

On the Rankin-Selberg integral of Kohnen and Skoruppa

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The Rankin-Selberg integral of Kohnen and Skoruppa produces the Spin L -function for holomorphic Siegel modular forms of genus two. In this paper, we reinterpret and extend their integral to apply to arbitrary cuspidal automorphic representations of PGSp_4 . We show that the integral is related to a non-unique model and analyze it using the approach of Piatetski-Shapiro and Rallis.

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1. Introduction

The Rankin-Selberg integral representation of Kohnen-Skoruppa [8] produces the Spin L -function for holomorphic Siegel modular cusp forms on GSp_4 . Their integral makes use of a special Siegel modular form $P_D(Z)$, which is in the Saito-Kurokawa or Maass subspace. Here Z is the variable in the Siegel upper half space of genus two, and D is a negative discriminant. The proof in [8] is classical, and involves global calculations with the Fourier coefficients of Siegel modular forms. In particular, Kohnen and Skoruppa make essential use of the Maass relations, which are identities satisfied by the Fourier coefficients of Siegel modular forms that are Saito-Kurokawa lifts.

The purpose of this paper is to reinterpret and extend the integral representation in [8]. We define a class of automorphic functions $P_D^\alpha(g)$ on GSp_4 depending on a nonzero square-free integer D (which is allowed to be positive) and auxiliary data α , generalizing the $P_D(Z)$ of [8]. Denote by Q a Klingen parabolic of GSp_4 , i.e. a maximal parabolic stabilizing a line in the defining four-dimensional representation of GSp_4 . For a cusp form ϕ in the space of an automorphic cuspidal representation π of GSp_4 with trivial central character, we define

$$I(\phi, s) = \int_{Z(\mathbf{A}) \mathrm{GSp}_4(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{A})} E(g, s, \Phi) P_D^\alpha(g) \phi(g) dg,$$

where $E(g, s, \Phi)$ is a Klingen Eisenstein series. We also define a second integral $J(\phi, s)$ that may be thought of as a degenerate version of $I(\phi, s)$. The definitions of $E(g, s, \Phi)$ and $J(\phi, s)$ are given in Section 2.3 below. We unfold these integrals and show that they are equal to the partial Spin L -function $L^S(\pi, \mathrm{Spin}, s)$ times an integral $I_S(\phi, s)$ or $J_S(\phi, s)$ over some set of bad places S .

It follows from a theorem of Li [10] that every cuspidal automorphic representation on GSp_4 possesses a nonzero Fourier coefficient of maximal rank for the Siegel parabolic. We combine this with calculations at ramified finite places and the Archimedean place to show that one can always choose D and the other data in the integral so that $I(\phi, s)$ is nonvanishing. In the special case of a representation attached to a holomorphic Siegel cusp form of level one, we prove a more precise statement than this nonvanishing: for a particular choice of data at the Archimedean place, $I(\phi, s)$ is equal to the completed L -function. This recovers the main result of Kohnen-Skoruppa [8].

The integral $I(\phi, s)$ unfolds to a *non-unique model*. In the majority of the Rankin-Selberg literature, the unfolded integral involves a model of the cuspidal representation π that is known to be unique, such as the Whittaker or Bessel model. This uniqueness allows one to factorize the integral (for suitable data) and check, place-by-place, that the local factor of the unfolded integral is equal to the local Langlands L -function. However, there are a handful of examples ([6], [13], [4], [7]) of Rankin-Selberg convolutions that unfold to a model that is not unique, yet are still known to represent L -functions. The integral in this paper is one more such example.

We are interested in $I(\phi, s)$ and $J(\phi, s)$ for several reasons. They have the unusual ability to represent the Spin L -function for all cuspidal automorphic representations (for suitable data), rather than either only generic or only holomorphic ones. They are also members of the limited class of Rankin-Selberg integrals that can potentially be reinterpreted in terms of geometric objects on a Shimura variety, and so could prove useful in answering arithmetic questions. There are higher rank generalizations of both these integrals to GSp_6 that again give the Spin L -function [14]. The integrals in [14] should have connections to motivic questions.

We briefly describe the special functions P_D^α . Recall that there is an isomorphism between PGSp_4 and a split special orthogonal group $\mathrm{SO}(V_5, q)$. Here V_5 is a certain 5-dimensional rational vector space and q is a quadratic form on V_5 . Let D be nonzero and square-free, and choose v_D in $V_5(\mathbf{Q})$ such that $q(v_D) = D$. Let $\alpha = \prod_v \alpha_v$ be a factorizable function in $\mathcal{S}(V_5(\mathbf{A}))$, the space of Schwartz functions on $V_5(\mathbf{A})$, equal to the characteristic function of $V_5(\mathbf{Z}_p)$ almost everywhere. In order to make the calculations cleaner, we assume that GSp_4 acts on V_5 on the right. The special automorphic function P_D^α is

$$P_D^\alpha(g) = \sum_{\delta \in \mathrm{Stab}(v_D)(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q})} \alpha(v_D \delta g),$$

where $\mathrm{Stab}(v_D)$ is the stabilizer of v_D in GSp_4 . (See Section 2.2 for more details. When $D \equiv 1 \pmod{4}$, we will make slightly different choices to avoid issues at the prime 2.) Actually, the condition that α_∞ be Schwartz is too restrictive; to treat holomorphic Siegel modular forms it will behoove us to take α_∞ decaying polynomially at infinity.

One important difference between this paper and [8] is in our definition of the special function P_D^α . In [8], P_D is defined as the lift to the Maass subspace of a certain Poincare series for Jacobi forms. The authors then use Hecke operators on Jacobi forms and the special properties of the Fourier-Jacobi expansion of elements in the Maass subspace as their key tools to

relate $I(\phi, s)$ to the Spin L -function. However, it is not all that difficult to check that the P_D of [8] may be expressed as a sum of the above type; hence our definition of P_D^α . By expressing P_D^α as a sum, we can use it to help unfold the global integral. Thus our unfolding is different from that in [8], as is our proof that the unfolded integral represents the Spin L -function.

When D is negative, the data α may be chosen so that $\overline{P_D^\alpha(g)}$ corresponds to a classical Siegel modular form $\overline{P_D^\alpha(Z)}$, where $\bar{\cdot}$ denotes complex conjugation. In this case, $\overline{P_D^\alpha(Z)}$ has an expression as a sum

$$(1.1) \quad \overline{P_D^\alpha(Z)} = \sum_{v \in V_5(\mathbf{Z}), q(v) = -|D|} \frac{1}{Q_v(Z)^r}.$$

Here Z is in the Siegel upper half-space, Q_v is a certain quadratic form on the space of complex two-by-two symmetric matrices, and $r \geq 6$ is an integer. (See Section 6.2.) Modular forms similar to (1.1) have been considered previously by many authors in different contexts. Zagier [16] appears to have first defined the analogous functions for modular forms ($\mathrm{PGL}_2 \simeq \mathrm{SO}_3$) and Hilbert modular forms over a real quadratic field ($\mathrm{PGL}_{2, \mathbf{Q}(\sqrt{D})}^* \simeq \mathrm{SO}_4$).

Finally, let us mention some other integrals for the Spin L -function on symplectic groups. There are integrals of Novodvorsky [11] and Piatetski-Shapiro [12] (following Andrianov [1]) on GSp_4 , which unfold to the Whittaker and Bessel models, respectively. There are also integrals of Bump-Ginzburg [5] on $\mathrm{GSp}_6, \mathrm{GSp}_8$ and GSp_{10} , which unfold to the Whittaker models on these groups. The integral in this paper draws upon similar techniques to [14], in which we give a new Rankin-Selberg integral for the Spin L -function on GSp_6 . Although the integral in [14] does not apply to holomorphic Siegel modular forms, it uses the same special function P_D^α on GSp_4 used here, and also unfolds to a non-unique model.

The contents of the various sections are as follows. In Section 2, we define the groups, the global Rankin-Selberg integrals, and the special functions that we use. In Section 3, we unfold the integrals, discuss the general technique of Piatetski-Shapiro and Rallis to analyze non-unique models, and state the main theorems. In Section 4 we give the unramified calculation, and in Section 5 we control the behavior at ramified finite primes and the Archimedean place. In Section 6 we choose data appropriate for Siegel modular forms and precisely calculate a corresponding Archimedean integral $I_\infty(\phi, s)$ in terms of Γ -functions.

2. Global constructions

In this section we define the global Rankin-Selberg integrals considered in this paper.

2.1. Groups and embeddings

We define

$$J_4 = \begin{pmatrix} & \mathbf{1}_2 \\ -\mathbf{1}_2 & \end{pmatrix}$$

and $\mathrm{GSp}_4 = \{g \in \mathrm{GL}_4 \mid gJ_4^t g = \nu(g)J_4\}$, where $\nu : \mathrm{GSp}_4 \rightarrow \mathrm{GL}_1$ is the similitude and $\mathbf{1}_n$ denotes the $n \times n$ identity matrix. The letter Z is used to denote the diagonal center of GSp_4 . Denote by $W_4 = \mathbf{Q}^4$ the defining 4-dimensional representation of GSp_4 and by $\langle \cdot, \cdot \rangle$ the symplectic pairing. We let GSp_4 act on W_4 on the right. That is, we think of W_4 as row vectors, and the action of GSp_4 is the usual right action of matrices on row vectors. Denote the ordered basis of W_4 by $\{e_1, e_2, f_1, f_2\}$, so that $\langle e_i, f_j \rangle = \delta_{i,j}$.

Let

$$L = \mathbf{Q}[x]/(x^2 - D)$$

be a quadratic étale \mathbf{Q} -algebra, where D is square-free and nonzero. Let $\mathrm{GL}_{2,L}$ denote the restriction of scalars $\mathrm{Res}_{\mathbf{Q}}^L \mathrm{GL}_2$, and denote by $\mathrm{GL}_{2,L}^*$ the subgroup of $\mathrm{GL}_{2,L}$ with determinant in $\mathbf{G}_m \subseteq \mathrm{Res}_{\mathbf{Q}}^L \mathbf{G}_m$. (In particular, $\mathrm{GL}_{2,L}^*(\mathbf{Q})$ is the set of invertible 2×2 matrices with entries in L and determinant in \mathbf{Q}^\times .) Suppose first that $D \not\equiv 1 \pmod{4}$. One can define an embedding $\mathrm{GL}_{2,L}^* \rightarrow \mathrm{GSp}_4$ as follows. We consider the L -vector space L^2 and write c_1 and c_2 for its two standard basis elements. There is an L -valued symplectic form $\langle \cdot, \cdot \rangle_L$ on L^2 given by

$$\langle u_1 c_1 + u_2 c_2, v_1 c_1 + v_2 c_2 \rangle_L = u_1 v_2 - u_2 v_1.$$

Suppose that α denotes the image of x in $L = \mathbf{Q}[x]/(x^2 - D)$, so that α is a square root of D . We write $\bar{\cdot}$ for the action of the nontrivial Galois automorphism on L . Define a \mathbf{Q} -valued symplectic form $\langle \cdot, \cdot \rangle'$ on L^2 by $\langle u, v \rangle' = \mathrm{tr}_{L/\mathbf{Q}}(\alpha \langle u, v \rangle_L) / (2D)$. Consider the \mathbf{Q} -basis

$$(2.1) \quad e_1 = c_1, \quad e_2 = \alpha c_1, \quad f_1 = -\bar{\alpha} c_2, \quad f_2 = c_2$$

of L^2 . Then $\langle e_i, f_j \rangle' = \delta_{ij}$, and $\langle e_i, e_j \rangle' = \langle f_i, f_j \rangle' = 0$. Hence the basis (2.1) yields an identification

$$(2.2) \quad (L^2, \langle \cdot, \cdot \rangle') \simeq (W_4, \langle \cdot, \cdot \rangle).$$

If $g \in \text{GL}_{2,L}$ and $u, v \in L^2$, then

$$\langle ug, vg \rangle_L = \det(g) \langle u, v \rangle_L.$$

Hence if $g \in \text{GL}_{2,L}^*$, then $\langle ug, vg \rangle' = \det(g) \langle u, v \rangle'$. Thus (2.2) yields the inclusion $\text{GL}_{2,L}^* \rightarrow \text{GSp}_4$.

Suppose that $g = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \in \text{GL}_{2,L}^*$, where $u_i = \epsilon_i + \alpha \eta_i$ for ϵ_i, η_i in \mathbf{Q} . Then under the mapping $\text{GL}_{2,L}^* \rightarrow \text{GSp}_4$, the image of g is the matrix

$$(2.3) \quad \left(\begin{array}{cc|cc} \epsilon_1 & \eta_1 & \eta_2 & \epsilon_2 \\ D\eta_1 & \epsilon_1 & \epsilon_2 & D\eta_2 \\ \hline D\eta_3 & \epsilon_3 & \epsilon_4 & D\eta_4 \\ \epsilon_3 & \eta_3 & \eta_4 & \epsilon_4 \end{array} \right).$$

If $D \equiv 1 \pmod{4}$ and one uses the minimal polynomial $x^2 + x + \frac{1-D}{4}$ of $\frac{-1+\sqrt{D}}{2}$ in place of $x^2 - D$ and adjusts the definition of the pairing to $\langle u, v \rangle' = \text{tr}_{L/\mathbf{Q}}((2\alpha + 1)\langle u, v \rangle_L) / D$, the preceding discussion (with α the image of x) leads in exactly the same way to the mapping of $g = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ to

$$(2.4) \quad \left(\begin{array}{cc|cc} \epsilon_1 & \eta_1 & \eta_2 & \epsilon_2 - \eta_2 \\ \frac{D-1}{4}\eta_1 & \epsilon_1 - \eta_1 & \epsilon_2 - \eta_2 & -\epsilon_2 + \eta_2 \frac{D+3}{4} \\ \hline \epsilon_3 + \frac{D-1}{4}\eta_3 & \epsilon_3 & \epsilon_4 & \frac{D-1}{4}\eta_4 \\ \epsilon_3 & \eta_3 & \eta_4 & \epsilon_4 - \eta_4 \end{array} \right).$$

We remark that the image of $\text{GL}_{2,L}^*$ has an alternative description as the centralizer of the matrix $\begin{pmatrix} \epsilon_D & \\ & \iota_{\epsilon_D} \end{pmatrix}$ inside GSp_4 , where we take $\epsilon_D = \begin{pmatrix} D & \\ & 1 \end{pmatrix}$ if $D \equiv 2, 3 \pmod{4}$ and $\epsilon_D = \begin{pmatrix} \frac{D-1}{4} & 1 \\ & -1 \end{pmatrix}$ if $D \equiv 1 \pmod{4}$.

If $D = 1$, so that $L \cong \mathbf{Q} \times \mathbf{Q}$, then $\text{GL}_{2,L}^*$ is isomorphic to the subgroup of $\text{GL}_2 \times \text{GL}_2$ consisting of the (g_1, g_2) with $\det(g_1) = \det(g_2)$. For later use, we record the following explicit realization of this identity. Set $e'_1 = e_1 + e_2$,

$e'_2 = e_2$, $f'_1 = f_1$, and $f'_2 = f_1 - f_2$. Then if $D = 1$, the matrix in (2.4) sends

$$(2.5) \quad \begin{aligned} e'_1 &\mapsto \epsilon_1 e'_1 + \epsilon_2 f'_1 \\ f'_1 &\mapsto \epsilon_3 e'_1 + \epsilon_4 f'_1 \\ e'_2 &\mapsto (\epsilon_1 - \eta_1) e'_2 + (\eta_2 - \epsilon_2) f'_2 \\ f'_2 &\mapsto (\eta_3 - \epsilon_3) e'_2 + (\epsilon_4 - \eta_4) f'_2. \end{aligned}$$

2.2. The special function P_D^α

We now define the special function P_D^α to be used in the integral representations. Recall that ν denotes the similitude character. Define

$$\wedge_0^2(W_4) = \ker\{\wedge^2 W_4 \xrightarrow{v \wedge w \mapsto \langle v, w \rangle} 1 \otimes \nu\}$$

and define a 5-dimensional representation V_5 of GSp_4 as the diagonal action on $V_5 = \wedge_0^2(W_4) \otimes \nu^{-1}$. As $\wedge^4 W_4 \otimes \nu^{-2}$ is the trivial representation of GSp_4 , the exterior product

$$\wedge : V_5 \otimes V_5 \rightarrow \wedge^4 W_4 \otimes \nu^{-2}$$

defines the invariant quadratic form q on V_5 . More precisely, we first define (\cdot, \cdot) via

$$v \wedge w = (v, w) e_1 \wedge e_2 \wedge f_1 \wedge f_2,$$

where $e_1 \wedge e_2 \wedge f_1 \wedge f_2$ is here considered as an element of $\wedge^4 W_4 \otimes \nu^{-2}$. The triviality of $\wedge^4 W_4 \otimes \nu^{-2}$ shows that (\cdot, \cdot) is invariant. We then define the quadratic form $q(v) = \frac{1}{2}(v, v)$. If $D \not\equiv 1 \pmod{4}$, we define an integral structure on V_5 by choosing the integral basis

$$\{e_1 \wedge f_1 - e_2 \wedge f_2, e_1 \wedge e_2, e_1 \wedge f_2, e_2 \wedge f_1, f_1 \wedge f_2\}$$

and define

$$(2.6) \quad v_D = D e_1 \wedge f_2 + e_2 \wedge f_1.$$

Then $q(v_D) = (v_D, v_D) = D$, and one can check that the stabilizer of v_D is $\mathrm{GL}_{2,L}^*$ in the embedding described by (2.3). If $D \equiv 1 \pmod{4}$, we replace

$e_1 \wedge f_1 - e_2 \wedge f_2$ by $\frac{1}{2}(e_1 \wedge f_1 - e_2 \wedge f_2)$ in the integral basis and set

$$(2.7) \quad v_D = \frac{D-1}{4}e_1 \wedge f_2 + \frac{1}{2}(e_1 \wedge f_1 - e_2 \wedge f_2) + e_2 \wedge f_1$$

so that $q(D) = \frac{D}{4}$. Once again, the stabilizer is $\mathrm{GL}_{2,L}^*$ in its embedding from (2.4). To see that the chosen integral structure is $\mathrm{GSp}_4(\mathbf{Z})$ -stable, we observe that since $e_1 \wedge f_1 + e_2 \wedge f_2$ is fixed by the action of g (due to the twisting by ν^{-1} in the definition of V_5), we have

$$\begin{aligned} \frac{1}{2}(e_1 \wedge f_1 - e_2 \wedge f_2)g &= \frac{1}{2}(e_1 \wedge f_1 + e_2 \wedge f_2 - 2e_2 \wedge f_2)g \\ &= \frac{1}{2}(e_1 \wedge f_1 + e_2 \wedge f_2) - (e_2 \wedge f_2)g \\ &= \frac{1}{2}(e_1 \wedge f_1 - e_2 \wedge f_2) + e_2 \wedge f_2 - (e_2 \wedge f_2)g, \end{aligned}$$

which is in $V_5(\mathbf{Z})$.

Now take $\alpha = \prod_v \alpha_v$ to be factorizable in the Schwartz space $\mathcal{S}(V_5(\mathbf{A}))$. For all but finitely many finite primes p , we require that α_p is the characteristic function of $V_5(\mathbf{Z}_p)$ (using the integral structure defined above). We define

$$P_D^\alpha(g) = \sum_{\delta \in \mathrm{GL}_{2,L}^*(\mathbf{Q}) \setminus \mathrm{GSp}_4(\mathbf{Q})} \alpha(v_D \delta g).$$

In Section 6.2 we will choose an α_∞ that is very convenient for computing with holomorphic Siegel modular forms, although this α_∞ will not be a Schwartz function.

2.3. The global integrals

We now define the two global Rankin-Selberg convolutions to be studied in this paper. The integral representations use the Klingen Eisenstein series on GSp_4 . We define the Klingen parabolic, denoted Q , to be the stabilizer of the line $\mathbf{Q}f_2$ via the right action. Alternatively, Q is the subgroup of GSp_4 consisting of elements of the form

$$g = \begin{pmatrix} * & 0 & * & * \\ * & a & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & d \end{pmatrix}.$$

For such a g , $\nu(g) = ad$, $f_2g = df_2$, and $\delta_Q(g) = \left|\frac{a}{d}\right|^2$.

If $\Phi = \prod_v \Phi_v \in \mathcal{S}(W_4(\mathbf{A}))$ is a factorizable Schwartz-Bruhat function, define

$$(2.8) \quad f_v^\Phi(g, s) = |\nu(g)|_v^{2s} \int_{\mathrm{GL}_1(\mathbf{Q}_v)} \Phi_v(tfg) |t|_v^{4s} dt$$

and

$$(2.9) \quad f^\Phi(g, s) = |\nu(g)|^{2s} \int_{\mathrm{GL}_1(\mathbf{A})} \Phi(tfg) |t|^{4s} dt = \prod_v f_v^\Phi(g_v, s).$$

Then $f^\Phi(g, s) \in \mathrm{Ind}_{Q(\mathbf{A})}^{\mathrm{GSp}_4(\mathbf{A})}(\delta_Q^s)$, where δ_Q is the modulus character of the Klingen parabolic. The Eisenstein series is

$$E(g, s, \Phi) = \sum_{\gamma \in Q(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q})} f^\Phi(\gamma g, s).$$

Remark 2.1. If Φ_p is the characteristic function of $W_4(\mathbf{Z}_p)$ for $p < \infty$ and $\Phi_\infty(v)$ is the Gaussian $e^{-\pi\|v\|^2}$, then $E(g, s, \Phi)$ is the normalized spherical Eisenstein series. Indeed, denote by $f_v^0(g, s)$ the unique element of

$$\mathrm{Ind}_{Q(\mathbf{Q}_v)}^{\mathrm{GSp}_4(\mathbf{Q}_v)}(\delta_Q^s)$$

that satisfies $f_v^0(k, s) = 1$ for all $k \in K_v$. Then for every v , with this choice of Φ_v , $f_v^\Phi(k, s) = \zeta_v(4s)$ for $k \in K_v$, where $\zeta_{\mathbf{R}}(s) = \Gamma_{\mathbf{R}}(s) = \pi^{-s/2} \Gamma(s/2)$. Hence by the Iwasawa decomposition, $f_v^\Phi(g, s) = \zeta_v(4s) f_v^0(g, s)$. Langlands' theorem of the constant term and functional equation then imply, by an intertwining operator calculation, that this $E(g, s, \Phi)$ has finitely many poles and satisfies the functional equation $E(g, s, \Phi) = E(g, 1 - s, \Phi)$. For general Schwartz-Bruhat functions Φ , one can obtain a bound on the poles of $E(g, s, \Phi)$ and its functional equation by using Poisson summation. Since we will not use these facts, we omit the proof.

Suppose π is a cuspidal automorphic representation on GSp_4 with trivial central character and ϕ is a cusp form in the space of π . We define our global integrals as follows:

$$(2.10) \quad I(\phi, s) = \int_{Z(\mathbf{A}) \mathrm{GSp}_4(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{A})} E(g, s, \Phi) P_D^\alpha(g) \phi(g) dg$$

and

$$(2.11) \quad J(\phi, s) = \int_{Z(\mathbf{A}) \mathrm{GL}_{2,L}^*(\mathbf{Q}) \backslash \mathrm{GL}_{2,L}^*(\mathbf{A})} E(g, s, \Phi) \phi(g) dg.$$

The integrals $I(\phi, s)$ and $J(\phi, s)$ depend on D and Φ , but we omit this data from the notation. The main theorems of this paper, spelled out in the next section, relate both of these integrals to the partial Spin L -function on GSp_4 , $L^S(\pi, \mathrm{Spin}, 2s - \frac{1}{2})$.

The integrals $I(\phi, s)$ and $J(\phi, s)$ are very closely related. In the proof of Proposition 3.2 below, after unfolding the sum defining P_D^α in $I(\phi, s)$, we find that the integral $J(\phi, s)$ appears as an inner integral of this partially unfolded $I(\phi, s)$. Note also that $J(\phi, s)$ is similar to the integrals of Andrianov [1] and Piatetski-Shapiro [12] for the Spin L -function on GSp_4 , both of which unfold to the Bessel model. The main difference is that the integrals of [1] and [12] use an Eisenstein series for the group $\mathrm{GL}_{2,L}^*$ as opposed to one for GSp_4 . The benefit of studying both integrals is that the integral $I(\phi, s)$ provides a more flexible construction than $J(\phi, s)$, while $J(\phi, s)$ makes the connection to the geometry of the Shimura variety (and with this the potential to answer motivic questions) more direct.

3. Main theorems

In this section we unfold the global integrals and state the main theorems of the paper. The calculations proving these theorems are given in Sections 4, 5, and 6. As we mentioned above, the integral unfolds to a model that is not unique. The general strategy for analyzing such integrals – which we follow – was developed by Piatetski-Shapiro and Rallis [13], and is somewhat well-known [4, 7]. For the convenience of the reader, we briefly review this strategy.

3.1. Unfolding

Define the standard Siegel parabolic P on GSp_4 to be the stabilizer of the isotropic subspace of W_4 spanned by f_1, f_2 . Denote by U its unipotent radical, and by M the Levi consisting of elements of GSp_4 whose lower left and upper right two-by-two blocks are zero. We denote by N the unipotent radical of the upper triangular Borel in $\mathrm{GL}_{2,L}^*$. Then, under the embedding $\mathrm{GL}_{2,L}^* \rightarrow \mathrm{GSp}_4$, N is contained in U . This can be seen immediately from the matrices (2.3) and (2.4).

In fact, define a unitary character $\chi : U(\mathbf{A}) \rightarrow \mathbf{C}^\times$ as follows. As is usual, pick once and for all an additive character $\psi : \mathbf{Q} \backslash \mathbf{A} \rightarrow \mathbf{C}^\times$, with conductor

equal to one at all finite places, and $\psi_\infty(x) = e^{2\pi ix}$ for $x \in \mathbf{R}$. If

$$u = \left(\begin{array}{c|cc} 1 & u_{11} & u_{12} \\ \hline & u_{12} & u_{22} \\ \hline & 1 & \\ & & 1 \end{array} \right),$$

then set

$$\chi(u) = \begin{cases} \psi(-Du_{11} + u_{22}) & D \not\equiv 1 \pmod{4} \\ \psi(\frac{1-D}{4}u_{11} + u_{12} + u_{22}) & D \equiv 1 \pmod{4} \end{cases}.$$

Note that χ is trivial on N . With this definition of χ , we define functions ϕ_χ and α_χ on GSp_4 by the absolutely convergent integrals

$$(3.1) \quad \begin{aligned} \phi_\chi(g) &= \int_{U(\mathbf{Q}) \backslash U(\mathbf{A})} \chi^{-1}(u) \phi(ug) \, du \quad \text{and} \\ \alpha_\chi(g) &= \int_{N(\mathbf{A}) \backslash U(\mathbf{A})} \chi(u) \alpha(v_D ug) \, du. \end{aligned}$$

The second integral is factorizable since $\chi = \prod_p \chi_p$ and α are, so we may write $\alpha_\chi = \prod_p \alpha_{\chi,p}$, where

$$(3.2) \quad \alpha_{\chi,p}(g_p) = \int_{N(\mathbf{Q}_p) \backslash U(\mathbf{Q}_p)} \chi_p(u) \alpha_p(v_{D,p} ug_p) \, du.$$

Here p denotes either an infinite or finite place of \mathbf{Q} .

Lemma 3.1. *The integral for α_χ in (3.1) is absolutely convergent.*

Proof. Note that the function $\alpha' : V_5(\mathbf{A}) \rightarrow \mathbf{C}$ defined by $\alpha'(v) = \alpha(vg)$ is still a Schwartz-Bruhat function. Set

$$n_{24}(z) = \left(\begin{array}{c|c} 1 & z \\ \hline & 1 \\ \hline & & 1 \end{array} \right).$$

Then $v_D n_{24}(z) = v_D + z f_2 \wedge f_1$ for all D . We obtain

$$\begin{aligned}
 (3.3) \quad \alpha_\chi(g) &= \int_{N(\mathbf{A}) \backslash U(\mathbf{A})} \chi(u) \alpha(v_D u g) \, du \\
 &= \int_{\mathbf{G}_a(\mathbf{A})} \psi(z) \alpha'(v_D n_{24}(z)) \, dz \\
 &= \int_{\mathbf{G}_a(\mathbf{A})} \psi(z) \alpha'(v_D + z f_2 \wedge f_1) \, dz.
 \end{aligned}$$

Since α' is a Schwartz-Bruhat function, it is easily seen that the function $z \mapsto \alpha'(v_D + z f_2 \wedge f_1)$ is a Schwartz-Bruhat function on \mathbf{A} . Hence the integral (3.3) is absolutely convergent. \square

Proposition 3.2. *The global integrals I and J unfold as*

$$(3.4) \quad I(\phi, s) = \int_{U(\mathbf{A}) Z(\mathbf{A}) \backslash \mathrm{GSp}_4(\mathbf{A})} f^\Phi(g, s) \phi_\chi(g) \alpha_\chi(g) \, dg$$

and

$$(3.5) \quad J(\phi, s) = \int_{N(\mathbf{A}) Z(\mathbf{A}) \backslash \mathrm{GL}_{2,L}^*(\mathbf{A})} f^\Phi(g, s) \phi_\chi(g) \, dg.$$

Proof. We first unfold $I(\phi, s)$. Assume for now that L is a field. Unfolding P_D^α , one obtains

$$(3.6) \quad I(\phi, s) = \int_{\mathrm{GL}_{2,L}^*(\mathbf{Q}) Z(\mathbf{A}) \backslash \mathrm{GSp}_4(\mathbf{A})} E(g, s, \Phi) \phi(g) \alpha(v_D g) \, dg.$$

Although it is not needed below, note that the integral $J(\phi, s)$ appears if one performs an inner integration in (3.6) over the group $\mathrm{GL}_{2,L}^*(\mathbf{A})$, since v_D is stabilized by $\mathrm{GL}_{2,L}^*(\mathbf{A})$.

Next, we unfold the Eisenstein series. Set $B'(\mathbf{Q}) = Q(\mathbf{Q}) \cap \mathrm{GL}_{2,L}^*(\mathbf{Q})$. We claim that the double coset space $Q(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q}) / \mathrm{GL}_{2,L}^*(\mathbf{Q})$ is a singleton, and thus

$$(3.7) \quad E(g, s, \Phi) = \sum_{\gamma \in B'(\mathbf{Q}) \backslash \mathrm{GL}_{2,L}^*(\mathbf{Q})} f^\Phi(\gamma g, s).$$

Assuming this fact for the moment, one obtains

$$(3.8) \quad I(\phi, s) = \int_{B'(\mathbf{Q}) Z(\mathbf{A}) \backslash \mathrm{GSp}_4(\mathbf{A})} \alpha(v_D g) f^\Phi(g, s) \phi(g) \, dg$$

by inserting (3.7) into (3.6) and unfolding the sum over $\mathrm{GL}_{2,L}^*(\mathbf{Q})$. To check the claim, note that the map $Q(\mathbf{Q})g \mapsto \mathbf{Q}f_2g$ defines a bijection between $Q(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q})$ and lines in W_4 . Moreover, $\mathrm{GL}_{2,L}^*$ acts transitively on the lines in W_4 since $\mathrm{SL}_{2,L}$ acts transitively on the nonzero elements of its two-dimensional representation L^2 . Hence $Q(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q}) / \mathrm{GL}_{2,L}^*(\mathbf{Q})$ is indeed a singleton.

Using the notation (2.3) or (2.4), B' consists of the elements of GSp_4 with $\epsilon_3 = \eta_1 = \eta_3 = \eta_4 = 0$. Now

$$(3.9) \quad \int_{N(\mathbf{Q}) \backslash N(\mathbf{A})} \phi(ng) \, dn = \sum_{\gamma \in \mathrm{GL}_1(\mathbf{Q})} \phi_\chi(\gamma g),$$

where $\mathrm{GL}_1(\mathbf{Q})$ denotes the matrices $\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix}$, $\lambda \in \mathbf{Q}^\times$. Integrating the right-hand side of (3.8) over $N(\mathbf{Q}) \backslash N(\mathbf{A})$, and then applying the Fourier expansion (3.9) into (3.8), we obtain

$$\int_{N(\mathbf{A})Z(\mathbf{A}) \backslash \mathrm{GSp}_4(\mathbf{A})} f^\Phi(g, s) \phi_\chi(g) \alpha(v_D g) \, dg.$$

The proposition follows in this case.

Now suppose that $D = 1$, so that $L \cong \mathbf{Q} \times \mathbf{Q}$ is split. Then $\mathrm{GL}_{2,L}^*$ acts on the lines in W_4 with three orbits, represented by $\mathbf{Q}f_2$, $\mathbf{Q}(f'_1)$, and $\mathbf{Q}(f'_2)$. (Recall the notation (2.5).) Indeed, consider the action of $\mathrm{GL}_{2,L}^*$ on W_4 in terms of the basis (2.5). The action described in (2.5) shows that $\mathbf{Q}f'_1$ and $\mathbf{Q}f'_2$ represent the orbits of lines in $\mathrm{span}(e'_1, f'_1)$ and $\mathrm{span}(e'_2, f'_2)$ respectively, while the fact that $\mathrm{SL}_2(\mathbf{Q})$ acts transitively on $\mathbf{Q}^2 \setminus \{0\}$ shows that $\mathbf{Q}(f'_1 - f'_2) = \mathbf{Q}f_2$ represents an orbit containing all other lines in W_4 . Fix $\gamma_1, \gamma_2 \in \mathrm{GSp}_4(\mathbf{Q})$ satisfying $f_2\gamma_1 = f'_1$, $f_2\gamma_2 = f'_2$. We have just computed that $Q(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q}) / \mathrm{GL}_{2,L}^* = \{1, \gamma_1, \gamma_2\}$. Denote by B'_i the subgroup of $\mathrm{GL}_{2,L}^*$ that stabilizes the line $\mathbf{Q}f'_i$.

Using these double coset representatives to unfold (3.6), we obtain

$$I(\phi, s) = I_{f_2}(\phi, s) + I_{f'_1}(\phi, s) + I_{f'_2}(\phi, s),$$

where $I_{f_2}(\phi, s)$ is the integral on the right-hand side of (3.8), and

$$I_{f'_i}(\phi, s) = \int_{B'_i(\mathbf{Q})Z(\mathbf{A}) \backslash \mathrm{GSp}_4(\mathbf{A})} \alpha(v_D g) f^\Phi(\gamma_i g, s) \phi(g) \, dg.$$

The unfolding of the integral $I_{f_2}(\phi, s)$ proceeds as above. We check that the integrals $I_{f'_i}(\phi, s)$ vanish by the cuspidality of ϕ . Assume $i = 1$; the case

$i = 2$ is identical. Using the ordered symplectic basis e'_1, e'_2, f'_2, f'_1 of W_4 , B'_1 is the subgroup of GSp_4 of matrices with the form

$$(3.10) \quad \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Denote by U_0 the one-dimensional unipotent subgroup

$$\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of GSp_4 . Let \bar{U} be the abelian group U/U_0 , where U is the unipotent subgroup of GSp_4 consisting of the matrices

$$u(x_1, x_2) \triangleq \begin{pmatrix} 1 & x_1 & x_2 & * \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & -x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, let $B''_1 \subseteq B'_1$ denote the matrices in GSp_4 that are of the form

$$\begin{pmatrix} a & 0 & 0 & * \\ 0 & a & * & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

We integrate ϕ over $U_0(\mathbf{Q}) \backslash U_0(\mathbf{A})$ and Fourier expand along \bar{U} to obtain the identity

$$(3.11) \quad \phi_{U_0}(g) \triangleq \int_{U_0(\mathbf{Q}) \backslash U_0(\mathbf{A})} \phi(ug) \, du = \sum_{\gamma \in B''_1(\mathbf{Q}) \backslash B'_1(\mathbf{Q})} \phi_{(1,0,0)}(\gamma g),$$

where

$$\phi_{(1,0,0)}(g) \triangleq \int_{\bar{U}(\mathbf{Q}) \backslash \bar{U}(\mathbf{A})} \psi(x_1) \phi_{U_0}(u(x_1, x_2)g) \, du.$$

Next, we perform an inner integration over $U_0(\mathbf{Q}) \backslash U_0(\mathbf{A})$ in $I_{f_1'}(\phi, s)$ and apply the Fourier expansion (3.11) to obtain

$$(3.12) \quad I_{f_1'}(\phi, s) = \int_{B_1'(\mathbf{Q})U_0(\mathbf{A})Z(\mathbf{A}) \backslash \mathrm{GSp}_4(\mathbf{A})} \alpha(v_D g) f^\Phi(\gamma_1 g, s) \phi_{(1,0,0)}(g) dg.$$

Denote by N_0 the one-dimensional unipotent subgroup of GSp_4 consisting of the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We perform an inner integration in (3.12) over the group $N_0(\mathbf{Q}) \backslash N_0(\mathbf{A})$, which shows that $I_{f_1'}(\phi, s)$ vanishes by the cuspidality of ϕ along the unipotent group

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now we unfold $J(\phi, s)$. The unfolding proceeds identically to the unfolding of $I(\phi, s)$ starting from the step (3.6). For example, assume L is a field. The coset and stabilizer computations for $Q(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q}) / \mathrm{GL}_{2,L}^*(\mathbf{Q})$ above show that

$$(3.13) \quad J(\phi, s) = \int_{B'(\mathbf{Q})Z(\mathbf{A}) \backslash \mathrm{GSp}_4(\mathbf{A})} f^\Phi(g, s) \phi(g) dg.$$

We integrate over $N(\mathbf{Q}) \backslash N(\mathbf{A})$ and use the Fourier expansion (3.9) to obtain

$$J(\phi, s) = \int_{N(\mathbf{A})Z(\mathbf{A}) \backslash \mathrm{GL}_{2,L}^*(\mathbf{A})} f^\Phi(g, s) \phi_\chi(g) dg.$$

If instead $D = 1$, we first use the coset computation

$$Q(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q}) / \mathrm{GL}_{2,L}^*(\mathbf{Q}) = \{1, \gamma_1, \gamma_2\}$$

above to obtain

$$J(\phi, s) = J_{f_2}(\phi, s) + J_{f_1'}(\phi, s) + J_{f_2}(\phi, s),$$

where $J_{f_2}(\phi, s)$ is the integral on the right-hand side of (3.13), and

$$J_{f'_i}(\phi, s) = \int_{B'_i(\mathbf{Q})Z(\mathbf{A}) \backslash \mathrm{GL}_{2,L}^*(\mathbf{A})} f^\Phi(\gamma_i g, s) \phi(g) dg.$$

The same manipulations used to check the vanishing of $I_{f'_i}(\phi, s)$ show that the integrals $J_{f'_1}(\phi, s)$ and $J_{f'_2}(\phi, s)$ vanish by the cuspidality of ϕ . The unfolding of $J_{f,2}(\phi, s)$ is the same as when L is a field, so the proposition follows. \square

3.2. Non-unique models

The Fourier coefficient ϕ_χ does not factorize for general cusp forms ϕ , so it does not follow from the unfolding calculation that the integrals I and J have the structure of an Euler product. However, Piatetski-Shapiro and Rallis in [13] introduced a technique that can sometimes factorize the global integral in this situation.

Suppose (π_v, V) is an irreducible admissible representation of $\mathrm{GSp}_4(\mathbf{Q}_v)$. Define a (U, χ) -functional to be a linear map $\ell : V \rightarrow \mathbf{C}$ that satisfies

$$\ell(\pi(u)v) = \chi(u)\ell(v)$$

for all $u \in U(\mathbf{Q}_v)$ and $v \in V$. If one replaced U in this definition by U_B , the unipotent radical of a Borel of GSp_4 , and χ by a nondegenerate character of U_B , then one would get the definition of the Whittaker functional. However, unlike the Whittaker functional, the space of (U, χ) -functionals will usually be infinite dimensional. Consequently, the Fourier coefficient ϕ_χ will not factorize. However, the following theorem allows one to recover the Euler product nature of the global integral.

Theorem 3.3. *Suppose that p is finite, π_p is unramified, Φ_p is the characteristic function of $W_4(\mathbf{Z}_p)$, and α_p is the characteristic function of $V_5(\mathbf{Z}_p)$. Let v_0 denote a nonzero spherical vector in the representation π_p . Then for all (U, χ) -functionals ℓ ,*

$$\begin{aligned} I_p(\ell, s) &\triangleq \int_{U(\mathbf{Q}_p)Z(\mathbf{Q}_p) \backslash \mathrm{GSp}_4(\mathbf{Q}_p)} f_p^{\Phi_p}(g, s) \alpha_{\chi,p}(g) \ell(\pi(g)v_0) dg \\ &= \ell(v_0) L\left(\pi_p, \mathrm{Spin}, 2s - \frac{1}{2}\right) \end{aligned}$$

for $\operatorname{Re}(s)$ sufficiently large. Similarly,

$$\begin{aligned} J_p(\ell, s) &\triangleq \int_{N(\mathbf{Q}_p)Z(\mathbf{Q}_p)\backslash \operatorname{GL}_{2,L}^*(\mathbf{Q}_p)} f_p^{\Phi_p}(g, s)\ell(\pi(g)v_0) dg \\ &= \ell(v_0)L\left(\pi_p, \operatorname{Spin}, 2s - \frac{1}{2}\right) \end{aligned}$$

for $\operatorname{Re}(s)$ sufficiently large.

We will prove this theorem in the next section. Note that when $\ell(v_0) = 0$, Theorem 3.3 says that the integrals $I_p(\ell, s)$ and $J_p(\ell, s)$ appearing in the theorem are 0.

For a finite set of places Ω , define

$$(3.14) \quad I_\Omega(\phi, s) \triangleq \int_{U(\mathbf{A}_\Omega)Z(\mathbf{A}_\Omega)\backslash \operatorname{GSp}_4(\mathbf{A}_\Omega)} f_\Omega^\Phi(g, s)\alpha_{\chi,\Omega}(g)\phi_\chi(g) dg,$$

where for a linear algebraic group H , we write $H(\mathbf{A}_\Omega) = \prod_{v \in \Omega} H(\mathbf{Q}_v)$, and where f_Ω and $\alpha_{\chi,\Omega}$ respectively denote the product of the factors of f^Φ and α_χ at the places in Ω . We now apply the argument of the Basic Lemma of [13, pg. 117] to deduce the following corollary. Write $H : \pi \rightarrow \mathbf{C}$ for the functional $\phi \mapsto \phi_\chi(1)$.

Corollary 3.4. *Suppose that the cusp form ϕ corresponds to a factorizable element under some choice of isomorphism $\pi \cong \otimes_p \pi_p$. Suppose that S is a finite set of places of \mathbf{Q} containing all the places excluded in the statement of Theorem 3.3. That is, if $p \notin S$, then p is finite, Φ_p is the characteristic function of $W_4(\mathbf{Z}_p)$, α_p is the characteristic function of $V_5(\mathbf{Z}_p)$, and π_p is spherical. Then if ϕ is invariant under $\operatorname{GSp}_4(\mathbf{Z}_p)$ for all $p \notin S$, $I(\phi, s) = I_S(\phi, s)L^S(\pi, \operatorname{Spin}, 2s - \frac{1}{2})$.*

Proof. The argument we use is well-known: see [4, Theorem 1.2] or [7, pg. 157]. For the reader's convenience, we sketch the proof.

Let Ω denote a finite set of places of \mathbf{Q} containing S . Then it is the definition of the adelic integral on the right-hand side of (3.4) that

$$I(\phi, s) = \varinjlim_{\Omega \supseteq S} I_\Omega(\phi, s)$$

in the range of absolute convergence of the integral. Now,

$$\begin{aligned}
 (3.15) \quad & I_{\Omega \cup \{p\}}(\phi, s) \\
 &= \int_{U(\mathbf{A}_{\Omega \cup \{p\}})Z(\mathbf{A}_{\Omega \cup \{p\}}) \backslash \mathrm{GSp}_4(\mathbf{A}_{\Omega \cup \{p\}})} f_{\Omega \cup \{p\}}^{\Phi}(g, s) \alpha_{\chi, \Omega \cup \{p\}}(g) \phi_{\chi}(g) dg \\
 &= \int_{U(\mathbf{A}_{\Omega})Z(\mathbf{A}_{\Omega}) \backslash \mathrm{GSp}_4(\mathbf{A}_{\Omega})} f_{\Omega}^{\Phi}(g, s) \alpha_{\chi, \Omega}(g) \\
 &\quad \times \left(\int_{U(\mathbf{Q}_p)Z(\mathbf{Q}_p) \backslash \mathrm{GSp}_4(\mathbf{Q}_p)} f_p^{\Phi_p}(g_p, s) \alpha_{\chi, p}(g_p) \phi_{\chi}(g_p g) dg_p \right) dg
 \end{aligned}$$

since f^{Φ} and α_{χ} are factorizable.

Fix g in $\mathrm{GSp}_4(\mathbf{A}_{\Omega})$ and denote by V_{π_p} the space of the representation π_p . Since ϕ corresponds to a factorizable element of the space of π under some choice of isomorphism $\pi \cong \otimes_p \pi_p$ and is $\mathrm{GSp}_4(\mathbf{Z}_p)$ -invariant, there is a spherical vector v_0 of V_{π_p} and a π_p -equivariant map ι_p from V_{π_p} into the space of cusp forms on $\mathrm{GSp}_4(\mathbf{A})$ that sends v_0 to ϕ .

Consider the functional $\ell : V_{\pi_p} \rightarrow \mathbf{C}$ given by $\ell(v) = H(\pi(g)\iota_p(v))$. Since the elements in $\mathrm{GSp}_4(\mathbf{A}_{\Omega})$ and $\mathrm{GSp}_4(\mathbf{Q}_p)$ commute, ℓ is a (U, χ) -functional on V_{π_p} . Furthermore, $\phi_{\chi}(g_p g) = \ell(\pi(g_p)v_0)$. Hence by Theorem 3.3, the inner integral in (3.15) is equal to $L(\pi_p, \mathrm{Spin}, 2s - \frac{1}{2})\ell(v_0)$. But $\ell(v_0) = \phi_{\chi}(g)$, so $I_{\Omega \cup \{p\}}(\phi, s) = I_{\Omega}(\phi, s)L(\pi_p, \mathrm{Spin}, 2s - \frac{1}{2})$. Taking the limit over $\Omega \supseteq S$ yields

$$I(\phi, s) = I_S(\phi, s)L^S\left(\pi, \mathrm{Spin}, 2s - \frac{1}{2}\right)$$

as desired. □

For a finite prime p , denote by \mathcal{H}_p the Hecke algebra of locally constant, compactly supported functions on $\mathrm{GSp}_4(\mathbf{Q}_p)$. The integration over the bad finite places in I_S may be controlled using the following proposition.

Proposition 3.5. *For any v_p in V_{π_p} , there exists a Schwartz function Φ_p on $W_4(\mathbf{Q}_p)$, a Schwartz function α_p on $V_5(\mathbf{Q}_p)$, and an element β_p in \mathcal{H}_p such that for all (U, χ) -functionals ℓ ,*

$$\int_{U(\mathbf{Q}_p)Z(\mathbf{Q}_p) \backslash \mathrm{GSp}_4(\mathbf{Q}_p)} f_p^{\Phi_p}(g, s) \alpha_{\chi, p}(g) \ell(\pi(g)(\beta_p * v_p)) dg = \ell(v_p).$$

The proposition is proved in Section 5. One also has the analogous statement for the integral J . As in Corollary 3.4, one can use this proposition to control $I_S(\phi, s)$.

Corollary 3.6. *Suppose the cusp form ϕ and the finite set of places S are as in Corollary 3.4. Then there exists $\beta = \otimes_p \beta_p$ in $\otimes_{p \in S \setminus \{\infty\}} \mathcal{H}_p$ together with Φ_p and α_p for all $p \in S \setminus \{\infty\}$ such that $I_S(\beta * \phi, s) = I_\infty(\phi, s)$.*

Proof. Since ϕ corresponds to a factorizable element under some choice of isomorphism $\pi \cong \otimes_p \pi_p$, for each p in $S \setminus \{\infty\}$ there exists a π_p -equivariant map from V_{π_p} into the space of cusp forms on $\mathrm{GSp}_4(\mathbf{A})$ and an element v_p in V_{π_p} such that $\iota_p(v_p) = \phi$. Let β_p, Φ_p , and α_p be functions that satisfy the conclusion of Proposition 3.5 for this v_p .

Suppose that Ω is a finite set of places contained in S and containing ∞ , and set $\beta_\Omega = \prod_{v \in \Omega \setminus \{\infty\}} \beta_v$. We will show $I_{\Omega \cup \{p\}}(\beta_{\Omega \cup \{p\}} * \phi, s) = I_\Omega(\beta_\Omega * \phi, s)$. Granted this equality, the corollary follows by descending induction on Ω .

As in the proof of Corollary 3.4, one has

(3.16)

$$I_{\Omega \cup \{p\}}(\beta_{\Omega \cup \{p\}} * \phi, s) = \int_{U(\mathbf{A}_\Omega)Z(\mathbf{A}_\Omega) \backslash \mathrm{GSp}_4(\mathbf{A}_\Omega)} f_\Omega^\Phi(g, s) \alpha_{\chi, \Omega}(g) \\ \times \left(\int_{U(\mathbf{Q}_p)Z(\mathbf{Q}_p) \backslash \mathrm{GSp}_4(\mathbf{Q}_p)} f_p^{\Phi_p}(g_p, s) \alpha_{\chi, p}(g_p) (\beta_{\Omega \cup \{p\}} * \phi)_\chi(g_p g) dg_p \right) dg.$$

Fix $g \in \mathrm{GSp}_4(\mathbf{A}_\Omega)$. Define a functional $\ell : V_{\pi_p} \rightarrow \mathbf{C}$ by

$$\ell(v) = H(\pi(g)\beta_\Omega * \iota_p(v)).$$

Then ℓ is a (U, χ) -functional on V_{π_p} and

$$(\beta_{\Omega \cup \{p\}} * \phi)_\chi(g_p g) = \ell(\pi_p(g_p)\beta_p * v_p).$$

Since β_p, Φ_p , and α_p satisfy the conclusion of Proposition 3.5, the inner integral in (3.16) is $\ell(v_p) = (\beta_\Omega * \phi)_\chi(g)$. Hence

$$I_{\Omega \cup \{p\}}(\beta_{\Omega \cup \{p\}} * \phi, s) = I_\Omega(\beta_\Omega * \phi, s),$$

completing the proof. □

It follows immediately from Corollary 3.4 and Corollary 3.6 that the data (Φ, ϕ', α) for the integral may be chosen so that

$$I(\phi', s) = I_\infty(\phi, s) L^S \left(\pi, \mathrm{Spin}, 2s - \frac{1}{2} \right).$$

We also verify in Section 5 that the integral $J(\phi, s)$ can be made non-vanishing for any cuspidal automorphic representation. Namely, we prove the following proposition.

Proposition 3.7. *Suppose that $\phi = v_\infty \otimes v_f$ is factorizable under some choice of factorization $\pi \cong \pi_\infty \otimes \pi_f$ and that $H(\phi) \neq 0$. Then there is a smooth vector v'_∞ in the space of π_∞ together with a Schwartz function Φ_∞ in $\mathcal{S}(W_4(\mathbf{R}))$ such that if $\phi' = v'_\infty \otimes v_f$, then the Archimedean integral*

$$(3.17) \quad J_\infty(\phi', s) = \int_{N(\mathbf{R})Z(\mathbf{R}) \backslash \mathrm{GL}_{2,L}^*(\mathbf{R})} H(\pi(g)\phi') f_\infty^{\Phi_\infty}(g, s) dg$$

is not identically zero.

One can prove the analogous result for the integral $I(\phi, s)$.

3.3. Main theorems

We summarize the findings above in the first main theorem.

Theorem 3.8. *Suppose π is an automorphic cuspidal representation of $\mathrm{PGSp}_4(\mathbf{A})$, and ϕ is a cusp form in the space of π . Then the data α_f , Φ , and $\phi' \in V_\pi$ may be chosen so that*

$$I(\phi', s) = I_\infty(\phi, s) L^S \left(\pi, \mathrm{Spin}, 2s - \frac{1}{2} \right)$$

for a sufficiently large set of finite primes S . Similarly, the data may be chosen so that

$$J(\phi', s) = J_\infty(\phi, s) L^S \left(\pi, \mathrm{Spin}, 2s - \frac{1}{2} \right)$$

for a sufficiently large set of finite primes S .

In Theorem 3.8, the Archimedean integral $I_\infty(\phi, s)$ is given by (3.14) with $\Omega = \{\infty\}$, and similarly for $J_\infty(\phi, s)$. Observe that these integrals involve the actual cusp form ϕ rather than an arbitrary (U, χ) -functional ℓ .

If ϕ comes from a level one holomorphic Siegel modular form, then π_p is spherical for all finite primes p . (See Section 6 for a precise statement.) Thus we can select Φ_p and α_p for all finite primes p so that the set of bad places S in Theorem 3.8 is empty. Furthermore, we can even choose α_∞ and Φ_∞ in such a way that we can calculate $I_\infty(\phi, s)$ explicitly in terms of Γ -functions.

Theorem 3.9. *Suppose ϕ comes from a level one holomorphic Siegel modular form f_ϕ of weight¹ $r \geq 6$, as made precise in Section 6. Set $T = \begin{pmatrix} -D & \\ & 1 \end{pmatrix}$ if $D \not\equiv 1 \pmod{4}$ and $T = \begin{pmatrix} \frac{1-D}{4} & \frac{1}{2} \\ & 1 \end{pmatrix}$ if $D \equiv 1 \pmod{4}$. Denote by $a(T)$ the Fourier coefficient of f_ϕ corresponding to the two-by-two symmetric matrix T . Then $\alpha_\infty, \Phi_\infty$ may be chosen so that*

$$I_\infty(\phi, s) = a(T)\pi^{-2s}(4\pi)^{-(2s+r-2)}\Gamma(2s)\Gamma(2s+r-2).$$

Consequently, with this choice of data,

$$I(\phi, s) = a(T)\pi^{-2s}(4\pi)^{-(2s+r-2)}\Gamma(2s)\Gamma(2s+r-2)L\left(\pi, \text{Spin}, 2s - \frac{1}{2}\right).$$

This theorem will be proved in Section 6.

4. Unramified calculation

In this section, we carry out a purely local non-Archimedean calculation to prove Theorem 3.3. Thus p in this section always denotes a fixed finite place. Both here as well as in Section 5, we will find it convenient to have a notation for the characteristic functions of sets. If \mathcal{P} is a condition on, say, the set of four-by-four matrices, then $\text{char}(\mathcal{P})$ denotes the characteristic function of the set where \mathcal{P} is satisfied. For instance, $\text{char}(\det(g) \neq 0)$ denotes the characteristic function of GL_4 inside M_4 . Since we will use $\text{char}(g \in M_4(\mathbf{Z}_p))$ very frequently, we write just $\text{char}(g)$ for this characteristic function of $M_4(\mathbf{Z}_p)$. The exact domain of definition of the functions $\text{char}(\mathcal{P})$ will always be clear from the context.

The general strategy for proving results such as Theorem 3.3 goes back to Piatetski-Shapiro and Rallis [13]. Define

$$(4.1) \quad \Delta^s(g) = |\nu(g)|^s \text{char}(g),$$

where $s \in \mathbf{C}$ and $g \in \text{GSp}_4(\mathbf{Q}_p)$. One starts with the explicit determination of $L(\pi_p, \text{Spin}, s)$ in terms of Hecke operators, which is codified in the following statement that relates this local L -function to Δ^s .

¹In the presence of our level one assumption, the condition $r \geq 6$ is redundant, since the smallest weight of a level one holomorphic Siegel modular cusp form on GSp_4 is 10.

Proposition 4.1. *Suppose $\Omega_p(g)$ is the Macdonald spherical function for the unramified representation π_p , normalized so that $\Omega_p(1) = 1$. Then*

$$\int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \Omega_p(g) dg = (1 - \omega_\pi(p) p^{2-2s}) L\left(\pi_p, \mathrm{Spin}, s - \frac{3}{2}\right),$$

where ω_π is the central character of π_p .

This is a classical result of Shimura [15, Theorem 2] in the theory of the rank two symplectic group. A proof can be found in [2, Proposition 3.53]. Since the statement and proof in [2] is in a semi-classical framework, we give a self-contained argument in Appendix A as well as definitions of Ω_p and the local Spin L -function.

Suppose now that ℓ is a (U, χ) -functional as defined in Section 3.2. If v_0 is a spherical vector for π_p , it follows from Proposition 4.1 that

$$\begin{aligned} & \int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \ell(\pi(g)v_0) dg \\ &= \int_{\mathrm{GSp}_4(\mathbf{Z}_p) \backslash \mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \int_{\mathrm{GSp}_4(\mathbf{Z}_p)} \ell(\pi(kg)v_0) dk dg \\ &= \ell(v_0) \int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \Omega_p(g) dg \\ &= \ell(v_0) (1 - \omega_\pi(p) p^{2-2s}) L\left(\pi_p, \mathrm{Spin}, s - \frac{3}{2}\right), \end{aligned}$$

where we have used the invariance of Δ under left $\mathrm{GSp}_4(\mathbf{Z}_p)$ -multiplication in the first equality and, in the second, the fact that

$$(4.2) \quad \ell(v_0) \Omega_p(g) = \int_{\mathrm{GSp}_4(\mathbf{Z}_p)} \ell(\pi(kg)v_0) dk.$$

To see the validity of (4.2), note that the right-hand side of this equation defines a bi- K -invariant function on $\mathrm{GSp}_4(\mathbf{Q}_p)$, whose value at $g = 1$ is $\ell(v_0)$. Since the eigenvalues of this function under the action of the spherical Hecke algebra of $\mathrm{GSp}_4(\mathbf{Q}_p)$ agree with those of a nonzero spherical vector in the space of π_p , the equality follows. (When $\ell(v_0) = 0$, this argument shows that both sides of (4.2) are 0.)

Thus, to prove $I_p(\ell, s) = \ell(v_0) L(\pi_p, \mathrm{Spin}, 2s - \frac{1}{2})$, it suffices to prove

$$(4.3) \quad (1 - \omega_\pi(p) p^{2-2s}) I_p\left(\ell, \frac{s-1}{2}\right) = \int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \ell(\pi(g)v_0) dg.$$

We will prove this by explicitly calculating both sides.

Recall the definition of $\alpha_{\chi,p}$ from (3.2). We now calculate $\alpha_{\chi,p}$ when α_p is the characteristic function of $V_5(\mathbf{Z}_p)$. To succinctly express the result, we first define functions Δ_0 and Δ'_0 as follows. First, by the Iwasawa decomposition, any $m \in \mathrm{GL}_2(\mathbf{Q}_p)$ can be transformed by right $\mathrm{GL}_2(\mathbf{Z}_p)$ -multiplication to a matrix of the form $\begin{pmatrix} m_{11} & m_{12} \\ & m_{22} \end{pmatrix}$. We then set

$$(4.4) \quad \Delta'_0(m) = \begin{cases} 1 & \text{if } m_{22} | m_{11}, m_{22} | m_{12}, \text{ and } \left(\frac{m_{12}}{m_{22}}\right)^2 \equiv D \pmod{\frac{m_{11}}{m_{22}}} \\ 0 & \text{otherwise} \end{cases}$$

if $D \not\equiv 1 \pmod{4}$ and

$$(4.5) \quad \Delta'_0(m) = \begin{cases} 1 & \text{if } m_{22} | m_{11}, m_{22} | 2m_{12}, \text{ and } \left(\frac{m_{12}}{m_{22}}\right)^2 - \frac{m_{12}}{m_{22}} \equiv \frac{D-1}{4} \pmod{\frac{m_{11}}{m_{22}}} \\ 0 & \text{otherwise} \end{cases}$$

if $D \equiv 1 \pmod{4}$. This definition is independent of the choice of m_{11}, m_{12} , and m_{22} . For a matrix

$$m = \begin{pmatrix} m_t & \\ & m_b \end{pmatrix}$$

in the Levi of the Siegel parabolic of GSp_4 , so that m_t, m_b are 2×2 matrices, we define

$$(4.6) \quad \Delta_0(m) = \mathrm{char} \left(\left| \frac{\det(m_t)}{\det(m_b)} \right| \leq 1 \right) \left| \frac{\det(m_t)}{\det(m_b)} \right|^{1/2} \Delta'_0(m_b).$$

Proposition 4.2. *Suppose that α_p is the characteristic function of $V_5(\mathbf{Z}_p)$. Then if m is in the Levi of the Siegel parabolic, $\alpha_{\chi,p}(m) = \Delta_0(m)$.*

Proof. Suppose

$$(4.7) \quad m = \left(\begin{array}{c|c} \lambda & \\ \hline & \lambda \\ \hline & 1 \\ & 1 \end{array} \right) \left(\begin{array}{cc|c} a & b & \\ \hline c & d & \\ \hline & d & -c \\ & -b & a \end{array} \right) \quad \text{and}$$

$$n_{24}(z) = \left(\begin{array}{c|c} 1 & \\ \hline & 1 \\ \hline & z \\ & 1 \end{array} \right).$$

Then if $D \not\equiv 1 \pmod{4}$, we have

$$\begin{aligned} v_{Dn_{24}}(z)m &= \delta^{-1}(cd - Dab)(e_1 \wedge f_1 - e_2 \wedge f_2) + \delta^{-1}(Da^2 - c^2)e_1 \wedge f_2 \\ &\quad + \delta^{-1}(d^2 - Db^2)e_2 \wedge f_1 - \frac{z}{\lambda}f_1 \wedge f_2, \end{aligned}$$

where

$$\delta = ad - bc.$$

Since $\alpha_{\chi,p}(m)$ is invariant under right $\mathrm{GSp}_4(\mathbf{Z}_p)$ -multiplication, we may assume $b = 0$. Then

$$\begin{aligned} v_{Dn_{24}}(z)m &= \frac{c}{a}(e_1 \wedge f_1 - e_2 \wedge f_2) + \frac{Da^2 - c^2}{ad}e_1 \wedge f_2 + \frac{d}{a}e_2 \wedge f_1 \\ &\quad - \frac{z}{\lambda}f_1 \wedge f_2. \end{aligned}$$

Thus, under our assumption on α_p , for the integral

$$\int_{\mathbf{Q}_p} \psi(z)\alpha_p(v_{Dn_{24}}(z)m) dz = \alpha_{\chi,p}(m)$$

to be nonzero, it is necessary to have $\frac{c}{a}$, $\frac{Da^2 - c^2}{ad}$, and $\frac{d}{a}$ in \mathbf{Z}_p . If these hold, by orthogonality of characters on the compact subgroup of \mathbf{Q}_p where $\alpha_p(v_{Dn_{24}}(z)m)$ is supported, the integral is nonzero exactly when $|\lambda| \leq 1$ and in this case equal to $|\lambda|$. The conditions for the first three coefficients are exactly the conditions defining $\Delta'_0(m_b)$. Since $|\lambda| = \left| \frac{\det(m_a)}{\det(m_b)} \right|^{1/2}$, the proposition follows.

If $D \equiv 1 \pmod{4}$, we have

$$\begin{aligned} v_{Dn_{24}}(z)m &= \left(\delta^{-1}cd - \frac{D-1}{4}\delta^{-1}ab + \frac{1}{2} \right) (e_1 \wedge f_1 - e_2 \wedge f_2) \\ &\quad + \delta^{-1} \left(\frac{D-1}{4}a^2 - ac - c^2 \right) e_1 \wedge f_2 \\ &\quad + \delta^{-1} \left(d^2 + db - \frac{D-1}{4}b^2 \right) e_2 \wedge f_1 - \frac{z}{\lambda}f_1 \wedge f_2. \end{aligned}$$

When $b = 0$, this is

$$\begin{aligned} &\left(\frac{2c}{a} + 1 \right) \frac{1}{2}(e_1 \wedge f_1 - e_2 \wedge f_2) + \frac{\frac{D-1}{4} - \frac{c}{a} - \left(\frac{c}{a}\right)^2}{\frac{d}{a}}e_1 \wedge f_2 + \frac{d}{a}e_2 \wedge f_1 \\ &- \frac{z}{\lambda}f_1 \wedge f_2. \end{aligned}$$

Reasoning as above and using the different integral structure on $V_5(\mathbf{Z}_p)$ in this case yields the proposition. \square

Recall the formula (2.8) for the section $f_p^\Phi(g, s)$ appearing in the definition of the Eisenstein series. Applying (2.8), we now compute

$$\begin{aligned} I_p(\ell, s) &= \int_{U(\mathbf{Q}_p)Z(\mathbf{Q}_p)\backslash \mathrm{GSp}_4(\mathbf{Q}_p)} f_p^\Phi(g, s, \Phi)\alpha_\chi(g)\ell(\pi(g)v_0) dg \\ &= \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{2s}\Phi(f_2g)\alpha_\chi(g)\ell(\pi(g)v_0) dg \\ &= \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{2s}\Delta_0(g)\mathrm{char}(g)\ell(\pi(g)v_0) dg. \end{aligned}$$

For the last equality, it follows easily from the definition in (4.6) that if $\Delta_0(g) \neq 0$, then $\Phi(f_2g) \neq 0$ if and only if $\mathrm{char}(g) \neq 0$. For an element g of $\mathrm{GL}_4(\mathbf{Q}_p)$, we define $\mathrm{val}_p(g)$ to be the unique integer such that $g = p^{\mathrm{val}_p(g)}g_0$ with g_0 in $M_4(\mathbf{Z}_p) \setminus pM_4(\mathbf{Z}_p)$. Then

$$\begin{aligned} (4.8) \quad I_p(\ell, s) &= \sum_{k \geq 0} (|p|^{4s}\omega_\pi(p))^k \\ &\quad \times \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{2s}\Delta_0(g)\mathrm{char}(\mathrm{val}_p(g) = 0)\ell(\pi(g)v_0) dg \\ &= (1 - \omega_\pi(p)p^{-4s})^{-1} \\ &\quad \times \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{2s}\Delta_0(g)\mathrm{char}(\mathrm{val}_p(g) = 0)\ell(\pi(g)v_0) dg. \end{aligned}$$

Shifting the s parameter, we deduce that

$$\begin{aligned} (4.9) \quad &(1 - \omega_\pi(p)p^{2-2s})I_p\left(\ell, \frac{s-1}{2}\right) \\ &= \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g)|\nu(g)|^{s-1}\Delta_0(g)\mathrm{char}(\mathrm{val}_p(g) = 0)\ell(\pi(g)v_0) dg. \end{aligned}$$

We now consider the right-hand side of (4.3), which is

$$\int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta^s(g)\ell(\pi(g)v_0) dg.$$

By the Iwasawa decomposition, we may rewrite this integral as

$$\int_{M(\mathbf{Q}_p)} \delta_p^{-1}(g) |\nu(g)|^s \ell(\pi(g)v_0) \left\{ \int_{U(\mathbf{Q}_p)} \chi(u) \text{char}(ug) du \right\} dg.$$

We study the conditions under which the inner integral can be nonvanishing.

Lemma 4.3. *Define $K_M = \text{GSp}_4(\mathbf{Z}_p) \cap M(\mathbf{Q}_p)$, let g be in $M(\mathbf{Q}_p)$, and suppose that g is equivalent by right K_M -multiplication to a matrix with bottom right 2×2 block of the form $\begin{pmatrix} p^\alpha & b \\ & p^\delta \end{pmatrix}$. Then*

$$(4.10) \quad \int_{U(\mathbf{Q}_p)} \chi(u) \text{char}(ug) du$$

is 0 unless $\delta = 0$, $\alpha \geq 0$, and $b^2 \equiv D \pmod{p^\alpha}$ or $b^2 - b \equiv \frac{D-1}{4} \pmod{p^\alpha}$ in the respective cases that $D \not\equiv 1 \pmod{4}$ or $D \equiv 1 \pmod{4}$. When these conditions hold, the integral is p^α .

Proof. The set U_g defined by $U_g = \{u \in U(\mathbf{Q}_p) \mid \text{char}(ug) \neq 0\}$ forms a compact abelian group. Hence by orthogonality of characters, the integral vanishes unless χ is identically 1 on U_g . If $\delta > 0$ then the matrix with upper right block $\begin{pmatrix} 0 & 0 \\ 0 & p^{-1} \end{pmatrix}$, whose image is nontrivial under χ , is in U_g , so the integral vanishes. So we must have $\delta = 0$, and $\alpha \geq 0$ is forced by the integrality of ug . The matrix $u \in U(\mathbf{Q}_p)$ with upper right block $\begin{pmatrix} p^{-\alpha} & -bp^{-\alpha} \\ -bp^{-\alpha} & b^2p^{-\alpha} \end{pmatrix}$ is in U_g . As $\chi(u)$ is $\psi(\frac{b^2-D}{p^\alpha})$ or $\psi(\frac{b^2-b+\frac{1-D}{4}}{p^\alpha})$ if $D \not\equiv 1 \pmod{4}$ or $D \equiv 1 \pmod{4}$ respectively, we obtain the third condition. Finally, when all the conditions are satisfied, suppose u has upper right block $\begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix}$. Then one sees that for u to be in U_g we must have $u_{11} \in p^{-\alpha}\mathbf{Z}_p$, and then that u_{12}, u_{22} are determined modulo \mathbf{Z}_p . The third condition forces χ to be trivial on U_g , so the integral is the measure of U_g , which is p^α . \square

Although Lemma 4.3 treats g of a special form, the integral (4.10) is invariant by right K_M -multiplication, so there is in fact no loss of generality. Note also that when p is inert in L , the only 2×2 matrix satisfying the conditions given in Lemma 4.3 is the identity.

We next need to understand the relationship of the model with the various integrality conditions that appeared in (4.9).

Lemma 4.4. *Suppose that $g = \begin{pmatrix} m_t & \\ & m_b \end{pmatrix} \in M(\mathbf{Q}_p)$, where $m_t, m_b \in M_2(\mathbf{Q}_p)$. If $\ell(\pi(g)v_0) \neq 0$, then $|\frac{\det(m_t)}{\det(m_b)}| \leq 1$.*

Proof. For $u \in U(\mathbf{Z}_p)$, we have

$$\ell(\pi(g)v_0) = \ell(\pi(g)uv_0) = \chi(gug^{-1})\ell(\pi(g)v_0).$$

Thus if $\ell(\pi(g)v_0) \neq 0$, we must have $\chi(gug^{-1}) = 1$ for all $u \in U(\mathbf{Z}_p)$. Define $X = \begin{pmatrix} -D & \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} \frac{1-D}{4} & \frac{1}{2} \\ & 1 \end{pmatrix}$ if $D \not\equiv 1 \pmod{4}$ or $D \equiv 1 \pmod{4}$, respectively. Then if $u = \begin{pmatrix} 1 & n_u \\ & 1 \end{pmatrix}$ for $n_u \in M_2(\mathbf{Z}_p)$, we have

$$\chi(gug^{-1}) = \psi(\text{tr}(m_t n_u m_b^{-1} X)) = \psi(\text{tr}(n_u m_b^{-1} X m_t)).$$

Hence $\ell(\pi(g)v_0) \neq 0$ implies $m_b^{-1} X m_t$ is half-integral. We will use this to show that $|\frac{\det(m_t)}{\det(m_b)}| \leq 1$.

Assume that g is written in the form specified by (4.7). We may assume without loss of generality that m_b is upper-triangular, i.e. $b = 0$, since the hypotheses and results are insensitive to right-multiplication by an element of $M(\mathbf{Z}_p)$. Then we define Y to be the matrix

$$(4.11) \quad Y = \lambda^{-1} m_b^{-1} X m_t = \begin{pmatrix} \frac{(\frac{c}{a})^2 - D}{\frac{d}{a}} & \frac{c}{a} \\ & \frac{d}{a} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{1-D}{4} + \frac{c}{a} + (\frac{c}{a})^2}{\frac{d}{a}} & \frac{1}{2} + \frac{c}{a} \\ & \frac{d}{a} \end{pmatrix}$$

in the respective cases $D \not\equiv 1 \pmod{4}$ or $D \equiv 1 \pmod{4}$. We write $x = \frac{d}{a}$ and $y = \frac{c}{a}$ so that

$$(4.12) \quad Y = \begin{pmatrix} \frac{y^2 - D}{x} & y \\ & x \end{pmatrix} \text{ or } \begin{pmatrix} \frac{y^2 + y + \frac{1-D}{4}}{x} & \frac{1}{2} + y \\ & x \end{pmatrix}.$$

Denote the set of half-integral 2×2 symmetric matrices by $\text{Sym}_2(\mathbf{Z}_p)^\vee$. Above we showed that $m_b^{-1} X m_t = \lambda Y \in \text{Sym}_2(\mathbf{Z}_p)^\vee$. Since $|\frac{\det(m_t)}{\det(m_b)}| = |\lambda|^2$, we need $|\lambda| \leq 1$. Thus it suffices to check that $Y \notin p \text{Sym}_2(\mathbf{Z}_p)^\vee$.

First assume $D \not\equiv 1 \pmod{4}$. If $\frac{y^2 - D}{x}$ and $x \in p\mathbf{Z}_p$, then $y^2 - D \equiv 0 \pmod{p^2}$. It follows that if $p \nmid 2D$, y is a unit. Hence $Y \notin p \text{Sym}_2(\mathbf{Z}_p)^\vee$. If $p \mid 2D$, the congruence $y^2 - D \equiv 0 \pmod{p^2}$ has no solution since D is square-free and $D \not\equiv 1 \pmod{4}$. We again obtain $Y \notin p \text{Sym}_2(\mathbf{Z}_p)^\vee$.

Now assume $D \equiv 1 \pmod{4}$. As above, if $\frac{y^2 + y + \frac{1-D}{4}}{x}, x \in p\mathbf{Z}_p$, we have $y^2 + y \equiv \frac{D-1}{4} \pmod{p^2}$. In particular, y is p -integral. Thus if $p = 2$, the entry $\frac{1}{2} + y$ of Y shows that $Y \notin p \text{Sym}_2(\mathbf{Z}_p)^\vee$. If $p > 2$, we rearrange to obtain $(y + \frac{1}{2})^2 \equiv D \pmod{p^2}$. Just as in the $D \not\equiv 1 \pmod{4}$ case, if $p \mid D$ then this congruence has no solution since D is square free, and if $p \nmid D$ then $y + \frac{1}{2}$ is a p -adic unit. So $Y \notin p \text{Sym}_2(\mathbf{Z}_p)^\vee$ in all cases, finishing the proof. \square

We can now prove the following.

Lemma 4.5. *If $g \in M(\mathbf{Q}_p)$ and $\ell(\pi(g)v_0) \neq 0$, then*

$$\int_{U(\mathbf{Q}_p)} \chi(u) \text{char}(ug) \, du = |\nu(g)|^{-1} \Delta_0(g) \text{char}(\text{val}_p(g) = 0)$$

Proof. Write $g = \begin{pmatrix} m_t & \\ & m_b \end{pmatrix}$, and assume without loss of generality that $m_b = \begin{pmatrix} p^\alpha & b \\ & p^\delta \end{pmatrix}$. Then when both sides are non-zero, Lemma 4.3 shows that they are equal. It remains to check that the vanishing conditions on the two sides are the same. If the right-hand side is non-zero, $m_b \in M_2(\mathbf{Z}_p)$. Since $\ell(\pi(g)v_0) \neq 0$, Lemma 4.4 verifies that $|\frac{\det(m_t)}{\det(m_b)}| \leq 1$. It follows from this inequality and the condition $\text{val}_p(g) = 0$ that some entry of m_b is a p -adic unit. The conditions defining $\Delta_0(g)$ then force $\delta = 0$ and either the congruence $b^2 \equiv D \pmod{p^\alpha}$ or $b^2 + b \equiv \frac{D-1}{4} \pmod{p^\alpha}$. Conversely, these conditions verify that $m_b \in M_2(\mathbf{Z}_p)$ and provide a unital entry in m_b . Using Lemma 4.4, we have $|\frac{\det(m_t)}{\det(m_b)}| \leq 1$, so m_t is integral. It follows that $\text{val}_p(g) = 0$ and $\Delta_0(g) \neq 0$. \square

We may now compute

$$\begin{aligned} & \int_{\text{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \ell(\pi(g)v_0) \, dg \\ &= \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^s \ell(\pi(g)v_0) \left\{ \int_{U(\mathbf{Q}_p)} \chi(u) \text{char}(ug) \, du \right\} dg \\ &= \int_{M(\mathbf{Q}_p)} \delta_P^{-1}(g) |\nu(g)|^s |\nu(g)|^{-1} \Delta_0(g) \text{char}(\text{val}_p(g) = 0) \ell(\pi(g)v_0) \, dg. \end{aligned}$$

This last line is what we obtained in (4.9) for $(1 - \omega_\pi(p)p^{2-2s})I_p(\ell, \frac{s-1}{2})$. Thus

$$\int_{\text{GSp}_4(\mathbf{Q}_p)} \Delta^s(g) \ell(\pi(g)v_0) \, dg = (1 - \omega_\pi(p)p^{2-2s})I_p\left(\ell, \frac{s-1}{2}\right),$$

completing the unramified calculation for the integral I .

We now treat the integral J by reducing the unramified calculation to the one completed above. We have already computed the right-hand side

of (4.3). We thus need to compute

$$J_p \left(\ell, \frac{s-1}{2} \right) = \int_{N(\mathbf{Q}_p)Z(\mathbf{Q}_p) \backslash \mathrm{GL}_{2,L}^*(\mathbf{Q}_p)} f_p^\Phi \left(g, \frac{s-1}{2} \right) \ell(\pi(g)v_0) dg.$$

Substituting the definition (2.8), applying the Iwasawa decomposition, and rearranging, we obtain

$$J_p \left(\ell, \frac{s-1}{2} \right) = \int_{T_L^*(\mathbf{Q}_p)} \delta_{B_L^*}^{-1}(g) |\nu(g)|^{s-1} \Phi(f_2g) \ell(\pi(g)v_0) dg.$$

Here T_L^* and B_L^* denote the torus and Borel of $\mathrm{GL}_{2,L}^*$. The explicit description of g in (2.3) or (2.4) shows that if $\Phi(f_2g) = 1$, the bottom right block of g is integral (and conversely). If $\ell(\pi(g)v_0) \neq 0$, Lemma 4.4 ensures that $|\frac{\det(m_t)}{\det(m_b)}| \leq 1$ and thus $g \in M_4(\mathbf{Z}_p)$. We may reduce to the case that $\mathrm{val}_p(g) = 0$ by the same process used in (4.8), which gives

$$\begin{aligned} J_p \left(\ell, \frac{s-1}{2} \right) &= (1 - \omega_\pi(p)p^{2-2s})^{-1} \\ &\quad \times \int_{T_L^*(\mathbf{Q}_p)} \delta_{P^*}^{-1}(g) |\nu(g)|^{s-1} \mathrm{char}(\mathrm{val}_p(g) = 0) \ell(\pi(g)v_0) dg. \end{aligned}$$

The unramified calculation now reduces to checking the following proposition.

Proposition 4.6. *We have*

$$(4.13) \quad \int_{M(\mathbf{Q}_p)} \delta_{P^*}^{-1}(g) |\nu(g)|^{s-1} \Delta_0(g) \mathrm{char}(\mathrm{val}_p(g) = 0) \ell(\pi(g)v_0) dg$$

$$(4.14) \quad = \int_{T_L^*(\mathbf{Q}_p)} \delta_{B_L^*}^{-1}(g) |\nu(g)|^{s-1} \mathrm{char}(\mathrm{val}_p(g) = 0) \ell(\pi(g)v_0) dg.$$

Proof. We claim that if $h \in \mathrm{GL}_2(\mathbf{Q}_p)$, then the matrix Y defined by $Y = h^{-1}X(\det(h)^t h^{-1})$ is half-integral if and only if $\Delta'_0(h) \neq 0$, where X is the matrix defined in Lemma 4.4. To check this, it suffices to assume that $h = \begin{pmatrix} d & -c \\ & a \end{pmatrix}$ is upper-triangular. Then Y was computed in (4.11), and an inspection of the matrix on the right-hand side of (4.11) verifies the claim.

We next use the claim to verify that if the integrand in (4.14) is non-zero, then so is the integrand in (4.13). This amounts to checking that if $\begin{pmatrix} m_t & \\ & m_b \end{pmatrix}$ is in $T_L^*(\mathbf{Q}_p)$, then $\Delta'_0(m_b) \neq 0$. Using the explicit form of the matrix m_b specified in (2.3) or (2.4), one checks immediately that $m_b X(\det(m_b)^t m_b^{-1}) = X$ is half-integral, which implies by the claim that $\Delta'_0(m_b) \neq 0$.

We claim that in fact there is a bijection, induced by the embedding $T_L^* \subseteq M$, from the elements of $T_L^*(\mathbf{Q}_p)/T_L^*(\mathbf{Z}_p)$ for which the integrand on the right is nonzero to elements of $M(\mathbf{Q}_p)/M(\mathbf{Z}_p)$ for which the integrand on the left is nonzero. We have just observed the existence of an injective map and need only check the surjectivity. Maintain the notation of (4.7). By Lemma 4.4, if $g \in M(\mathbf{Q}_p)$ satisfies $\ell(\pi(g)v_0) \neq 0$ and $\text{val}_p(g) = 0$, we have $|\frac{\det(m_t)}{\det(m_b)}| \leq 1$, so some entry of the lower right 2×2 block of g is a unit. As in the proof of Lemma 4.5, $\Delta_0(g) \neq 0$ forces g to be equivalent by right $M(\mathbf{Z}_p)$ -multiplication to an element of $M_4(\mathbf{Z}_p) \cap M(\mathbf{Q}_p)$ with $m_b = \begin{pmatrix} p^\alpha & b \\ & 1 \end{pmatrix}$, where $b^2 \equiv D \pmod{p^\alpha}$ or $b^2 - b \equiv \frac{D-1}{4} \pmod{p^\alpha}$ depending on $D \pmod{4}$ as before. The coset in $M(\mathbf{Q}_p)/M(\mathbf{Z}_p)$ is determined by α, b , and λ . (Note that changing b by a multiple of p^α does not change the coset.) Given choices of α, b , and λ meeting the aforementioned conditions, we will produce an element of $T_L^*(\mathbf{Q}_p)$ equivalent to m by right $M(\mathbf{Z}_p)$ -multiplication.

Assume that $D \not\equiv 1 \pmod{4}$. We will choose an element $m'_b \in M_2(\mathbf{Z}_p)$ such that if one sets $m'_t = \lambda \det(m'_b)^t (m'_b)^{-1}$, the matrix $m' = \begin{pmatrix} m'_t & \\ & m'_b \end{pmatrix}$ is in $T_L^*(\mathbf{Q}_p) \cap mM(\mathbf{Z}_p)$. If $p|D$, the relation $b^2 \equiv D \pmod{p^\alpha}$ forces $\alpha = 0$ or $\alpha = 1$ and (possibly after changing m within its coset) $b = 0$. Then $m'_b = \begin{pmatrix} D & \\ & 1 \end{pmatrix}$ puts m' in the same coset as m . If $p = 2$ and D is odd, the condition $D \not\equiv 1 \pmod{4}$ again forces $\alpha = 0$ or $\alpha = 1$. The case $\alpha = 0$ is trivial, and if $\alpha = 1$ we may assume $b = 1$. Setting $m'_b = \begin{pmatrix} 1 & D \\ & 1 \end{pmatrix}$ again puts m' in the same coset as m . Finally, we assume $p \nmid 2D$. We define an element $m'(\gamma) \in T_L^*(\mathbf{Q}_p)$, where $\gamma \in \mathbf{Z}_p$, by defining the lower right block to be $m'_b(\gamma) = \begin{pmatrix} b+p^\alpha\gamma & D \\ 1 & b+p^\alpha\gamma \end{pmatrix}$ and the upper left to be $m'_t(\gamma) = \lambda \det(m'_b(\gamma))^t (m'_b(\gamma))^{-1}$. Then after applying column operations, m_b transforms to $\begin{pmatrix} D-(b+\gamma p^\alpha)^2 & b+\gamma p^\alpha \\ & 1 \end{pmatrix}$. We have

$$\begin{aligned} D - (b + \gamma p^\alpha)^2 &= D - b^2 - \gamma^2 p^{2\alpha} + 2b\gamma p^\alpha \\ &= (\gamma' - \gamma^2 p^\alpha + 2b\gamma) p^\alpha, \end{aligned}$$

where $\gamma' p^\alpha = D - b^2$. Since $p \nmid 2D$, $2b$ is a unit. Then for any γ' , one can choose γ so that $\gamma' + 2b\gamma$ and thus $\gamma' - \gamma^2 p^\alpha + 2b\gamma$ is a unit. Multiplying the first column of $\begin{pmatrix} D-(b+\gamma p^\alpha)^2 & b+\gamma p^\alpha \\ & 1 \end{pmatrix}$ by the inverse of this unit and then subtracting γ times the first column from the second yields $\begin{pmatrix} p^\alpha & b \\ & 1 \end{pmatrix}$ as needed.

Now assume that $D \equiv 1 \pmod{4}$. Examining (2.4), the matrix m'_b must now be of the form $\begin{pmatrix} \epsilon & \frac{D-1}{4}\eta \\ \eta & \epsilon-\eta \end{pmatrix}$. Consider the matrix $m'_b(\gamma) \in M_2(\mathbf{Z}_p)$ defined for $\gamma \in \mathbf{Z}_p$ by $m'_b(\gamma) = \begin{pmatrix} b+\gamma p^\alpha & \frac{D-1}{4} \\ 1 & b+\gamma p^\alpha-1 \end{pmatrix}$. By applying column operations one

can transform this into $\begin{pmatrix} (b+\gamma p^\alpha)^2 - (b+\gamma p^\alpha) + \frac{1-D}{4} & b \\ 1 & 1 \end{pmatrix}$. We have

$$\begin{aligned} (b + \gamma p^\alpha)^2 - (b + \gamma p^\alpha) + \frac{1 - D}{4} &= b^2 - b + \frac{1 - D}{4} + (2b - 1)\gamma p^\alpha + \gamma^2 p^{2\alpha} \\ &= (\gamma' + (2b - 1)\gamma + \gamma^2 p^\alpha)p^\alpha, \end{aligned}$$

where $\gamma' p^\alpha = b^2 - b + \frac{1-D}{4}$. If $2b - 1$ is a unit, then γ may be chosen so that $\gamma' + (2b - 1)\gamma + \gamma^2 p^\alpha$ is a unit and the argument proceeds as above. If $p|(2b - 1)$ (which means $p \neq 2$), the transformed relation $4b^2 - 4b + 1 = (2b - 1)^2 \equiv D \pmod{p^\alpha}$ forces either $\alpha = 0$ or $p|D$ and $\alpha = 1$. The former case is trivial, and in the latter case we claim that we may use $\begin{pmatrix} b & \frac{D-1}{4} \\ 1 & b-1 \end{pmatrix}$. Indeed, if we subtract $b - 1$ times the left column from the right, switch columns, and multiply by the left column by -1 , we obtain $\begin{pmatrix} b^2 - b + \frac{1-D}{4} & b \\ 1 & 1 \end{pmatrix}$. We have $\text{ord}_p(b^2 - b + \frac{1-D}{4}) = \text{ord}_p(4b^2 - 4b + 1 - D) = 1$ since $p^2|(2b - 1)^2$ and p exactly divides D .

It remains to check that for each $m \in T_L^*(\mathbf{Q}_p)$ for which the right-hand side integrand is non-zero, the two integrands are identical when evaluated at m . In other words, we need $\delta_P^{-1}(m) \left| \frac{\det(m_t)}{\det(m_b)} \right|^{1/2} = \delta_{B_L^*}^{-1}(m)$, which is clear by direct calculation. □

5. Ramified calculation

In this section we will prove Propositions 3.5 and 3.7. Let p be a place of \mathbf{Q} , either finite or infinite. Suppose that ℓ is a (U, χ) -functional on V_{π_p} and Φ_p is a Schwartz function on $W_4(\mathbf{Q}_p)$. (We use the notation $\mathbf{Q}_\infty = \mathbf{R}$.) For a smooth vector v_1 in V_{π_p} , consider the integral

$$(5.1) \quad J_p(\ell, v_1, s) = \int_{N(\mathbf{Q}_p)Z(\mathbf{Q}_p)\backslash\text{GL}_{2,L}^*(\mathbf{Q}_p)} f_p^{\Phi_p}(g, s)\ell(\pi(g)v_1) dg.$$

We decompose the domain of the right-hand side integral in (5.1) as follows. Let

$$T' = \{t \in \text{GSp}_4 \mid t = \text{diag}(y, y, 1, 1), y \in \mathbf{G}_m\}.$$

We have

$$\begin{aligned}
 J_p(\ell, v_1, s) &= \int_{N(\mathbf{Q}_p) \backslash \mathrm{GL}_{2,L}^*(\mathbf{Q}_p)} |\nu(g)|^{2s} \Phi_p(f_2 g) \ell(\pi(g)v_1) dg \\
 (5.2) \quad &= \int_{T'(\mathbf{Q}_p)} |\nu(t)|^{2s} \int_{N(\mathbf{Q}_p) \backslash \mathrm{SL}_{2,L}(\mathbf{Q}_p)} \Phi_p(f_2 xt) \ell(\pi(xt)v_1) dx dt \\
 (5.3) \quad &= \int_{T'(\mathbf{Q}_p)} |\nu(t)|^{2s-2} \int_{N(\mathbf{Q}_p) \backslash \mathrm{SL}_{2,L}(\mathbf{Q}_p)} \Phi_p(f_2 x) \ell(\pi(tx)v_1) dx dt \\
 (5.4) \quad &= \int_{N(\mathbf{Q}_p) \backslash \mathrm{SL}_{2,L}(\mathbf{Q}_p)} \Phi_p(f_2 x) \int_{T'(\mathbf{Q}_p)} |\nu(t)|^{2s-2} \ell(\pi(tx)v_1) dt dx.
 \end{aligned}$$

Here, to go from (5.2) to (5.3), we applied the change of variables $x \mapsto txt^{-1}$ and used $f_2 t = f_2$.

Before proving Propositions 3.5 and 3.7, we consider one more preparatory fact. Let v be a smooth vector in V_{π_p} , let η be a Schwartz function on \mathbf{Q}_p , and set

$$u(z) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ & 1 & z \\ \hline & & 1 \\ & & & 1 \end{array} \right).$$

Suppose

$$(5.5) \quad v_1 = \int_{\mathbf{G}_a(\mathbf{Q}_p)} \eta(z) \pi(u(z)) v dz.$$

Then if $t = \mathrm{diag}(y, y, 1, 1)$,

$$\begin{aligned}
 \ell(\pi(t)v_1) &= \left\{ \int_{\mathbf{G}_a(\mathbf{Q}_p)} \eta(z) \chi(tu(z)t^{-1}) dz \right\} \ell(tv) \\
 (5.6) \quad &= \left\{ \int_{\mathbf{G}_a(\mathbf{Q}_p)} \eta(z) \psi(yz) dz \right\} \ell(tv) \\
 &= \hat{\eta}(y) \ell(tv).
 \end{aligned}$$

Here, $\hat{\eta}$ is the Fourier transform of η , defined to be the expression in braces in (5.6).

The following lemma gives the analogue of Proposition 3.5 for the integral J . Recall that \mathcal{H}_p denotes the Hecke algebra of locally constant compactly supported functions on $\mathrm{GSp}_4(\mathbf{Q}_p)$.

Proposition 5.1. *Suppose p is a finite place. For any v in V_{π_p} , there exists a Schwartz function Φ_p on $W_4(\mathbf{Q}_p)$ and an element β_p of \mathcal{H}_p such that for all (U, χ) functionals ℓ*

$$J_p(\ell, \beta_p * v, s) = \int_{N(\mathbf{Q}_p)Z(\mathbf{Q}_p)\backslash\mathrm{GL}_{2,L}^*(\mathbf{Q}_p)} f_p^{\Phi_p}(g, s)\ell(\pi(g)(\beta_p * v)) dg = \ell(v).$$

Proof. Denote by $K_N = \{\mathbf{1}_4 + p^N M_4(\mathbf{Z}_p)\} \cap \mathrm{GSp}_4(\mathbf{Q}_p)$ the congruence subgroup of level p^N of $\mathrm{GSp}_4(\mathbf{Z}_p)$. The vector v is stabilized by some K_N , $N \gg 0$. We may choose η in $C_c^\infty(\mathbf{Q}_p)$ so that $\hat{\eta}(y) = \mathrm{char}(y \in 1 + p^N \mathbf{Z}_p)$. In fact, $\eta(z) = \psi(-z)|p|^N \mathrm{char}(p^N z \in \mathbf{Z}_p)$ is the unique such η .

The vector v_1 , defined as in (5.5), is stable by some congruence subgroup K_{N_1} . The stabilizer of the action $f_2 \mapsto f_2 g$ for g in $\mathrm{SL}_{2,L}(\mathbf{Q}_p)$ is exactly $N(\mathbf{Q}_p)$, so we may select Φ_p such that the function $x \mapsto \Phi_p(f_2 x)$ is the restriction to $\mathrm{SL}_{2,L}(\mathbf{Q}_p)$ of the characteristic function of $N(\mathbf{Q}_p)K_{N_1}$. Choosing such a Φ_p and putting it into (5.3), we obtain

$$\begin{aligned} & J_p(\ell, v_1, s) \\ &= \int_{T'(\mathbf{Q}_p)} |\nu(t)|^{2s-2} \int_{N(\mathbf{Q}_p)\backslash\mathrm{SL}_{2,L}(\mathbf{Q}_p)} \mathrm{char}(x \in N(\mathbf{Q}_p)K_{N_1})\ell(\pi(tx)v_1) dx dt \\ &= C' \int_{T'(\mathbf{Q}_p)} |\nu(t)|^{2s-2} \ell(\pi(t)v_1) dt \\ &= C \int_{\mathrm{GL}_1(\mathbf{Q}_p)} |y|^{2s-2} \mathrm{char}(y \in 1 + p^N \mathbf{Z}_p)\ell(\pi(ty)v) dy \\ &= C\ell(v) \end{aligned}$$

for some positive constants C' and C that do not depend on ℓ . Here $t_y = \mathrm{diag}(y, y, 1, 1)$. Choosing β_p so that $\beta_p * v = C^{-1}v_1$ gives the proposition. \square

Proof of Proposition 3.5. Pick α_p so that $\alpha_p(v_D g) = \mathrm{char}(g \in \mathrm{GL}_{2,L}^* K_N)$ for N sufficiently large. Then we are reduced to the case in Proposition 5.1. \square

The following lemma immediately implies Proposition 3.7.

Lemma 5.2. *Suppose that v is a smooth vector in V_{π_∞} , ℓ is a (U, χ) -functional on the space of such smooth vectors, and $\ell(v) \neq 0$. Then there exists a smooth vector v' in V_{π_∞} and a Schwartz function Φ_∞ so that the integral $J_\infty(\ell, v', s)$ is not identically zero.*

Proof. Choose a Schwartz function η on \mathbf{R} with $\hat{\eta}$ compactly supported near $y = 1$, and define v_1 as in (5.5). Set $t_y = \text{diag}(y, y, 1, 1)$. Then

$$\int_{T'(\mathbf{R})} |\nu(t)|^{2s-2} \ell(\pi(t)v_1) dt = \int_{\text{GL}_1(\mathbf{R})} |y|^{2s-2} \hat{\eta}(y) \ell(\pi(t_y)v) dy$$

can be made nonzero for an appropriate η since $\ell(v) \neq 0$. With this choice of v_1 , the inner integral in (5.4) is nonzero when $x = 1$. As in the proof of Proposition 5.1, given any smooth function Φ' on $\text{SL}_{2,L}(\mathbf{R})$ that is left $N(\mathbf{R})$ -invariant and compactly supported on $N(\mathbf{R}) \backslash \text{SL}_{2,L}(\mathbf{R})$, we can pick Φ_∞ such that $\Phi_\infty(f_2x) = \Phi'(x)$. For $\text{Re}(s) \gg 0$, the integrand in (5.4) is continuous as a function of $x \in \text{SL}_{2,L}(\mathbf{R})$. Hence we can choose Φ_∞ so that (5.4) is nonzero. Setting $v' = v_1$ proves the lemma. \square

6. Holomorphic Siegel modular forms

The purpose of this section is to prove Theorem 3.9. We choose the data α_∞ for P_D^α and Φ_∞ for $E(g, s, f^\Phi)$ so that the Archimedean integral $I_\infty(\phi, s)$ may be computed explicitly when ϕ comes from a level one holomorphic Siegel modular form. We will see that for such a ϕ and with our choice of data, the Archimedean integral is equal to $\pi^{-2s} (4\pi)^{-(2s+r-2)} \Gamma(2s) \Gamma(2s+r-2)$ up to a constant. The discriminant D is negative throughout this section, so $L = \mathbf{Q}(\sqrt{D})$ is an imaginary quadratic field. We use the notation $u(X) = \begin{pmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{pmatrix}$ for a symmetric 2×2 matrix X in this section. The paper [9] of Kudla was very helpful for us in making the correct choice of α_∞ .

6.1. Preliminaries

Denote by \mathcal{H}_2 the upper half space of symmetric 2×2 complex matrices with positive definite imaginary part. We write $\text{GSp}_4^+(\mathbf{R})$ for the elements of $\text{GSp}_4(\mathbf{R})$ with positive similitude. Then $\text{GSp}_4^+(\mathbf{R})$ acts transitively on \mathcal{H}_2 via the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} z = (Az + B)(Cz + D)^{-1}.$$

We set $i = \sqrt{-1}$ and, abusing notation, also denote by i the 2×2 matrix $i\mathbf{1}_2 \in \mathcal{H}_2$. Denote by K_∞ the subgroup of $\text{Sp}_4(\mathbf{R})$ that stabilizes i . Then

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A + iB \in U(2) \right\}.$$

For $\gamma \in \mathrm{GSp}_4(\mathbf{R})$, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $z \in \mathcal{H}_2$, we write $j(\gamma, z) = \det(Cz + D)$. In this section, we assume our cusp form ϕ satisfies the following two conditions.

- 1) We have $\phi(gk_fk_\infty) = j(k_\infty, i)^{-r} \phi(g)$ for all $k_f \in \prod_{p < \infty} \mathrm{GSp}_4(\mathbf{Z}_p)$ and $k_\infty \in K_\infty$.
- 2) The function $f_\phi : \mathcal{H}_2 \rightarrow \mathbf{C}$ defined by

$$f_\phi(g_\infty(i)) = \nu(g_\infty)^{-r} j(g_\infty, i)^r \phi(g_\infty)$$

for all $g_\infty \in \mathrm{GSp}_4^+(\mathbf{R})$ is a classical holomorphic Siegel modular form.

Assumption (1) implies that the function f_ϕ is well-defined. Assumption (2) ensures that f_ϕ is a holomorphic function on \mathcal{H}_2 . It is then a consequence of assumption (1) that it is a classical Siegel modular form of level one and weight r . For a general discussion about the relationship between Siegel modular forms and automorphic representations, see [3, §4].

Since f_ϕ is a level one Siegel modular form, it has a Fourier expansion of the form

$$f_\phi(Z) = \sum_{S > 0} a(S) e^{2\pi i \operatorname{tr}(SZ)}.$$

Here the sum is taken over 2×2 symmetric matrices S that are half-integral and positive definite, and the Fourier coefficients $a(S)$ are in \mathbf{C} . Set

$$(6.1) \quad T = \begin{cases} \begin{pmatrix} -D & 1 \\ \frac{1-D}{4} & \frac{1}{2} \end{pmatrix} & D \not\equiv 1 \pmod{4} \\ \begin{pmatrix} \frac{1-D}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} & D \equiv 1 \pmod{4} \end{cases}.$$

The matrix T is associated to χ in the sense that $\chi(u(X)) = \psi(\operatorname{tr}(TX))$. It follows from assumptions (1) and (2) that for $g \in \mathrm{GSp}_4^+(\mathbf{R})$,

$$(6.2) \quad \phi_\chi(g_\infty) = a(T) \nu(g)^r j(g, i)^{-r} e^{2\pi i \operatorname{tr}(Tg(i))}.$$

Indeed, suppose $u(X) = u(X_\infty)u(X_f)$, and $u(X_f) \in \mathrm{GSp}_4(\widehat{\mathbf{Z}})$. Then applying the two assumptions,

$$(6.3) \quad \nu(g)^{-r} j(g, i)^r \phi(u(X)g) = \sum_{S > 0} e^{2\pi i \operatorname{tr}(SX_\infty)} a(S) e^{2\pi i \operatorname{tr}(Sg \cdot i)}.$$

The natural inclusion

$$U_P(\mathbf{Z}) \backslash U_P(\mathbf{R}) U_P(\widehat{\mathbf{Z}}) \rightarrow U_P(\mathbf{Q}) \backslash U_P(\mathbf{A})$$

is a bijection. Applying (6.3), one obtains

$$\begin{aligned}
 (6.4) \quad \nu(g)^{-r} j(g, i)^r \phi_\chi(g_\infty) &= \int_{U_P(\mathbf{Z}) \backslash U_P(\mathbf{R}) U_P(\widehat{\mathbf{Z}})} \psi^{-1}(\text{tr}(TX)) \\
 &\quad \times \left(\sum_{S>0} e^{2\pi i \text{tr}(SX_\infty) a(S) e^{2\pi i \text{tr}(Sg \cdot i)}} \right) dX \\
 &= a(T) e^{2\pi i \text{tr}(Tg \cdot i)}.
 \end{aligned}$$

If $g_\infty \in \text{GSp}_4(\mathbf{R})$ has $\nu(g_\infty) < 0$, then our assumptions imply $\phi_\chi(g_\infty) = 0$. To see this identity, first set $g'_\infty = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}_\infty g_\infty$, so that $\nu(g'_\infty) > 0$. Then

$$\begin{aligned}
 \phi_\chi(g_\infty) &= \int_{U_P(\mathbf{Q}) \backslash U_P(\mathbf{A})} \chi^{-1}(u(X)) \phi(u(X)g_\infty) dX \\
 &= \int_{U_P(\mathbf{Q}) \backslash U_P(\mathbf{A})} \chi^{-1}\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} u(X) \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\right) \\
 &\quad \times \phi\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} u(X) \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g_\infty\right) dX \\
 &= \int_{U_P(\mathbf{Q}) \backslash U_P(\mathbf{A})} \chi^{-1}(u(-X)) \phi(u(X)g'_\infty) dX \\
 &= \int_{U_P(\mathbf{Q}) \backslash U_P(\mathbf{A})} \psi^{-1}(\text{tr}(-TX)) \phi(u(X)g'_\infty) dX,
 \end{aligned}$$

where we have used that ϕ is invariant on the right by $\text{GSp}_4(\widehat{\mathbf{Z}})$. The vanishing of the quantity on the last line follows as in the computation (6.4) since $f_\phi(Z)$ has no negative definite Fourier coefficients.

We define the K_∞ -invariant inner product $\{\cdot, \cdot\}_4$ on $W_4(\mathbf{R})$ to be the unique inner product for which e_1, e_2, f_1, f_2 is an orthonormal basis. Denote by $\{\cdot, \cdot\}_5$ the inner product on $\wedge^2 W_4(\mathbf{R}) \supseteq V_5(\mathbf{R})$ induced by $\{\cdot, \cdot\}_4$. That is, if u_1, u_2, v_1, v_2 are in $W_4(\mathbf{R})$, then we set

$$\{u_1 \wedge u_2, v_1 \wedge v_2\}_5 = \{u_1, v_1\}_4 \{u_2, v_2\}_4 - \{u_1, v_2\}_4 \{u_2, v_1\}_4$$

and extend $\{\cdot, \cdot\}_5$ by linearity. Since $\{\cdot, \cdot\}_5$ is induced by $\{\cdot, \cdot\}_4$, $\{\cdot, \cdot\}_5$ is also K_∞ -invariant. Finally, denote by $\|\cdot\|$ the norms on $W_4(\mathbf{R})$ and $V_5(\mathbf{R})$ associated to these inner products. That is, $\|v\|^2 = \{v, v\}_4$ or $\{v, v\}_5$. One has

$$\|a_1 e_1 + a_2 e_2 + b_1 f_1 + b_2 f_2\|^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2$$

and

$$\begin{aligned} & \|Ae_1 \wedge e_2 + B_1e_1 \wedge f_2 + B_2(e_1 \wedge f_1 - e_2 \wedge f_2) + B_3f_1 \wedge e_2 + Cf_1 \wedge f_2\|^2 \\ &= A^2 + B_1^2 + 2B_2^2 + B_3^2 + C^2. \end{aligned}$$

6.2. Definition of and results on P_D^α

We now define α_∞ . Let $r \geq 6$ be an integer. Define a vector

$$(6.5) \quad w = -(e_1 - if_1) \wedge (e_2 - if_2) \in V_5(\mathbf{C}).$$

Then for $g \in \mathrm{GSp}_4(\mathbf{R})$ we define

$$(6.6) \quad \alpha_\infty(g) = \overline{(w, v_D g)}^{-r}.$$

Recall that v_D is defined by (2.6) or (2.7) depending on $D \pmod{4}$ and that (\cdot, \cdot) is the GSp_4 -invariant linear form on V_5 defined in Section 2.1. We will see shortly that the hypothesis $q(v) = \frac{1}{2}(v, v) < 0$ implies $(w, v) \neq 0$, and hence α_∞ is well-defined.

We have assumed that $\alpha = \prod_v \alpha_v$ is factorizable. Define $\alpha_f = \prod_{v < \infty} \alpha_v$ to be the finite part so that $\alpha = \alpha_f \otimes \alpha_\infty$. With this definition of α_∞ , one obtains

$$P_D^\alpha(g) = \sum_{\gamma \in \mathrm{GL}_{2,L}^*(\mathbf{Q}) \backslash \mathrm{GSp}_4(\mathbf{Q})} \alpha_f(v_D \gamma g_f) \overline{(w, v_D \gamma g_\infty)}^{-r}.$$

Note that α_∞ is not a Schwartz function on $V_5(\mathbf{R})$, but we will shortly see that the condition $r \geq 6$ implies that P_D^α is absolutely convergent and defines a function of moderate growth. The following lemma summarizes the properties of w that we need.

Lemma 6.1. *The element $w = -(e_1 - if_1) \wedge (e_2 - if_2) \in V_5(\mathbf{C})$ has the following properties.*

- 1) $wk_\infty = j(k_\infty, i)^{-1}w$ for all $k_\infty \in K_\infty$.
- 2) For $v \in V_5(\mathbf{R})$, $|(w, v)|^2 = \|v\|^2 - (v, v)$, and hence (w, v) is not zero when $q(v) < 0$.
- 3) Suppose that $g \in \mathrm{GSp}_4^\dagger(\mathbf{R})$ and $v \in V_5(\mathbf{R})$. Then $j(g, i)^{-1}\nu(g)(w, vg)$, as a function of g , is right-invariant under $Z(\mathbf{R})K_\infty$, so it descends

to a function of \mathcal{H}_2 . If

$$(6.7) \quad v = Ae_1 \wedge e_2 + B_1e_1 \wedge f_2 + B_2(e_1 \wedge f_1 - e_2 \wedge f_2) + B_3f_1 \wedge e_2 + Cf_1 \wedge f_2,$$

and $Z = g(i)$, then

$$(6.8) \quad j(g, i)^{-1}\nu(g)(w, vg) = -A \det(Z) + \operatorname{tr} \left(\begin{pmatrix} -B_1 & B_2 \\ B_2 & -B_3 \end{pmatrix} Z \right) - C.$$

Proof. The first item is a simple calculation. For the second, assume v is written in the form (6.7). Then one computes

$$(w, v) = (C - A) + i(B_1 - B_3) \quad \text{and} \quad (v, v) = 2B_2^2 - 2B_1B_3 + 2AC,$$

from which the desired equality follows.

For the third item, we observe using the first part that $\nu(g)j(g, i)^{-1}wg^{-1}$ is a right $Z(\mathbf{R})K_\infty$ -invariant function and thus descends to \mathcal{H}_2 . Hence to compute it, we may assume g is in the Siegel parabolic, i.e. has the lower left 2×2 block equal to zero. We will compare the expressions $g(i) = Z$ and $\nu(g)j(g, i)^{-1}wg^{-1}$ to obtain the formula

$$(6.9) \quad \nu(g)j(g, i)^{-1}wg^{-1} = -\det(Z)f_1 \wedge f_2 + z_{11}f_1 \wedge e_2 + z_{12}(e_1 \wedge f_1 - e_2 \wedge f_2) + z_{22}e_1 \wedge f_2 - e_1 \wedge e_2.$$

The third item follows immediately from this equality. To prove (6.9), we first write a general element of the Siegel parabolic in the form $g = nm$, where these matrices are defined by

$$g = \underbrace{\begin{pmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{pmatrix}}_n \cdot \underbrace{\begin{pmatrix} \lambda \mathbf{1}_2 & \\ & \mathbf{1}_2 \end{pmatrix}}_m \cdot \underbrace{\begin{pmatrix} a & b & & \\ c & d & & \\ & & d & -c \\ & & -b & a \end{pmatrix}}_m$$

for $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}$. Let $\delta = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $g(i) = X + iY = Z$, where

$$Y = \frac{\lambda}{\delta} \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}.$$

We also have $\nu(g) = \nu(m) = \lambda\delta$ and

$$j(g, i) = \det \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \delta,$$

so $j(g, i)^{-1}\nu(g) = \lambda$.

We need to calculate

$$\nu(g)j(g, i)^{-1}wg^{-1} = \lambda wm^{-1}n^{-1}$$

in terms of a, b, c, d, X, δ , and λ . Recalling from Section 2.2 that $V_5 = \wedge_0^2 W_4 \otimes \nu^{-1}$, we calculate the action of m^{-1} on $w \in V_5$ as

$$wm^{-1} = -((e_1 - if_1) \wedge (e_2 - if_2)m^{-1}) \cdot \nu(m^{-1})^{-1},$$

where m^{-1} acts on e_i and f_i by right multiplication. In the following, we will substantially simplify the calculation by writing W for the coefficient of $f_1 \wedge f_2$ since it can be identified using the other coefficients at the end. Expanding the right hand side, we have

$$\begin{aligned} wm^{-1} &= -((\lambda\delta)^{-1}(de_1 - be_2) - i\delta^{-1}(af_1 + cf_2)) \wedge ((\lambda\delta)^{-1}(-ce_1 + ae_2) \\ &\quad - i\delta^{-1}(bf_1 + df_2))\nu(m) \\ &= -(\delta\lambda)^{-1}((de_1 - be_2) \wedge (-ce_1 + ae_2) \\ &\quad - i\lambda(af_1 + cf_2) \wedge (-ce_1 + ae_2) - i\lambda(de_1 - be_2) \wedge (bf_1 + df_2) \\ &\quad + Wf_1 \wedge f_2) \\ &= -(\delta\lambda)^{-1}(\delta(e_1 \wedge e_2) - i\lambda(c^2 + d^2)e_1 \wedge f_2 \\ &\quad - i\lambda(ac + bd)(e_1 \wedge f_1 - e_2 \wedge f_2) + i\lambda(a^2 + b^2)(e_2 \wedge f_1) \\ &\quad + Wf_1 \wedge f_2) \\ &= -(\delta\lambda)^{-1}(\delta(e_1 \wedge e_2) - i\delta y_{22}e_1 \wedge f_2 - i\delta y_{12}(e_1 \wedge f_1 - e_2 \wedge f_2) \\ &\quad + i\delta y_{11}(e_2 \wedge f_1) + Wf_1 \wedge f_2), \end{aligned}$$

where in the third equality we cancel many terms with opposite signs. We now calculate

$$\begin{aligned}
 & \lambda w m^{-1} n^{-1} \\
 &= - (e_1 \wedge e_2 - i y_{22} e_1 \wedge f_2 - i y_{12} (e_1 \wedge f_1 - e_2 \wedge f_2) + i y_{11} e_2 \wedge f_1 \\
 & \quad + W f_1 \wedge f_2) \cdot n^{-1} \\
 &= - (e_1 - x_{11} f_1 - x_{12} f_2) \wedge (e_2 - x_{12} f_1 - x_{22} f_2) \\
 & \quad + i y_{22} (e_1 - x_{11} f_1 - x_{12} f_2) \wedge f_2 + i y_{12} ((e_1 - x_{11} f_1 - x_{12} f_2) \wedge f_1 \\
 & \quad - (e_2 - x_{12} f_1 - x_{22} f_2) \wedge f_2) + i y_{11} f_1 \wedge (e_2 - x_{12} f_1 - x_{22} f_2) \\
 & \quad + W f_1 \wedge f_2 \\
 &= W f_1 \wedge f_2 + z_{11} f_1 \wedge e_2 + z_{12} (e_1 \wedge f_1 - e_2 \wedge f_2) + z_{22} e_1 \wedge f_2 - e_1 \wedge e_2,
 \end{aligned}$$

where the last step involves many cancellations.

To determine W , we observe that $(w, w) = 0$ and thus $\nu(g)j(g, i)^{-1}wg^{-1}$ is also isotropic for the pairing (\cdot, \cdot) . Using the coefficients of the other terms in the expression for $\nu(g)j(g, i)^{-1}wg^{-1}$, we obtain $-2W + 2z_{12}^2 - 2z_{11}z_{22} = 0$, or $W = -\det Z$. The third item now follows. \square

Using Lemma 6.1.(2), we can check that the sum defining P_D^α converges absolutely and defines a function of moderate growth. First note that there is a Schwartz-Bruhat function α'_f on $V_5(\mathbf{A}_f)$ that satisfies

- $|\alpha_f(v)| \leq \alpha'_f(v)$ and
- α'_f is a constant times the characteristic function of an open compact subset of $V_5(\mathbf{A}_f)$.

Suppose that $g = g_f g_\infty$. Consider the set $\Lambda'(g_f)$ consisting of those elements v of $V_5(\mathbf{Q})$ that satisfy $\alpha'_f(vg_f) \neq 0$. Then $\Lambda'(g_f)$ is a lattice in $V_5(\mathbf{R})$. Hence

$$(6.10) \quad |P_D^\alpha(g)| \leq \sum_{v \in \Lambda'(g_f)} \frac{1}{(\|vg_\infty\|^2 + |D|)^{\frac{r}{2}}}.$$

The right-hand side of (6.10) converges for $r \geq 6$, and this expression also shows that $P_D^\alpha(g_f g_\infty)$ is a function of moderate growth.

Suppose that $\alpha_f = \prod_{v < \infty} \alpha_v$ is the characteristic function of $V_5(\widehat{\mathbf{Z}})$ and $\overline{P_D(Z)}$ denotes the associated Siegel modular form. (We drop the α from the notation to emphasize that we have made a particular special choice of

α .) Then in classical notation we have

$$\overline{P_D(Z)} = \sum_{\substack{v \in V_5(\mathbf{Z}) \\ q(v) = -|D|}} \frac{1}{Q_v(Z)^r}$$

with Q_v defined as follows. When v is written in the form given in (6.7) above (in terms of A, B_1, B_2, B_3 , and C), then we set

$$Q_v(Z) = -A \det(Z) + \operatorname{tr} \left(\begin{pmatrix} -B_1 & B_2 \\ B_2 & -B_3 \end{pmatrix} Z \right) - C,$$

which is the right-hand side of (6.8). The function $\overline{P_D(Z)}$ is the analogue on GSp_4 of the modular and Hilbert modular forms defined by Zagier in [16].

The following lemma is the key to the calculation of $I_\infty(\phi, s)$ below. It is the Archimedean analogue of Proposition 4.2.

Lemma 6.2. *Assume that $g \in \operatorname{GSp}_4^+(\mathbf{R}) \cap M(\mathbf{R})$ and define $Y = g(i) = \operatorname{Im}(g(i))$. Recall that α_∞ is defined in (6.6) and the function $\alpha_{\chi, \infty}$ is defined in terms of α_∞ in (3.2). Also recall the 2×2 symmetric matrix T defined in (6.1). We have*

$$(6.11) \quad \overline{\alpha_{\chi, \infty}(g)} = \nu(g)^r j(g, i)^{-r} \frac{(-2\pi i)^r}{(r-1)!} e^{-2\pi \operatorname{tr}(TY)}.$$

Proof. By definition,

$$\overline{\alpha_{\chi, \infty}(g)} = \int_{N(\mathbf{R}) \backslash U(\mathbf{R})} e^{-2\pi i \operatorname{tr}(TX)} (w, v_D u(X)g)^{-r} dX.$$

We have

$$\begin{aligned} & \nu(g)^{-r} j(g, i)^r \int_{N(\mathbf{R}) \backslash U(\mathbf{R})} e^{-2\pi i \operatorname{tr}(TX)} (w, v_D u(X)g)^{-r} dX \\ &= \int_{N(\mathbf{R}) \backslash U(\mathbf{R})} e^{-2\pi i \operatorname{tr}(TX)} (w, v_D u(X)g)^{-r} \nu(u(X)g)^{-r} j(u(X)g, i)^r dX \end{aligned}$$

since $\nu(u(X)) = 1$ and $j(u(X)g, i) = j(g, i)$. By the third part of Lemma 6.1,

$$(w, v_D u(X)g)^{-r} \nu(u(X)g)^{-r} j(u(X)g, i)^r = \operatorname{tr}(T(u(X)g)(i))^{-r}.$$

We define $z = \text{tr}(T(u(X)g)(i))$ and define x and y by $z = x + iy$, noting that $y > 0$. Then we have

$$\begin{aligned} & \frac{j(g, i)^r}{\nu(g)^r} \int_{N(\mathbf{R}) \backslash U(\mathbf{R})} e^{-2\pi i \text{tr}(TX)} (w, v_D u(X)g)^{-r} dX \\ &= \int_{\mathbf{R}} e^{-2\pi i x} \frac{1}{(x + iy)^r} dx = e^{-2\pi \text{tr}(TY)} \int_{\text{Im}(z)=y} e^{-2\pi i z} z^{-r} dz \\ &= e^{-2\pi \text{tr}(TY)} \frac{(-2\pi i)^r}{(r - 1)!}. \end{aligned}$$

The last equality is standard and comes from computing the residue of the integrand around 0. The lemma follows. \square

6.3. The Eisenstein series and the calculation of I_∞

For $w \in W_4(\mathbf{R})$, we define $\Phi_\infty(w) = e^{-\pi \|w\|^2}$. Then for $g \in \text{GSp}_4(\mathbf{R})$,

$$f_\infty^\Phi(g, s) = |\nu(g)|^{2s} \int_{\text{GL}_1(\mathbf{R})} \Phi_\infty(tf_2g) |t|^{4s} dt.$$

Then f_∞^Φ is right invariant under $Z(\mathbf{R})K_\infty$, hence descends to a function on \mathcal{H}_2 for g in $\text{GSp}_4^+(\mathbf{R})$. In fact, if $g \in \text{GSp}_4^+(\mathbf{R})$ and $g(i) = X + iY$, then

$$(6.12) \quad f_\infty^\Phi(g, s) = \pi^{-2s} \Gamma(2s) |y_{11}|^{-2s} |\det(Y)|^{2s},$$

where y_{ij} denotes the entry of Y in position (i, j) .

We now have all the ingredients necessary to compute the Archimedean integral

$$I_\infty(\phi, s) = \int_{Z(\mathbf{R})N(\mathbf{R}) \backslash \text{GSp}_4(\mathbf{R})} f_\infty^\Phi(g, s) \alpha_\infty(g) \phi_\chi(g) dg.$$

We will compute it up to a nonzero constant. Integrating over $N(\mathbf{R}) \backslash U(\mathbf{R})$ this becomes

$$\int_{Z(\mathbf{R}) \backslash M(\mathbf{R})} \delta_P^{-1}(m) f_\infty^\Phi(m, s) \alpha_{\chi, \infty}(m) \phi_\chi(m) dm.$$

As all the terms in the integrand are right invariant under K_∞ , and since $\phi_\chi(m) = 0$ if $\nu(m) < 0$, we may rewrite this as an integral over the purely imaginary part of the upper half space. We first use the formulas (6.2),

(6.12), and (6.11) to obtain

$$a(T) \frac{(2\pi i)^r}{(r-1)!} \pi^{-2s} \Gamma(2s) \times \int_{Z(\mathbf{R}) \setminus M^+(\mathbf{R})} \delta_P^{-1}(m) |\det(Y)/y_{11}|^{2s} \left| \frac{\nu(m)}{j(m, i)} \right|^{2r} e^{-4\pi \operatorname{tr}(TY)} dm.$$

Set $dY = dy_{11} dy_{12} dy_{22}$. Then we can replace $\delta_P(m) dm$ with dY and use the relations $\delta_P(m) = |\det(Y)|^{3/2}$ and $|\nu(m)|^{2r} |j(m, i)|^{-2r} = |\det(Y)|^r$ to transform $I_\infty(\phi, s)$ into, up to a nonzero constant,

$$a(T) \pi^{-2s} \Gamma(2s) \int_Y |\det(Y)/y_{11}|^{2s+r-3} y_{11}^{r-3} e^{-4\pi \operatorname{tr}(TY)} dY$$

where the integration is over the set of two-by-two real symmetric positive-definite matrices. Following Kohnen-Skoruppa [8, pg. 548], we make the variable change $t = \det(Y)/y_{11} = y_{22} - y_{12}^2/y_{11}$. We obtain

$$a(T) \pi^{-2s} \Gamma(2s) \int_{t>0, y_{11}>0, y_{12} \in \mathbf{R}} t^{2s+r-2} e^{-4\pi(t+y_{12}^2/y_{11}+|D|y_{11})} dy_{11} dy_{12} \frac{dt}{t}$$

or

$$a(T) \pi^{-2s} \Gamma(2s) \int_{t>0, y_{11}>0, y_{12} \in \mathbf{R}} t^{2s+r-2} e^{-4\pi(t+y_{12}^2/y_{11}+y_{12}+|D|y_{11})} dy_{11} dy_{12} \frac{dt}{t}$$

if $D \not\equiv 1 \pmod{4}$ or $D \equiv 1 \pmod{4}$, respectively. Up to a nonzero constant (depending on $|D|$), this is

$$a(T) \pi^{-2s} (4\pi)^{-(2s+r-2)} \Gamma(2s) \Gamma(2s+r-2).$$

This completes the proof of Theorem 3.9.

Appendix A. The local Spin L -function

In this appendix we recall the definition of the local Spin L -function and prove Proposition 4.1. Since we work primarily with a Borel subgroup of GSp_4 , we reorder the symplectic basis so that we may choose the Borel to be upper-triangular. Namely, we use the ordered basis (e_1, e_2, f_2, f_1) , where as usual the pairing is defined by $\langle e_i, f_j \rangle = \delta_{ij}$. The subgroup B of upper-triangular matrices in GSp_4 is a Borel. Denote by T and N the diagonal torus and unipotent radical of B , respectively.

Suppose that α is an unramified character of T . Also denote by α the extension of this character to B . Let π be the unramified irreducible sub-quotient of $\text{Ind}_{B(\mathbf{Q}_p)}^{\text{GSp}_4(\mathbf{Q}_p)}(\delta_B^{1/2}\alpha)$, where δ_B denotes the modulus character of B . We define four elements of $T(\mathbf{Q}_p)$ of similitude p by

$$\begin{aligned} t_A &= \text{diag}(p, p, 1, 1), & t_B &= \text{diag}(p, 1, p, 1), \\ t_C &= \text{diag}(1, p, 1, p), & \text{and } t_D &= \text{diag}(1, 1, p, p). \end{aligned}$$

The local Spin L -function of π is defined by

$$(A.1) \quad L(\pi, \text{Spin}, s) = (1 - \alpha(t_A)|p|^s)^{-1} (1 - \alpha(t_B)|p|^s)^{-1} \\ \times (1 - \alpha(t_C)|p|^s)^{-1} (1 - \alpha(t_D)|p|^s)^{-1}.$$

Since the Weyl group of GSp_4 permutes $\{t_A, t_B, t_C, t_D\}$, the right-hand side of (A.1) is Weyl-invariant. It follows that $L(\pi, \text{Spin}, s)$ depends only on π rather than α .

Denote by ϕ_α the unique spherical vector in $\text{Ind}_{B(\mathbf{Q}_p)}^{\text{GSp}_4(\mathbf{Q}_p)}(\delta_B^{1/2}\alpha)$ that satisfies the normalization $\phi_\alpha(1) = 1$. The Macdonald spherical function Ω_α is defined by

$$(A.2) \quad \Omega_\alpha(g) = \int_{\text{GSp}_4(\mathbf{Z}_p)} \phi_\alpha(kg) dk.$$

Proof of Proposition 4.1. Recall the function $\Delta^s(g) = |\nu(g)|^s \text{char}(g)$. Since Δ^s is invariant on the left by $\text{GSp}_4(\mathbf{Z}_p)$,

$$(A.3) \quad \int_{\text{GSp}_4(\mathbf{Q}_p)} \phi_\alpha(g)\Delta^s(g) dg = \int_{\text{GSp}_4(\mathbf{Q}_p)} \Omega_\alpha(g)\Delta^s(g) dg.$$

Applying the Iwasawa decomposition to the left-hand side of (A.3), one obtains

$$(A.4) \quad \int_{\text{GSp}_4(\mathbf{Q}_p)} \phi_\alpha(g)\Delta^s(g) dg = \int_{T(\mathbf{Q}_p)} \delta_B^{-\frac{1}{2}}(t)\alpha(t)I_B(t)|\nu(t)|^s dt$$

where

$$I_B(t) = \int_{N(\mathbf{Q}_p)} \text{char}(nt) dn.$$

We will compute $I_B(t)$ explicitly.

We first compute a simple function related to I_B . Let P , U , and M denote the Siegel parabolic, its unipotent radical, and its Levi, respectively.

For $m = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in M(\mathbf{Q}_p)$, we define

$$I_P(m) = \int_{U(\mathbf{Q}_p)} \text{char}(um) \, du.$$

We write $u(X)$ for the matrix $\begin{pmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{pmatrix}$ for $X \in M_2(\mathbf{Q}_p)$. If $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, the condition $u(X) \in U(\mathbf{Q}_p)$ is equivalent to $x_{11} = x_{22}$. We define a function $\Delta_1 : M_2(\mathbf{Q}_p) \rightarrow \mathbf{R}_{\geq 0}$ by $\Delta_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \sup\{|a|, |b|, |c|, |d|\}$.

Lemma A.1. *Suppose $m = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \in M_4(\mathbf{Z}_p)$. Then*

$$I_P(m) = |\det(m_2)|^{-1} \Delta_1(m_2)^{-1}.$$

Proof. Since $m \in M_4(\mathbf{Z}_p)$, $I_P(m)$ is the measure of the set $H(m_2)$ defined by

$$H(m_2) = \{X \in M_2(\mathbf{Q}_p) : u(X) \in U(\mathbf{Q}_p) \text{ and } Xm_2 \in M_2(\mathbf{Z}_p)\}.$$

It is clear that $H(m_2k') = H(m_2)$ if $k' \in \text{GL}_2(\mathbf{Z}_p)$. If $k \in \text{GL}_2(\mathbf{Z}_p)$ and ${}^a k$ denotes interchanging the top left and bottom right entries, then the map $X \mapsto {}^a kXk$ defines a measure-preserving bijection $H(km_2) \rightarrow H(m_2)$. Suppose $m_2 = k \begin{pmatrix} p^a & \\ & p^b \end{pmatrix} k'$ with k, k' in $\text{GL}_2(\mathbf{Z}_p)$ and $a \geq b \geq 0$ integers. It follows that $I_P(m_2) = I_P\left(\begin{pmatrix} p^a & \\ & p^b \end{pmatrix}\right)$. The set

$$H\left(\begin{pmatrix} p^a & \\ & p^b \end{pmatrix}\right) = \begin{pmatrix} p^{-b}\mathbf{Z}_p & p^{-b}\mathbf{Z}_p \\ p^{-a}\mathbf{Z}_p & * \end{pmatrix}$$

has measure p^{a+2b} , so we have $I_P(m) = p^{a+2b} = |\det(m)|^{-1} p^b$. Since $\Delta_1(m)$ is a bi- $\text{GL}_2(\mathbf{Z}_p)$ -invariant function on $M_2(\mathbf{Q}_p)$, $\Delta_1(m_2) = p^{-b}$. The lemma follows. □

The integrals I_B and I_P are related by

$$(A.5) \quad I_B(t) = \int_{U(\mathbf{Q}_p) \setminus N(\mathbf{Q}_p)} I_P(nt) \, dn.$$

Suppose

$$t = \text{diag}(\lambda t_1, \lambda t_2, t_1, t_2) \in M_4(\mathbf{Z}_p) \cap T(\mathbf{Q}_p).$$

Identifying $U(\mathbf{Q}_p)\backslash N(\mathbf{Q}_p)$ with $\mathbf{G}_a(\mathbf{Q}_p)$ in the right-hand-side of (A.5) and applying Lemma A.1, we obtain

$$\begin{aligned}
 \text{(A.6)} \quad I_B(t) &= \int_{\mathbf{G}_a(\mathbf{Q}_p)} |t_1 t_2|^{-1} \sup\{|t_1|, |t_2|, |xt_2|\}^{-1} \text{char}(|x\lambda t_2| \leq 1) \\
 &\quad \times \text{char}(|xt_2| \leq 1) dx \\
 &= |t_1 t_2^2|^{-1} \int_{\mathbf{G}_a(\mathbf{Q}_p)} \sup\{|t_1|, |t_2|, |y|\}^{-1} \text{char}(|\lambda y| \leq 1) \text{char}(|y| \leq 1) dy \\
 &= \delta_B(t)^{1/2} |\nu(t)|^{-3/2} \int_{\mathbf{G}_a(\mathbf{Q}_p)} \sup\{|t_1|, |t_2|, |y|\}^{-1} \text{char}(|\lambda y| \leq 1) \\
 &\quad \times \text{char}(|y| \leq 1) dy.
 \end{aligned}$$

For the second equality, we use the change of variables $xt_2 = y$. We have used $\delta_B(t) = |\lambda|^3 |t_1/t_2|$ and $\nu(t) = \lambda t_1 t_2$ for the third equality.

If $\mu \in \mathbf{Q}_p^\times$ and $|\mu| \leq 1$, then

$$\begin{aligned}
 \text{(A.7)} \quad \int_{\mathbf{Q}_p} \sup\{|\mu|, |y|\}^{-1} \text{char}(|y| \leq 1) dy &= 1 + \text{ord}_p(\mu)(1 - |p|) \\
 &= \text{ord}_p(p\mu) - |p| \text{ord}_p(\mu).
 \end{aligned}$$

For $t \in T$, denote by $\text{val}_p(t)$ the largest integer such that $p^{-\text{val}_p(t)} t \in M_4(\mathbf{Z}_p)$. To evaluate (A.6), we consider the cases $|\lambda| > 1$ and $|\lambda| \leq 1$. If $|\lambda| \leq 1$, then by applying (A.7) to (A.6) one obtains

$$I_B(t) = \delta_B(t)^{1/2} |\nu(t)|^{-3/2} (\text{val}_p(pt) - |p| \text{val}_p(t)).$$

If $|\lambda| > 1$, then

$$\begin{aligned}
 \text{(A.8)} \quad I_B(t) &= \delta_B(t)^{1/2} |\nu(t)|^{-3/2} \\
 &\quad \times \int_{\mathbf{G}_a(\mathbf{Q}_p)} \sup\{|t_1|, |t_2|, |y|\}^{-1} \text{char}(|\lambda y| \leq 1) dy \\
 \text{(A.9)} \quad &= \delta_B(t)^{1/2} |\nu(t)|^{-3/2} \\
 &\quad \times \int_{\mathbf{G}_a(\mathbf{Q}_p)} \sup\{|\lambda t_1|, |\lambda t_2|, |z|\}^{-1} \text{char}(|z| \leq 1) dz \\
 &= \delta_B(t)^{1/2} |\nu(t)|^{-3/2} (\text{val}_p(pt) - |p| \text{val}_p(t))
 \end{aligned}$$

using the variable change $z = \lambda y$ to go from (A.8) to (A.9). Hence

$$\text{(A.10)} \quad I_B(t) = \delta_B(t)^{1/2} |\nu(t)|^{-3/2} (\text{val}_p(pt) - |p| \text{val}_p(t))$$

in both cases.

Inserting (A.10) into (A.4), we obtain

$$\begin{aligned}
 \text{(A.11)} \quad & \int_{\text{GSp}_4(\mathbf{Q}_p)} \phi_\alpha(g) \Delta^s(g) dg \\
 &= \int_{T(\mathbf{Q}_p)} \alpha(t) \text{char}(t) (\text{val}_p(pt) - |p| \text{val}_p(t)) |\nu(t)|^{s-3/2} dt.
 \end{aligned}$$

Let ω_α denote the central character of α . Then

$$\begin{aligned}
 \text{(A.12)} \quad & \int_{T(\mathbf{Q}_p)} \alpha(t) \text{char}(t) \text{val}_p(t) |\nu(t)|^{s-3/2} dt \\
 &= \int_{T(\mathbf{Q}_p)} \alpha(pt) \text{char}(pt) \text{val}_p(pt) |\nu(pt)|^{s-3/2} dt \\
 &= \omega_\alpha(p) |p|^{2s-3} \int_{T(\mathbf{Q}_p)} \alpha(t) \text{char}(pt) \text{val}_p(pt) |\nu(t)|^{s-3/2} dt \\
 &= \omega_\alpha(p) |p|^{2s-3} \int_{T(\mathbf{Q}_p)} \alpha(t) \text{char}(t) \text{val}_p(pt) |\nu(t)|^{s-3/2} dt.
 \end{aligned}$$

In the last line we have used $\text{char}(pt) \text{val}_p(pt) = \text{char}(t) \text{val}_p(pt)$, which holds because $\text{val}_p(pt) = 0$ when $pt \in M_4(\mathbf{Z}_p)$ and $t \notin M_4(\mathbf{Z}_p)$. Inserting (A.12) into (A.11), we obtain

$$\begin{aligned}
 & \int_{\text{GSp}_4(\mathbf{Q}_p)} \phi_\alpha(g) \Delta^s(g) dg \\
 &= (1 - \omega_\alpha(p) |p|^{2s-2}) \int_{T(\mathbf{Q}_p)} \alpha(t) \text{char}(t) \text{val}_p(pt) |\nu(t)|^{s-3/2} dt.
 \end{aligned}$$

Lemma A.2. *Suppose $t \in T(\mathbf{Q}_p) \cap M_4(\mathbf{Z}_p)$. If $\text{val}_p(t) = k$, then up to the action of $T(\mathbf{Z}_p)$, t can be written as a product of t_A, t_B, t_C and t_D in $k + 1$ ways.*

Proof. The set of elements of $T(\mathbf{Q}_p) \cap M_4(\mathbf{Z}_p)/T(\mathbf{Z}_p)$ with $\text{val}_p(t) = k$ is given by

$$\{\text{diag}(p^{u_1}, p^{u_2}, p^{u_3}, p^{u_4}) : u_1 + u_4 = u_2 + u_3, \min\{u_1, u_2, u_3, u_4\} = k\}.$$

For such an element, we are asked to find the number of 4-tuples of nonnegative integers $(\alpha, \beta, \gamma, \delta)$ so that

$$\text{(A.13)} \quad t_A^\alpha t_B^\beta t_C^\gamma t_D^\delta = \text{diag}(p^{u_1}, p^{u_2}, p^{u_3}, p^{u_4}).$$

Set $u'_i = u_i - k$. There is a unique solution $(\alpha', \beta', \gamma', \delta')$ to the equation

$$t_A^{\alpha'} t_B^{\beta'} t_C^{\gamma'} t_D^{\delta'} = \text{diag}(p^{u'_1}, p^{u'_2}, p^{u'_3}, p^{u'_4})$$

since some $u'_i = 0$, which determines two of α', β', γ' , and δ' , and the other two are determined by two of the remaining values of u'_i . We now prove by induction on k that the solutions are

$$(A.14) \quad (t_A t_D)^x (t_B t_C)^y t_A^{\alpha'} t_B^{\beta'} t_C^{\gamma'} t_D^{\delta'} = \text{diag}(p^{u_1}, p^{u_2}, p^{u_3}, p^{u_4}),$$

where $x, y \geq 0$ and $x + y = k$. Assume the case $\text{val}_p(t) = k$ is known, and consider the case $\text{val}_p(t) = k + 1$. Then for each tuple $(\alpha, \beta, \gamma, \delta)$, we must have either both $\alpha \geq 1$ and $\delta \geq 1$ or both $\beta \geq 1$ and $\gamma \geq 1$. Dividing by $t_A t_D$ or $t_B t_C$ in these respective cases, we find by our inductive hypothesis that every solution must be the product of either $t_A t_D$ or $t_B t_C$ with the left-hand side of (A.14). The $k + 1$ unique tuples obtainable in this way are the ones enumerated in (A.14). \square

Applying Lemma A.2, one obtains

$$\begin{aligned} & \int_{T(\mathbf{Q}_p)} \alpha(t) \text{char}(t) \text{val}_p(pt) |\nu(t)|^{s-3/2} dt \\ &= \left(\sum_{k \geq 0} \alpha(t_A) |p|^{k(s-3/2)} \right) \left(\sum_{k \geq 0} \alpha(t_B) |p|^{k(s-3/2)} \right) \\ & \quad \times \left(\sum_{k \geq 0} \alpha(t_C) |p|^{k(s-3/2)} \right) \left(\sum_{k \geq 0} \alpha(t_D) |p|^{k(s-3/2)} \right). \end{aligned}$$

The proposition follows. \square

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