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On necklaces inside thin subsets of $\mathbb{R}^d$

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We study occurrences of point configurations in subsets of $\mathbb{R}^d$, $d \geq 3$. Given a finite collection of points, a well-known question is: How high does the Hausdorff dimension $\dim_{\mathcal{H}}(E)$ of a compact set $E \subset \mathbb{R}^d$ need to be to ensure that E contains some similar copy of this configuration? We study a related problem, showing that, for $\dim_{\mathcal{H}}(E)$ sufficiently large, E must contain a continuum of point configurations which we call *k-necklaces of constant gap*. Rather than a single geometric shape, a constant-gap k-necklace encompasses a family of configurations and may be viewed as a higher dimensional generalization of equilateral triangles and rhombuses in the plane. Our results extend and complement those in [1, 3], where related questions were recently studied.

1. Introduction

The study of finite point configurations in sets of various sizes spans analysis, ergodic theory, number theory and combinatorics. A corollary (due to Steinhaus) of the Lebesgue density theorem states that any measurable set in $\mathbb{R}^d$ with positive Lebesgue measure contains a similar copy of any finite configuration of points. There are many variations on this result. For instance, instead of sets of positive Lebesgue measure, one can consider an unbounded set $E \subset \mathbb{R}^d$ of positive upper Lebesgue density, in the sense that

$$\limsup_{R\to\infty} \frac{\left|E \cap [-R,R]^d\right|}{(2R)^d} > 0.$$

Here $|\cdot|$ denotes the d-dimensional Lebesgue measure. Bourgain [2] proved that E contains similar copies of any non-degenerate $(k+1)$-point configuration, i.e., a k-simplex, $k \leq d$, for all sufficiently large dilation parameters. (See also Furstenberg, Katznelson and Weiss [6].) Ziegler [19] generalized this result for $k \geq d$, but the sufficiently large copies of the configuration are only shown to be contained in an arbitrarily small neighborhood of E, rather than in E itself. In particular, results of this type show that we can recover every simplex similarity type inside a subset of $\mathbb{R}^d$ that is "thick", either in the sense of positive Lebesgue measure or of positive upper Lebesgue density. A question of interest is whether similar conclusions continue to hold even if such thickness assumptions are weakened. However, the following result due to Maga [10] shows that the conclusion in general fails for Lebesgue null sets in $\mathbb{R}^d$, *even if the set under consideration is of full Hausdorff dimension.* Let $\dim_{\mathcal{H}}(E)$ denote the Hausdorff dimension of a set $E \subset \mathbb{R}^d$.

Theorem 1.1. *(Maga [10]) The following conclusions hold.*

(a) For any $d \geq 2$, there exists a compact set $A \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}}(A) = d$ such that A does not contain the vertices of any parallelogram.

(b) If $d = 2$, then given any nondegenerate triple of points x^1, x^2, x^3 in $\mathbb{R}^2$, there exists a compact set $A \subset \mathbb{R}^2$ with $\dim_{\mathcal{H}}(A) = 2$ such that A does not contain the vertices of any triangle similar to $\triangle x^1x^2x^3$.

In view of Maga's result, it is reasonable to ask whether interesting specific point configurations can be found inside thin sets under additional structural hypotheses. This question has been recently addressed by Chan, Łaba and Pramanik [3], where the authors establish the existence of certain finite point configurations in sets of sufficiently high Hausdorff dimension and carrying a Borel measure with decaying Fourier transform. (The measure should also satisfy certain size bounds for Euclidean balls.) The point configurations obtained in [3] were required to obey appropriate nondegeneracy constraints when expressed as a linear system, and included both geometric and algebraic patterns such as corners in the plane or parallelograms in $\mathbb{R}^d$. However, some natural configurations do not satisfy the non-degeneracy assumption of [3]. For example, *corners* in $\mathbb{R}^3$, defined as 4-tuples (x, y, z, w) of points in $\mathbb{R}^3$ such that

$$\begin{aligned} (x-y) \perp (x-z), \quad (x-y) \perp (x-z), \quad (x-z) \perp (x-w), \\ |x-y| = |x-y| = |x-w|, \end{aligned} \tag{1.1}$$

are not expressible as a linear system and hence this configuration not covered by the setup of [3]. The same conclusion applies to a *nonplanar* (i.e., not necessarily planar) *rhombus* in $\mathbb{R}^3$, defined as a 4-tuple (x, y, z, w) such that $|x-y| = |y-z| = |z-w| = |w-x|$.

It is also interesting to ask which point configurations can be recovered without extra assumption on the Fourier decay. In view of Maga's results, one cannot hope to prove nontrivial results of this type for configurations

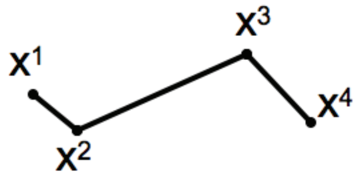

Figure 1: A 3-chain

that contain a planar loop. However it still seems plausible that we may be able to handle tree-like point configurations and loops that are not contained in a plane and hence enjoy greater directional freedom. This question is partially addressed in [1]. To present this result, we need the following definition.

Definition 1.2. A *k-chain* in $E \subset \mathbb{R}^d$ *with gaps* $\{t_i\}_{i=1}^k$ is a sequence

$$\{x^1, x^2, \ldots, x^{k+1} : x^j \in E,\ \ |x^{i+1} - x^i| = t_i > 0,\ \ 1 \leq i \leq k\}.$$

The k-chain has *constant gap* $t > 0$ if all the $t_i = t$. Finally, we say that the chain is *non-degenerate* if all the points x^i are distinct.

See Fig. 1 for an example of a 3-chain.

Theorem 1.3. *(Bennett, Iosevich and Taylor [1]) Suppose that $E \subset \mathbb{R}^d$ is a compact set, $d \geq 2$, and that $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$. Then for any $k \geq 1$, there exists an open interval $I \subset \mathbb{R}$, such that for each $t \in I$ there exists a non-degenerate k-chain in E with constant gap t.*

The idea behind the proof of Thm. 1.3 was to construct a measure on all k-chains, naturally induced from a Frostman measure μ on E, and consider its Radon-Nikodym derivative. It is bounded from above in all cases, and from below in the case when all the gaps are in a suitable interval. The lower bound was obtained using the continuity of the distance measure in appropriate dimensional regimes. An upper bound was shown using a fractal variant of the classical Parseval identity recently established by Iosevich, Sawyer, Taylor and Uriarte-Tuero [7], based on an earlier result of Strichartz [17].

In practice, this approach amounts to obtaining upper and lower bounds on the quantity

$$C_k^\epsilon(\mu) = \int \cdots \int \prod_{j=1}^{k} \sigma_t^\epsilon(x^{j+1} - x^j) d\mu(x^j) \tag{1.2}$$

that are uniform in ϵ. Here and throughout the paper, σ_t will denote the normalized surface measure on the sphere of radius t and $\sigma_t^\epsilon = \sigma_t * \rho_\epsilon$, with $\rho \geq 0$ a smooth cut-off function, $\int \rho(x) = 1$ and $\rho_\epsilon(x) = \epsilon^{-d}\rho\left(\frac{x}{\epsilon}\right)$.

An analogous multilinear form, expressed in terms of the Fourier transforms of measures rather than the measures themselves, was used in [3] as well. There, a finite upper bound on the form justified its existence and definition; a nontrivial lower bound then established the existence of the linear configurations. We follow a similar approach here. One also needs to ascertain that the configurations thus obtained are indeed nontrivial, by verifying that the size of the contributions from degenerate trivial configurations to the multilinear form is small.

While the results in [1, 3] were focused on point configurations that do not contain loops, we shall see that both the lower bound and the upper bound idea in [1], combined with the generalized three-lines lemma approach in [7], allow us to capture configurations that were inaccessible by these previous methods. In particular, we are able to establish the existence of a continuum of nonplanar rhombuses (with sides lengths ranging over an interval) in dimensions three and higher, as well as more complicated closed loops. We now turn to a precise formulation of our results.

2. Statement of results

Definition 2.1. A *k-necklace* in $E \subset \mathbb{R}^d$, $d \geq 2$, *with gaps* $\tilde{\mathbf{t}} = (t_1, t_2, \ldots, t_k)$, $t_j > 0$, is a finite sequence $x^1, x^2, \ldots, x^k$, $x^j \in E$, such that $|x^j - x^{j+1}| = t_j$, $1 \leq j \leq k-1$ and $|x^k - x^1| = t_k$. We say that this necklace is *non-degenerate* if $x^i \neq x^j$ for any $1 \leq j \leq k$, and *has constant gap* $t > 0$ if $t_1 = \cdots = t_k = t$.

Remark 2.2. Thus, a k-necklace is a closed k-chain (see Fig. 2), and being of constant gap t is the same as all edges being of equal length.

Definition 2.3. For $d \geq 3$, a *nonplanar rhombus* of side length $t > 0$ in $\mathbb{R}^d$ is a non-degenerate 4-necklace of constant gap $t > 0$. (Note that generically a nonplanar rhombus does not lie in a plane, but that possibility is allowed.)

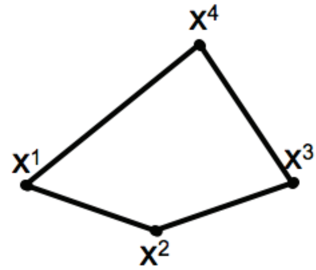

Figure 2: A 4-necklace

Remark 2.4. In general, a k-necklace with a given gap vector $\tilde{\mathbf{t}}$ is a member of a union of a continuum of similarity classes of k-simplices, rather than being a similar copy of a specific k-simplex.

We can now state our main result.

Theorem 2.5. *Let E be a compact subset of $\mathbb{R}^d$, $d \geq 3$.*

i) Suppose that $d \geq 4$, $k \geq 4$ is even and $dim_{\mathcal{H}}(E) > \frac{d+3}{2}$, without any additional assumptions on measures carried by E. Then there exists a non-empty open interval I such that for every $t \in I$, E contains some k-necklace with constant gap t.

ii) Suppose that $d \geq 3$. Suppose that for some $\delta > 0$, $\dim_{\mathcal{H}}(E) > d - \delta$ and there exists a Borel measure μ supported on E obeying both a ball condition and a Fourier decay condition, namely

$$\mu(B(x;r)) \leq Cr^{d-\delta} \text{ for all } r > 0, \text{ and} \tag{2.1}$$

$$|\widehat{\mu}(\xi)| \leq C|\xi|^{-1-\frac{\delta}{2}}, \quad \forall \xi \in \mathbb{R}^d. \tag{2.2}$$

Then there exists a non-empty open interval I such that for every $t \in I$, E contains a nonplanar rhombus of side length t.

Remark 2.6. We note that our argument does not give any quantitative information on the size of the interval I. This is analogous to the situation that arises in the proof of the celebrated Steinhaus theorem, which says that the difference set of a set of positive Lebesgue measure contains an open ball.

Remark 2.7. It would be interesting to extend Thm. 2.5(i) to cover the case of k odd. At least in the case $k = 3$, i.e., triangles, the conclusion is

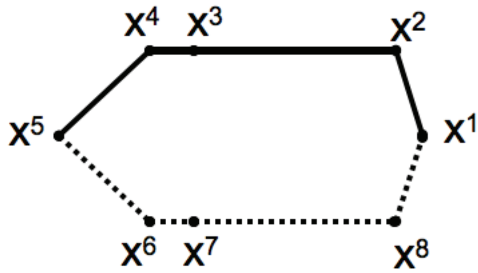

Figure 3: A 4-chain $x^1, \ldots, x^5$ generates an 8-necklace $x^1, \ldots, x^8$.

certainly false for $d = 2$ in view of Maga's counterexample Thm. 1.1(b); however, there are recent positive results for *equilateral* triangles in dimensions $d \geq 4$ [8] .

Remark 2.8. If the $\dim_{\mathcal{H}}(E) = s$, then in (2.2), $1 + \frac{\delta}{2} \leq \frac{s}{2}$. In particular, if E is a Salem set [11] in $\mathbb{R}^d$ of dimension $s > \frac{d+2}{2}$ supporting a sequence of measures that simultaneously realize the Hausdorff and Fourier dimensions arbitrarily closely, then E contains the vertices of a rhombus.

Remark 2.9. While Thm. 2.5 is stated for necklaces with constant gaps, a careful examination of the proof shows that we can say a bit more:

Definition 2.10. A non-degenerate $(n-1)$-chain in $\mathbb{R}^d$, with vertices $x^1, x^2, \ldots, x^n$, *generates* a non-degenerate $(2n-2)$-*necklace*, with vertices $x^1, x^2, \ldots, x^{2n-2}$, if

$$|x^{j+1} - x^{j+2}| = |x^{2n-j} - x^{2n-1-j}| \quad \text{for } 2 \leq j \leq n,$$
$$\text{and} \quad |x^1 - x^2| = |x^{2n-2} - x^1|.$$

(See Fig. 3 for an example of a 4-chain generating an 8-necklace).

The proof of Thm. 2.5 (i) shows that in fact the conclusion holds for *any necklace with an even number of vertices which is generated by a non-degenerate chain.*

3. Proof of main theorem

3.1. Preliminary results

For the proof of Thm. 2.5, we need the following two items: a theorem and a lemma/definition. The first is from [7], stated in the form needed here.

Theorem 3.1. *([7, Thm 1.1]) Let $K \in \mathcal{S}'(\mathbb{R}^d)$ be a tempered distribution satisfying*

$$|\widehat{K}(\xi)| \leq A_0 |\xi|^{-\gamma}, \ \gamma \in \left(0, \frac{d}{2}\right).$$

For $\epsilon > 0$, let $K^\epsilon = K * \rho_\epsilon$, where ρ_ϵ is the approximation to the identity defined below (1.2). Suppose that ϕ, ψ are compactly supported Borel measures on $\mathbb{R}^d$ satisfying

$$\phi(B(x,r)) \leq C_0 r^{s_\phi}, \qquad \psi(B(x,r)) \leq C_0 r^{s_\psi} \qquad \text{for all } r > 0, \tag{3.1}$$

with $s_\phi, s_\psi > 0$. Let $T_{K^\epsilon} f := K^\epsilon * (f\phi)$. Suppose that $\gamma > d - s$, where $s = \frac{s_\phi + s_\psi}{2}$. Then

$$||T_{K^\epsilon} f||_{L^2(\psi)} \leq C_0 A_0 C ||f||_{L^2(\phi)}$$

where C is a constant that depends on d, s_ϕ and s_ψ but not on ϵ, C_0, A_0.

Secondly, we need the following. The dimension assumptions in our main theorem will be exploited using measures obeying conditions of the form (3.1), the existence of which are ensured by Frostman's Lemma (see [11]):

Lemma 3.2. *Given a compact Borel set $E \subset \mathbb{R}^d$ of Hausdorff dimension s, and any $\eta > 0$, there exists a Borel probability measure μ_η supported on E and a constant $C_\eta > 0$ such that $\mu_\eta(B(x,r)) \leq C_\eta r^{s-\eta}$ for all $0 < r \leq$ diam(E).*

We will call a generic member of the collection $\{\mu_\eta : \eta > 0\}$ a *Frostman measure* and denote it by μ, with the understanding that the analysis in the sequel is carried out for $\mu = \mu_\eta$, with arbitrarily small η, in order to arrive at the conclusion of the theorem.

3.2. Proof of Theorem 2.5 (i) for $k \geq 6$

We start the proof of Thm. 2.5 by noting that, since k in its statement is even, we may write $k = 2n - 2$ with $n \geq 3$ an integer. For the moment, assume that $k \geq 6$, so that $n \geq 4$; the case $k = 4$ will be treated in §§3.3.

Let μ be a Frostman measure supported on the set E. Define

$$\mathcal{N}^\epsilon(\mu) = \int \cdots \int \left\{ \prod_{j=1}^{k-1} \sigma_t^\epsilon(x^{j+1} - x^j) d\mu(x^j) \right\} \sigma_t^\epsilon(x^k - x^1) d\mu(x^k), \tag{3.2}$$

which can be interpreted as the Radon-Nikodym derivative of a natural measure on the k-necklace with gaps $\equiv t$. Since σ_t^ϵ is a bounded function, $\mathcal{N}^\epsilon$ is well-defined and finite, with an upper bound that depends apriori on ϵ.

Using $k = 2n - 2$, observe that we can write

$$\mathcal{N}^\epsilon(\mu) = \iint \left\{ \int \cdots \int \prod_{j=1}^{n-1} \sigma_t^\epsilon(x^{j+1} - x^j) \prod_{j=2}^{n-1} d\mu(x^j) \right\}^2 d\mu(x^1) d\mu(x^n). \tag{3.3}$$

We will obtain both upper and lower bounds for $\mathcal{N}^\epsilon(\mu)$ as ϵ tends to 0, and we must show that at least some of the necklaces obtained this way are non-degenerate. More precisely, we show that $\mathcal{N}^\epsilon(\mu)$ is finite and bounded from above independently of ϵ, and that the $\liminf_{\epsilon \to 0}$ of this quantity is bounded from below. Furthermore, the configurations thus obtained are non-degenerate in the sense that all the vertices are distinct. The upper bound is established in §§3.2.1. The lower bound and non-degeneracy is already established in [1] and a further comment on this is in §§3.4.1.

3.2.1. Upper bound. Define $\mathcal{N}^{\epsilon,\alpha}$ by the formula

$$\mathcal{N}^{\epsilon,\alpha} = \iint |F^{\epsilon,\alpha}(x^1, x^n)|^2 d\mu(x^1) d\mu(x^n), \tag{3.4}$$

where, for $\alpha \in \mathbb{C}$,

$$F^{\epsilon,\alpha}(x^1, x^n) = \int \cdots \int \sigma_t^{\epsilon,-\alpha}(x^2 - x^1) \sigma_t^{\epsilon,-\alpha}(x^n - x^{n-1}) \times \prod_{j=2}^{n-2} \sigma_t^{\epsilon,\alpha}(x^{j+1} - x^j) \prod_{j=2}^{n-1} d\mu(x^j), \tag{3.5}$$

and

$$\sigma^{\epsilon,\alpha} = \sigma^\alpha * \rho_\epsilon, \text{ with } \sigma^\alpha(x) = \frac{1}{\Gamma(\alpha)} (1 - |x|^2)_+^{\alpha - 1}. \tag{3.6}$$

Recall the well-known fact (see e.g. [14, 16]) that $\sigma^0 = \sigma$ and

$$|\widehat{\sigma}_t^\alpha(\xi)| \le C_a |\xi|^{-\frac{d-1}{2} - a} \quad \text{for } \alpha = a + ib, \ |a| \le 1. \tag{3.7}$$

$\mathcal{N}^{\epsilon,\alpha}$ is an analytic function of α obeying appropriate growth conditions (to be proved below) on an infinite vertical strip straddling the imaginary axis, and that $\mathcal{N}^{\epsilon,0} = \mathcal{N}^{\epsilon}$, making it amenable to Hirschman's version of the three lines lemma and Stein's analytic interpolation theorem [15]. The claimed upper bound is therefore a consequence of the following estimate: there exists a small $\delta_0 > 0$ and a finite constant $C = C(n, \delta_0)$ independent of ϵ such that for all $\delta_0 \le \delta \le 1$,

$$\left|\mathcal{N}^{\epsilon,1-\delta+iu}\right| + \left|\mathcal{N}^{\epsilon,-1+\delta+iu}\right| \le Ce^{C|u|}. \tag{3.8}$$

We first consider the case $\mathrm{Re}(\alpha) = 1-\delta$, $\delta_0 \le \delta \le 1$. Let $\alpha = 1-\delta-iu$. Observe that in this case $|\sigma_t^{\epsilon,\alpha}|$ is uniformly bounded in x, with $|\sigma_t^{\alpha}(x)| \le C/|\Gamma(\alpha)|$ where the constant C is uniformly bounded for t away from zero. It follows from Stirling's formula that

$$\begin{aligned}
|\mathcal{N}^{\epsilon,\alpha}(\mu)| \le \left(\frac{C}{|\Gamma(\alpha)|}\right)^{n-4} &\iint \left\{\int \cdots \int G(x^1, x^2, x^{n-1}, x^n) \prod_{j=2}^{n-1} d\mu(x^j)\right\}^2 \\
&\times d\mu(x^1)\, d\mu(x^n) \\
&\le C_n e^{C_n|u|} \iint \left\{\iint G(x^1, x^2, x^{n-1}, x^n)\, d\mu(x^2) d\mu(x^{n-1})\right\}^2 \\
&\times d\mu(x^1)\, d\mu(x^n),
\end{aligned}$$

where

$$G = G(x^1, x^2, x^{n-1}, x^n) = |\sigma_t^{\epsilon,-1+\delta+iu}(x^2 - x^1)||\sigma_t^{\epsilon,-1+\delta+iu}(x^n - x^{n-1})|.$$

Let us denote $|\sigma_t^{\epsilon,\alpha}(x)| =: \lambda^{\epsilon,\alpha}(x)$, and observe that

$$|\sigma_t^{\alpha}(x)| = \frac{|\Gamma(Re(\alpha))|}{|\Gamma(\alpha)|}\sigma_t^{Re(\alpha)}(x); \tag{3.9}$$

Therefore,

$$\begin{aligned}
&\iint G(x^1, x^2, x^{n-1}, x^n) d\mu(x^2)\, d\mu(x^{n-1}) \\
&= \lambda^{\epsilon,-1+\delta+iu} * \mu(x^1) \lambda^{\epsilon,-1+\delta+iu} * \mu(x^n).
\end{aligned}$$

Substituting this into the bound for $\mathcal{N}^{\epsilon,\alpha}$ leads to the following estimate:

$$\left|\mathcal{N}^{\epsilon,1-\delta-iu}\right| \le C_n e^{C_n|u|} \left\|\lambda^{\epsilon,-1+\delta} * \mu\right\|_{L^2(\mu)}^2 \le C_n e^{C_n|u|}.$$

The last inequality is a consequence of

$$\int \left(\lambda^{\epsilon,-1+\delta+iu} * \mu(x)\right)^2 d\mu(x) \le C_n e^{C_n|u|}, \tag{3.10}$$

where μ is a Frostman measure on a set of Hausdorff dimension $> \frac{d+3}{2} - \delta$. Since the relations (3.7) and (3.9) imply that

$$|\widehat{\lambda^{\epsilon,-1+\delta+iu}}(\xi)| \le C_n e^{C_n|u|} |\xi|^{-\frac{d-3}{2}-\delta}, \tag{3.11}$$

the claim follows from Thm. 3.1 with $\gamma = \frac{d-3}{2} + \delta$.

It remains to consider the case $\mathrm{Re}(\alpha) = -1+\delta$, $\delta_0 \le \delta \le 1$. In order to estimate $\mathcal{N}^{\epsilon,-1+\delta+iu}$, we first apply the uniform pointwise bound on $\sigma_t^{\epsilon,1-\delta-iu}$ used in the previous case, arriving at

$$\begin{aligned} &C_n^{-1} e^{-C_n|u|} |\mathcal{N}^{\epsilon,-1+\delta+iu}(\mu)| \\ &\le \iint \left\{ \int \cdots \int \prod_{j=2}^{n-2} |\sigma_t^{\epsilon,-1+\delta+iu}(x^{j+1}-x^j)| \prod_{j=2}^{n-1} d\mu(x^j) \right\}^2 d\mu(x^1) d\mu(x^n) \\ &\le \iint \left\{ \int \cdots \int \prod_{j=2}^{n-2} \lambda^{\epsilon,-1+\delta+iu}(x^{j+1}-x^j) \prod_{j=2}^{n-1} d\mu(x^j) \right\}^2 d\mu(x^1) d\mu(x^n). \end{aligned} \tag{3.12}$$

Let $g_1(x) = \lambda^{\epsilon,-1+\delta+iu} * \mu(x)$ and define inductively $g_j(x) = \lambda^{\epsilon,-1+\delta+iu} * (g_{j-1}\mu)(x)$. By inspection, the final expression in (3.12) equals the left side of

$$\begin{aligned} &\iint \left(\int g_{n-3}(x^{n-1}) d\mu(x^{n-1}) \right)^2 d\mu(x^n) d\mu(x^1) \\ &\le \int |g_{n-3}(x^{n-1})|^2 d\mu(x^{n-1}), \end{aligned} \tag{3.13}$$

with the domination by the right side being obtained using Cauchy-Schwarz and, twice, the fact that μ is a probability measure.

Given a measurable function h, let $Th(x) = \lambda^{\epsilon,-1+\delta+iu} * h(x)$. Then the right hand side of (3.13) equals

$$\int |Tg_{n-2}(x)|^2 d\mu(x).$$

Applying Thm. 3.1 repeatedly, recalling (3.11) and using that μ is a Frostman measure, we see that this expression is $\leq C_n e^{C_n|u|}||g_1||^2_{L^2(\mu)}$, provided that

$$\dim_{\mathcal{H}}(E) > d - \frac{d-3}{2} - \delta = \frac{d+3}{2} - \delta,$$

which is holds given our hypothesis that $\dim_{\mathcal{H}}(E) > \frac{d+3}{2}$. Applying Thm. 3.1 one last time, we see that $||g_1||_{L^2(\mu)}$ is finite and the proof of the upper bound when $n \geq 4$ is completed, as explained earlier, by applying the three lines lemma and analytic interpolation [15].

The proof of Thm. 2.5(i) will be complete once we address the upper bound in the case $k = 4$ and prove that at least some of the k-necklaces obtained are non-degenerate.

3.3. Proof of upper bound for $k = 4$

Consider

$$\mathcal{N}^{\epsilon,\alpha} = \iint \left\{ \int \sigma_t^{\epsilon,\alpha}(x-z)\sigma_t^{\epsilon,-\alpha}(y-z)d\mu(z) \right\}^2 d\mu(x)d\mu(y). \tag{3.14}$$

Suppose that $\text{Re}(\alpha) = 1 - \delta$. Then $|\mathcal{N}^{\epsilon,1-\delta+iu}|$ is bounded, upto a constant factor of exponential growth in u, by

$$\begin{aligned} &\iint \left\{ \int |\sigma_t^{\epsilon,-\alpha}(y-z)|d\mu(z) \right\}^2 d\mu(x)d\mu(y) \\ &= \iint \left\{ \int \frac{1}{|\Gamma(\alpha)|}(1-|y-z|^2)_+^{-Re(\alpha)-1} d\mu(z) \right\}^2 d\mu(x)d\mu(y). \end{aligned} \tag{3.15}$$

This quantity is bounded above by the proof of the case $Re(\alpha) = 1 - \delta, n \geq 4$ above. By symmetry, we arrive at the same expression for $\text{Re}(\alpha) = -1 + \delta$, reversing the roles of the variables x and y. Thus analytic interpolation takes care of the upper bound.

3.4. Proof of Theorem 2.5 (ii)

Rewrite the expression in (3.15) above in the form

$$\iint \left|\lambda^{\epsilon,-1+iu} * \mu(y)\right|^2 d\mu(y)d\mu(x) = \int \left|\lambda^{\epsilon,-1+iu} * \mu(x)\right|^2 d\mu(x).$$

Before we apply Thm. 3.1, we need a simple calculation. Treating $\lambda^{\epsilon,-1+iu}$ as a measure, we observe that

$$|\lambda^{\epsilon,-1+iu}(B(x,r))| \leq Cr^{d-2}.$$

The proof follows by a direct calculation. We now apply Thm. 3.1 with the roles of μ and λ reversed, i.e., by setting $K = \mu$, $\phi = \lambda^{\epsilon,-1+iu}$ and $\psi = \mu$. We are assuming that

$$|\widehat{\mu}(\xi)| \leq C|\xi|^{-\gamma}$$

for some $\gamma > 0$. It follows that the $L^2(\phi) \to L^2(\psi)$ bound holds, with $f \equiv 1$ if

$$\gamma > d - \frac{d-2+s}{2} = \frac{d}{2} + 1 - \frac{s}{2}.$$

In particular, this means that if $s = d - \delta$, for some $\delta > 0$, then

$$\gamma > 1 + \frac{\delta}{2}.$$

3.4.1. Lower bound and non-degeneracy. Applying Cauchy-Schwarz to (3.3) we see that it suffices to obtain a lower bound for

$$\int \cdots \int \left\{ \prod_{j=1}^{n-1} \sigma_t^{\epsilon}(x^{j+1} - x^j) d\mu(x^j) \right\} d\mu(x^n). \tag{3.16}$$

(In other words, the Cauchy-Schwarz inequality turns a chain into a necklace. The case $n = 5$ is depicted in Fig. 3 above.)

Observe that the quantity in (3.16) equals $\int d\mu_n^{\epsilon}(x)$. The fact that $\liminf_{\epsilon \to 0}$ of this quantity is bounded from below is established in [1, §§2.2], building upon previous results in [12] and [9]. The fact that the resulting chain is non-degenerate is established in §3 of the same paper, so that the non-degeneracy of the necklaces follows as a special case.

4. Concluding remarks

We close by putting the methods of this paper into perspective and describing their limitations. The general approach of this paper can be described as follows. One uses Cauchy-Schwarz to associate a chain to a given necklace with an even number of vertices. This procedure allows one to obtain an immediate lower bound on the Radon-Nikodym derivative of the natural candidate for the measure on the set of necklaces with prescribed gaps.

We then obtain an upper bound on the Radon-Nikodym derivative using analytic interpolation theorem, thus completing the proof of the assertion that vertices of the necklace can be found inside a compact subset of $\mathbb{R}^d$ of a sufficiently large Hausdorff dimension.

The method of proof described above suggests that further progress may be possible if we use the results of this paper and then create more elaborate point configuration by the means of the Cauchy-Schwarz or Hölder's inequalities. What types of configuration can we hope to obtain in this way? In order to see a general framework for such results, start with a 4-necklace and apply the Cauchy-Schwarz inequality in the x^1, x^2, x^3-variables. We obtain

$$\left[\int_{E^4} \prod_{j=1}^{3} \sigma^\epsilon(x^j - x^{j+1})\sigma^\epsilon(x^4 - x^1) \prod_{j=1}^{4} d\mu(x^j)\right]^2$$

$$\leq \int_E \left\{\int_{E^3} \prod_{j=1}^{3} \sigma^\epsilon(x^j - x^{j+1})\sigma^\epsilon(x^4 - x^1) \prod_{j=1}^{3} d\mu(x^j)\right\}^2 d\mu(x^4) \tag{4.1}$$

$$\leq \int_{E^7} \sigma^\epsilon(x^1 - x^4) \cdot \prod_{j=1}^{3} \sigma^\epsilon(x^{j+1} - x^j) \tag{4.2}$$

$$\times\, \sigma^\epsilon(x^4 - x^7) \cdot \prod_{j=4}^{6} \sigma^\epsilon(x^{j+1} - x^j) \prod_{j=1}^{7} d\mu(x^j),$$

which is the Radon-Nikodym derivative of the natural measure on two 4-necklaces sharing the vertex x^4. See Fig 4.

Obtaining an upper bound for (4.2) is possible with some work. We outline the argument because it leads to interesting harmonic analysis and illustrates the rich set of connections between geometric problems and harmonic analytic inequalities that these questions foster. Recalling the idea behind (3.14), we can express (4.1) in the form

$$\iiint F^2(x^2, x^4) G^2(x^7, x^4) d\mu(x^2) d\mu(x^4) d\mu(x^6), \tag{4.3}$$

where

$$F(x^2, x^4) = \int \sigma^\epsilon(x^2 - x^1)\sigma^\epsilon(x^4 - x^1) d\mu(x^1), \text{ and}$$

$$G(x^7, x^4) = \int \sigma^\epsilon(x^7 - x^4)\sigma^\epsilon(x^7 - x^6) d\mu(x^7).$$

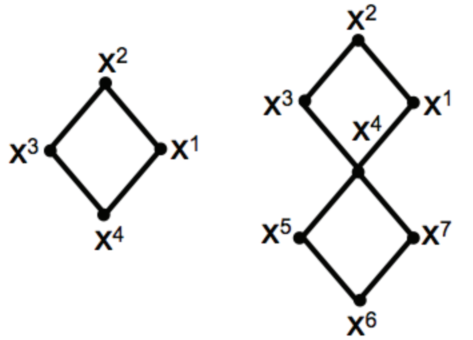

Figure 4: Cauchy-Schwarz turns a necklace into two necklaces sharing a point

Applying Cauchy-Schwarz yet again reduces matters to bounding the quantity

$$\iint \left\{ \int \sigma^\epsilon(x^2 - x^1)\sigma^\epsilon(x^4 - x^1)d\mu(x^1) \right\}^4 d\mu(x^2)d\mu(x^4). \tag{4.4}$$

We point out a difference between this quantity and (3.14), the expression we needed to bound to handle the rhombus (4-necklace). In (3.14) the inner expression in (4.4) is raised to the power of 2 instead of the power of 4. This naturally leads us to consider the L^4 version of Thm. 3.1, which can be obtained, with a worse yet still non-trivial lower bound on exponents s_ϕ and s_ψ, corresponding to the dimensional restriction, by a rather straightforward modification of the proof.

By the same method we can start with any necklace with an even number of vertices and, by applying Hölder's inequality with the integer exponent $m \geq 2$, obtain m necklaces sharing a common vertex. While it would be difficult to classify succinctly all the point configurations that can be obtained by starting with a chain and successively applying Hölder's inequality, this example is quite representative and also illustrates the limitations of our method.

There exist geometric configurations which cannot be handled either by the methods of this paper, or those in [3]. For example, the three-dimensional corner, described in (1.1,) appears to be outside the reach of both techniques, and new ideas would be required to handle this configuration.

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