Math. Res. Lett.
Volume 24, Number 1, 153–172, 2017

Free logarithmic derivation modules over factorial domains

Cleto B. Miranda-Neto

We introduce and characterize the class of *tangentially free ideals*, which are (not necessarily principal) ideals whose logarithmic derivation module is free, in (not necessarily regular) factorial domains essentially of finite type over a field of characteristic zero. This yields an extension of Saito's celebrated theory of free divisors in smooth manifolds. Examples are worked out, for instance a non-principal, tangentially free ideal in the coordinate ring of the so-called E_8-singularity. Further, we notice a connection to the classical Zariski-Lipman conjecture in the open case of surfaces.

1. Introduction

The influential theory protagonized by the so-called *free divisors* in smooth ambient spaces (complex manifolds) has its cornerstone in K. Saito's classical paper [16], which has been extensively studied and opened wide research activity (for a piece of its legacy, see [1], [4], [13], [14], [15], [20], [21], [22], [25], [26]). From a purely algebraic viewpoint, Saito's theory relies primarily on detecting freeness of logarithmic derivation modules of *principal* ideals, over polynomial rings or *regular* domains essentially of finite type over the complex number field.

In the present paper, with both language and tools from Commutative Algebra, our main goal is to extend and generalize the theory of free divisors, by developing an investigation of not necessarily principal, suitable ideals called *tangentially free ideals* — *free ideals*, for short — together with a closely related class of *pseudo-free ideals*, in factorial domains essentially of

---

2010 Mathematics Subject Classification: Primary 13N15, 13C05, 13C10, 32M25.
Key words and phrases: Derivation, logarithmic derivation, free divisor, Zariski-Lipman conjecture.
This paper is based on the author's doctoral thesis (Federal University of Pernambuco, Brazil).

finite type over a field of characteristic zero. Singularities are allowed herein, so that our ambient factorial rings may be non-regular.

Let $\mathscr{O}$ be an algebra essentially of finite type over a field k containing the rational numbers, and let $\mathscr{I} \subset \mathscr{O}$ be an ideal. If $D_{\mathscr{O}/k}$ stands for the module of ordinary k-derivations of $\mathscr{O}$, consider the submodule formed by the logarithmic k-derivations of $\mathscr{I} \subset \mathscr{O}$,

$$T_{\mathscr{O}/k}(\mathscr{I}) \ = \ \{d \in D_{\mathscr{O}/k} \mid d(\mathscr{I}) \subset \mathscr{I}\},$$

which we could denote also by $T_{\mathscr{Y}/k}(\mathscr{X})$, thinking geometrically on the closed subscheme $\mathscr{X} = \mathrm{Spec}(\mathscr{O}/\mathscr{I})$ of the affine k-scheme $\mathscr{Y} = \mathrm{Spec}(\mathscr{O})$. Vaguely speaking, in the case where $\mathscr{Y}$ is an affine or projective space (say, over $k = \mathbf{C}$) and $\mathscr{X} \subset \mathscr{Y}$ is an algebraic variety, this module encodes much of geometric information concerning vector fields and foliations on $\mathscr{Y}$ leaving $\mathscr{X}$ invariant, that is, tangent to $\mathscr{X}$ (see [3], [5], [13], [18] and their recommended references on the theme). Further, much attention has been devoted to the important case where $\mathscr{X}$ defines a hyperplane arrangement (cf., *e.g.*, [15], [21], [25], [26]).

In the spirit of Saito's original idea for divisors, the central question we consider herein is: when is the $\mathscr{O}$-module $T_{\mathscr{O}/k}(\mathscr{I})$ free? Except for the standard case of principal ideals, this problem seems to be untouched up to now. In this paper, we solve it when $\mathscr{O}$ is factorial and $\mathscr{I} \subset \mathscr{O}$ is an arbitrary ideal. More precisely, within a suitable setup, we prove that $T_{\mathscr{O}/k}(\mathscr{I})$ is free if and only if $\mathscr{I}$ has the form $\mathscr{I} = f\mathscr{J}$, for some ideal $\mathscr{J} \subset \mathscr{O}$ and some *abstract free divisor* $f \in \mathscr{O}$ such that $\tau_f^{(j)}(\mathscr{J}) \subset \mathscr{J}, \ \ j = 1, \ldots, s$, where $\{\tau_f^{(1)}, \ldots, \tau_f^{(s)}\}$ is any set of generators for the (free) logarithmic derivation module of f, which we compute explicitly from the syzygies of the *abstract Jacobian ideal* of f. In this case, the ideal $\mathscr{I}$ is said to be *free*, and the — often higher codimensional — ideal $\mathscr{J}$ is dubbed *pseudo-free* (with respect to f). As we shall clarify, this can be done essentially by means of a module-theoretic approach since our strategy for investigating the freeness of $T_{\mathscr{O}/k}(\mathscr{I})$ landed on searching first its reflexiveness as an $\mathscr{O}$-module. Further, the theory will be illustrated through detailed examples, in virtue of the effective side of our characterization.

We now explain the content of the paper. Section 2 is divided into two parts, and essentially generalizes some well-known results of Saito and Terao on the principal ideal case. In the first part 2.1, given an extension $k \subset \mathscr{O}$ of Noetherian rings, we exploit some general features of the module $T_{\mathscr{O}/k}(f)$ of logarithmic derivations of a divisor $f \in \mathscr{O}$. After introducing the notion of *abstract Jacobian ideal* (Definition 2.1), we describe generators for the

$\mathscr{O}$-module $T_{\mathscr{O}/k}(f)$ (Proposition 2.3), and under mild conditions we prove that it is reflexive (Proposition 2.4). In the second part 2.2, the *abstract free divisors* (Definition 2.5) are characterized (Propositions 2.7 and 2.8), for instance by means of perfectness of abstract Jacobian ideals if the ambient derivation module is free (up to now, the subring k may have arbitrary characteristic and is not even assumed to contain a field). In Example 2.11, an abstract free divisor in the coordinate ring of the twisted cubic curve in complex projective 3-space is exhibited.

Section 3 contains the core of the theory. Our main result (Theorem 3.1) furnishes reflexiveness criteria for the $\mathscr{O}$-module $T_{\mathscr{O}/k}(\mathscr{I})$, where k is a field containing the rationals and $\mathscr{O}$ is a factorial domain of finite type or essentially of finite type over k. Freeness criteria follow automatically (Corollary 3.2), putting us in a position to introduce the class of *tangentially free ideals* (Definition 3.3), or simply *free ideals*, thus providing a (purely algebraic) generalization of free divisor theory. As we have mentioned above, the key structural point is that, in order for an ideal to be free, it must have an abstract free divisor f as a component, together with a suitable component that must be invariant under the logarithmic derivations of f (computed via Proposition 2.3). Example 3.5 provides a non-principal free ideal in the factorial coordinate ring of the non-smooth complex surface $x^2 + y^3 + z^5 = 0$, the so-called E_8-singularity; this example is, then, a particular bonus of our theory with respect to Saito's traditional one, established specifically for hypersurfaces in smooth ambient spaces. A well-structured family of free ideals is described in Corollary 3.7.

Pseudo-free ideals are introduced in Definition 3.10. In the polynomial case, they yield algebraic varieties — here dubbed *pseudo-free varieties* — that typically have codimension at least 2. For instance, the (reduced) singular locus of a polynomial free divisor is a pseudo-free variety (Corollary 3.11). It is also observed that the radical of a pseudo-free ideal is again pseudo-free (Remark 3.12).

We finish the paper with Section 4, which brings a connection to the long-standing *Zariski-Lipman conjecture* in the open 2-dimensional case. In Theorem 4.3 we show that it is true provided there exists an abstract free divisor whose abstract Jacobian ideal is non-principal and *full* relative to the maximal ideal. As a crucial ingredient, besides our Proposition 2.7, we used a nice result of Goto and Hayasaka. We believe that the technique employed herein may contribute to the solution of this classical problem in its full form.

Conventions. Throughout this paper, all rings are assumed to be commutative and with identity 1. If $k \subset \mathscr{O}$ is a ring extension and $d \in D_{\mathscr{O}/k}$, then a given ideal $\mathscr{I} \subset \mathscr{O}$ is *invariant under* d if $d(\mathscr{I}) \subset \mathscr{I}$; in this case, we also say that d is *logarithmic* for $\mathscr{I}$, or that d *preserves* $\mathscr{I}$. By a *factorial* domain we mean a domain with the unique factorization property. For technical convenience, we shall implicitly assume the mild condition that $D_{\mathscr{O}/k} \neq 0$, which holds essentially in all situations of interest, for instance, if k is a field of characteristic zero and $\mathscr{O}$ is a k-algebra essentially of finite type either having positive Krull dimension or being 0-dimensional but not a direct product of separable algebraic field extensions of k.

2. Logarithmic derivations and free divisors

Logarithmic vector fields and free divisors have been studied, traditionally, in the local complex analytic setup, but the well-known fact that projective modules over polynomial rings are free allows us to adapt these concepts into a global algebro-geometric context.

Let $\mathscr{O}$ stand provisorily for a polynomial ring over an algebraically closed field k of characteristic zero, and consider a reduced hypersurface $\mathscr{F} = \mathscr{V}(f) \subset \mathbf{A}_k^n = \mathbf{A}^n$, for some non-constant $f \in \mathscr{O}$ (we point out that the projective context is allowed as well). We may look at the $\mathscr{O}$-module $T_{\mathbf{A}^n/k}(\mathscr{F})$, quite often denoted $\mathrm{Der}(-\log \mathscr{F})$, formed by the *logarithmic vector fields* of $\mathscr{F}$; these are the vector fields defined globally on $\mathbf{A}^n$ and tangent to $\mathscr{F}$ at a suitable Zariski dense open subset. We may realize them as being the k-derivations $\tau \colon \mathscr{O} \to \mathscr{O}$ with the property that the rational function $\tau(f)/f$ is in fact polynomial, that is, $T_{\mathbf{A}^n/k}(\mathscr{F})$ is the module of logarithmic derivations of the principal ideal $(f) \subset \mathscr{O}$. The (embedded) hypersurface $\mathscr{F} \subset \mathbf{A}^n$ is said to be a *free divisor* (or a *Saito divisor*) if $T_{\mathbf{A}^n/k}(\mathscr{F})$ is a free module, necessarily of rank n. For more on this notion in the polynomial setup, see [13, Section 4], [14] and [20, Subsection 3.2].

Now, let $\mathscr{O}$ be a factorial domain which is a localization of a finitely generated k-algebra, where k is a field of characteristic zero. As explained in the Introduction, the aim of this paper is to characterize the ideals $\mathscr{I} \subset \mathscr{O}$ such that the associated $\mathscr{O}$-module of logarithmic derivations $T_{\mathscr{O}/k}(\mathscr{I}) = \{d \in D_{\mathscr{O}/k} \mid d(\mathscr{I}) \subset \mathscr{I}\}$ is free. This will follow automatically from our investigation of reflexiveness, which will be done in Section 3. In the present section, within the more general setting where $k \subset \mathscr{O}$ is an extension of Noetherian (local) rings, what we do is to treat the case where $\mathscr{I} \subset \mathscr{O}$ is a *principal* ideal, as it will be a central ingredient into our main results in Section 3, and also since it represents, in more generality, the classical subject founded by Saito and expanded by many specialists.

2.1. The module of logarithmic derivations of a divisor

Let $k \subset \mathscr{O}$ be an extension of Noetherian rings and let $\mathscr{I} = (f) \subset \mathscr{O}$ be a principal ideal. We may write $T_{\mathscr{O}/k}(\mathscr{I}) = T_{\mathscr{O}/k}(f)$, the module of logarithmic k-derivations of f. In this part, we shall be focused mainly on establishing the reflexiveness of $T_{\mathscr{O}/k}(f)$, provided f is $\mathscr{O}$-regular (that is, a non-zero-divisor in $\mathscr{O}$) and $\mathscr{O}$ satisfies certain mild conditions (for instance, to be Gorenstein locally in depth 1). First, we consider the following notion:

Definition 2.1. Let $k \subset \mathscr{O}$ be a general ring extension. For any element $f \in \mathscr{O}$, denote by $\mathscr{G}_f$ the ideal generated by the image of the natural $\mathscr{O}$-linear map $D_{\mathscr{O}/k} \to \mathscr{O}$ given by evaluation at f. We define $\mathscr{J}_f = (\mathscr{G}_f, f)$, the *abstract Jacobian ideal* of f (over k). Thus, we may write $\mathscr{J}_f = (d_\alpha(f))_{\alpha \in \mathcal{A}} + (f)$ for any given set of generators $\{d_\alpha\}_{\alpha \in \mathcal{A}}$ of $D_{\mathscr{O}/k}$ (notice that there is no dependence with respect to the choice of the generating set). For instance, if $D_{\mathscr{O}/k}$ admits a finite generating set $\{d_1, \ldots, d_n\}$ as an $\mathscr{O}$-module, then $\mathscr{J}_f = (d_1(f), \ldots, d_n(f), f)$. Clearly, a concrete typical situation where $\mathscr{J}_f = \mathscr{G}_f$ is when f is a quasi-homogeneous polynomial with coefficients in a field.

Remark 2.2. Let $\mathscr{A}$ be an algebra of finite type over a field k, and let $f \in \mathscr{A}$ be a non-invertible divisor with abstract Jacobian ideal $\mathscr{J}_f$ as defined above. The ring $\mathscr{A}/(f)$ has a presentation $\mathscr{O}/\mathscr{I}$, for some ideal $\mathscr{I}$ in a polynomial ring $\mathscr{O}$ over k, so that we can speak about the "true" Jacobian ideal $J(\mathscr{A}/(f))$ of $\mathscr{A}/(f)$. However, in general, it does *not* coincide with the ideal $\mathscr{J}_f/(f) \subset \mathscr{A}/(f)$ — which also may be seen in $\mathscr{O}/\mathscr{I}$. This will be illustrated latter at the end of Example 2.11. In particular, a natural question arises as to when the equality $J(\mathscr{A}/(f)) = \mathscr{J}_f/(f)$ holds, and whether this condition reflects back to $\mathscr{A}$ or $\mathscr{A}/(f)$ (for instance, concerning smoothness).

The proposition below gives generators for the logarithmic derivation module of a divisor. It will be crucial in order to reveal the *effective* side of our main results in Section 3.

Proposition 2.3. *Let $k \subset \mathscr{O}$ be an extension of Noetherian rings such that $D_{\mathscr{O}/k}$ is finitely generated as an $\mathscr{O}$-module, and let $\{d_1, \ldots, d_n\}$ be a finite set of generators. For an element $f \in \mathscr{O}$, fix the* (*ordered, signed*) *generating set $\{d_1(f), \ldots, d_n(f), f\}$ of its abstract Jacobian ideal $\mathscr{J}_f$, together with a* *free presentation*

$$\mathscr{O}^s \xrightarrow{\phi_f} \mathscr{O}^{n+1} \longrightarrow \mathscr{J}_f \longrightarrow 0 \qquad\qquad \phi_f = (f_{ij})_{i=1,\ldots,n+1}^{j=1,\ldots,s}$$

with respect to the canonical bases of $\mathscr{O}^s$ *and* $\mathscr{O}^{n+1}$*. Then, the* $\mathscr{O}$*-module* $T_{\mathscr{O}/k}(f)$ *of logarithmic derivations of* f *is generated by the* k*-derivations*

$$\tau_f^{(j)} \;=\; \sum_{i=1}^{n} f_{ij} d_i, \qquad j \;=\; 1, \ldots, s$$

obtained from the column-vectors of the matrix ϕ_f *after deletion of its last row.*

Proof. Pick $\tau \in T_{\mathscr{O}/k}(f)$. Writing $\tau = \sum_{i=1}^n g_i d_i$, for certain g_i's in $\mathscr{O}$, we obtain $\sum_{i=1}^n g_i d_i(f) + gf = 0$, for some $g \in \mathscr{O}$. This means that $\mathfrak{r} = (g_1, \ldots, g_n, g) \in \mathscr{O}^{n+1}$ is a relation of $\mathscr{J}_f$. But the module of first-order syzygies is generated by the column-vectors of ϕ_f, hence there exist $h_1, \ldots, h_s \in \mathscr{O}$ such that $\mathfrak{r} = \sum_{j=1}^s h_j \mathfrak{r}_j$, where $\mathfrak{r}_j = (f_{1j}, \ldots, f_{nj}, f_{n+1,j})$, which yields

$$\tau = \sum_{i=1}^{n} g_i d_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{s} h_j f_{ij} \right) d_i = \sum_{j=1}^{s} h_j \left(\sum_{i=1}^{n} f_{ij} d_i \right) = \sum_{j=1}^{s} h_j \tau_f^{(j)},$$

thus showing the inclusion $T_{\mathscr{O}/k}(f) \subset \sum_{j=1}^s \mathscr{O}\tau_f^{(j)}$. To get the equality, it suffices to check that each $\tau_f^{(j)}$ is logarithmic for f. This is clear, since $\sum_{i=1}^n f_{ij} d_i(f) + f_{n+1,j} f = 0, \quad j = 1, \ldots, s$, which may be rewritten as $\tau_f^{(j)}(f) = -f_{n+1,j} f$. □

At this point, we recall a few notions and basic facts. Let $\mathscr{O}$ be a Noetherian ring. A finitely generated $\mathscr{O}$-module E is *reflexive* if the canonical map from E into its bidual $\mathrm{Hom}_{\mathscr{O}}(\mathrm{Hom}_{\mathscr{O}}(E, \mathscr{O}), \mathscr{O})$ is an isomorphism. We say that E satisfies the "Serre type" $\widetilde{S}_n$ condition, for a given positive integer n, if $\operatorname{depth} E_p \geq \min\{n, \operatorname{depth} \mathscr{O}_p\}$ for every prime ideal $p \subset \mathscr{O}$, where the depth is taken with respect to the maximal ideal of $\mathscr{O}_p$. If $\mathscr{O}$ is a Gorenstein ring locally in depth 1 (in the sense that $\mathscr{O}_p$ is a Gorenstein local ring for every prime ideal $p \subset \mathscr{O}$ with $\operatorname{depth} \mathscr{O}_p \leq 1$), then it is well-known that E is reflexive if and only if E has the $\widetilde{S}_2$ condition; if E is torsion-free, it suffices to check that $\operatorname{depth} E_p \geq 2$ whenever $\operatorname{depth} \mathscr{O}_p \geq 2$. For details on such standard notions, we refer to [2].

The result below makes use of such observations, and generalizes the first part of [16, Corollary 1.7]. Further, it will be used later into the proof of Theorem 3.1.

Proposition 2.4. *Let $k \subset \mathscr{O}$ be an extension of Noetherian rings such that $D_{\mathscr{O}/k}$ is a finitely generated $\mathscr{O}$-module with the $\widetilde{S}_2$ condition. Assume that $\mathscr{O}$ is a Gorenstein ring (e.g., the ring of a complete intersection) locally in depth* 1*. If $f \in \mathscr{O}$ is (either zero or) $\mathscr{O}$-regular, then $T_{\mathscr{O}/k}(f)$ is a reflexive $\mathscr{O}$-module.*

Proof. We may assume that $\mathscr{O}$ is local with $\operatorname{depth} \mathscr{O} \geq 2$. Also, we may suppose that $\mathscr{J}_f \neq (f)$ (in particular, $f \neq 0$), since otherwise the $\mathscr{O}$-module $T_{\mathscr{O}/k}(f)$ would coincide with $D_{\mathscr{O}/k}$, which in the present situation is reflexive. In order for the torsion-free module $T_{\mathscr{O}/k}(f)$ to be reflexive, we need to show that its depth is at least 2.

The ideal $\mathscr{G}_f$ (see Definition 2.1) fits into a surjective homomorphism $D_{\mathscr{O}/k} \to \mathscr{G}_f$, given by evaluation at f. By composition with the projection $\mathscr{G}_f \to (\mathscr{G}_f, f)/(f)$, we get a surjective $\mathscr{O}$-linear map $D_{\mathscr{O}/k} \to \mathscr{J}_f/(f)$ whose kernel is easily seen to be $T_{\mathscr{O}/k}(f)$. Hence, putting $\widetilde{\mathscr{J}_f} = \mathscr{J}_f/(f)$ (which is non-zero), we obtain a short exact sequence of $\mathscr{O}$-modules

$$0 \longrightarrow T_{\mathscr{O}/k}(f) \longrightarrow D_{\mathscr{O}/k} \longrightarrow \widetilde{\mathscr{J}_f} \longrightarrow 0.$$

We now proceed by depth-chasing. If $\operatorname{depth} \widetilde{\mathscr{J}_f} \geq \operatorname{depth} D_{\mathscr{O}/k}$, then $\operatorname{depth} T_{\mathscr{O}/k}(f) \geq \operatorname{depth} D_{\mathscr{O}/k}$ and therefore $\operatorname{depth} T_{\mathscr{O}/k}(f) \geq 2$ in virtue of the $\widetilde{S}_2$ condition of $D_{\mathscr{O}/k}$. Thus we may assume that $\operatorname{depth} \widetilde{\mathscr{J}_f} < \operatorname{depth} D_{\mathscr{O}/k}$, in which case $\operatorname{depth} T_{\mathscr{O}/k}(f) = \operatorname{depth} \widetilde{\mathscr{J}_f} + 1$. It now suffices to check that $\operatorname{depth} \widetilde{\mathscr{J}_f} > 0$. Assume the contrary. Thus, $\operatorname{depth} \mathscr{J}_f > \operatorname{depth} \widetilde{\mathscr{J}_f}$ since f is a non-zero-divisor. But then, as $\widetilde{\mathscr{J}_f}$ is the cokernel of the "multiplication by f" injection $\mathscr{O} \hookrightarrow \mathscr{J}_f$, we would get $\operatorname{depth} \mathscr{O} = 1$, a contradiction. □

2.2. Abstract free divisors

We are now concerned with the question as to when the module of logarithmic derivations of a given divisor is free, at least in the case where the ambient derivation module is free. First, in the spirit of the typical definition of free divisor, it seems natural to introduce the following *abstract* version:

Definition 2.5. With respect to a ring extension $k \subset \mathscr{O}$, an element $f \in \mathscr{O}$ is said to be an *abstract free divisor* — or, simply, *free divisor* — if $T_{\mathscr{O}/k}(f)$

is a free $\mathscr{O}$-module. Of course, despite the global, general aspect of this definition, it may be dealt with in the local setup, more concretely in the case of local algebras essentially of finite type (as well as standard graded of finite type) over a field.

It turns out that Proposition 2.4 recovers, algebraically, a well-known result of Saito stating that plane curves are free (the second part of [16, Corollary 1.7]), as follows:

Proposition 2.6. *Let $\mathscr{O}$ be a 2-dimensional regular local ring that is essentially of finite type over a perfect field k. Then, every divisor $f \in \mathscr{O}$ is free.*

Proof. In this situation, $D_{\mathscr{O}/k}$ is free and $T_{\mathscr{O}/k}(f)$ has finite homological dimension over $\mathscr{O}$. Proposition 2.4 yields depth $T_{\mathscr{O}/k}(f) \geq 2$, but on the other hand this number is at most 2 (the dimension of $\mathscr{O}$). By the Auslander-Buchsbaum formula, we conclude that $T_{\mathscr{O}/k}(f)$ is free. □

If $k \subset \mathscr{O}$ is an extension of Noetherian local rings, we shall say that a given $f \in \mathscr{O}$ is *quasi-smooth* (over k) if $\mathscr{J}_f$ is free as an $\mathscr{O}$-module, that is, if the ideal $\mathscr{J}_f$ is principal generated by a non-zero-divisor. For instance, any *smooth* $f \in \mathscr{O}$, in the sense that $\mathscr{J}_f = \mathscr{O}$, is automatically quasi-smooth. If E is a finitely generated $\mathscr{O}$-module, we denote by $\mathrm{hd}_{\mathscr{O}} E$ the homological dimension of E over $\mathscr{O}$. We are going to show that, in a suitable setting, an $\mathscr{O}$-regular divisor $f \in \mathscr{O}$ is free if and only if $\mathrm{hd}_{\mathscr{O}} \mathscr{J}_f \leq 1$ (this fact will be used later in Theorem 4.3).

Proposition 2.7. *Let $k \subset \mathscr{O}$ be an extension of Noetherian local rings such that $D_{\mathscr{O}/k}$ is a free $\mathscr{O}$-module of finite rank. Let $f \in \mathscr{O}$ be a non-zero-divisor. Then, f is a free divisor if and only if either f is quasi-smooth or its abstract Jacobian ideal $\mathscr{J}_f$ has a minimal free resolution of the form*

$$0 \longrightarrow \mathscr{O}^n \longrightarrow \mathscr{O}^{n+1} \longrightarrow \mathscr{J}_f \longrightarrow 0$$

for some integer $n \geq 1$.

Proof. Write $\widetilde{\mathscr{J}_f} = \mathscr{J}_f/(f)$, and assume that f is a free divisor that is not quasi-smooth. In particular, $\widetilde{\mathscr{J}_f}$ is non-zero. As we have noticed in the proof of Proposition 2.4, there is a short exact sequence of $\mathscr{O}$-modules

$$0 \longrightarrow T_{\mathscr{O}/k}(f) \longrightarrow D_{\mathscr{O}/k} \longrightarrow \widetilde{\mathscr{J}_f} \longrightarrow 0.$$

It follows that $\mathrm{hd}_{\mathscr{O}} \widetilde{\mathscr{J}_f} \leq 1$ and hence necessarily $\mathrm{hd}_{\mathscr{O}} \widetilde{\mathscr{J}_f} = 1$, as f is $\mathscr{O}$-regular and annihilates $\widetilde{\mathscr{J}_f}$. But then, by chasing homological dimension along the exact sequence

$$0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{J}_f \longrightarrow \widetilde{\mathscr{J}_f} \longrightarrow 0$$

we conclude that $\mathrm{hd}_{\mathscr{O}} \mathscr{J}_f \leq 1$, which must be exactly 1 since f is not quasi-smooth. For the converse, we may assume that $\widetilde{\mathscr{J}_f} \neq 0$. If $\mathrm{hd}_{\mathscr{O}} \mathscr{J}_f \leq 1$ then, by the latter exact sequence, we again must have $\mathrm{hd}_{\mathscr{O}} \widetilde{\mathscr{J}_f} = 1$, and hence, by the former, $T_{\mathscr{O}/k}(f)$ must be free. □

The result below is a variation of Proposition 2.7 and extends a result of Terao (cf. [21, Proposition 2.4]), who looked at the question of characterizing certain free divisors by means of Cohen-Macaulayness of gradient ideals in the complex analytic setting (for the polynomial version, see [13, Corollary 4.4] and [20, Proposition 3.7]). If $\mathscr{O}$ is a Noetherian local ring, the height of a proper ideal $\mathscr{I} \subset \mathscr{O}$ is denoted $\mathrm{ht}\mathscr{I}$, and $\mathscr{I}$ is said to be a *perfect ideal* if $\mathrm{hd}_{\mathscr{O}}(\mathscr{O}/\mathscr{I})$ is finite and equal to $\mathrm{ht}\mathscr{I}$; for instance, complete intersections (ideals generated by regular sequences) are perfect.

Proposition 2.8. *Let $k \subset \mathscr{O}$ be an extension of Noetherian local rings such that $D_{\mathscr{O}/k}$ is a free $\mathscr{O}$-module of finite rank. Let $f \in \mathscr{O}$ be a non-smooth non-zero-divisor such that* $\mathrm{ht}\,\mathscr{J}_f \geq 2$. *Then, f is a free divisor if and only if $\mathscr{J}_f$ is a perfect ideal with* $\mathrm{ht}\,\mathscr{J}_f = 2$.

Proof. If f is a non-smooth $\mathscr{O}$-regular divisor with $\mathrm{ht}\,\mathscr{J}_f \geq 2$ (in particular, f is not quasi-smooth), then the result follows readily from Proposition 2.7 together with the well-known Hilbert-Burch theorem. □

Remarks 2.9. (i) In the setup of Propositions 2.7 and 2.8, let f be a free divisor and let $\phi\colon \mathscr{O}^n \hookrightarrow \mathscr{O}^{n+1}$ be the minimal free resolution of the abstract Jacobian ideal $\mathscr{J}_f$, supposed to have height at least 2. If E^* denotes the $\mathscr{O}$-dual $\mathrm{Hom}_{\mathscr{O}}(E, \mathscr{O})$ of an $\mathscr{O}$-module E, and if $\phi^*\colon (\mathscr{O}^{n+1})^* \to (\mathscr{O}^n)^*$ is the dual map of ϕ, then we can identify

$$\mathscr{J}_f \;=\; \text{image}\left(\bigwedge^n(\mathscr{O}^{n+1})^* \stackrel{\wedge^n\phi^*}{\longrightarrow} \bigwedge^n(\mathscr{O}^n)^* \simeq \mathscr{O}\right),$$

so that $\mathscr{J}_f$ may be generated by the maximal subdeterminants of a matrix representing ϕ. Of course, this follows from the Hilbert-Burch theorem.

(ii) As we know, the hypothesis of freeness of $D_{\mathscr{O}/k}$ is fulfilled if k is a perfect field and $\mathscr{O}$ is a regular local ring essentially of finite type over k. Propositions 2.7 and 2.8 may be applied into the polynomial setup as well, in which case we may also resort to the well-known *Saito's freeness criterion* ([16]): if $f \in \mathscr{O} = \mathbf{C}[x_1, \ldots, x_n]$ is a (reduced, quasi-homogeneous) polynomial and if

$$\tau_j \;=\; \sum_{i=1}^{n} f_{ij}\frac{\partial}{\partial x_i}, \qquad j \;=\; 1, \ldots, n$$

are logarithmic derivations of f with the property that the determinant of the matrix (f_{ij}) is of the form cf, for some $c \in \mathbf{C} \setminus \{0\}$, then f must be a free divisor, and $\{\tau_1, \ldots, \tau_n\}$ is a basis for $T_{\mathscr{O}/\mathbf{C}}(f)$ as a free $\mathscr{O}$-module.

Examples 2.10. (i) The cubic $f = x^3 + y^3 + z^3 - 3xyz$ is a free divisor in $\mathscr{O} = \mathbf{C}[x, y, z]$. Indeed, Saito's freeness criterion yields that a basis for $T_{\mathscr{O}/\mathbf{C}}(f)$ corresponds to the vectors $(x, y, z),\ (y, z, x),\ (z, x, y) \in \mathscr{O}^3$.

(ii) If k is a field with $\mathrm{char} k = 3$, then $f = x^2y + xyz + z^3$ is a free divisor in $\mathscr{O} = k[x, y, z]$. This follows from Proposition 2.7, since $\mathscr{J}_f = (yz - xy,\ x^2 + xz,\ xy)$ has a minimal free resolution of the form

$$0 \longrightarrow \mathscr{O}^2 \longrightarrow \mathscr{O}^3 \longrightarrow \mathscr{J}_f \longrightarrow 0.$$

(iii) As we should expect, product of free divisors is *not* necessarily a free divisor. For instance, in the polynomial ring $\mathbf{C}[x, y, z]$, pick the free divisors $f = x + y + z$ (a smooth divisor) and $g = xyz$ (a normal crossing). The product $fg = x^2yz + xy^2z + xyz^2$ is not free, since the ideal $\mathscr{J}_{fg}$ is not perfect.

(iv) We wish to apply Proposition 2.8 into a non-polynomial example. Let k be a field of characteristic 3 and let $\mathscr{O} = \mathscr{O}_{\mathscr{X},o}$ be the local ring of the surface

$$\mathscr{X} \;=\; \mathscr{V}(z^3 - y^3 - xy) \;\subset\; \mathbf{A}_k^3$$

at the singularity $o = (0,0,0) \in \mathscr{X}$. In this case, the $\mathscr{O}$-module $D_{\mathscr{O}/k}$ is free, a basis being $\{\overline{d}_1, \overline{d}_2\}$, where $\overline{d}_1, \overline{d}_2$ are, respectively, the images in $D_{\mathscr{O}/k}$ of the polynomial vector fields $d_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, $d_2 = \frac{\partial}{\partial z}$. Let ρ be the image of $x + z^2$ in $\mathscr{O}$. Its abstract Jacobian ideal $\mathscr{J}_\rho = (x,z)\mathscr{O}$ is a complete intersection of height 2. By Proposition 2.8, ρ must be a free divisor.

For a reduced algebra $\mathscr{O}$ essentially of finite type (or $\mathbf{N}$-graded of finite type) over a field k containing the rationals, the derivation module $D_{\mathscr{O}/k}$ is not free in general but it has (generic, constant) rank equal to the Krull dimension d of $\mathscr{O}$. Thus, we may check whether a given algebraic divisor $f \in \mathscr{O}$ is free by resorting to Proposition 2.3 and extracting a minimal generating set of $T_{\mathscr{O}/k}(f)$; if d elements suffice, then f must be a free divisor.

Example 2.11. Let us detect an abstract free divisor in the homogeneous coordinate ring

$$\mathscr{O}_{\mathscr{C}} \;=\; \mathscr{O}/\mathscr{P} \;=\; \mathbf{C}[x,y,z,w]/(y^2 - xz, yz - xw, z^2 - yw)$$

of the twisted cubic curve $\mathscr{C} \subset \mathbf{P}^3$. Consider the derivation module $D_{\mathscr{O}_{\mathscr{C}}/\mathbf{C}}$ of $\mathscr{O}_{\mathscr{C}}$, seen as a submodule of $\mathscr{O}_{\mathscr{C}}^4$. It is minimally generated by the 4 derivations $\overline{d}_1, \overline{d}_2, \overline{d}_3, \overline{\epsilon}$, which are the images, modulo $\mathscr{P}^{\oplus 4}$, of the column-vectors d_1, d_2, d_3, ϵ of the matrix

$$\begin{pmatrix} 0 & 0 & 3y & x \\ x & y & 2z & y \\ 2y & 2z & w & z \\ 3z & 3w & 0 & w \end{pmatrix}.$$

Notice that the fourth column gives the Euler derivation ϵ. Writing $\overline{f}$ for the image in $\mathscr{O}_{\mathscr{C}}$ of any given polynomial $f \in \mathscr{O}$, we claim that $\overline{y}$ is a free divisor in $\mathscr{O}_{\mathscr{C}}$. After computing a presentation matrix of the abstract Jacobian ideal $\mathscr{J}_{\overline{y}}$ with respect to its generating set $\{\overline{d}_1(\overline{y}), \overline{d}_2(\overline{y}), \overline{d}_3(\overline{y}), \overline{\epsilon}(\overline{y}), \overline{y}\} = \{\overline{x}, \overline{y}, 2\overline{z}, \overline{y}, \overline{y}\}$, and applying Proposition 2.3, we get that $\{\overline{\epsilon}, \overline{d}_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8\}$ is a set of generators for $T_{\mathscr{O}_{\mathscr{C}}/\mathbf{C}}(\overline{y})$, where $\tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8$ are the images, modulo $\mathscr{P}^{\oplus 4}$, of the global vector fields given respectively by $xd_3 - 2zd_1$, $yd_3 - 2zd_2$, $zd_3 - 2wd_2$, $xd_2 - yd_1$, $yd_2 - zd_1$, $zd_2 - wd_1$. Using the defining equations of $\mathscr{C}$, we may write $\tau_3 = 3\overline{y}(\overline{\epsilon} - \overline{d}_2)$, $\tau_4 = 3\overline{z}(\overline{\epsilon} - \overline{d}_2)$, $\tau_5 = 3\overline{w}(\overline{\epsilon} - \overline{d}_2)$. Moreover, the derivations τ_6, τ_7, τ_8 vanish identically on $\mathscr{O}_{\mathscr{C}}$.

This shows that the set $\{\overline{\epsilon}, \overline{d}_2\}$ generates $T_{\mathscr{O}_{\mathscr{C}}/\mathbf{C}}(\overline{y})$. As $\mathscr{O}_{\mathscr{C}}$ is 2-dimensional, the freeness of $\overline{y}$ follows:

$$T_{\mathscr{O}_{\mathscr{C}}/\mathbf{C}}(\overline{y}) \ = \ \mathscr{O}_{\mathscr{C}}\overline{\epsilon} \ \oplus \ \mathscr{O}_{\mathscr{C}}\overline{d}_2.$$

Finally, we want to revisit the warning made in Remark 2.2. In the present example, the ideal $\mathscr{J}_{\overline{y}}/(\overline{y}) \subset \mathscr{O}_{\mathscr{C}}/(\overline{y})$ may be identified with $(x, y, z)/\mathscr{I} \subset \mathscr{O}/\mathscr{I}$, where $\mathscr{I} = (y, \mathscr{P}) = (y, xz, xw, z^2)$. On the other hand, a direct calculation gives that the "true" Jacobian ideal $J(\mathscr{O}_{\mathscr{C}}/(\overline{y}))$, seen in $\mathscr{O}/\mathscr{I}$, equals $(x^2, zw, \mathscr{I})/\mathscr{I}$. Therefore, $\mathscr{J}_{\overline{y}}/(\overline{y}) \neq J(\mathscr{O}_{\mathscr{C}}/(\overline{y}))$.

Remark 2.12. Let $\mathscr{O} = \bigoplus_{i\geq 0} \mathscr{O}_i$ be a standard graded algebra of finite type over a field $\mathscr{O}_0 = k$, and let $\ell \in \mathscr{O}_1$ be a free divisor. Example 2.11 above illustrates a situation where a basis for $T_{\mathscr{O}/k}(\ell)$ (with $\ell = \overline{y}$) is a subset of a set of generators of $D_{\mathscr{O}/k}$. However, this is far from being true in general: easy counterexamples are linear forms in polynomial rings. For a non-polynomial instance, consider the graded k-algebra $\mathscr{O} = k[x, y, z]/(xy + z^2)$, with $\mathrm{char} k = 2$. The $\mathscr{O}$-module $D_{\mathscr{O}/k}$ is free, a basis being $\{\overline{d}_1, \overline{d}_2\}$, where $\overline{d}_1, \overline{d}_2$ are represented, respectively, by the polynomial derivations $d_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $d_2 = \frac{\partial}{\partial z}$. Thus, the image $\ell = \overline{z} \in \mathscr{O}$ of the variable z is a (smooth) free divisor, with logarithmic derivation module given by

$$T_{\mathscr{O}/k}(\ell) \ = \ \mathscr{O}\overline{d}_1 \ \oplus \ \mathscr{O}\ell\overline{d}_2.$$

3. Free and pseudo-free ideals

This section contains the core of our paper. In order to derive freeness criteria for the module of logarithmic derivations of an ideal, our strategy will be to embark first on furnishing necessary and sufficient conditions for reflexiveness.

Let us invoke a very useful fact: if $\mathscr{O}$ is a Noetherian normal domain and $E \subset E'$ are finitely generated $\mathscr{O}$-modules, with E reflexive and E' torsion-free, such that the height of the ideal $E\colon E' = \{g \in \mathscr{O} \mid gE' \subset E\}$ is at least 2 — that is, E and E' coincide locally in height 1 — then $E = E'$. This follows easily from a classical result of Samuel ([17, Proposition 1]), which guarantees that the reflexive module E may be expressed as the intersection (taken in the vector space $E \otimes_{\mathscr{O}} \mathscr{K}$, where $\mathscr{K}$ is the fraction field of $\mathscr{O}$) of the localizations of E at the height 1 prime ideals of $\mathscr{O}$. It can be shown that this happens in more generality, but, as we shall need the factorial hypothesis, the present case suffices for our purposes. Recall the standard fact that a height 1 prime ideal of a Noetherian factorial domain must be principal.

Theorem 3.1. *Let $\mathscr{O}$ be a factorial domain of finite type or essentially of finite type over a field k of characteristic zero, and let $\mathscr{I} \subset \mathscr{O}$ be an ideal. The following assertions are equivalent:*

(i) $T_{\mathscr{O}/k}(\mathscr{I})$ *is a reflexive $\mathscr{O}$-module.*

(ii) $\mathscr{I} \subset (f)$, *for some $f \in \mathscr{O}$ such that $T_{\mathscr{O}/k}(\mathscr{I}) = T_{\mathscr{O}/k}(f)$.*

(iii) $\mathscr{I} = f\mathscr{J}$, *for some ideal $\mathscr{J} \subset \mathscr{O}$ and some $f \in \mathscr{O}$ such that $\tau_f^{(j)}(\mathscr{J}) \subset \mathscr{J}$, $j = 1, \ldots, s$, where $\tau_f^{(1)}, \ldots, \tau_f^{(s)}$ are the derivations described in Proposition 2.3.*

Proof. In this situation, the $\mathscr{O}$-module $D_{\mathscr{O}/k}$ is finitely generated and reflexive. To prove the implication (i)⇒(ii), assume that $T_{\mathscr{O}/k}(\mathscr{I})$ is reflexive and consider, first, the case $\operatorname{ht}\mathscr{I} \geq 2$. Since $\mathscr{I} D_{\mathscr{O}/k} \subset T_{\mathscr{O}/k}(\mathscr{I}) \subset D_{\mathscr{O}/k}$, we get $\mathscr{I} \subset T_{\mathscr{O}/k}(\mathscr{I})\colon D_{\mathscr{O}/k}$, so that $\operatorname{ht}(T_{\mathscr{O}/k}(\mathscr{I})\colon D_{\mathscr{O}/k}) \geq 2$. Thus, as in particular $\mathscr{O}$ is a normal domain, Samuel's result quoted above guarantees that the reflexive modules $T_{\mathscr{O}/k}(\mathscr{I})$ and $D_{\mathscr{O}/k}$ must be globally equal and hence we can simply take $f = 1$.

Assume now that $\operatorname{ht}\mathscr{I} = 1$, the case $\mathscr{I} = (0)$ being clear. Then, since $\mathscr{O}$ is factorial, $\mathscr{I}$ must be contained in a (prime) principal ideal, that is, we may write $\mathscr{I} = f\mathscr{J}$, for some non-zero element $f \in \mathscr{O}$ and some ideal $\mathscr{J} = (f_1, \ldots, f_s) \subset \mathscr{O}$, which we may assume to be proper. Absorbing a greatest common divisor of the f_i's into f if necessary, we may suppose that $\operatorname{ht}\mathscr{J} \geq 2$. Therefore, as every associated prime ideal of $\mathscr{O}/(f)$ has height 1 in $\mathscr{O}$ (by the normality of $\mathscr{O}$), there exists $f_0 \in \mathscr{J}$ which is $\mathscr{O}/(f)$-regular.

Now, let $d \in D_{\mathscr{O}/k}$ be such that the ideal $\mathscr{I} = f\mathscr{J}$ is invariant under d. We claim that both (f) and $\mathscr{J}$ are invariant too. In fact, we have $d(ff_0) \in \mathscr{I} \subset (f)$; by Leibniz's rule, $f_0 d(f) \in (f)$, which implies $d(f) \in (f)$ since f_0 is regular modulo (f). If $g \in \mathscr{J}$, we get $fg \in \mathscr{I} \Rightarrow fd(g) + gd(f) \in \mathscr{I} \Rightarrow fd(g) \in \mathscr{I}$, which implies $d(g) \in \mathscr{J}$ since $f \neq 0$ and $\mathscr{O}$ is a domain. This shows the claim. Note that the converse is clear: if $d \in D_{\mathscr{O}/k}$ is such that both (f) and $\mathscr{J}$ are invariant under d, then, for any $g \in \mathscr{J}$, we have $fd(g), gd(f) \in \mathscr{I}$, whence $d(fg) \in \mathscr{I}$. Thus, we have shown $T_{\mathscr{O}/k}(\mathscr{I}) = T_{\mathscr{O}/k}(\mathscr{J}) \cap T_{\mathscr{O}/k}(f) \subset T_{\mathscr{O}/k}(f)$. Further, it is clear that $\mathscr{J} \subset T_{\mathscr{O}/k}(\mathscr{I}) : T_{\mathscr{O}/k}(f)$, and hence $\operatorname{ht}(T_{\mathscr{O}/k}(\mathscr{I})\colon T_{\mathscr{O}/k}(f)) \geq 2$. Again by Samuel's result, the reflexive $\mathscr{O}$-module $T_{\mathscr{O}/k}(\mathscr{I})$ must coincide with $T_{\mathscr{O}/k}(f)$, as needed.

Now, assume (ii), that is, $\mathscr{I} \subset (f)$ with $T_{\mathscr{O}/k}(\mathscr{I}) = T_{\mathscr{O}/k}(f)$. By the principal ideal case established in Proposition 2.4, it follows that $T_{\mathscr{O}/k}(\mathscr{I})$ is reflexive. Thus, (i) and (ii) are equivalent.

Finally, if $\mathscr{I}$ is a product $f\mathscr{J}$, then, exactly as above, we get an equality $T_{\mathscr{O}/k}(\mathscr{I}) = T_{\mathscr{O}/k}(\mathscr{J}) \cap T_{\mathscr{O}/k}(f)$, and therefore (ii) is immediately seen to be equivalent to an inclusion $T_{\mathscr{O}/k}(f) \subset T_{\mathscr{O}/k}(\mathscr{J})$, which clearly means that $\mathscr{J}$ is invariant under the $\tau_f^{(j)}$'s (generators of the $\mathscr{O}$-module $T_{\mathscr{O}/k}(f)$), as in (iii). □

Freeness criteria follow immediately:

Corollary 3.2. *Let $\mathscr{O}$ be a factorial domain of finite type or essentially of finite type over a field k of characteristic zero, and let $\mathscr{I} \subset \mathscr{O}$ be an ideal. The following assertions are equivalent:*

(i) *The $\mathscr{O}$-module $T_{\mathscr{O}/k}(\mathscr{I})$ is free.*

(ii) *$\mathscr{I} \subset (f)$, for some free divisor $f \in \mathscr{O}$ such that $T_{\mathscr{O}/k}(\mathscr{I}) = T_{\mathscr{O}/k}(f)$.*

(iii) *$\mathscr{I} = f\mathscr{J}$, for some ideal $\mathscr{J} \subset \mathscr{O}$ and some free divisor $f \in \mathscr{O}$ such that $\tau_f^{(j)}(\mathscr{J}) \subset \mathscr{J}$, $j = 1, \ldots, s$, where $\tau_f^{(1)}, \ldots, \tau_f^{(s)}$ are the derivations described in Proposition 2.3.*

Corollary 3.2 allows us to introduce a class of ideals that extends, algebraically, the celebrated class of free divisors. In spite of the hypothesis of factoriality imposed above, we propose our (global) definition in full generality.

Definition 3.3. With respect to a ring extension $k \subset \mathscr{O}$, an ideal $\mathscr{I} \subset \mathscr{O}$ is said to be a *tangentially free ideal* — *free ideal*, for short — if $T_{\mathscr{O}/k}(\mathscr{I})$ is a free $\mathscr{O}$-module.

Remark 3.4. In the setting of Corollary 3.2, if $\mathscr{I} = f\mathscr{J} \subset \mathscr{O}$ is a free ideal, with f a free divisor, then it follows from item (ii) that $T_{\mathscr{O}/k}(\mathscr{I}) = \bigoplus_{j=1}^{d} \mathscr{O}\tau_j \simeq \mathscr{O}^d$, where $\{\tau_1, \ldots, \tau_d\}$ is a basis of $T_{\mathscr{O}/k}(f)$ $(= T_{\mathscr{O}/k}(\mathscr{I}))$, which may be extracted from the generating set furnished by Proposition 2.3.

Next, we provide an example of a *non-principal* free ideal in the coordinate ring of a *non-smooth* arithmetically factorial variety. Like the family of free ideals to be furnished in Corollary 3.7, this example is a particular bonus from our theory, in the sense that it could not be treated by means of standard free divisor theory.

Example 3.5. Let $\mathscr{O}_{\mathscr{X}}$ be the coordinate ring of the singular complex surface

$$\mathscr{X} \;=\; \mathscr{V}(x^2+y^3+z^5) \subset \mathbf{A}^3_{\mathbf{C}},$$

which is the so-called E_8-singularity (referring to the origin). It is well-known that $\mathscr{O}_{\mathscr{X}}$ is a factorial domain. Seen as a submodule of $\mathscr{O}^3_{\mathscr{X}}$, the (non-free) $\mathscr{O}_{\mathscr{X}}$-module $D_{\mathscr{O}_{\mathscr{X}}/\mathbf{C}}$ is generated by $\overline{d}_1, \overline{d}_2, \overline{d}_3, \overline{d}_4$, which are the images (modulo the defining equation of $\mathscr{X}$) of the column-vectors d_1, d_2, d_3, d_4 of the matrix

$$\begin{pmatrix} 15x & 3y^2 & 5z^4 & 0 \\ 10y & -2x & 0 & 5z^4 \\ 6z & 0 & -2x & -3y^2 \end{pmatrix}.$$

If $f \in \mathscr{O} = \mathbf{C}[x,y,z]$, denote by $\overline{f}$ its image in $\mathscr{O}_{\mathscr{X}}$. We first claim that $\overline{z}$ is a free divisor. Its abstract Jacobian ideal may be conveniently written as $\mathscr{J}_{\overline{z}} = (6\overline{z}, \overline{0}, -2\overline{x}, -3\overline{y}^2, \overline{z}) \subset \mathscr{O}_{\mathscr{X}}$. Now, apply the recipe described in Proposition 2.3. We get that $\{\overline{d}_1, \overline{d}_2, \tau_3, \tau_4, \tau_5, \tau_6\}$ is a set of generators for $T_{\mathscr{O}_{\mathscr{X}}/\mathbf{C}}(\overline{z})$, where $\tau_3, \tau_4, \tau_5, \tau_6$ are the images in $D_{\mathscr{O}_{\mathscr{X}}/\mathbf{C}}$ of the derivations of $\mathscr{O}$ respectively given by $xd_1 + 3zd_3$, $\;y^2d_1 + 2zd_4$, $\;3y^2d_3 - 2xd_4$, $\;z^4d_1 - 3xd_3 - 2yd_4$. But $\tau_3 = -5\overline{y}\overline{d}_2$, $\tau_4 = 5\overline{x}\overline{d}_2$, $\tau_5 = 5\overline{z}^4\overline{d}_2$, and moreover τ_6 vanishes identically on $\mathscr{O}_{\mathscr{X}}$, so that $\overline{z}$ is a free divisor, as claimed, and a basis for $T_{\mathscr{O}_{\mathscr{X}}/\mathbf{C}}(\overline{z})$ is $\{\overline{d}_1, \overline{d}_2\}$. Since clearly $\mathscr{J}_{\overline{z}}$ is invariant under both $\overline{d}_1$ and $\overline{d}_2$, we obtain an inclusion $T_{\mathscr{O}_{\mathscr{X}}/\mathbf{C}}(\overline{z}) \subset T_{\mathscr{O}_{\mathscr{X}}/\mathbf{C}}(\mathscr{J}_{\overline{z}})$. Thus, we can apply the characterization given in Corollary 3.2 in order to conclude that the ideal $\overline{z}\mathscr{J}_{\overline{z}} = (\overline{xz},\ \overline{y}^2\overline{z},\ \overline{z}^2) \subset \mathscr{O}_{\mathscr{X}}$ is tangentially free, with

$$T_{\mathscr{O}_{\mathscr{X}}/\mathbf{C}}(\overline{z}\mathscr{J}_{\overline{z}}) \;=\; \mathscr{O}_{\mathscr{X}}\overline{d}_1 \;\oplus\; \mathscr{O}_{\mathscr{X}}\overline{d}_2.$$

Proposition 3.6 below is well-known and is also true for non-principal ideals (at least in the polynomial setup). We shall employ it for detecting a well-structured class of free ideals, to wit, those in the form $f\mathscr{J}_f$, with f a free divisor. Recall that $D_{\mathscr{O}/k}$ is naturally a Lie algebra, with the usual Lie bracket: $[d_1, d_2] = d_1d_2 - d_2d_1$, for $d_1, d_2 \in D_{\mathscr{O}/k}$.

Proposition 3.6. *Let $k \subset \mathscr{O}$ be a ring extension and let $f \in \mathscr{O}$ be such that each Lie bracket $[d_i, d_j]$ vanishes at f, for some finite generating set $\{d_1, \ldots, d_n\}$ of $D_{\mathscr{O}/k}$. Then $T_{\mathscr{O}/k}(f) \subset T_{\mathscr{O}/k}(\mathscr{J}_f)$.*

Proof. Take $\tau = \sum_{i=1}^n g_i d_i \in T_{\mathscr{O}/k}(f)$. It suffices to show that τ sends $d_1(f)$ into $\mathscr{J}_f = (d_1(f), \ldots, d_n(f), f)$. Set $f_1 = \sum_{i=1}^n g_i d_1(d_i(f))$, which by Leibniz's rule may be written as $f_1 = d_1(\tau(f)) - \sum_{i=1}^n d_1(g_i) d_i(f)$. Since $\tau(f) =$

hf for some $h \in \mathscr{O}$, we have $d_1(\tau(f)) \in \mathscr{J}_f$. Hence $f_1 \in \mathscr{J}_f$. On the other hand, by the condition on the Lie brackets, we may write $f_1 = \sum_{i=1}^n g_i d_i(d_1(f)) = \tau(d_1(f))$, which concludes the proof. □

Corollary 3.7. *In the setting of Corollary 3.2, let $f \in \mathscr{O}$ be a free divisor such that each Lie bracket $[d_i, d_j]$ vanishes at f, for some generating set $\{d_1, \ldots, d_n\}$ of $D_{\mathscr{O}/k}$. Then $T_{\mathscr{O}/k}(f\mathscr{J}_f) = T_{\mathscr{O}/k}(f)$, and therefore $f\mathscr{J}_f$ is a free ideal.*

Proof. This follows readily from Proposition 3.6 together with Corollary 3.2. □

Corollary 3.8. *Let $\mathscr{O} = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k of characteristic zero, and let $f \in \mathscr{O}$ be any free divisor. Then, the ideal $\left(f\frac{\partial f}{\partial x_1}, \ldots, f\frac{\partial f}{\partial x_n}, f^2\right) \subset \mathscr{O}$ is tangentially free.*

Proof. Apply Corollary 3.7 with $d_i = \frac{\partial}{\partial x_i}$ for every i. □

Remark 3.9. We point out that free ideals are *not* necessarily of the form $f\mathscr{J}_f$. For instance, in the polynomial ring $\mathscr{O} = \mathbf{C}[x, y, z]$, pick the free divisor $f = xyz$, a basis of $T_{\mathscr{O}/\mathbf{C}}(f)$ being $\mathscr{B} = \{x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\}$. Consider the ideals $\mathscr{J} = (xz, yz)$ and $\mathscr{P} = (x, y)$ of $\mathscr{O}$. Then $\mathscr{J} \subsetneq \mathscr{J}_f \subsetneq \mathscr{P}$ and we easily check that each element of $\mathscr{B}$ preserves both $\mathscr{J}$ and $\mathscr{P}$. Hence, by Corollary 3.2, $f\mathscr{J}$ and $f\mathscr{P}$ are (distinct) free ideals. Also note that z is a free divisor but the ideal $\mathscr{J} = z\mathscr{P}$ is not free, since $T_{\mathscr{O}/\mathbf{C}}(z) \nsubseteq T_{\mathscr{O}/\mathbf{C}}(\mathscr{P})$; in fact, unlike $\mathscr{P}$, the ideal (z) is invariant under $\frac{\partial}{\partial x}$.

Definition 3.10. We say that an ideal $\mathscr{J} \subset \mathscr{O}$ is a *pseudo-free ideal* if there exists an $\mathscr{O}$-regular free divisor $f \in \mathscr{O}$ such that $f\mathscr{J}$ is a free ideal; in this case, if f is known, we say that $\mathscr{J}$ is *pseudo-free with respect to f*. For instance, in Remark 3.9, the ideals $\mathscr{J}$ and $\mathscr{P}$ are (non-free) pseudo-free ideals with respect to $f = xyz$. Moreover, Corollary 3.7 yields that, under the conditions of Corollary 3.2, abstract Jacobian ideals of free divisors are pseudo-free. If $\mathscr{O}$ is a polynomial ring over a field and $\mathscr{J} \subset \mathscr{O}$ is a radical pseudo-free ideal, we say that the corresponding algebraic variety $\mathscr{V}(\mathscr{J})$ is a *pseudo-free variety.*

Corollary 3.11. *Keep the setting of Corollary 3.8. If $f \in \mathscr{O}$ is a non-constant free divisor, then $(\mathrm{Sing}\mathscr{V}(f))_{\mathrm{red}} \subset \mathbf{A}_k^n$ — the reduced singular locus of the hypersurface defined by f — is a pseudo-free variety.*

Proof. We have $(\mathrm{Sing}\mathscr{V}(f))_{\mathrm{red}} = \mathscr{V}(\sqrt{\mathscr{J}_f})$, as in this polynomial setup the ideal $\mathscr{J}_f \subset \mathscr{O}$ coincides with the usual lifted Jacobian ideal of f. Corollary 3.8 yields that $\mathscr{J}_f$ is pseudo-free, so we have to prove this property for its radical. First, recall the well-known fact (cf. [11, Page 12]) that if $\mathscr{I} \subset \mathscr{O}$ is an arbitrary ideal that is invariant under a k-derivation of $\mathscr{O}$, then so is its radical $\sqrt{\mathscr{I}}$. Applied to $\mathscr{J}_f$, this fact may be expressed as an inclusion $T_{\mathscr{O}/k}(\mathscr{J}_f) \subset T_{\mathscr{O}/k}(\sqrt{\mathscr{J}_f})$. On the other hand, Corollary 3.2(iii) implies that $T_{\mathscr{O}/k}(f) \subset T_{\mathscr{O}/k}(\mathscr{J}_f)$ (by the freeness of $f\mathscr{J}_f$) and hence, by the same token, $f\sqrt{\mathscr{J}_f}$ must be free, as needed. □

Remark 3.12. Clearly, an argument completely similar to the one used in the proof above shows that, in the setting of Corollary 3.2, the radical of a pseudo-free ideal is pseudo-free as well (with respect to the same free divisor).

4. On the Zariski-Lipman conjecture for surfaces.

Freeness of derivation modules is the crucial point in the long-standing *Zariski-Lipman conjecture*. Let $\mathscr{O}$ be a local ring which is an algebra essentially of finite type over a field k of characteristic zero. The conjecture predicts that $\mathscr{O}$ is regular if the $\mathscr{O}$-module $D_{\mathscr{O}/k}$ is free. It has been settled in the affirmative in several cases (cf. [6], [10], [12], [19], [23]), so that, essentially, the remaining open case is when $\dim\mathscr{O} = 2$. It is known (cf. [12]) that a ring satisfying the conditions of the conjecture must be a normal domain. A nice overview on this and other conjectures can be found in the survey [9].

Now, recall that a proper ideal $\mathscr{I}$ of a Noetherian local ring $(\mathscr{O}, \mathscr{M})$ is said to be *full relative to* $\mathscr{M}$, or simply $\mathscr{M}$*-full*, if there exists $g \in \mathscr{M}$ such that $\mathscr{M}\mathscr{I} : (g) = \mathscr{I}$. It is well-known, for instance, that *integrally closed ideals — e.g.*, radical ideals — are $\mathscr{M}$-full; see [7, Theorem 2.4]. For generalizations of this notion we refer to Vasconcelos' book [24].

Question 4.1. Let $(\mathscr{O}, \mathscr{M})$ be a Noetherian local domain. When does there exist a non-zero free divisor $f \in \mathscr{M}$ whose abstract Jacobian ideal $\mathscr{J}_f$ is non-principal and $\mathscr{M}$-full?

This question admits an easy solution in some situations, for instance in case $\mathscr{O} = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$, the local ring of affine n-space at the origin. Simply, take f to be the product of at least two (distinct) x_i's. Then, f is easily seen to be free (a normal-crossing), and $\mathscr{J}_f$ is non-principal and radical. It seems tempting to guess that a solution for the question in generality could be given in terms of suitable products of parameters.

The interest in Question 4.1, as Theorem 4.3 will clarify, is that it may lead to a solution of the Zariski-Lipman conjecture in the 2-dimensional case. As a crucial ingredient, we shall use the beautiful fact below observed by Goto and Hayasaka (see [8, Remark 3.3]).

Lemma 4.2. *Let $(\mathscr{O}, \mathscr{M})$ be a Noetherian local domain. If there exists an $\mathscr{M}$-full ideal $\mathscr{I} \subset \mathscr{O}$ with finite homological dimension over $\mathscr{O}$, such that $\mathscr{M}$ is an associated prime ideal of $\mathscr{O}/\mathscr{I}$, then $\mathscr{O}$ is a regular local ring.*

Theorem 4.3. *Let $(\mathscr{O}, \mathscr{M})$ be a 2-dimensional local domain essentially of finite type over a field k of characteristic zero. Assume that there exists a non-zero $f \in \mathscr{M}$ such that $\mathscr{J}_f$ is non-principal and $\mathscr{M}$-full. Then, the following conditions are equivalent:*

(i) *f is a free divisor and $D_{\mathscr{O}/k}$ is a free $\mathscr{O}$-module.*

(ii) *$\mathscr{O}$ is a regular local ring.*

Proof. For the implication (ii) $\Rightarrow$ (i), notice that the freeness of f follows from Proposition 2.6, and the freeness of $D_{\mathscr{O}/k}$ (which is the $\mathscr{O}$-dual of the module of Kähler k-differentials of $\mathscr{O}$) is consequence of the usual Jacobian criterion of regularity.

Now, assume (i). Then, Proposition 2.7 yields that the homological dimension (over $\mathscr{O}$) of $\mathscr{J}_f$ is at most 1, which must be exactly 1 since $\mathscr{J}_f$ is non-principal, and hence the $\mathscr{O}$-module $\mathscr{O}/\mathscr{J}_f$ has homological dimension 2. By the Auslander-Buchsbaum formula, the depth of $\mathscr{O}/\mathscr{J}_f$ must be zero, or equivalently, $\mathscr{M}$ is an associated prime of $\mathscr{O}/\mathscr{J}_f$. We are then in a position to apply Lemma 4.2 in order to conclude that $\mathscr{O}$ is regular. $\square$

Acknowledgements. The author thanks: J. Lipman, for kindly reading the manuscript and suggesting an improvement in the presentation of Theorem 3.1; W. V. Vasconcelos, for his stimulating support in earlier stages of this work and for his inspiring personality; and A. Simis (author's thesis advisor), for suggesting free divisor theory as the environment for the author's Ph.D. thesis, on which most of this paper is based.

References

[1] J. W. Bruce, *Vector fields on discriminants and bifurcation varieties*, Bull. London Math. Soc. **17** (1985), 257–262.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1993.

[3] D. Cerveau and A. Lins Neto, *Holomorphic foliations in* **CP**(2) *having an invariant algebraic curve*, Ann. Inst. Fourier (Grenoble) **41** (1991), no. 4, 883–903.

[4] J. Damon, *On the legacy of free divisors: discriminants and Morse type singularities*, Amer. J. Math. **120** (1998), 453–492.

[5] E. Esteves, *The Castelnuovo-Mumford regularity of an integral variety of a vector field on projective space*, Math. Res. Lett. **9** (2002), 1–15.

[6] H. Flenner, *Extendability of differential forms on non-isolated singularities*, Invent. Math. **94** (1988), 317–326.

[7] S. Goto, *Integral closedness of complete-intersection ideals*, J. Algebra **108** (1987), 151–160.

[8] S. Goto and F. Hayasaka, *Finite homological dimension and primes associated to integrally closed ideals*, Proc. Amer. Math. Soc. **130** (2002), 3159–3164.

[9] J. Herzog, *The module of differentials*, Workshop on Commutative Algebra and its Relation to Combinatorics and Computer Algebra, Trieste, 1994.

[10] M. Hochster, *The Zariski-Lipman conjecture in the graded case*, J. Algebra **47** (1977), 411–424.

[11] I. Kaplansky, *An Introduction to Differential Algebra*, Hermann, Paris, 1957.

[12] J. Lipman, *Free derivation modules on algebraic varieties*, Amer. J. Math. **87** (1965), 874–898.

[13] C. B. Miranda-Neto, *Vector fields and a family of linear type modules related to free divisors*, J. Pure Appl. Algebra **215** (2011), 2652–2659.

[14] C. B. Miranda-Neto, *Graded derivation modules and algebraic free divisors*, J. Pure Appl. Algebra **219** (2015), 5442–5466.

[15] I. Peeva, *Hyperplane arrangements and linear strands in resolutions*, Trans. Amer. Math. Soc. **355** (2002), no. 2, 609–618.

[16] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 265–291.

[17] P. Samuel, *Anneaux gradués factoriels et modules réflexifs*, Bull. Soc. Math. France **92** (1964), 237–249.

[18] J. B. Sancho de Salas, *Tangent algebraic subvarieties of vector fields*, Trans. Amer. Math. Soc. **357** (2004), no. 9, 3509–3523.

[19] G. Scheja and U. Storch, *Differentielle Eigenschaften der Lokalisierungen analytischer Algebren*, Math. Ann. **197** (1972), 137–170.

[20] A. Simis, *Differential idealizers and algebraic free divisors*, Commutative Algebra, 211–226, Lect. Notes Pure Appl. Math., vol. 244, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[21] H. Terao, *Arrangements of hyperplanes and their freeness I, II*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 293–320.

[22] H. Terao, *The bifurcation set and logarithmic vector fields*, Math. Ann. **263** (1983), 313–321.

[23] W. V. Vasconcelos, *Modules of differentials of symmetric algebras*, Arch. Math. **54** (1990), 436–442.

[24] W. V. Vasconcelos, *Integral closure. Rees Algebras, Multiplicities, Algorithms*, Springer Monogr. Math., Springer, Berlin, 2005.

[25] M. Yoshinaga, *Characterization of a free arrangement and conjecture of Edelman and Reiner*, Invent. Math. **157** (2004), 449–454.

[26] G. M. Ziegler, *Matroid representations and free arrangements*, Trans. Amer. Math. Soc. **320** (1990), 525–541.

Departamento de Matemática, Universidade Federal da Paraíba
58051-900 João Pessoa, PB, Brazil
E-mail address: cleto@mat.ufpb.br

Received May 21, 2014