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On the expansion of certain vector-valued characters of $U_q(\mathfrak{gl}_n)$ with respect to the Gelfand-Tsetlin basis

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Macdonald polynomials are an important class of symmetric functions, with connections to many different fields. Etingof and Kirillov showed an intimate connection between these functions and representation theory: they proved that Macdonald polynomials arise as (suitably normalized) vector-valued characters of irreducible representations of quantum groups. In this paper, we provide a branching rule for these characters. The coefficients are expressed in terms of skew Macdonald polynomials with plethystic substitutions. We use our branching rule to give an expansion of the characters with respect to the Gelfand-Tsetlin basis. Finally, we study in detail the $q = 0$ case, where the coefficients factor nicely, and have an interpretation in terms of certain p-adic counts.

1. Introduction

Macdonald polynomials were originally discovered in the 1980s [9, 10], and have found a variety of uses in mathematics, appearing in a number of disparate fields (mathematical physics, combinatorics, representation theory and number theory, among others). These polynomials have the key property of being invariant under all permutations of their n variables. They are indexed by partitions λ with length at most n, and form an orthogonal basis for the ring of symmetric polynomials with coefficients in $\mathbb{C}(q, t)$ with respect to a certain density function. The existence of such polynomials was proved by exhibiting particular difference operators which have these polynomials as their eigenfunctions. Macdonald polynomials contain many important families as particular degenerations of the parameters q and t. In particular, the ubiquitous Schur functions are obtained by setting $q = t$; crucially, these are

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characters of irreducible representations of GL_n. Hall-Littlewood polynomials are recovered in the limit $q = 0$, and these have interpretations as zonal spherical functions on p-adic groups. Some other important subfamilies are the monomial, elementary, and power sum symmetric functions.

Given the various connections to representation theory, one might ask whether Macdonald polynomials arise as characters of certain irreducible representations. Etingof and Kirillov discovered such a realization in [2], where they demonstrate that Macdonald polynomials are ratios of vector-valued characters of representations of the quantum group $U_q(\mathfrak{gl}_n)$. Recall that the finite-dimensional, irreducible representations V_λ of $U_g(\mathfrak{gl}_n)$ are indexed by $\lambda \in \mathcal{P}_+^{(n)} = \{(\lambda_1, \dots, \lambda_n) : \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+\}$. Note that elements of $\mathcal{P}_+^{(n)}$ can be written as $(a, a, \dots, a) + \widetilde{\lambda}$, where $a \in \mathbb{C}$ and $\widetilde{\lambda}$ is a standard partition of length n. Let $k \in \mathbb{N}$ be fixed, then they show the existence of an intertwining operator (unique up to scaling):

$$\phi_\lambda^{(k)} : V_{\lambda+(k-1)\rho} \to V_{\lambda+(k-1)\rho} \otimes U, \tag{1}$$

where $U \simeq V_{(k-1)\cdot(n-1,-1,\dots,-1)}$ and $\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2}\right)$. Note that U has the special property that all weight subspaces are one-dimensional. Fix the normalization of $\phi_\lambda^{(k)}$ so that $v_{\lambda+(k-1)\rho} \to v_{\lambda+(k-1)\rho} \otimes u_0 + \cdots$, where $v_{\lambda+(k-1)\rho}$ is a fixed non-zero highest weight vector for $V_{\lambda+(k-1)\rho}$ and u_0 is a fixed non-zero vector in the (one-dimensional) weight zero subspace of U. Consider the corresponding trace function of this operator:

$$\Phi_\lambda^{(k)}(x_1, \dots, x_n) = \mathrm{Tr}(\phi_\lambda^{(k)} \cdot x^h) \in \mathbb{C}(q)[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

where $x^h \cdot v = x_1^{w_1} \cdots x_n^{w_n} \cdot v$ for v a weight vector with weight $(w_1, \dots, w_n)$. Then Etingof and Kirillov proved the following intimate connection between these trace functions and Macdonald polynomials:

Theorem 1.1. *[2] There are formal power series*

$$\widetilde{\Phi}_\lambda(x; q, t) \in \mathbb{C}(q, t)[[x_1, \dots, x_n]]$$

uniquely determined by

$$\widetilde{\Phi}_\lambda(x_1, \dots, x_n; q, q^k) = \frac{1}{x^{(k-1)\rho_n}} \Phi_\lambda^{(k)}(x_1, \dots, x_n) \qquad (\textit{for all } k \in \mathbb{N}),$$

and the Macdonald polynomial $P_\lambda(x;q^2,t^2)$ is given by the ratio

$$P_\lambda(x;q^2,t^2)=\frac{\widetilde{\Phi}_\lambda(x;q,t)}{\widetilde{\Phi}_0(x;q,t)}.$$

In particular, for $k\in\mathbb{N}$ we have

$$P_\lambda(x;q^2,q^{2k})=\frac{\Phi^{(k)}_\lambda(x)}{\Phi^{(k)}_0(x)}.$$

The two-parameter family $\widetilde{\Phi}_\lambda(x;q,t)$ is constructed in [2] as the trace function of an intertwining operator analogous to (1), where $V_{\lambda+(k-1)\rho}$ and U are replaced by suitable infinite-dimensional, irreducible representations of $U_q(\mathfrak{gl}_n)$. Note that we renormalize the functions from [2] by a factor of $\frac{1}{x^{(k-1)\rho_n}}$ where k is a formal parameter satisfying $t=q^k$.

In this paper we consider the expansion of the trace function $\Phi^{(k)}_\lambda(x)$ with respect to the Gelfand-Tsetlin basis of $V_{\lambda+(k-1)\rho}$. We give explicit combinatorial formulas for the diagonal coefficients of the intertwining operator $\phi^{(k)}_\lambda$ with respect to this basis, as sums of products of well-known rational functions appearing in symmetric function theory, specialized to $t=q^k$. Inspired by Kashiwara's theory of crystal bases [4], we consider the $q\to 0$ limit of these coefficients (with algebraically independent t). We find that for $t=p^{-1}$ with p an odd prime, this limit is proportional to a count of certain chains of groups of p-adic type. While it is well-known that Hall-Littlewood polynomials are intimately related to p-adic representation theory, it is interesting to find p-adic quantities arising from quantum group constructions. Other connections between p-adic quantities and quantum groups have been provided in the literature, for example, we refer the interested reader to [11] for an earlier work which deals with an interpolation between real and p-adic limits using the quantum Grassmannian.

We will now state our results more precisely. First recall that a Gelfand-Tsetlin pattern of shape $\lambda\in\mathcal{P}^{(n)}_+$ is a sequence $\lambda=\lambda^{(0)}\succeq\lambda^{(1)}\succeq\cdots\succeq\lambda^{(n-1)}$, where $\lambda^{(i)}\in\mathcal{P}^{(n-i)}_+$ and $\succeq$ denotes the interlacing relation: $\lambda^{(i+1)}_j-\lambda^{(i)}_{j+1}\in\mathbb{Z}_+$, $\lambda^{(i)}_{j+1}-\lambda^{(i+1)}_{j+1}\in\mathbb{Z}_+$. This may be visualized as an array consisting of n rows with the parts of λ in the first row, parts of $\lambda^{(1)}$ in the second row, etc. There is a canonical basis of V_λ which is indexed by $GT(\lambda)$, the Gelfand-Tsetlin patterns of shape λ.

Our aim is to compute the expansion of the trace function $\Phi^{(k)}_\lambda$ in the Gelfand-Tsetlin basis for $V_{\lambda+(k-1)\rho_n}$. We would like to index these patterns in a uniform way with respect to the parameter $k \in \mathbb{N}$. Conveniently, there is a canonical way of doing this for the Gelfand-Tsetlin patterns whose coefficient in the expansion of $\Phi_\lambda^{(k)}$ is non-zero:

Definition 1.2. Let $\lambda \in \mathcal{P}_+^{(n)}$, and let $\lambda = \mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)}$ be such that:

1) $\mu^{(i)} \in \mathcal{P}_+^{(n-i)}$.
2) $\mu_j^{(i)} - \mu_j^{(i+1)} \in \mathbb{Z}_+$, $1 \leq j \leq n-i-1$.
3) $\mu_j^{(i)} - \mu_{j-1}^{(i+1)} \leq k-1$, $2 \leq j \leq n-i$.

Define $\overline{\mu}^{(i)} = \mu^{(i)} + (k-1)\rho_{n-i} + (k-1)\cdot(i/2,\ldots,i/2)$, then $(\overline{\mu}^{(0)} \succeq \overline{\mu}^{(1)} \succeq \cdots \succeq \overline{\mu}^{(n-1)})$ is a Gelfand-Tsetlin pattern of shape $\lambda + (k-1)\rho_n$.

We can now give our formula for the coefficients in the expansion of $\Phi_\lambda^{(k)}(x)$ in the Gelfand-Tsetlin basis, along with a new branching formula for the functions $\widetilde{\Phi}(x;q,t)$. This will be expressed in terms of functions $\psi_{\gamma/\delta}, \Omega_{\gamma/\delta}$, which are plethystic substitutions of skew Macdonald polynomials, and d_α, which is the norm with respect to a particular inner product, see Section 2 for more details.

Theorem 1.3. *For $\mu \subset \lambda$ with $\lambda \in \mathcal{P}_+^{(n)}$, $\mu \in \mathcal{P}_+^{(n-1)}$, define*

$$c_{\lambda,\mu}(q,t) = d_\mu(q^2,t^2) \sum_{\substack{\beta \in \mathcal{P}_+^{(n-1)} \\ \mu \subseteq \beta \preceq \lambda}} \frac{\psi_{\lambda/\beta}(q^2,t^2)}{d_\beta(q^2,t^2)} \Omega_{\beta/\mu}(q^2,t^2). \tag{2}$$

Then the trace functions $\widetilde{\Phi}(x;q,t)$ satisfy the branching rule:

$$\widetilde{\Phi}_\lambda^{(n)}(x;q,t) = \sum_{\substack{\mu \in \mathcal{P}_+^{(n+1)} \\ \mu \subset \lambda}} c_{\lambda,\mu}(q,t) \cdot \widetilde{\Phi}_\mu^{(n-1)}(x;q,t) \cdot x_n^{|\lambda|-|\mu|},$$

where $|\lambda| = \sum \lambda_i$.

Moreover, with respect to the Gelfand-Tsetlin basis of $V_{\lambda+(k-1)\rho_n}$, the diagonal coefficient of the intertwining operator $\phi_\lambda^{(k)}$ corresponding to the Gelfand-Tsetlin pattern $\Lambda = (\lambda^{(0)} \succeq \cdots \succeq \lambda^{(n-1)}) \in GT(\lambda + (k-1)\rho_n)$ is *equal to*

$$c_\Lambda(q,q^k) = \begin{cases} \prod_{1\le i\le n-1} c_{\overline{\mu}^{(i-1)},\overline{\mu}^{(i)}}(q,q^k), & \exists\, (\mu^{(0)} \supseteq \cdots \supseteq \mu^{(n-1)}) \\ & s.t.\ \lambda^{(i)} = \overline{\mu}^{(i)} \\ 0, & otherwise \end{cases}$$

Kashiwara's crystal bases [4] allow one to interpret finite-dimensional representations of $U_q(\mathfrak{gl}_n)$ in the "crystal limit" $q \to 0$. Remarkably, there is a rich combinatorial structure in the crystal limit. Since the Gelfand-Tsetlin basis yields a crystal basis for V_λ, it seems natural to consider the limit as $q \to 0$ of the coefficients in Theorem 1.3. A priori this limit need not even exist, but in fact we are able to obtain a simple closed formula, in a factorized form:

Theorem 1.4. *Let $\mu \subset \lambda$ with $\lambda \in \mathcal{P}_+^{(n)}$, $\mu \in \mathcal{P}_+^{(n-1)}$, then we have*

$$\lim_{q\to 0} c_{\lambda,\mu}(q,t) = \frac{b_\mu(t^2)}{b_\lambda(t^2)}(1-t^2)sk_{\lambda/\mu}(t^2) = t^{2\sum_j \binom{\lambda'_j-\mu'_j}{2}} \prod_{j\ge 1} \binom{\lambda'_j - \mu'_{j+1}}{\lambda'_j - \lambda'_{j+1}}_{t^2}. \tag{3}$$

Here the coefficients $sk_{\lambda/\mu}(t)$ are those studied in [7, 13] in the context of Pieri rules. A precise formula for $b_\mu(t^2)$ is found in the next section of the paper, in (9). Note that when $t = p^{-1}$ for an odd prime p,

$$sk_{\lambda/\mu}(t) = t^{n(\lambda)-n(\mu)}\alpha_\lambda(\mu;p),$$

where $\alpha_\lambda(\mu;p)$ is the number of subgroups of type μ in a finite abelian p-group of type λ. For $S = (\mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)})$ with $\mu^{(i)} \in \mathcal{P}_+^{(n-i)}$, we define the coefficient

$$sk_S(t) = sk_{\mu^{(0)}/\mu^{(1)}}(t)sk_{\mu^{(1)}/\mu^{(2)}}(t)\cdots sk_{\mu^{(n-2)}/\mu^{(n-1)}}(t). \tag{4}$$

Note that when $t = p^{-1}$, $sk_S(t)$ is (up to a power of t) the number of nested chains of subgroups with types specified by the sequence S. We also let

$$wt(S) = \big(|\mu^{(n-1)}|, |\mu^{(n-2)}| - |\mu^{(n-1)}|, \ldots, |\mu^{(1)}| - |\mu^{(2)}|, |\mu^{(0)}| - |\mu^{(1)}|\big).$$

Theorem 1.5. *Let $\lambda \in \mathcal{P}_+^{(n)}$. Then*

$$\lim_{q\to 0} \widetilde{\Phi}_\lambda(x;q,t) = \frac{(1-t^2)^n}{b_\lambda(t^2)} \sum_{\substack{S=(\lambda=\mu^{(0)}\supset\mu^{(1)}\supset\cdots\supset\mu^{(n-1)}) \\ \mu^{(i)}\in\mathcal{P}_+^{(n-i)}}} sk_S(t^2)x^{wt(S)}. \tag{5}$$

The simple combinatorial structure of our formula in the limit $q \to 0$ seems to suggest a possible connection to crystal bases, but we have not found a direct link and leave it as an open question to describe this relation more precisely.

The remainder of the paper is organized as follows. In Section 2 we include background definitions and results from symmetric function theory, most of this material is from [10]. Section 3 contains the proof of Theorem 1.3. Finally, in Section 4 we analyze the $q \to 0$ limit and prove Theorems 1.4 and 1.5.

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2. Background on symmetric function theory

As in the introduction, we let $P_\lambda(x;q,t)$ denote the Macdonald polynomials in parameters q,t. When unclear from context, we shall write $P_\lambda^{(n)}(x;q,t)$ to specify that these are polynomials in n-variables $(x_1,\ldots,x_n)$. We will also use this superscript similarly for other functions to denote the number of variables. We will now set up some notation from symmetric function theory that will be relevant throughout the paper.

Recall that $\lambda = (\lambda_1,\ldots,\lambda_n) \in (\mathbb{Z}_+)^n$ is a partition if $\lambda_i \geq \lambda_{i+1}$. The length of a partition, $l(\lambda)$, is the number of nonzero parts and the weight, $|\lambda|$, is the sum of the parts. We let $m_i(\lambda)$ denote the number of parts of λ equal to i. For $1 \leq k \leq n$, we will write $\lambda|_k$ to denote the restriction to the first k parts, i.e., the string $(\lambda_1,\ldots,\lambda_k)$.

There is a partial order on partitions defined by $\lambda > \mu$ if and only if $\sum \lambda_i = \sum \mu_i$ and for some $k < n$ we have $\lambda_i = \mu_i$ for all $i \leq k$ and $\lambda_{k+1} > \mu_{k+1}$. We will work with polynomials of n variables, i.e., over $\mathbb{C}[x_1,\ldots,x_n]$. For $\lambda \in \mathbb{Z}^n$, we let $x^\lambda = x_1^{\lambda_1}\cdots x_n^{\lambda_n}$.

We fix $k \in \mathbb{N}$, and set $t = q^k$. Let $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2})$ be half the sum of the positive roots; we will also write ρ_n when it is not clear from context. Note that

$$\rho_n - \rho_{n-1} = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2}\right) - \left(\frac{n-2}{2}, \frac{n-4}{2}, \dots, \frac{2-n}{2}, 0\right) = \left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1-n}{2}\right), \tag{6}$$

where the subtraction should be taken over n-tuples by appending a zero at the end of ρ_{n-1}.

We recall the q-Pochhamer symbol

$$(a;q)_k = \begin{cases} \prod_{j=0}^{k-1}(1 - aq^j), & \text{if } k > 0 \\ \prod_{j=1}^{|k|}(1 - aq^{-j})^{-1}, & \text{if } k < 0 \\ 1, & \text{if } k = 0. \end{cases}$$

We also let

$$(a;q) = \prod_{k \geq 0}(1 - aq^k).$$

We now define a number of different coefficients arising from symmetric function theory, see [10]; we also review some relevant results from the literature.

Definition 2.1. Define the functions $g(\gamma; q^2, t^2)$ for $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n_{\geq 0}$ by

$$g(\gamma; q^2, t^2) = t^{2|\gamma|} \frac{(t^{-2}q^2; q^2)_{\gamma_1} \cdots (t^{-2}q^2; q^2)_{\gamma_n}}{(q^2; q^2)_{\gamma_1} \cdots (q^2; q^2)_{\gamma_n}}.$$

Definition 2.2. Let $c^{\beta}_{\gamma\mu}(q,t)$ be the coefficients in the following Pieri rule:

$$m^{(n)}_{\gamma}(x) P^{(n)}_{\mu}(x; q, t) = \sum_{\delta} c^{\beta}_{\gamma\mu}(q,t) P^{(n)}_{\beta}(x; q, t),$$

where $m^{(n)}_{\gamma}(x)$ denotes the monomial symmetric functions.

We note that the coefficients $c^{\delta}_{\gamma\mu}$ can be determined via the change of basis coefficients $\{m^{(n)}_{\gamma}(x)\} \to \{P^{(n)}_{\eta}(x;q,t)\}$ in conjunction with the Pieri coefficients that express the product $P^{(n)}_{\eta}(x;q,t)P^{(n)}_{\mu}(x;q,t)$ in the Macdonald polynomial basis.

Definition 2.3. Let $\Omega_{\beta/\mu}(q^2, q^{2k})$ be the coefficient on $P^{(n-1)}_{\beta}(x;q^2,q^{2k})$ in the expansion of

$$P^{(n-1)}_{\mu}(x;q^2,q^{2k}) \prod_{i=1}^{n-1} \frac{(q^2 x_i;q^2)_\infty}{(q^{2k}x_i;q^2)_\infty}$$

in the basis $\{P^{(n-1)}_{\beta}(x;q^2,q^{2k})\}_{\beta}$.

We now recall the following result which provides the coefficients when one expands a Macdonald polynomial in n-variables over the $(n-1)$-variable basis.

Theorem 2.4. *We have the following branching rule for Macdonald polynomials*

$$P^{(n)}_{\lambda}(x;q,t) = \sum_{\mu \preceq \lambda} x_n^{|\lambda|-|\mu|} \psi_{\lambda/\mu}(q,t) P^{(n-1)}_{\mu}(x;q,t).$$

Proof. See (1.7) of [8] for example. □

Remark. There is a product formula for the coefficients $\psi_{\lambda/\mu}(q,t)$ appearing above ([10] p 342)

$$\psi_{\lambda/\mu}(q,t) = \prod_{1 \le i \le j \le l(\mu)} \frac{f(q^{\mu_i-\mu_j}t^{j-i})f(q^{\lambda_i-\lambda_{j+1}}t^{j-i})}{f(q^{\lambda_i-\mu_j}t^{j-i})f(q^{\mu_i-\lambda_{j+1}}t^{j-i})},$$

where $f(a) = (at)_\infty/(aq)_\infty$ with $(a)_\infty = \prod_{i \ge 0}(1-aq^i)$.

Proposition 2.5. *We have*

$$\lim_{q\to 0} \psi_{\lambda/\mu}(q,t) = \prod_{\substack{\{j:\lambda'_j=\mu'_j \\ \text{and } \lambda'_{j+1}=\mu'_{j+1}+1\}}} (1-t^{m_j(\mu)})$$

if λ/μ is a horizontal strip, and zero otherwise.

Proof. This follows from the branching rule for Hall-Littlewood polynomials (see for example [10] p228 (5.5'), (5.14')). □

Definition 2.6. Let

$$\phi_{\lambda/\mu}(t) = \prod_{\substack{\{j:\lambda'_j=\mu'_j+1 \\ \text{and } \lambda'_{j+1}=\mu'_{j+1}\}}} (1-t^{m_j(\lambda)}),$$

if λ/μ is a horizontal strip, and zero otherwise.

Note that these coefficients are the $q \to 0$ limiting case of certain coefficients $\phi_{\lambda/\mu}(q,t)$ in parameters q and t which also arise as branching coefficients (see [13] for an explicit formula, and also the remark below).

Remark. The functions $\phi_{\lambda/\beta}(q,t), \Omega_{\beta/\bar{\mu}}(q,t)$ have interpretations in terms of the skew Macdonald polynomials $Q_{\lambda/\beta}(X;q,t)$ with plethystic substitutions. In particular, we have $\phi_{\lambda/\beta}(q,t) = Q_{\lambda/\beta}(1)$ and $\Omega_{\beta/\bar{\mu}}(q,t) = Q_{\beta/\bar{\mu}}\left(\frac{t-q}{1-t}\right) = t^{|\beta/\bar{\mu}|}Q_{\beta/\bar{\mu}}\left(\frac{1-q/t}{1-t}\right)$, where the evaluations denote plethystic substitutions, and both these quantities have nice factorized forms. The reader may refer to [13] for more details on skew Macdonald polynomials and plethystic subsitutions, as well as these identities.

We will write $\psi_{\lambda/\mu}(t)$, $g(\gamma;t^2)$, etc. to denote the limit $q \to 0$ of these functions.

Definition 2.7. [5, 7] For any skew shape λ/μ, define the coefficients

$$sk_{\lambda/\mu}(t) = t^{\sum_j \binom{\lambda'_j-\mu'_j}{2}} \prod_{j\geq 1} \binom{\lambda'_j-\mu'_{j+1}}{m_j(\mu)}_t .$$

Theorem 2.8. *[5, 7] For a partition λ and $r \geq 0$, we have*

$$P^{(n)}_\lambda(x;t)s^{(n)}_r(x) = \sum_{\lambda+} sk_{\lambda+/\lambda}(t)P^{(n)}_{\lambda+}(x;t),$$

with the sum over partitions $\lambda \subset \lambda+$ for which $|\lambda+/\lambda| = r$.

We now recall two inner products that will appear throughout the paper. We let $\langle\cdot,\cdot\rangle_n$ denote the Macdonald inner product (defined via integration over the n-torus). In particular, letting λ and μ be partitions of length $\leq n$, we have

$$\begin{aligned}(7)\qquad & \langle P_\lambda^{(n)}(x;q,t), P_\mu^{(n)}(x;q,t)\rangle_n \\ &= \int_{T_n} P_\lambda^{(n)}(x;q,t)P_\mu^{(n)}(x^{-1};q,t)\tilde{\Delta}_S^{(n)}(x;q,t)dT \\ &= \delta_{\lambda,\mu}\frac{1}{d_\lambda(q,t)},\end{aligned}$$

where an explicit formula for $d_\lambda(q,t)$ can be found in [10]. Note that the coefficient $d_\lambda(q,t)$ depends on n; in keeping with the notation in [10], we supress the superscript n for $d_\lambda(q,t)$ and other coefficients. Here $\tilde{\Delta}_S^{(n)}(x;q,t)$ is the symmetric density

$$\tilde{\Delta}_S^{(n)}(x;q,t) = \prod_{1\leq i\neq j\leq n}\frac{(x_i/x_j;q)}{(tx_i/x_j;q)},$$

expressed in terms of the (infinite) q-Pochhammer symbol $(a;q)$.

We now briefly recall another scalar product that is relevant to this discussion. Let p_λ denote the power sum symmetric functions. Then $\langle\cdot,\cdot\rangle'$ is defined on this basis as follows:

$$\langle p_\lambda, p_\mu\rangle' = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{l(\lambda)}\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

where

$$z_\lambda = \prod_{i\geq 1} i^{m_i(\lambda)} m_i(\lambda)!.$$

We define the coefficients $b_\lambda(q,t)$ via the following normalization:

$$\langle P_\lambda^{(n)}(x;q,t), P_\lambda^{(n)}(x;q,t)\rangle' = \frac{1}{b_\lambda(q,t)}.$$

and $Q_\mu^{(n)}(x;q,t) = b_\mu(q,t)P_\mu^{(n)}(x;q,t)$ are scalar multiples of the Macdonald polynomials. We have

$$(8)\qquad \langle P_\lambda^{(n)}(x;q,t), Q_\mu^{(n)}(x;q,t)\rangle' = \delta_{\lambda,\mu}.$$

Note that this inner product is independent of n, and we have $\lim_{n\to\infty} \langle\cdot,\cdot\rangle_n = \langle\cdot,\cdot\rangle'$. We have

$$b_\lambda(t) = \prod_{i\geq 1} \phi_{m_i(\lambda)}(t), \tag{9}$$

where $m_i(\lambda)$ denotes the number of times i occurs as a part of λ and

$$\phi_r(t) = (1-t)(1-t^2)\cdots(1-t^r).$$

We also have

$$d_\lambda(t) = \frac{1}{(1-t)^n} \prod_{i\geq 0} \phi_{m_i(\lambda)}(t),$$

so that if $l(\lambda) = n$, $b_\lambda(t)(1-t)^n = d_\lambda(t)$.

We recall the following fact relating the branching coefficients $\phi_{\lambda/\beta}$ and $\psi_{\lambda/\beta}$ [10].

Proposition 2.9. *We have*

$$\phi_{\lambda/\beta}(q,t)/b_\lambda(q,t) = \psi_{\lambda/\beta}(q,t)/b_\beta(q,t).$$

Note that, using (7) and (8), the coefficients $c^\delta_{\lambda\mu}$ and $sk_{\lambda+/\lambda}$ may be defined in terms of inner products. We have

$$\begin{aligned} c^\delta_{\lambda\mu}(q,t) &= \langle m^{(n)}_\gamma(x) P^{(n)}_\mu(x;q,t), Q^{(n)}_\delta(x;q,t)\rangle' \\ &= d_\delta(q,t)\langle m^{(n)}_\gamma(x) P^{(n)}_\mu(x;q,t), P^{(n)}_\delta(x;q,t)\rangle \end{aligned}$$

and similarly

$$\langle P^{(n)}_\lambda(x;t) s^{(n)}_r(x), Q^{(n)}_{\lambda+}(x;t)\rangle' = sk_{\lambda+/\lambda}(t).$$

3. The Gelfand-Tsetlin basis expansion

In this section, we fix $k \in \mathbb{N}$ and set $t = q^k$. We will prove Theorem 1.3 of the introduction. Namely, we will expand the trace function $\Phi^{(k)}_\lambda(x_1,\ldots,x_n)$ in the Gelfand-Tsetlin basis and compute the diagonal coefficients $c_\Lambda(q,t)$. We will use the multiplicity-one decomposition of $V_{\lambda+(k-1)\rho}$ as a $U_q(\mathfrak{gl}_{n-1})$-module, and iterate, in order to do this.

Etingof and Kirillov [2] provide the following closed form for the trace function at $\lambda = 0$:

Proposition 3.1.

$$\Phi_0^{(k)}(x_1,\dots,x_n) = \prod_{i=1}^{k-1}\prod_{\alpha\in R^+}(x^{\alpha/2} - q^{2i}x^{-\alpha/2})$$
$$= x^{(k-1)\rho}\prod_{i=1}^{k-1}\prod_{n\geq l>m\geq 1}(1 - q^{2i}x_l/x_m)$$

We use this to provide the following formula for the following ratio:

Proposition 3.2.

$$\frac{\Phi_0^{(k)}(x_1,\dots,x_n)}{\Phi_0^{(k)}(x_1,\dots,x_{n-1})} = x^{(k-1)(\rho_n-\rho_{n-1})}\sum_{\substack{\gamma\in\mathbb{Z}_{\geq 0}^{n-1}\\ \text{a partition}}} g(\gamma;q^2,q^{2k})x_n^{|\gamma|}m_{-\gamma}(x_1,\dots,x_{n-1}),$$

where $m_{-\gamma}(x_1,\dots,x_{n-1})$ *denotes the monomial symmetric function* m_γ *evaluated at* $x_1^{-1},\dots,x_{n-1}^{-1}$.

Proof. By Proposition 3.1, we have

$$\frac{\Phi_0^{(k)}(x_1,\dots,x_n)}{\Phi_0^{(k)}(x_1,\dots,x_{n-1})} = x^{(k-1)(\rho_n-\rho_{n-1})}\prod_{i=1}^{k-1}\prod_{j=1}^{n-1}(1 - q^{2i}x_n/x_j).$$

Now note that, for fixed $1 \leq j \leq n-1$,

$$\prod_{i=1}^{k-1}(1 - q^{2i}x_n/x_j) = \frac{(x_n/x_j;q^2)_k}{(x_n/x_j;q^2)_1} = \frac{(x_n/x_j;q^2)_\infty}{(q^{2k}x_n/x_j;q^2)_\infty}\cdot\frac{(q^2x_n/x_j;q^2)_\infty}{(x_n/x_j;q^2)_\infty}$$
$$= \frac{(q^2x_n/x_j;q^2)_\infty}{(q^{2k}x_n/x_j;q^2)_\infty}.$$

Now, putting $t = q^k$ and using the q-binomial theorem,

$$\frac{(q^2x_n/x_j;q^2)_\infty}{(t^2x_n/x_j;q^2)_\infty} = \sum_{m=0}^{\infty}\frac{(t^{-2}q^2;q^2)_m}{(q^2;q^2)_m}(t^2x_n/x_j)^m.$$

Taking the product over all $1 \leq j \leq n-1$ yields

$$\sum_{l_1,\ldots,l_{n-1}=0}^{\infty} t^{2\sum l_i} \frac{(t^{-2}q^2;q^2)_{l_1}\cdots(t^{-2}q^2;q^2)_{l_{n-1}}}{(q^2;q^2)_{l_1}\cdots(q^2;q^2)_{l_{n-1}}} x_n^{\sum l_i} x_1^{-l_1}\cdots x_{n-1}^{-l_{n-1}}$$
$$= \sum_{\substack{\gamma\in\mathbb{Z}_{\geq 0}^{n-1} \\ \text{a partition}}} g(\gamma;q^2,q^{2k}) x_n^{|\gamma|} m_{-\gamma}(x_1,\ldots,x_{n-1}),$$

which completes the proof. □

Lemma 3.3. *Let λ be fixed with $l(\lambda)=n$. Then the map*

$$\mu \to \bar{\mu} = \mu + (k-1)\Big(\rho_{n-1} + \Big(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\Big)\Big) = \mu + (k-1)\rho_n|_{n-1}$$

is a bijection between:

- *$\mu \subset \lambda$, such that $\lambda_{j+1} - \mu_j \leq k-1$ for all j*
- *$\bar{\mu} \preceq \lambda + (k-1)\rho_n$, such that $\bar{\mu} - (k-1)\rho_n|_{n-1} \in \mathcal{P}_+$*

Proof. Follows from the definition of the interlacing condition and Equation 6. □

Proposition 3.4. *The following branching rule for trace functions holds:*

$$\Phi_\lambda^{(k)}(x_1,\ldots,x_n) = x^{(k-1)(\rho_n-\rho_{n-1})} \sum_{\mu\subset\lambda} x_n^{|\lambda|-|\mu|} a_{\lambda,\mu}(q) \Phi_\mu^{(k)}(x_1,\ldots,x_{n-1})$$

for some coefficients $a_{\lambda,\mu}(q)$.

Proof. One first notes the multiplicity-free decomposition of $V_{\lambda+(k-1)\rho_n}$ as a module over $U_q(gl_{n-1})$:

$$V_{\lambda+(k-1)\rho_n}|_{U_q(gl_{n-1})} = \oplus_{\bar{\mu}\preceq\lambda+(k-1)\rho_n} V_{\bar{\mu}}.$$

Thus, we have

$$\Phi_\lambda(x;q,q^k) = \mathrm{Tr}(\phi_\lambda \cdot x^h) = \sum_{\bar{\mu}\preceq\lambda+(k-1)\rho_n} \mathrm{Tr}(\phi_\lambda \cdot x^h|_{V_{\bar{\mu}}})$$
$$= \sum_{\bar{\mu}\preceq\lambda+(k-1)\rho_n} \mathrm{Tr}\Big((\mathrm{Proj}_{V_{\bar{\mu}}} \otimes Id) \circ \phi_\lambda \circ x^h|_{V_{\bar{\mu}}}\Big),$$

since trace only takes into account diagonal coefficients.

We have

$$\phi_\lambda|_{V_{\bar\mu}} : V_{\bar\mu} \to V_{\lambda+(k-1)\rho_n} \otimes U \simeq \oplus_{\alpha\preceq\lambda+(k-1)\rho_n} V_\alpha \otimes U,$$

thus

$$(\mathrm{Proj}_{V_{\bar\mu}} \otimes Id) \circ \phi_\lambda|_{V_{\bar\mu}} : V_{\bar\mu} \to V_{\bar\mu} \otimes U$$

is an intertwining operator. By [2], this implies that

$$(\mathrm{Proj}_{V_{\bar\mu}} \otimes Id) \circ \phi_\lambda|_{V_{\bar\mu}} = \begin{cases} \hat a_{\lambda,\bar\mu}(q) \cdot \phi_{\bar\mu-(k-1)\rho_{n-1}}, & \text{if } \bar\mu - (k-1)\rho_{n-1} \in \mathcal{P}_+ \\ 0, & \text{else,} \end{cases}$$

for some coefficients $\hat a_{\lambda,\bar\mu}(q)$. Thus, we have,

$$\begin{aligned} &\Phi^{(k)}_\lambda(x_1,\ldots,x_n) \\ &= \sum_{\substack{\bar\mu\preceq\lambda+(k-1)\rho_n \\ \bar\mu-(k-1)\rho_{n-1}\in\mathcal{P}_+}} \hat a_{\lambda,\bar\mu}(q) \cdot x_n^{|\lambda|-|\bar\mu|} \cdot \Phi^{(k)}_{\bar\mu-(k-1)\rho_{n-1}}(x_1,\ldots,x_{n-1}). \end{aligned}$$

Finally we reparametrize by setting $\mu = \bar\mu - (k-1)\rho_{n-1} - (k-1)(\frac{1}{2})^{n-1}$ and defining $a_{\lambda,\mu}(q) = \hat a_{\lambda,\bar\mu}(q)$ with the condition that $a_{\lambda,\mu} = 0$ if $\lambda_{j+1} - \mu_j \le k-1$ does not hold for all j. By the previous Lemma, we can rewrite this as

$$\begin{aligned} &\Phi^{(k)}_\lambda(x_1,\ldots,x_n) \\ &= \sum_{\substack{\bar\mu\preceq\lambda+(k-1)\rho_n \\ \bar\mu-(k-1)\rho_{n-1}\in\mathcal{P}_+}} a_{\lambda,\mu}(q) \cdot x_n^{|\lambda|-|\mu|+\frac{(k-1)(1-n)}{2}} \cdot \Phi^{(k)}_{\mu+(k-1)(\frac{1}{2})^{(n)}}(x_1,\ldots,x_{n-1}). \end{aligned}$$

Pulling out the factor of $(x_1\cdots x_{n-1})^{\frac{k-1}{2}} x_n^{\frac{(k-1)(1-n)}{2}} = x^{(k-1)(\rho_n-\rho_{n-1})}$ completes the proof. ☐

By iterating the branching rule of the previous propostion and recalling that the Gelfand-Tsetlin basis is also obtained by iterating the multiplicity-free decomposition, we obtain the following result.

Proposition 3.5. *We have the following formula for $\Phi^{(n)}_\lambda(x;q,q^k)$ as a sum over Gelfand-Tsetlin patterns $(\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(n-1)})$ with $\lambda^{(0)} = \lambda + (k-1)\rho_n$:*

$$\Phi^{(k)}_\lambda(x_1,\ldots,x_n) = \sum_{\substack{\Lambda\in GT(\lambda+(k-1)\rho_n) \\ \Lambda=(\lambda^{(0)},\ldots,\lambda^{(n-1)})}} \prod_{1\le i\le n-1} a_{\lambda^{(i-1)},\lambda^{(i)}}(q) x^{wt(\Lambda)}$$

We will show that the coefficients $a_{\lambda,\mu}(q)$ are equal to $c_{\lambda,\mu}(q,q^k)$ defined in the introduction. We will prove this through a series of propositions. Recall the definitions of the functions $g(\cdot;\cdot,\cdot), \psi_{\cdot/\cdot}(\cdot,\cdot), c^{\cdot}_{\cdot,\cdot}(\cdot;\cdot)$ in the introduction.

Lemma 3.6. *For any $m \in \mathbb{C}$, the branching coefficients $a_{\lambda,\mu}(q)$ satisfy the shift invariance:*

$$a_{\lambda+m^n,\mu+m^{n-1}}(q) = a_{\lambda,\mu}(q).$$

Proof. The intertwining operator $\phi^{(k)}$, as well as the multiplicity one decomposition used in the proof of Proposition 3.4, is determined by the $U_q(\mathfrak{sl}_n)$-module structure. Indeed, $U_q(\mathfrak{gl}_n)$ differs only from $U_q(\mathfrak{sl}_n)$ by the addition of the central element $q^{\epsilon_1+\cdots+\epsilon_n}$. The result then follows easily from the observation that, as $U_q(\mathfrak{sl}_n)$-modules, $V_{\lambda+m^n}$ is isomorphic to V_λ for any partition λ and any $m \in \mathbb{C}$. □

Remark. By the previous Lemma, to compute $a_{\lambda,\mu}(q)$ for $\lambda \in \mathcal{P}_+^{(n)}$, $\mu \in \mathcal{P}_+^{(n-1)}$, we may assume that λ, μ are partitions with $l(\lambda) = n$, $l(\mu) = n-1$. We will make this assumption implicitly throughout the paper.

Proposition 3.7. *The branching coefficients satisfy the following formula:*

$$a_{\lambda,\mu}(q) = \sum_{\substack{\beta\preceq\lambda, l(\beta)\le n-1 \\ \gamma\in\mathbb{Z}_{\ge 0}^{n-1} \\ \text{a partition}}} g(\gamma; q^2, q^{2k})\psi_{\lambda/\beta}(q^2, q^{2k}) c^{\mu}_{-\gamma,\beta}(q^2, q^{2k}).$$

Proof. Combining Theorem 1.1 with Proposition 3.4 gives the following:

$$\begin{aligned}&\frac{\Phi_0^{(k)}(x_1,\ldots,x_n)}{\Phi_0^{(k)}(x_1,\ldots,x_{n-1})} P_\lambda^{(n)}(x;q^2,q^{2k}) \\ &= x^{(k-1)(\rho_n-\rho_{n-1})} \sum_{\mu\subset\lambda} x_n^{|\lambda|-|\mu|} a_{\lambda,\mu}(q) P_\mu^{(n-1)}(x;q^2,q^{2k}).\end{aligned}$$

We then use Theorem 2.4 to rewrite this as

$$\begin{aligned}&\frac{\Phi_0^{(k)}(x_1,\ldots,x_n)}{\Phi_0^{(k)}(x_1,\ldots,x_{n-1})} \sum_{\substack{\beta\preceq\lambda \\ l(\beta)\le n-1}} x_n^{|\lambda|-|\beta|}\psi_{\lambda/\beta}(q^2,q^{2k}) P_\beta^{(n-1)}(x;q^2,q^{2k}) \\ &= x^{(k-1)(\rho_n-\rho_{n-1})} \sum_{\mu\subset\lambda} x_n^{|\lambda|-|\mu|} a_{\lambda,\mu}(q) P_\mu^{(n-1)}(x;q^2,q^{2k}).\end{aligned} \tag{10}$$

Combining the equation above with Proposition 3.2, and cancelling out the common factor of $x^{(k-1)(\rho_n-\rho_{n-1})}$, we obtain the equation:

$$\sum_{\substack{\gamma\in\mathbb{Z}_{\geq 0}^{n-1} \\ \text{a partition} \\ \beta\preceq\lambda \\ l(\beta)\leq n-1}} x_n^{|\lambda|-|\beta|+|\gamma|} g(\gamma;q^2,q^{2k})\psi_{\lambda/\beta}(q^2,q^{2k}) m_{-\gamma}(x_1,\dots,x_{n-1}) P_\mu^{(n-1)}(x;q^2,q^{2k})$$
$$=\sum_{\mu\subset\lambda} x_n^{|\lambda|-|\mu|} a_{\lambda,\mu}(q) P_\mu^{(n-1)}(x;q^2,q^{2k}).$$

Next we use Definition (2.2) to rewrite this as

$$\sum_{\substack{\gamma\in\mathbb{Z}_{\geq 0}^{n-1} \\ \beta\preceq\lambda, l(\beta)\leq n-1 \\ \mu,\gamma \text{ partitions}}} x_n^{|\lambda|-|\beta|+|\gamma|} g(\gamma;q^2,q^{2k})\psi_{\lambda/\beta}(q^2,q^{2k}) c^\mu_{-\gamma,\beta}(q^2,q^{2k}) P_\mu^{(n-1)}(x;q^2,q^{2k})$$
$$=\sum_{\mu\subset\lambda} x_n^{|\lambda|-|\mu|} a_{\lambda,\mu}(q) P_\mu^{(n-1)}(x;q^2,q^{2k}).$$

Since both sides of the above equation are expansions in the Macdonald polynomial basis, the corresponding coefficients must be equal. (Note that $c^\mu_{-\gamma,\beta}=0$ unless $|\gamma|=|\beta|-|\mu|$, so the powers on x_n on both sides of the equation do agree, however we will find it convenient to avoid explicitly restricting the summation to $|\gamma|=|\beta|-|\mu|$.)

We then have

$$a_{\lambda,\mu}(q)=\sum_{\substack{\beta\preceq\lambda, l(\beta)\leq n-1 \\ \gamma\in\mathbb{Z}_{\geq 0}^{n-1} \\ \text{a partition}}} g(\gamma;q^2,q^{2k})\psi_{\lambda/\beta}(q^2,q^{2k}) c^\mu_{-\gamma,\beta}(q^2,q^{2k}),$$

as desired. □

Proposition 3.8. *Let $\mu\subset\lambda$ with $l(\mu)\leq n-1$. Then we have*

$$a_{\lambda,\mu}(q)=\sum_{\substack{\beta\preceq\lambda, l(\beta)\leq n-1 \\ l(\gamma)\leq n-1}} g(\gamma;q^2,q^{2k})\psi_{\lambda/\beta}(q^2,q^{2k}) \frac{d_\mu(q^2,q^{2k})}{d_\beta(q^2,q^{2k})} c^\beta_{\gamma,\mu}(q^2,q^{2k}).$$

Proof. Using standard facts about integration over $\mathbb{T}_n$, we have

$$\begin{aligned} c^{\mu}_{-\gamma,\beta}(q^2,q^{2k}) &= \langle m_{-\gamma}(x)P_\beta(x;q^2,q^{2k}), P_\mu(x;q^2,q^{2k})\rangle d_\mu(q^2,q^{2k}) \\ &= d_\mu(q^2,q^{2k})\langle m_\gamma(x)P_\mu(x;q^2,q^{2k}), P_\beta(x;q^2,q^{2k})\rangle \\ &= \frac{d_\mu(q^2,q^{2k})}{d_\beta(q^2,q^{2k})} c^{\beta}_{\gamma,\mu}(q^2,q^{2k}). \end{aligned}$$

Combining this with the previous theorem gives the result. □

We will now show that the branching coefficients $a_{\lambda,\mu}(q)$ are given by suitable specializations of $c_{\lambda,\mu}(q,t)$ as defined in Theorem 1.3.

Proposition 3.9. *The branching coefficients satisfy* $a_{\lambda,\mu}(q) = c_{\lambda,\mu}(q,q^k)$.

Proof. By the previous proposition, we have

$$\begin{aligned} a_{\lambda,\mu}(q) &= d_\mu(q^2,q^{2k}) \\ &\quad\times \sum_{\substack{\beta\preceq\lambda, l(\beta)\le n-1 \\ l(\gamma)\le n-1}} g(\gamma;q^2,q^{2k})\psi_{\lambda/\beta}(q^2,q^{2k})\frac{1}{d_\beta(q^2,q^{2k})} c^{\beta}_{\gamma,\mu}(q^2,q^{2k}). \end{aligned}$$

Now, note that for fixed β, we have

$$\begin{aligned} &\sum_{l(\gamma)\le n-1} g(\gamma;q^2,q^{2k}) c^{\beta}_{\gamma,\mu}(q^2,q^{2k}) \\ &= \sum_{l(\gamma)\le n-1} q^{2k|\gamma|}\frac{(q^{-2k}q^2;q^2)_{\gamma_1}\cdots(q^{-2k}q^2;q^2)_{\gamma_{n-1}}}{(q^2;q^2)_{\gamma_1}\cdots(q^2;q^2)_{\gamma_{n-1}}} c^{\beta}_{\gamma,\mu}(q^2,q^{2k}) \\ &= \sum_{l(\gamma)\le n-1} q^{2k|\gamma|}\frac{(q^{-2k}q^2;q^2)_{\gamma_1}\cdots(q^{-2k}q^2;q^2)_{\gamma_{n-1}}}{(q^2;q^2)_{\gamma_1}\cdots(q^2;q^2)_{\gamma_{n-1}}} \\ &\qquad\times \langle m^{(n-1)}_\gamma(x)P^{(n-1)}_\mu(x;q^2,q^{2k}), Q^{(n-1)}_\beta(x;q^2,q^{2k})\rangle' \\ &= \Bigg\langle\Bigg(\sum_{l(\gamma)\le n-1} q^{2k|\gamma|}\frac{(q^{-2k}q^2;q^2)_{\gamma_1}\cdots(q^{-2k}q^2;q^2)_{\gamma_{n-1}}}{(q^2;q^2)_{\gamma_1}\cdots(q^2;q^2)_{\gamma_{n-1}}} m^{(n-1)}_\gamma(x)\Bigg) \\ &\qquad\times P^{(n-1)}_\mu(x;q^2,q^{2k}), Q^{(n-1)}_\beta(x;q^2,q^{2k})\Bigg\rangle'. \end{aligned}$$

Recall the following q,t-generalizations of the complete homogeneous symmetric functions $h_r(x)$, which are defined by the generating series (see [10, p.311]):

$$\sum_{r\geq 0} g_r(x;q,t)y^r = \prod_{i\geq 1}\frac{(tx_iy;q)_\infty}{(x_iy;q)_\infty}.$$

We have the following identity from [10, p314]:

$$g_r(x;q^2,t^2) = \sum_{|\gamma|=r}\frac{(t^2;q^2)_\gamma}{(q^2;q^2)_\gamma}m_\gamma(x).$$

Using this, we may write the previous equation as

$$\sum_{l(\gamma)\leq n-1} g(\gamma;q^2,q^{2k})c^{\beta}_{\gamma,\bar{\mu}}(q^2,q^{2k})$$
$$= \left\langle \left(\sum_{r\geq 0} q^{2kr} g_r^{(n-1)}(x;q^2,q^{-2k}q^2)\right) P_\mu^{(n-1)}(x;q^2,q^{2k}), Q_\beta^{(n-1)}(x;q^2,q^{2k})\right\rangle'.$$

From the definition of $g_r(x;q,t)$, we then we have

$$\sum_{l(\gamma)\leq n-1} g(\gamma;q^2,q^{2k})c^{\beta}_{\gamma,\mu}(q^2,q^{2k})$$
$$= \left\langle \prod_{i\geq 1}\frac{(q^2x_i;q^2)_\infty}{(q^{2k}x_i;q^2)_\infty} P_\mu^{(n-1)}(x;q^2,q^{2k}), Q_\beta^{(n-1)}(x;q^2,q^{2k})\right\rangle'$$
$$= \Omega_{\beta/\mu}(q^2,q^{2k}),$$

by the definition of $\Omega_{\beta/\mu}(q^2,q^{2k})$.

Combining this with the original sum, and comparing with the formula for $c_{\lambda,\mu}(q,q^k)$ from (2), completes the proof. □

Remarks. The coefficients $c_{\lambda,\mu}(q,t)$ do not appear to factor nicely at the q-level, due to the restriction on length in the sum. For example, for $\lambda = (2,1)$ and $\mu = (1)$ one obtains $(1-t)(1-qt^2)(1-q-q^2+t)/(1-qt)$, and the term $(1-q-q^2+t)$ cannot be expressed as a product of $(1-q^it^j)$.

We are now prepared to complete the proof of Theorem 1.3

Proof of Theorem 1.3. The formula for the diagonal coefficients with respect to the Gelfand-Tsetlin basis follows immediately from Propositions 3.5 and 3.9 above. We will now complete the proof of the branching formula for $\widetilde{\Phi}_\lambda^{(n)}$.

By Propositions 3.4 and 3.9 above, we have

$$\Phi_\lambda^{(k)}(x_1,\dots,x_n) = x^{(k-1)(\rho_n-\rho_{n-1})} \times \sum_{\substack{\mu\in\mathcal{P}_+^{(n+1)}\\ \mu\subset\lambda}} c_{\lambda,\mu}(q,q^k)\cdot\Phi_\mu^{(k)}(x_1,\dots,x_{n-1})\cdot x_n^{|\lambda|-|\mu|}$$

By Theorem 1.1, it follows that the equation

$$\widetilde{\Phi}_\lambda^{(n)}(x;q,t) = \sum_{\substack{\mu\in\mathcal{P}_+^{(n+1)}\\ \mu\subset\lambda}} c_{\lambda,\mu}(q,t)\cdot\widetilde{\Phi}_\mu^{(n-1)}(x;q,t)\cdot x_n^{|\lambda|-|\mu|}$$

holds for $t=q^k$ for any $k\in\mathbb{N}$. Noting that on both sides of the equation the coefficients on a fixed monomial in $x_1,\dots,x_n$ are rational functions of q,t, it follows that equation must hold for generic t. $\square$

4. The $q\to 0$ limit

We will look at the $q\to 0$ limit of the coefficients $c_{\lambda,\mu}(q,t)$. We find that the formula has a nice product form, in terms of certain p-adic counts. The simplification of these coefficients at $q=0$ may be related to the crystal basis structure of the Gelfand-Tsetlin basis, although we have not investigated a direct link.

The goal of this section is to prove Theorems 1.4 and 1.5 mentioned in the introduction.

Definition 4.1. We let $c_{\lambda,\mu}(t)$ denote $\lim_{q\to 0} c_{\lambda,\mu}(q,t)$.

Theorem 4.2. *Let λ be a partition of length n, and $\mu\subset\lambda,\mu\in\mathcal{P}_+^{(n-1)}$. Then retaining the notation of the previous sections, we have*

$$c_{\lambda,\mu}(t) = \frac{b_\mu(t^2)}{b_\lambda(t^2)} \sum_{\substack{\beta\preceq\lambda\\ l(\beta)\le n-1}} \phi_{\lambda/\beta}(t^2)t^{2|\beta/\mu|}sk_{\beta/\mu}(t^2).$$

Proof. We use Theorem 1.3, the functions there admit the limit $q\to 0$. We also use that

$$\Omega_{\beta/\mu}(q,t) = Q_{\beta/\mu}\Big(\frac{t-q}{1-t}\Big) = t^{|\beta/\mu|}Q_{\beta/\mu}\Big(\frac{1-q}{1-t}\Big)$$

and

$$\lim_{q\to 0} Q_{\beta/\mu}\Big(\frac{1-q}{1-t}\Big) = sk_{\beta/\mu}(t),$$

where the skew Macdonald polynomials are taken with respect to the parameters (q,t). This gives the following

$$c_{\lambda,\mu}(t) = d_\mu(t^2) \sum_{\substack{\beta\preceq\lambda \\ l(\beta)\leq n-1}} \frac{\psi_{\lambda/\beta}(t^2)}{d_\beta(t^2)} t^{2|\beta/\mu|} sk_{\beta/\mu}(t^2).$$

Finally, one notes that

$$d_\beta(t^2) = (1-t)^{n-1} b_\beta(t^2) \text{ and } d_\mu(t^2) = (1-t)^{n-1} b_\mu(t^2),$$

and by the $q \to 0$ limit of Proposition 2.9 we have

$$\phi_{\lambda/\beta}(t)/b_\lambda(t) = \psi_{\lambda/\beta}(t)/b_\beta(t);$$

using this in the previous equation gives the result. □

Our next goal is to obtain a nice factorized product form for $c_{\lambda,\mu}(t)$. We use a $q \to 0$ specialization of Rains' q-Pfaff-Saalschütz formula:

Theorem 4.3 ([12, Corollary 4.9]). *Let $\mu \subset \lambda$ be partitions, then for arbitrary parameters a, b, c we have the following identity:*

$$\sum_\beta \frac{(a)_\beta}{(c)_\beta} Q_{\lambda/\beta}\Big(\frac{a-b}{1-t}\Big) Q_{\beta/\mu}\Big(\frac{b-c}{1-t}\Big) = \frac{(a)_\mu (b)_\lambda}{(b)_\mu (c)_\lambda} Q_{\lambda/\mu}\Big(\frac{a-c}{1-t}\Big).$$

Proposition 4.4. *Let λ be a partition of length n and let $\mu \leq \lambda$ with $l(\mu) = n-1$. Then*

1)

$$\sum_{\beta\preceq\lambda} \phi_{\lambda/\beta}(t^2) t^{2|\beta/\mu|} sk_{\beta/\mu}(t^2) = sk_{\lambda/\mu}(t^2)$$

2)

$$\sum_{\substack{\beta\preceq\lambda \\ l(\beta)\leq n-1}} \phi_{\lambda/\beta}(t^2) t^{2|\beta/\mu|} sk_{\beta/\mu}(t^2) = (1-t^2)\cdot sk_{\lambda/\mu}(t^2)$$

Proof. Take $a = q$, $b = qt$, $c = q^2$ in in Theorem 4.3:

$$\sum_{\beta} \frac{(q)_\beta}{(q^2)_\beta} Q_{\lambda/\beta}\left(\frac{q-qt}{1-t}\right) Q_{\beta/\mu}\left(\frac{qt-q^2}{1-t}\right) = \frac{(q)_\mu (qt)_\lambda}{(qt)_\mu (q^2)_\lambda} Q_{\lambda/\mu}\left(\frac{q-q^2}{1-t}\right).$$

Using the relation $Q_{\lambda/\mu}(\frac{aq-bq}{1-t}) = q^{|\lambda/\mu|} Q_{\lambda/\mu}(\frac{a-b}{1-t})$, we have

$$\sum_{\beta} \frac{(q)_\beta}{(q^2)_\beta} Q_{\lambda/\beta}(1) Q_{\beta/\mu}\left(\frac{t-q}{1-t}\right) = \frac{(q)_\mu (qt)_\lambda}{(qt)_\mu (q^2)_\lambda} Q_{\lambda/\mu}\left(\frac{1-q}{1-t}\right).$$

(1) then follows by taking the limit $q \to 0$.

To prove (2), it suffices by (1) to show that

$$\sum_{\substack{\beta \preceq \lambda \\ l(\beta)=n}} \phi_{\lambda/\beta}(t^2) t^{2|\beta/\mu|} sk_{\beta/\mu}(t^2) = t^2 \cdot sk_{\lambda/\mu}(t^2).$$

Reindex the sum by replacing $\beta = \beta' + 1^n$. Then we have

$$\sum_{\beta' \preceq \lambda - 1^n} \phi_{\lambda/(\beta'+1^n)}(t^2) t^{2|\beta'/\mu|} t^{2n} sk_{(\beta'+1^n)/\mu}(t^2)$$

We have the following two identities:

$$\begin{aligned} \phi_{\lambda/(\beta'+1^n)}(t^2) &= \phi_{(\lambda-1^n)/\beta'}(t^2) \\ sk_{(\beta'+1^n)/\mu}(t^2) &= sk_{\beta'/(\mu-1^{(n-1)})}(t^2), \end{aligned}$$

which can be seen by using the explicit formulas in Section 2. It follows that

$$\begin{aligned} &\sum_{\beta' \preceq (\lambda - 1^n)} \phi_{(\lambda-1^n)/\beta'}(t^2) t^{2|\beta'/(\mu-1^{n-1})|} t^2 sk_{\beta'/(\mu-1^{(n-1)})}(t^2) \\ &= t^2 \cdot sk_{(\lambda-1^n)/(\mu-1^{(n-1)})}(t^2). \end{aligned}$$

Applying the identity above again completes the proof. □

We now provide a proof of Theorem 1.4, mentioned in the introduction. The proof relies on the previous results of this section.

Proof of Theorem 1.4. By Proposition 4.4, we have

$$c_{\lambda,\mu}(t) = \frac{b_\mu(t^2)}{b_\lambda(t^2)} \sum_{\substack{\beta \preceq \lambda \\ l(\beta) \leq n-1}} \phi_{\lambda/\beta}(t^2) t^{2|\beta/\mu|} sk_{\beta/\mu}(t^2) = \frac{b_\mu(t^2)}{b_\lambda(t^2)}(1-t^2) sk_{\lambda/\mu}(t^2),$$

which gives the first equality. By the definitions of $b_\lambda(t), sk_{\lambda/\mu}(t)$, this is equal to

$$\begin{aligned}
&\frac{\prod_{i\geq 1} \phi_{m_i(\mu)}(t^2)}{\prod_{i \geq 1} \phi_{m_i(\lambda)}(t^2)} (1-t^2) t^{2\sum_j \binom{\lambda_j'-\mu_j'}{2}} \prod_{j\geq 1} \binom{\lambda_j' - \mu_{j+1}'}{\mu_j' - \mu_{j+1}'}_{t^2} \\
&= (1-t^2) t^{2\sum_j \binom{\lambda_j'-\mu_j'}{2}} \prod_{j \geq 1} \frac{\phi_{\lambda_j' - \mu_{j+1}'}(t^2)}{\phi_{\lambda_j' - \lambda_{j+1}'}(t^2) \phi_{\lambda_j' - \mu_j'}(t^2)} \\
&= \frac{(1-t^2)}{\phi_{\lambda_1' - \mu_1'}(t^2)} t^{2\sum_j \binom{\lambda_j'-\mu_j'}{2}} \prod_{j \geq 1} \frac{\phi_{\lambda_j' - \mu_{j+1}'}(t^2)}{\phi_{\lambda_j' - \lambda_{j+1}'}(t^2) \phi_{\lambda_{j+1}' - \mu_{j+1}'}(t^2)} \\
&= t^{2\sum_j \binom{\lambda_j'-\mu_j'}{2}} \prod_{j\geq 1} \binom{\lambda_j' - \mu_{j+1}'}{\lambda_j' - \lambda_{j+1}'}_{t^2},
\end{aligned}$$

where we have used $m_i(\mu) = \mu_i' - \mu_{i+1}'$ and $\lambda_1' - 1 = \mu_1'$ (because $l(\lambda) = n$ and $l(\mu) = n-1$). □

We recall that, as mentioned in the introduction, there is a p-adic interpretation for coefficients $sk_{\lambda/\mu}(t)$ and thus for $c_{\lambda,\mu}(t)$. More precisely,

$$sk_{\lambda/\mu}(t) = t^{n(\lambda)-n(\mu)} \alpha_\lambda(\mu; t^{-1});$$

where $\alpha_\lambda(\mu; p)$ is the number of subgroups of type μ in a finite abelian p-group of type λ, see [13] for example, and the references therein.

Proof of Theorem 1.5. Recall that for $S = (\mu^{(0)} \supset \mu^{(1)} \supset \cdots \supset \mu^{(n-1)})$ with $\mu^{(i)} \in \mathcal{P}_+^{(n-i)}$, we defined the coefficient $sk_S(t)$ as a product of $sk_{\mu^{(i-1)}/\mu^{(i)}}(t)$ in (4). By Definition 1.2, one can associate to S a Gelfand-Tsetlin array Λ. Thus, using Theorem 1.4, we have

$$\lim_{q \to 0} c_\Lambda(q,t) = \frac{(1-t^2)^n}{b_\lambda(t^2)} sk_S(t^2).$$

Using this along with Theorem 1.3 gives the result. □

Note that when $t = p^{-1}$ for p an odd prime, the coefficients appearing in both Theorems 1.4 and 1.5 are explicit p-adic counts.

Corollary 4.5. *Let λ be a partition. We have the following formula for the Hall-Littlewood polynomial:*

$$P_\lambda(x_1,\ldots,x_n;t^2) = \frac{1}{b_\lambda(t^2)} \frac{\displaystyle\sum_{\substack{S=(\lambda=\mu^{(0)}\supset\mu^{(1)}\supset\cdots\supset\mu^{(n-1)})\\ \mu^{(i)}\in\mathcal{P}_+^{(n-i)}}} sk_S(t^2)x^{wt(S)}}{\displaystyle\sum_{\substack{S'=(0^n=\mu^{(0)}\supset\mu^{(1)}\supset\cdots\supset\mu^{(n-1)})\\ \mu^{(i)}\in\mathcal{P}_+^{(n-i)}}} sk_{S'}(t^2)x^{wt(S')}}.$$

Proof. Follows from Theorem 1.1 along with Theorem 1.5. □

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