ -acyclic.*

A finite-dimensional \mathbb{F}_p -acyclic complex with p -subgroup isotropy is constructed in [7, Theorem 2.14]. But this construction also uses a mapping telescope so it does not have finitely generated $\mathbb{Z}_{(p)}$ -homology.

The construction we propose uses some of our earlier methods for constructing G -CW-complexes. In particular, we use chain complexes over the orbit category. Recall that the orbit category $\Gamma_G := \text{Or}_{\mathcal{H}} G$ over a family \mathcal{H} is the category with objects G/H , where $H \in \mathcal{H}$, and whose morphisms are given by G -maps $G/H \rightarrow G/K$. Given a commutative ring R with unity, an $R\Gamma_G$ -module is defined as contravariant functors from Γ_G to the abelian category of R -modules. For more details on $R\Gamma_G$ -modules we refer the reader to [4] (see also Lück [11, §9, §17] and tom Dieck [20, §10-11]).

Recall that for every family \mathcal{H} , there is a universal space $E_{\mathcal{H}}G$ such that isotropy subgroups of $E_{\mathcal{H}}G$ are in \mathcal{H} and for every $H \in \mathcal{H}$, the fixed point set $(E_{\mathcal{H}}G)^H$ is contractible. If $\mathbf{C} = \mathbf{C}(E_{\mathcal{H}}G^?; R)$ denote the cellular chain complex (over the orbit category) of the space $E_{\mathcal{H}}G$, then \mathbf{C} is a chain complex of free $R\Gamma_G$ -modules. Note that the augmented complex

$$\tilde{\mathbf{C}} : \cdots \rightarrow \mathbf{C}_n \xrightarrow{\partial_n} \mathbf{C}_{n-1} \xrightarrow{\partial_{n-1}} \mathbf{C}_{n-2} \rightarrow \cdots \rightarrow \mathbf{C}_1 \rightarrow \mathbf{C}_0 \rightarrow \underline{R} \rightarrow 0,$$

is an exact sequence, where \underline{R} denotes the constant functor. Hence \mathbf{C} is a projective resolution of \underline{R} as an $R\Gamma_G$ -module.

Lemma 2.3. *Let $\mathcal{H} = \mathcal{F}_p$, the family of all p -subgroups in G , and let $R = \mathbb{Z}_{(p)}$. Then there is a positive integer n such that $\ker \partial_{n-1}$ is a projective $R\Gamma_G$ -module.*

Proof. This follows from the fact that \underline{R} has a finite projective dimension as an $R\Gamma_G$ -module (see [4, Corollary 3.15]). Note that n can be taken as any integer greater or equal to the homological dimension of \underline{R} as an $R\Gamma_G$ -module. □

Now we are ready to prove Proposition 2.2.

Proof of Proposition 2.2. Let $\mathbf{C} = \mathbf{C}(E_{\mathcal{F}_p} G^?; R)$ and $P = \ker \partial_{n-1}$ denote the projective $R\Gamma_G$ -module for a suitably large n (as in Lemma 2.3). To avoid problems in low dimensions, we also assume $n \geq 3$. Let Q be a projective $R\Gamma_G$ -module such that $P \oplus Q$ is a free $R\Gamma_G$ -module. Using the Eilenberg swindle, we see that $\ker \partial_{n-1} \oplus F \cong F$, where $F = Q \oplus P \oplus Q \oplus \dots$ is an infinitely generated free $R\Gamma_G$ -module. Adding the chain complex

$$\dots \rightarrow 0 \rightarrow F \xrightarrow{id} F \rightarrow 0 \rightarrow \dots$$

to the truncated complex, we obtain a complex of free $R\Gamma_G$ -modules

$$0 \rightarrow F \xrightarrow{\varphi} \mathbf{C}_{n-1} \oplus F \xrightarrow{(\partial_{n-1}, 0)} \mathbf{C}_{n-2} \rightarrow \dots \rightarrow \mathbf{C}_1 \rightarrow \mathbf{C}_0 \rightarrow 0$$

where the map φ is defined as the composition $F \cong \ker \partial_{n-1} \oplus F \hookrightarrow \mathbf{C}_{n-1} \oplus F$. Note that this chain complex can be lifted to a chain complex of free $\mathbb{Z}\Gamma_G$ -modules

$$\mathbf{D} : 0 \rightarrow \mathbf{D}_n \rightarrow \mathbf{D}_{n-1} \rightarrow \mathbf{D}_{n-2} \rightarrow \dots \rightarrow \mathbf{D}_0 \rightarrow 0$$

where the resulting complex has homology groups that are (possibly infinitely generated) abelian groups with torsion coprime to p . Because of the special structure of the original $R\Gamma_G$ -complex, we can assume that the lifting \mathbf{D} is of the form

$$\mathbf{D} : 0 \rightarrow \widehat{F} \xrightarrow{\widehat{\varphi}} \widehat{\mathbf{C}}_{n-1} \oplus \widehat{F} \xrightarrow{(\partial_{n-1}, 0)} \widehat{\mathbf{C}}_{n-2} \rightarrow \dots \rightarrow \widehat{\mathbf{C}}_1 \rightarrow \widehat{\mathbf{C}}_0 \rightarrow 0$$

where $\widehat{\mathbf{C}}_i = \mathbf{C}_i(E_{\mathcal{F}_p} G^?; \mathbb{Z})$ and \widehat{F} is a free $\mathbb{Z}\Gamma_G$ -module such that $\widehat{F} \otimes R \cong F$.

The map $\widehat{\varphi}$ is obtained as follows: let $\{e_i\}$ be a basis for F as an $R\Gamma_G$ -module. For each i , there is an integer s_i , coprime to p , such that $\varphi(s_i e_i) \in \widehat{\mathbf{C}}_{n-1} \oplus \widehat{F}$. Let \widehat{F} be the $\mathbb{Z}\Gamma_G$ -submodule of F generated by $\{s_i e_i\}$ and $\widehat{\varphi}$ be the map induced by φ . It is easy to see from this that the reduced homology of this complex \mathbf{D} is zero except at dimension $n - 1$ and $H_{n-1}(\mathbf{D})$ is a torsion abelian group with torsion coprime to p (possibly infinitely generated).

Note that we can assume that \mathbf{D} is partially realized by the $(n - 1)$ -skeleton of the complex $E_{\mathcal{F}_p} G$. In fact, by attaching orbits of cells to $E_{\mathcal{F}_p} G$ with p -subgroup isotropy, we can assume that \mathbf{D} is realized for dimensions $\leq n - 1$. The last realization step can be done using [4, Lemma 8.1]. Note that for this step we need to assume $n \geq 3$.

Hence, we can conclude that for every finite group G , there is a finite-dimensional G -CW-complex X with isotropy in \mathcal{F}_p , such that

- (i) X is n -dimensional and $(n-2)$ -connected where $n = \max\{3, \text{homdim } \underline{R}\}$;
- (ii) for each $P \in \mathcal{F}_p$, the only nontrivial reduced homology of the fixed point subspace X^P is at dimension $n - 1$ and $H_{n-1}(X)$ is a torsion abelian group with torsion coprime to p .

In particular, for every $P \in \mathcal{F}_p$, the fixed point subspace X^P is $\mathbb{Z}_{(p)}$ -acyclic. Hence this completes the proof of Proposition 2.2. □

Proof of Theorem 2.1. In Proposition 2.2 we have constructed a $\mathbb{Z}_{(p)}$ -acyclic complex X_p of dimension n_p , for each $p \mid |G|$. Let X be the join $\ast X_p$ of all the X_p 's over all $p \mid |G|$. The reduced homology of X is nonzero only at dimension $n - 1$, where $n = \coprod n_p$, and

$$H_{n-1}(X) \cong \bigotimes_{p \mid |G|} H_{n_p-1}(X_p).$$

Since $H_{n_p-1}(X_p)$ is a torsion group coprime to p , the homology group $H_{n-1}(X)$ is a torsion abelian group with torsion coprime to $|G|$. Such an abelian group has two step free resolution. To see this, note that as a $\mathbb{Z}G$ -module $N = H_{n-1}(X)$ is cohomologically trivial since it is a torsion group with torsion coprime to the order of the group. If we take a free cover of N , then we get an exact sequence of the form

$$0 \rightarrow M \rightarrow F_0 \rightarrow N \rightarrow 0.$$

Note that the module M is both torsion free and cohomologically trivial. Hence by [2, Theorem 8.10, p. 152], M is a projective module. By an Eilenberg swindle argument, we can add free modules to M and F_0 to obtain a two step free resolution for N . This means, we can kill the last homology group at dimension $n - 1$ by adding free orbits of cells. By taking further joins if necessary, we can assume that X is simply connected, hence the resulting G -CW-complex is contractible. For each $1 \neq P \in \mathcal{F}_p$, we have $H_*(X^P; \mathbb{Z}_{(p)}) \cong H_*(X_p^P; \mathbb{Z}_{(p)}) \cong H_*(pt; \mathbb{Z}_{(p)})$, so the fixed subspace X^P is $\mathbb{Z}_{(p)}$ -acyclic for every $P \in \mathcal{F}_p$. □

3. G -equivariant fibrations

Let G be a finite group. In this section, we first give some necessary definitions related to G -fibrations and then construct a G -fibration over a contractible base space with prime power isotropy. For more details on this

material we refer the reader to [24, Section 2] and to some earlier references mentioned in that paper.

Definition 3.1. A G -fibration is a G -map $q: E \rightarrow B$ which satisfies the following homotopy lifting property for every G -space X : given a commuting diagram of G -maps

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{h} & E \\ \downarrow & & \downarrow q \\ X \times I & \xrightarrow{H} & B, \end{array}$$

there exists a G -map $\tilde{H}: X \times I \rightarrow E$ such that $\tilde{H}|_{X \times \{0\}} = h$ and $p \circ \tilde{H} = H$.

If $p: E \rightarrow B$ is a G -fibration, then for every $x \in B$, the isotropy subgroup $G_x \leq G$ acts on the fiber space $F_x = q^{-1}(x)$. So, F_x is a G_x -space.

Definition 3.2. Let \mathcal{H} be a family of subgroups of G and $\{F_H\}$ denote a family of H -spaces over all $H \in \mathcal{H}$. If for every $x \in B$, the isotropy subgroup G_x lies in \mathcal{H} and the fiber space F_x is G_x -homotopy equivalent to F_{G_x} , then $p: E \rightarrow B$ is said to have *fiber type* $\{F_H\}$.

Here and throughout the paper a *family of subgroups* always means a collection of subgroups which are closed under conjugation and taking subgroups. In general a G -fibration does not have to satisfy the above criteria: for $x, y \in B$ with $G_x = G_y = H$, it may happen that F_x and F_y are not H -homotopy equivalent. Throughout the paper we only consider G -fibrations which do have a fiber type.

We will construct G -equivariant spherical fibrations whose fiber type is given by a family of linear H -spheres. To start we assume that we are given a compatible family of representations.

Definition 3.3. Let \mathcal{H} be a family of subgroups of G and $\mathbf{V} = \{V_H\}$ denote a family of complex H -representations defined over $H \in \mathcal{H}$. We say \mathbf{V} is a *compatible family of representations* if $f^*(V_K) \cong V_H$ for every G -map $f: G/H \rightarrow G/K$. In this case, we call \mathbf{V} an \mathcal{H} -representation (see [6, Definition 3.1]).

Note that since $1 \in \mathcal{H}$, all the H -representations V_H in \mathbf{V} have the same dimension. We call this common dimension the dimension of \mathbf{V} . We have the following result as a main tool for constructions of G -fibrations which was first proved by Klaus [9, Proposition 2.7].

Theorem 3.4. *Let G be a finite group, with \mathcal{H} a family of subgroups. Let B be a finite-dimensional G -CW-complex such that the isotropy subgroup G_x lies in \mathcal{H} , for every $x \in B$. Given a compatible family of complex representations $\mathbf{V} = \{V_H\}$ defined over \mathcal{H} , there exists an integer $k \geq 1$ and a G -equivariant spherical fibration $q: E \rightarrow B$ such that the fiber type of q is $\{S(V_H^{\oplus k})\}$.*

Proof. See [24, Proposition 4.3]. □

We will apply this theorem to construct a G -fibration over a base space with prime power isotropy. As before, let \mathcal{P} denote the family of all subgroups of G with prime power order, and \mathcal{F}_p denote the family of all p -subgroups of G .

Lemma 3.5. *Let G be a finite group and $\{V_p\}$ be a G -invariant family of Sylow representations (see Definition 1.1). For each $H \in \mathcal{F}_p$, let V_H be the representation obtained from V_p via the map*

$$H \xrightarrow{c^g} gHg^{-1} \hookrightarrow G_p$$

where c^g denotes the conjugation map $h \mapsto ghg^{-1}$ and the second map is the inclusion map (the element $g \in G$ is chosen arbitrarily such that $gHg^{-1} \leq G_p$). Then the collection $\mathbf{V} = (V_H)_{H \in \mathcal{P}}$ is a compatible family of representations over \mathcal{P} .

Proof. We only need to check that when $H, K \leq G_p$ are such that $H = gKg^{-1}$ for some $g \in G$, then $(c^g)^*(V_H) \cong V_K$ as K -representations. Note that the isomorphism holds because for every $x \in K$, we have

$$(c^g)^*(\chi_p)(x) = \chi_p(gxg^{-1}) = \chi_p(x)$$

by the character formula given in Definition 1.1. This also shows that the compatible family $\{V_H\}$ does not depend on the elements $g \in G$ chosen to define it (up to isomorphism). □

Suppose that we are given a G -invariant family of Sylow representations $\{V_p\}$. Then by Lemma 3.5, this gives a compatible family of representations $\mathbf{V} = (V_H)$. Let B be the G -CW-complex constructed in Proposition 2.1. By applying Proposition 3.4 to the base space B with family \mathbf{V} , we obtain a G -equivariant spherical fibration $q: E \rightarrow B$ with fiber type $\{S(V_H^{\oplus k})\}_{H \in \mathcal{P}}$ for some $k \geq 1$.

The total space E satisfies certain properties which will be used in our construction of finite homotopy G -spheres.

Definition 3.6. A G -map $f: X \rightarrow Y$ between two G -spaces is called a p -local G -equivalence if for every subgroup $H \leq G$, the map on fixed point sets $f^H: X^H \rightarrow Y^H$ induces an isomorphism on $\mathbb{Z}_{(p)}$ -homology.

We say that two G -spaces X and Y are p -locally G -equivalent if for some k there is are G -spaces $\{X_i\}$ and $\{Y_i\}$, for $0 \leq i \leq k$, such that $X_0 = X$ and $Y_k = Y$, together with two families of G -maps $X_i \rightarrow Y_i$ for $i \geq 0$, and $X_i \rightarrow Y_{i-1}$ for $i > 0$, which are p -local G -equivalences.

Now we prove the main result of this section.

Proposition 3.7. *Let G be a finite group, and let $\{V_p\}$ be a G -invariant family of Sylow representations. Then there exists an integer $k \geq 1$ and a finite-dimensional G -CW-complex E , with isotropy in \mathcal{P} , satisfying the following properties:*

- (i) E is homotopy equivalent to the sphere S^{2kn-1} where $n = \dim V_p$;
- (ii) For every $H \in \mathcal{P}$, the fixed point subspace E^H is simply connected;
- (iii) For every $p \mid |G|$, there is a G_p -map $j_p: S(V_p^{\oplus k}) \rightarrow E$ which is a p -local G_p -equivalence.

Proof. Let B be a contractible G -CW-complex as in Theorem 2.1, and E be the total space of a fibration $q: E \rightarrow B$ with fiber type $\{S(V_H^{\oplus k})\}_{H \in \mathcal{P}}$ for some $k \geq 1$. By construction of the G -fibration, the total space E is a G -homotopy equivalent to a finite-dimensional G -CW-complex (see [24, Proposition 4.4]). Since B has isotropy in \mathcal{P} , the total space E has isotropy in \mathcal{P} . Since B is contractible, E is homotopy equivalent to S^{2kn-1} .

For every $H \leq G$, the induced map $q^H: E^H \rightarrow B^H$ on fixed subspaces is a fibration with fiber type F^H . We can assume that for every $P \in \mathcal{F}_p$, the fixed point subspace B^H is simply connected (if not we can replace B with $B * B$). We can also assume that the subspaces F^H are simply connected by replacing k with a larger integer if necessary. Using the long exact homotopy sequence for the fibration $F^H \rightarrow E^H \rightarrow B^H$, we obtain that E^H is simply connected for every $H \in \mathcal{P}$.

For the second statement, observe that for every $p \mid |G|$, the fixed point space B^{G_p} is non-empty, by P. A. Smith Theory. If we take $x \in B^{G_p}$, then the inclusion map $i_x: \{x\} \rightarrow B^{G_p}$ induces a G_p -map $j_x: F_x \rightarrow E$, where $F_x = q^{-1}(x)$. By the definition of fiber type, we have $F_x \simeq S(V_p^{\oplus k})$ as a G_p -space. We define j_p as the composite $S(V_p^{\oplus k}) \simeq F_x \xrightarrow{j_x} E$ which is a G_p -map. For

each subgroup $H \leq G_p$, we have a fibration diagram:

$$\begin{array}{ccc}
 F_x^H & \xlongequal{\quad} & F_x^H \\
 \downarrow & & \downarrow \\
 F_x^H & \xrightarrow{j_x^H} & E^H \\
 \downarrow & & \downarrow \\
 \{x\} & \xrightarrow{i_x^H} & B^H.
 \end{array}$$

Since i_x^H induces a $\mathbb{Z}_{(p)}$ -homology isomorphism, the map j_x^H also induces a $\mathbb{Z}_{(p)}$ -homology isomorphism. This can be seen easily by a spectral sequence argument. Note that B^H is simply connected, so the E_2 -term of the Serre spectral sequence for the second fibration is of the form $E_2^{i,j} = H^i(B^H; H^j(F_x^H, \mathbb{Z}_{(p)}))$ with untwisted coefficients. By comparing two spectral sequences, we see that j_x^H induces an isomorphism on $\mathbb{Z}_{(p)}$ -homology. This shows that j_p is a p -local G_p -equivalence. \square

4. p -local G -CW-surgery

Let G be a finite group, \mathcal{P} denote the family of subgroups of G with prime power order, and $\{V_p\}$ be a G -invariant family of Sylow representations $V_p: G_p \rightarrow U(n)$ over all primes p dividing the order of G . In Section 3, we proved that there is a finite-dimensional G -CW-complex E , with isotropy in \mathcal{P} , homotopy equivalent to S^{2kn-1} for some $k \geq 1$, satisfying some further fixed point properties.

To prove Theorem B we will need to replace E with a finite G -CW-complex X having properties similar to E , with possibly a larger $k \geq 1$. We will do this by applying the G -CW-surgery techniques introduced in [12] to a particular G -map (see also [22]).

By part (iii) of Proposition 3.7, there is a G_p -map $j_p: S(V_p^{\oplus k}) \rightarrow E$ which induces a $\mathbb{Z}_{(p)}$ -homology isomorphism on fixed subspaces, for every $p \mid |G|$. Using these maps we can define a G -map

$$f_0: \coprod_{p \mid |G|} G \times_{G_p} S(V_p^{\oplus k}) \rightarrow E$$

by taking $f_0(g, x) = gj_p(x)$ for every $g \in G$ and $x \in S(V_p^{\oplus k})$. It is clear that f_0 is well-defined and it is a G -map, where the G -action on $G \times_{G_p} S(V_p^{\oplus k})$

is by left multiplication. We will apply G -CW-surgery methods to this map to convert it to a homotopy equivalence.

The first step of this surgery method is to get a p -local homology equivalence on H -fixed subspaces for every nontrivial p -subgroup $H \leq G$. We will do this step-by-step by a downward induction starting from Sylow p -subgroups. At a particular step H we will need to attach cells to complete that step. The following proposition is the main result of this section and it states exactly what we will need to complete a particular step in the downward induction.

Proposition 4.1. *Let G be a finite group and $f: X \rightarrow Y$ be a G -map between two simply connected G -CW-complexes, with isotropy subgroups in \mathcal{F}_p , such that*

- (i) X is a finite complex and X^P is an odd-dimensional $\mathbb{Z}_{(p)}$ -homology sphere for every p -subgroup $1 \neq P \leq G$;
- (ii) Y is a finite-dimensional complex with finitely generated $\mathbb{Z}_{(p)}$ -homology;
- (iii) The Euler characteristic $\sum_i \dim_{\mathbb{Q}}(-1)^i [H_i(Y; \mathbb{Q})] = 0 \in R_{\mathbb{Q}}(G)$, the rational representation ring of G .

If for every p -subgroup $1 \neq P \leq G$, the induced map $f^P: X^P \rightarrow Y^P$ on fixed point sets is a $\mathbb{Z}_{(p)}$ -homology equivalence, then by attaching finitely many free G -orbits of cells to X , we can extend f to a $\mathbb{Z}_{(p)}$ -homology equivalence $f': X' \rightarrow Y$.

Given a G -map $f: X \rightarrow Y$ between two G -CW-complexes, we define the n -th homotopy group of f , denoted by $\pi_n(f)$, as the equivalence classes of pairs of maps (α, β) such that the diagram

$$\begin{array}{ccc}
 S^{n-1} & \xrightarrow{i} & D^n \\
 \downarrow \alpha & & \downarrow \beta \\
 X & \xrightarrow{f} & Y
 \end{array}$$

commutes, where $i: S^{n-1} \rightarrow D^n$ is the inclusion map of the boundary of D^n . The equivalence relation is given by a pair of homotopy that fits into a similar diagram. It is easy to show that $\pi_n(f)$ is isomorphic to the n -th homotopy group of the pair $\pi_n(Z_f, X)$, where Z_f denotes the mapping cylinder $(X \times I) \cup_f Y$. We consider X as a subspace by identifying X with $X \times \{0\}$.

In a similar way, we can define relative homology group of a G -map $f: X \rightarrow Y$ in coefficients in R as follows:

$$K_*(f; R) := H_*(Z_f, X; R) \cong \widetilde{H}_*(M_f; R),$$

following the notation in [12], where M_f denotes the mapping cone of f . We recall the relative Hurewicz theorem for homotopy groups of pairs.

Lemma 4.2. *Let $R = \mathbb{Z}$ or $\mathbb{Z}_{(p)}$ for some prime p , and let $f: X \rightarrow Y$ be a map between two simply connected spaces. For $n \geq 2$, if $\pi_i(f) \otimes R = 0$ for all $i < n$, then $K_n(f; R) = 0$ for all $i < n$ and the Hurewicz map $\pi_n(f) \otimes R \rightarrow K_n(f; R)$ is an isomorphism.*

Proof. See [14, Theorem 7.5.4]. □

The Hurewicz theorem allows us to realize homology classes as homotopy classes. We kill the corresponding homotopy class by attaching free orbits of cells to X and extending the map f . If the homotopy class is represented by a pair of maps (α, β) as above, then the space X' is defined as the space $X' = X \cup_\alpha D^n$ and the map $f': X' \rightarrow Y$ is defined by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in X \\ \beta(x) & \text{if } x \in D^n \end{cases}$$

In the homotopy group $\pi_n(f')$, the homotopy class for the pair (α, β) is now zero and this cell attachment does not introduce any more homotopy classes at dimensions $i \leq n$.

Let $f: X \rightarrow Y$ be a G -map as in Proposition 4.1. By applying this cell attachment method we can assume that f is extended to a map $f_1: X_1 \rightarrow Y$ such that $d := \dim X_1 > \dim Y$ and f_1 induces an $\mathbb{Z}_{(p)}$ -homology isomorphism in dimensions $i < d$. Since Y has finitely generated $\mathbb{Z}_{(p)}$ -homology, in the process only finitely many free G -orbits are attached to X . So X_1 is still a finite complex.

Note that $K_i(f_1; \mathbb{Z}_{(p)})$ is nonzero only at dimension $i = d + 1$, and

$$M := K_{d+1}(f_1; \mathbb{Z}_{(p)}) \cong H_d(X_1; \mathbb{Z}_{(p)}).$$

Since X_1 is a finite complex and $d = \dim X_1$, as a $\mathbb{Z}_{(p)}$ -module M is a finitely generated and torsion free. We claim that M is a free $\mathbb{Z}_{(p)}G$ -module. First we prove a lemma which shows, in particular, that M is projective.

Lemma 4.3. *Let $R = \mathbb{Z}$ or $\mathbb{Z}_{(p)}$, and let $f: X \rightarrow Y$ be a G -map such that $d := \dim X > \dim Y$ and f induces an R -homology isomorphism on dimensions $i < d$. Assume also that for every $1 \neq H \leq G$, the induced map $f^H: X^H \rightarrow Y^H$ on fixed point subspaces is an R -homology equivalence. Then $K_{d+1}(f; R)$ is a projective RG -module.*

Proof. Let $X^s = \cup_{1 \neq H \leq G} X^H$ and $f^s: X^s \rightarrow Y^s$ denote the restriction of f to the singular set. For every nontrivial subgroup $H \leq G$, the induced map $f^H: X^H \rightarrow Y^H$ is an R -homology equivalence. This gives, in particular, that $f^s: X^s \rightarrow Y^s$ is an R -homology equivalence. Let $Z_f^s := X \cup Z_{f^s}$. Consider the homology sequence for the triple (Z_f, Z_f^s, X) with coefficients in R :

$$\cdots \rightarrow H_i(Z_f^s, X) \rightarrow H_i(Z_f, X) \rightarrow H_i(Z_f, Z_f^s) \rightarrow H_{i-1}(Z_f^s, X) \rightarrow \cdots$$

We have

$$H_i(Z_f^s, X) = H_i(X \cup Z_{f^s}, X) \cong H_i(Z_{f^s}, X^s) = 0$$

for all i , because f^s is an R -homology equivalence. From this we obtain that

$$H_i(Z_f, Z_f^s) \cong H_i(Z_f, X),$$

hence $H_i(Z_f, Z_f^s; R) = 0$ for $i < d + 1$ and it is isomorphic to $K_{d+1}(f; R)$ when $i = d + 1$.

The chain complex for the pair (Z_f, Z_f^s) in R -coefficients gives an exact sequence of RG -modules

$$0 \rightarrow K_{d+1}(f; R) \rightarrow C_{d+1}(Z_f, Z_f^s; R) \rightarrow \cdots \rightarrow C_0(Z_f, Z_f^s; R) \rightarrow 0.$$

For all i , the modules $C_i(Z_f, Z_f^s; R)$ are free RG -modules, hence this exact sequence splits and $K_{d+1}(f; R)$ is a projective RG -module. \square

Applying this lemma to the map $f_1: X_1 \rightarrow Y$ constructed above, we obtain that $M = K_{d+1}(f_1; \mathbb{Z}_{(p)}) \cong H_d(X_1; \mathbb{Z}_{(p)})$ is a projective $\mathbb{Z}_{(p)}G$ -module. Now we show that M is a free $\mathbb{Z}_{(p)}G$ -module.

Lemma 4.4. *Let $f: X \rightarrow Y$ be a G -map as in Proposition 4.1 and $f_1: X_1 \rightarrow Y$ is the map obtained by attaching cells to X as above. Then, $K_{d+1}(f_1; \mathbb{Z}_{(p)})$ is a finitely-generated free $\mathbb{Z}_{(p)}G$ -module.*

Proof. By Lemma 4.3, $M = K_{d+1}(f_1; \mathbb{Z}_{(p)})$ is a projective $\mathbb{Z}_{(p)}G$ -module. Let $\mathbb{Q}M = \mathbb{Q} \otimes M$. By [12, Lemma 2.4], M is a free $\mathbb{Z}_{(p)}G$ -module if $\chi_{\mathbb{Q}M}(g) = 0$

for all $1 \neq g \in G$. Since $M \cong H_d(X_1; \mathbb{Z}_{(p)})$ and X_1 is a finite G -CW-complex, we can calculate $\chi_{\mathbb{Q}M}$ using the the chain complex of X_1 . Let

$$0 \rightarrow C_d(X_1; \mathbb{Q}) \rightarrow C_{d-1}(X_1; \mathbb{Q}) \rightarrow \cdots \rightarrow C_0(X_1; \mathbb{Q}) \rightarrow 0$$

be the chain complex for X_1 in \mathbb{Q} -coefficients. In rational representation ring of G , we have

$$(-1)^d [H_d(X_1; \mathbb{Q})] + \sum_{i=0}^{d-1} (-1)^i [H_i(X_1; \mathbb{Q})] = \sum_{i=1}^d (-1)^i [C_i(X_1; \mathbb{Q})]$$

Since f_1 induces $\mathbb{Z}_{(p)}$ -homology isomorphism at dimensions $i < d$, we get

$$\sum_{i=0}^{d-1} (-1)^i [H_i(X_1; \mathbb{Q})] = \sum_{i=0}^{d-1} (-1)^i [H_i(Y; \mathbb{Q})] = 0$$

by the assumption in Proposition 4.1. This gives that for every $1 \neq g \in G$,

$$\begin{aligned} (-1)^d \chi_{\mathbb{Q}M}(g) &= \sum_{i=1}^d (-1)^i \dim_{\mathbb{Q}} C_i(X_1^g; \mathbb{Q}) \\ &= \sum_{i=1}^d (-1)^i \dim_{\mathbb{Q}} H_i(X_1^g; \mathbb{Q}) = \chi(X_1^{(g)}) \end{aligned}$$

Since for every p -group $1 \neq H \leq G$, the fixed point set X_1^H is an odd dimensional $\mathbb{Z}_{(p)}$ -homology sphere, we have $\chi(X_1^H) = 0$ for every nontrivial p -subgroup $H \leq G$. When $1 \neq H \leq G$ is a p' -subgroup, then X_1^H is empty, so again the Euler characteristic is zero. Hence $\chi_{\mathbb{Q}M}(g) = 0$ for all $1 \neq g \in G$. We conclude that M is a free $\mathbb{Z}_{(p)}G$ -module. \square

Proof of Proposition 4.1. Let $f: X \rightarrow Y$ be a G -map as in Proposition 4.1, and let $f_1: X_1 \rightarrow Y$ be the G -map obtained by attaching cells, as described above, so that f_1 induces an $\mathbb{Z}_{(p)}$ -homology isomorphism in dimensions $i < d$. By Lemma 4.4, $M = K_{d+1}(f_1; \mathbb{Z}_{(p)})$ is a finitely-generated free $\mathbb{Z}_{(p)}G$ -module. By Lemma 4.2,

$$K_{d+1}(f_1; \mathbb{Z}_{(p)}) \cong \pi_{d+1}(f_1) \otimes \mathbb{Z}_{(p)},$$

and hence $\pi_{d+1}(f_1)$ contains a finitely-generated free $\mathbb{Z}G$ -module $M' \subseteq \pi_{d+1}(f_1)$ with index prime to p . We attach free orbits of G -cells to X_1 using the pairs of maps (α, β) representing homotopy classes of $\mathbb{Z}G$ -basis elements in M' . The resulting map $f': X' \rightarrow Y$ is a $\mathbb{Z}_{(p)}$ -homology equivalence. \square

5. Proof of main theorems

In this section we prove Theorem A and Theorem B as stated in the introduction. Theorem A will follow from Theorem B almost directly by applying a theorem by Jackson [8, Theorem 47].

Let G be a finite group, \mathcal{P} denote the family of all subgroups of G with prime power order. Suppose we are given a G -invariant family of Sylow representations $\{V_p\}$ over the primes dividing the order of G . We will construct a finite G -CW-complex $X \simeq S^{2kn-1}$ such that for every $p \mid |G|$, the restriction of X to G_p is p -locally G_p -equivalent to $S(V_p^{\oplus k})$, for some $k \geq 1$. We showed in Section 4 that there is a G -map $f_0: X_0 \rightarrow E$ where

$$X_0 = \coprod_{p \mid |G|} G \times_{G_p} S(V_p^{\oplus k})$$

and E is the total space of the fibration constructed in Section 3. The G -map f_0 is induced from the G_p -maps $j_p: S(V_p^{\oplus k}) \rightarrow E$ which were introduced in Proposition 3.7.

We will first show that by a downward induction and by attaching cells at each step, we can extend the map f_0 to a map $f_1: X_1 \rightarrow E$ such that $f_1^H: X_1^H \rightarrow E^H$ is a p -local homology equivalence for every nontrivial p -subgroup $H \leq G$. Since we work with unitary representations, the fixed point subspace E^H is an odd dimensional sphere with trivial $N_G(H)/H$ -action. This implies in particular that as an $N_G(H)/H$ -space the fixed point subspace E^H satisfies the Euler characteristic condition for the target space in Proposition 4.1.

To show that each step of the downward induction can be performed, suppose H is a nontrivial p -subgroup such that f_1^K is a p -local homology equivalence for every K with $|K| > |H|$. Consider the induced $N_G(H)/H$ action on X_1^H . By Proposition 4.1, we can add free $N_G(H)/H$ -orbits of cells to X_1^H to extend f_1^H to a p -local homology equivalence. In fact, by adding cells of orbit type G/H (instead of just $N_H(H)/H$ -orbits) to X_1 we can make X_1^L a mod- p equivalence for every $L \leq G$ conjugate to G . This shows that downward induction can be carried out until we reach to a map $f_1: X_1 \rightarrow E$ such that f_1^H is a p -local homology equivalence for every nontrivial p -subgroup $H \leq G$, for all the primes $p \mid |G|$.

As we did in the previous section, we can add free cells to X_1 and extend f_1 to a map $f_2: X_2 \rightarrow E$ such that f_2 induces a homotopy equivalence for dimensions $i < d$ where $d := \dim X_2 > \dim E$.

Lemma 5.1. *The module $\mathbb{Z}G$ -module $M := K_{d+1}(f_2) \cong H_d(X_2, \mathbb{Z})$ is a finitely-generated projective module.*

Proof. It is enough to show that for every $p \mid |G|$, the $\mathbb{Z}_{(p)}G_p$ -module $\text{Res}_{G_p}^G M \otimes \mathbb{Z}_{(p)}$ is projective. This follows from Lemma 4.3. \square

In general, M does not have to be a free $\mathbb{Z}G$ -module, but we will obtain this condition by taking further joins. To describe the obstructions for finiteness, we need to introduce more definitions.

Definition 5.2. Let X be a finite G -CW-complex which has integral homology of an m -dimensional (orientable) sphere for $i \leq m$ and for each $i \geq m + 1$, assume that $H_i(X, \mathbb{Z})$ is a projective $\mathbb{Z}G$ -module. Then we say X is a G -resolution of an m -sphere.

Let $\tilde{K}_0(\mathbb{Z}G)$ denote the Grothendieck ring of finitely generated projective $\mathbb{Z}G$ -modules, modulo finitely generated free modules. We define the finiteness obstruction of G -resolution of an m -sphere as follows:

Definition 5.3. Let X be a G -resolution of an m -sphere. The finiteness obstruction of X is defined as an element in $\tilde{K}_0(X)$ as follows:

$$\sigma(X) = \sum_{i=m+1}^{\dim X} (-1)^i [H_i(X)] \in \tilde{K}_0(\mathbb{Z}G).$$

We have the following observation:

Lemma 5.4. *Let X_1 and X_2 be G -resolutions of spheres of dimensions $m_1 - 1$ and $m_2 - 1$. Then the join space $X_1 * X_2$ is a resolution of a sphere of dimension $m_1 + m_2 - 1$. Moreover, we have $\sigma(X_1 * X_2) = (-1)^{m_2} \sigma(X_1) + (-1)^{m_1} \sigma(X_2)$.*

Proof. Since tensor product (over \mathbb{Z}) of a projective module with any torsion-free $\mathbb{Z}G$ -module is projective, it is easy to see that all the homology above the dimension $m_1 + m_2 - 1$ will be projective. So, $X_1 * X_2$ is a G -resolution. Moreover, the tensor product of any two finitely generated projective $\mathbb{Z}G$ -modules is stably free as a $\mathbb{Z}G$ -module (See [4, Proposition 7.7]). So the only homology groups that contribute nontrivially to $\sigma(X_1 * X_2)$ will be the homology modules of the form $H_i(X_1) \otimes H_{m_2-1}(X_2)$, with $i \geq m_1$, or of the form $H_{m_1-1}(X_1) \otimes H_i(X_2)$, with $i \geq m_2$. \square

By Swan [16, Prop. 9.1], the obstruction group $\tilde{K}_0(\mathbb{Z}G)$ is a finite abelian group, so we can apply the above lemma to conclude that there is a positive integer l , such that $\sigma(*_l X_2) = 0$. Note that f_2 induces a G -map

$$*_l f_2: *_l X_2 \rightarrow *_l E.$$

We need the following result to complete the proof of Theorem B.

Lemma 5.5. *Let X be a G -resolution of an $(m - 1)$ -dimensional sphere and let $f: X \rightarrow E$ be a G -map which induces a homotopy equivalence in dimensions $\leq m - 1$. If $\sigma(X) = 0$ in $\tilde{K}_0(\mathbb{Z}G)$, then by adding finitely many free cells to X , the G -map f can be extended to a G -map $f': X' \rightarrow E$ which induces an isomorphism on homology.*

Proof. By adding free cells to X above dimension $m - 1$, we can assume we have a map $f_1: X_1 \rightarrow E$ such that all the homology of X_1 is concentrated at $d = \dim X_1 > m - 1$. Then, it is easy to see that $(-1)^d[H_d(X_1)] = \sigma(X_1) = 0$, hence $H_d(X_1)$ is stably free. By adding free cells to X_1 at dimension d and $d - 1$, we can kill all the remaining homology and extend f to a G -map $f': X' \rightarrow E$ which induces an isomorphism on homology. \square

Proof of Theorem B. Starting from the map $f_0: X_0 \rightarrow E$, we first apply p -local surgery methods to get a map $f_1: X_1 \rightarrow E$ which induced a p -local homology equivalence on fixed points $X_1^H \rightarrow E^H$ for every nontrivial p -subgroup $H \leq G$. This is done by a downward induction as described above. Then we add free orbits of cells to X_1 to obtain a map $f_2: X_2 \rightarrow E$ where X_2 is a G -resolution. Here we use Lemma 5.1 to conclude that X_2 is indeed a G -resolution. Finally we use Lemma 5.4 and 5.5 to kill the remaining homology by taking further joins.

As a result of the above construction we obtain a finite G -CW-complex X and a G -map $f: X \rightarrow *_l E$ which induces a homotopy equivalence. Since $*_l E \simeq S^{2kln-1}$, it follows that X is homotopy equivalent to a sphere of dimension $2kln - 1$. For every $p \mid |G|$, we have G_p -maps $X \rightarrow E$ and $S(V_p^{\oplus lk}) \rightarrow E$ which induce p -local homology equivalences on fixed points. So $\text{Res}_{G_p}^G X$ and $S(V_p^{\oplus lk})$ are p -locally G_p -equivalent. \square

Before giving a proof for Theorem A, we recall the following definition.

Definition 5.6. A finite group G has a p -effective representation if it has a representation $V_p: G_p \rightarrow U(n)$ which respects fusion (see Definition 1.1) and satisfies $\langle V_p|_E, 1_E \rangle = 0$ for each elementary abelian p -subgroup $E \leq G$ with $\text{rank } E = \text{rank}_p G$.

Proof of Theorem A. Let G be a finite group of rank two which is $\text{Qd}(p)$ -free. By Jackson [8, Theorem 47], for each $p \mid |G|$, there is a p -effective representation V_p . By taking multiples of these representations if necessary, we can assume that they have a common dimension. This gives a G -invariant family $\{V_p\}$ such that $\langle V_p|_E, 1_E \rangle = 0$ for every elementary abelian p -subgroup $E \leq G$ with $\text{rank } E = 2$. Applying Theorem B to this G -invariant family, we obtain a finite G -CW-complex X homotopy equivalent to a sphere S^{2kn-1} , for some $k \geq 1$, such that $\text{Res}_{G_p}^G X$ is p -locally G_p -equivalent to $S(V_p^{\oplus k})$, for every $p \mid |G|$. In particular, for every p -subgroup $H \leq G$, the fixed point space X^H has the same p -local homological dimension as the fixed point sphere $S(V_p^{\oplus k})^H$. Since $S(V_p)^E = \emptyset$, we have $S(V_p^{\oplus k})^H = \emptyset$ for every subgroup $H \leq G$ with $\text{rank}(H) = 2$. Hence the isotropy subgroups of X are all rank one subgroups with prime power order. \square

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