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Irregular varieties of Albanese fiber dimension one with small volume

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In this paper, we generalize Horikawa's theorem on irregular surfaces with small c_1^2 to higher dimensional irregular varieties of Albanese fiber dimension one with small volume.

1. Introduction

This paper is motivated by the following theorem of Horikawa:

Theorem. *[7, Theorem 3.1] Let S be an irregular minimal surface of general type over $\mathbb{C}$.*

(1) *If $\frac{8}{3}\chi(\mathcal{O}_S) \le K_S^2 < 3\chi(\mathcal{O}_S)$, then the Albanese map $S \to \mathrm{Alb}(S)$ induces a hyperelliptic pencil of curves of genus 2 or 3.*

(2) *If $2\chi(\mathcal{O}_S) \le K_S^2 < \frac{8}{3}\chi(\mathcal{O}_S)$, then the Albanese map $S \to \mathrm{Alb}(S)$ induces a pencil of curves of genus 2.*

The above theorem characterizes the geometry of irregular surfaces with small volume. It plays a crucial role in the theory of algebraic surfaces. For example, it leads to the following important conjecture proposed by Reid in [10]:

Conjecture (Reid). *For $g = 2, 3, \ldots$, there exist rational numbers a_g and b_g with*

$$a_2 < a_3 < \cdots \quad \text{and} \quad \lim_{g\to\infty} a_g = 4$$

such that for every surface X of general type, if $K_X^2 \le a_g\chi(\mathcal{O}_X) - b_g$, then X has a pencil of curves of genus at most g.

For this conjecture, to our knowledge, only $a_2 = \frac{8}{3}$ and $a_3 = 3$ are known. Partial results are also known. We refer to [10, 12] for details about this conjecture.

In fact, it is well-known that the relation between the (canonical) volume and the holomorphic Euler characteristic $\chi(\mathcal{O}_X)$ is a fundamental problem in the study of algebraic varieties, in particular, in the geography of algebraic varieties. Our purpose in this paper is to generalize Horikawa's theorem to higher dimensions. Throughout this paper, we will work over an algebraically closed field of characteristic zero.

Our main theorem is the following:

Theorem 1.1. *Let X be an n-dimensional normal, irregular minimal $\mathbb{Q}$-Gorenstein variety of general type with Albanese fiber dimension one.*

(1) *If $\frac{8}{3}(n-1)!\chi(\omega_X) \leq K_X^n < 3(n-1)!\chi(\omega_X)$, then the Albanese map $X \to \mathrm{Alb}(X)$ induces a fibration of hyperelliptic curves of genus 2 or 3.*

(2) *If $2(n-1)!\chi(\omega_X) \leq K_X^n < \frac{8}{3}(n-1)!\chi(\omega_X)$, then the Albanese map $X \to \mathrm{Alb}(X)$ induces a fibration of curves of genus 2.*

When $n = 2$, the above theorem coincides with Horikawa's theorem.

Notice that for an irregular minimal surface S of general type, we always have $K_S^2 \geq 2\chi(\mathcal{O}_S)$ by Bombieri [4].[1] This result is recently generalized by Barja in [1, Remark 4.5], where he has shown that $K_X^n \geq 2(n-1)!\chi(\omega_X)$ for any Gorenstein variety X satisfying the hypothesis in Theorem 1.1. In fact, using the method in this paper, we can give another proof of Barja's result. See Section 6.

It is also worth pointing out that if X is of maximal Albanese dimension, it is recently proved by Barja [1] and independently by the author [14] that we have the generalized Severi inequality $K_X^n \geq 2n!\chi(\omega_X)$. When $n = 2$, it coincides with the original Severi inequality proved by Pardini [9].

Before outlining the proof of Theorem 1.1, we first recall the original proof of Horikawa's theorem. Horikawa's original proof is as follows: using the Castelnuovo inequality $K^2 \geq 3p_g - 7$, the problem can be reduced to hyperelliptic fibrations. Then by the double cover method, the ratio $K^2/\chi(\mathcal{O}_X)$ can be explicitly calculated with a lower bound $4 - \frac{4}{g}$, where g is the genus of Albanese fibers. A different treatment is to apply Xiao's slope inequality in [11], which also gives the same lower bound. Unfortunately, in the higher dimensional case, the double cover method seems to be hard to handle. Also, to the best of our knowledge, the slope inequality for higher dimensional families of curves is still unknown.

Our proof for arbitrary dimension is different from the above. We observe that, with the help of Pardini's covering method [9] and the generic

[1] In fact, we even have a stronger result $K_S^2 \geq 2p_g(S)$ due to Debarre [5].

vanishing theorem of Green and Lazarsfeld [3], we can reduce this question to the explicit comparison between $h^0(K_X)$ and K_X^n on X with a fibration of relative dimension one, which is Theorem 4.6. The first two inequalities in Theorem 4.6 concern the hyperelliptic case. While, the last one plays the role of the Castelnuovo inequality in the surface case, which helps exclude the non-hyperelliptic case.

To prove Theorem 4.6, we study a more general comparison between $h^0(L)$ and $(\sigma^*K_X)L^{n-1}$ for any nef Cartier divisor $L \leq \sigma^*K_X$ on X'. Here $\sigma : X' \to X$ is any birational morphism. First, we use Proposition 2.4 and numerical results in [13] to get the desired ratios between $h^0(L)$ and $(\sigma^*K_X)L$ under different genus restrictions on fibered surfaces, i.e., when $n = 2$ (see Theorem 3.2). Finally, by induction on the dimension, we can finish the proof.

Conventions

Throughout this paper, X is always a projective variety. We say that X is *minimal* $\mathbb{Q}$*-Gorenstein of general type*, if X has at worst terminal singularities and K_X is a nef and big $\mathbb{Q}$-Cartier divisor. We say X is *irregular*, if $h^1(\mathcal{O}_X) > 0$, i.e., X has nontrivial Albanese map. For X irregular, we say that X has *one dimensional Albanese fiber*, if its Albanese image has dimension $\dim X - 1$.

Let X be a projective variety of dimension $n \geq 2$. We say that there is a *family of hyperelliptic (resp. non-hyperelliptic,* $\mathbb{P}^1$*) curves* on X, if there exists a variety Y of dimension $n-1$ and a rational map $f : X \dashrightarrow Y$ such that the general fibers of f are hyperelliptic (resp. non-hyperelliptic, $\mathbb{P}^1$) curves. When $\dim Y = 1$, such a family is also called a *pencil*. It is called a *fibration*, if f is a morphism.

2. Preliminaries

Let X be a smooth projective variety of dimension $n \geq 2$. Let $f : X \to C$ be a fibration from X to a smooth projective curve C with a smooth general fiber F. For any nef Cartier divisor L on X, we can find a unique integer e_L such that

- $L - e_LF$ is not nef;
- $L - eF$ is nef for any integer $e < e_L$.

We call this number the *minimum of L with respect to F*. In particular, $e_L > 0$.

Theorem 2.1. *With the above notation. Let L be a nef and effective Cartier divisor on X. Then we have the following quadruples*

$$\{(X_i, L_i, Z_i, a_i), \quad i = 0, 1, \dots, N\}$$

with the following properties:

1) $(X_0, L_0, Z_0, a_0) = (X, L, 0, e_L)$.
2) *For any $i = 0, \dots, N-1$, $\pi_i : X_{i+1} \to X_i$ is a composition of blow-ups of X_i such that the proper transform of the movable part of $|L_i - a_iF_i|$ is base point free. Here $F_0 = F$, $F_{i+1} = \pi_i^* F_i$, and $a_i = e_{L_i}$ is the minimum of L_i with respect to F_i. Moreover, we have the decomposition*

$$\pi_i^*(L_i - a_iF_i) = L_{i+1} + Z_{i+1}$$

such that $|L_{i+1}|$ is base point free and $Z_{i+1} \geq 0$.

3) *We have $h^0(L_0) \geq h^0(L_1) > \cdots > h^0(L_N) > h^0(L_N - a_NF_N) = 0$. Here $a_N = e_{L_N}$.*

Proof. See [14, Theorem 2.3]. □

Remark 2.2. If $\dim X = 2$, by [13, Theorem 2.2], we can assume $X_i = X_0$ for any i, i.e., we do not need blow-ups. In this case, $|L_{i+1}|$ may not be base point free, but it has no fixed part (see [13, Theorem 2.2 (2)]). Hence its base locus has dimension 0 (or empty). Still by [13, Theorem 2.2 (4)], we know that

$$L_0F > L_1F > \cdots > L_NF \geq 0.$$

If $|L_0|$ itself has only isolated base points, then $|L_i|_F|$ is base point free. Thus

$$h^0(L_0|_F) > h^0(L_1|_F) > \cdots > h^0(L_N|_F) \geq 1.$$

In particular, it implies that

$$N \leq h^0(L_0|_F) - 1.$$

This will be applied in the proof of Theorem 3.2 in Section 3.

Write $\rho_i = \pi_0 \circ \cdots \circ \pi_{i-1} : X_i \to X_0$ for $i = 1, \ldots, N$ and let ρ_0 be the identity morphism from X_0 to itself. Denote

$$L_i' = L_i - a_i F_i, \quad r_i = h^0(L_i|_{F_i}).$$

Fix a nef $\mathbb{Q}$-Cartier divisor P on X and write $P_i = \rho_i^* P$. Denote

$$d_i = \begin{cases} (P_i|_{F_i})(L_i|_{F_i})^{n-2}, & n \geq 3; \\ P_i F_i, & n = 2. \end{cases}$$

Remark 2.3. It is worth noticing that if $n = 2$, then $d_0 = d_1 = d_2 = \cdots$. The reader can be also reminded of this observation during the proof of Theorem 3.2 in Section 3.

Proposition 2.4. *For any $j = 0, 1, \ldots, N$, we have the following numerical inequalities:*

(1) $h^0(L_0) \leq h^0(L_j') + \sum_{i=0}^{j} a_i r_i$;

(2) $P_0 L_0^{n-1} \geq (n-1) \sum_{i=0}^{j} a_i d_i - (n-1) d_0$.

Proof. In [14, Proposition 2.6], (1) is proved. For (2), since P_i, $L_i' + F_i$, and $L_{i+1}' + F_{i+1}$ are all nef, we have

$$\begin{aligned} P_i(L_i' + F_i)^{n-1} &= P_{i+1}(\pi_i^* L_i' + F_{i+1})^{n-1} \\ &\geq P_{i+1}[(L_{i+1}' + F_{i+1}) + a_{i+1} F_{i+1}]^{n-1} \\ &= P_{i+1}(L_{i+1}' + F_{i+1})^{n-1} + (n-1) a_{i+1} d_{i+1}. \end{aligned}$$

Summing over $i = 0, \ldots, j-1$, we have

$$P_0(L_0' + F_0)^{n-1} \geq (n-1) \sum_{i=1}^{j} a_i d_i + P_j(L_j' + F_j)^{n-1} \geq (n-1) \sum_{i=1}^{j} a_i d_i.$$

Note that

$$P_0(L_0' + F_0)^{n-1} = P_0 L_0^{n-1} - (n-1)(a_0 - 1) d_0.$$

This finishes the proof. ∎

Lemma 2.5. *Under the above setting, we have*

$$P_0L_0^{n-1} \geq d_0\left((n-1)(a_0-1)+\sum_{i=1}^{N} a_i\right) \geq d_0\left(\sum_{i=0}^{N} a_i - 1\right).$$

Proof. For $i = 0, \ldots, N-1$, denote by

$$\tau_i = \pi_i \circ \cdots \circ \pi_{N-1} : X_N \to X_i$$

the composition of blow-ups and let $\tau_N = \mathrm{id}_{X_N} : X_N \to X_N$.

Write $b = a_1 + \cdots + a_N$ and $Z = \tau_1^*Z_1 + \cdots + \tau_N^*Z_N$. We have the following numerical equivalence on X_N:

$$\tau_0^*L_0' \sim_{\mathrm{num}} L_N' + bF_N + Z.$$

Since $L_0' + F_0$ and $L_N' + F_N$ are both nef, it follows that

$$\begin{aligned} P_0(L_0'+F_0)^{n-1} &= P_N(\tau_0^*L_0'+F_N)^{n-2}(L_N'+F_N+bF_N+Z) \\ &\geq bP_N(\tau_0^*L_0'+F_N)^{n-2}F_N \\ &= bd_0. \end{aligned}$$

Note that

$$P_0L_0^{n-1} - P_0(L_0'+F_0)^{n-1} = (n-1)(a_0-1)d_0.$$

Hence the first inequality is proved. Since $a_0 \geq 1$, the second inequality holds. □

3. Linear series on algebraic surfaces

In this section, we set up some basic results about linear series on algebraic surfaces. Throughout this section, we will use the following assumptions:

(A3.1) S' is a minimal surface of general type, and $\sigma : S \to S'$ is a birational morphism.

(A3.2) L is a nef Cartier divisor on S, and P is a nef $\mathbb{Q}$-Cartier divisor on S such that $L \leq P \leq \sigma^*K_{S'}$.

Proposition 3.1. *Assume that S has no hyperelliptic pencil. Then*

$$(\sigma^*K_{S'})L \geq 3h^0(L) - 7.$$

Proof. First, we can assume that $h^0(L) \geq 3$. Otherwise, the result would become trivial. Replacing S by an appropriate resolution, we can further assume that $|L|$ is base point free. Denote by $\phi_L : S \to \mathbb{P}^{h^0(L)-1}$ the morphism induced by $|L|$.

If ϕ_L is generically finite and $L^2 < 3h^0(L) - 7$, then using exactly the same argument in [2, Théorème 5.5], we know that S is a double cover over a birationally ruled surface. This would imply that S has a hyperelliptic pencil, which is impossible. Hence in this case, we have

$$(\sigma^* K_{S'})L \geq L^2 \geq 3h^0(L) - 7.$$

When $\dim \phi_L(S) = 1$, we can write

$$L \sim_{\text{num}} rC,$$

where C is a general fiber of ϕ_L, and $r \geq h^0(L) - 1 \geq 2$. If $(\sigma^* K_{S'})C \geq 3$, then

$$(\sigma^* K_{S'})L \geq 3r > 3h^0(L) - 7.$$

If $(\sigma^* K_{S'})C \leq 2$, then $K_{S'}C' \leq 2$, where $C' = \sigma(C)$. On the other hand, since $L \leq \sigma^* K_{S'}$, we know that $K_{S'} - 2C'$ is pseudo-effective. Hence

$$C'^2 \leq \frac{K_{S'}C'}{2} \leq 1.$$

By parity, the only possibility is that $K_{S'}C' + C'^2 = 2$, i.e., $p_a(C') = 2$. But it is absurd, because C' can not be hyperelliptic. □

Theorem 3.2. *Assume that S has a hyperelliptic fibration of genus g curves and $PC = 2g - 2$, where C is a general fiber.*

(1) *If $g \geq 3$, then*

$$PL \geq \frac{8}{3}h^0(L) - (4g - 4).$$

(2) *If $g \geq 4$, then*

$$PL \geq 3h^0(L) - (4g - 4).$$

Proof. Replacing S by an appropriate resolution, we can assume that $|L|$ is base point free. Since $L \leq \sigma^* K_{S'} \leq K_S$, by adjunction, $L|_C \leq K_C$, i.e., $L|_C$ is a special divisor. Moreover, since C is hyperelliptic and $|L|_C|$ is base point free, by [13, Lemma 2.6], we know that LC has to be even.

Let us prove (1) first. Assume that $g \geq 3$. If $LC \geq 4$, then we have

$$PL \geq L^2 \geq \frac{4LC}{LC+2}h^0(L) - 2LC \geq \frac{8}{3}h^0(L) - (4g-4).$$

The second inequality is by [13, Theorem 1.1], and the third inequality follows from the fact that $LC \leq PC = 2g-2$. If $LC = 0$, we know that $L \sim_{\mathrm{num}} rC$, where $r \geq h^0(L) - 1$. Thus

$$PL = rPC \geq (2g-2)(h^0(L)-1) \geq 4h^0(L) - 4 > 4h^0(L) - (4g-4).$$

The only issue here is when $LC = 2$. In this case, since $r_0 = h^0(L|_C) = 2$, we can apply Theorem 2.1 and Remark 2.2 to L to get the following triples

$$\{(L_i, Z_i, a_i) : i = 0, \ldots, N\}$$

where $N \leq 1$. Assume first that we get two triples, i.e., $N = 1$. By Remark 2.2, we know that $r_1 = h^0(L_1|_C) = 1$ and that $|L_1|_C|$ is base point free. It forces $L_1|_C = \mathcal{O}_C$. We will divide the proof into two cases.

Case 1.1: $a_0 \leq a_1$. By Remark 2.3 and Proposition 2.4, we get

$$h^0(L) \leq a_0 r_0 + a_1 r_1 = 2a_0 + a_1, \tag{3.1}$$

$$PL \geq a_0 d_0 + a_1 d_1 - d_0 \geq 4a_0 + 4a_1 - (2g-2). \tag{3.2}$$

It follows that

$$\frac{PL + (4g-4)}{h^0(L)} > \frac{PL + (2g-2)}{h^0(L)} \geq \frac{4a_0 + 4a_1}{2a_0 + a_1} \geq \frac{8}{3}.$$

Case 1.2: $a_0 > a_1$. In this case, instead of using Proposition 2.4, we need to use [14, Proposition 2.3], by which we obtain

$$L^2 \geq 2a_0 d_0' + (d_0' + d_1')a_1 - 2d_0' = 4a_0 + 2a_1 - 4. \tag{3.3}$$

Here $d_i' = L_i C$ so that $d_0' = 2$ and $d_1' = 0$.

Notice that $L - (a_0 - 1)C$ is nef. Thus

$$\begin{aligned}(P-L)L &= (P-L)(L-(a_0-1)C) + (a_0-1)(P-L)C \\ &\geq (a_0-1)(2g-4) \geq 2a_0 - 2.\end{aligned}$$

Combine this with (3.3), and we get

$$PL = L^2 + (PL - L^2) \geq 6a_0 + 2a_1 - 6.$$

On the other hand, the inequality (3.1) in Case 1.1 still holds in this case. Hence

$$\frac{PL+(4g-4)}{h^0(L)} > \frac{PL+6}{h^0(L)} \geq \frac{6a_0+2a_1}{2a_0+a_1} > \frac{8}{3}.$$

Assume now that we only have one triple, i.e., $N=0$. In this case, the above proof still applies by letting $a_1=0$ and the result follows.[2]

Now let us prove (2). In fact, the strategy is quite similar but a bit more complicated. In the following, we will assume that $g \geq 4$.

When $LC \geq 6$, by [13, Theorem 1.1] again, we get

$$PL \geq L^2 \geq \frac{4LC}{LC+2}h^0(L) - 2LC \geq 3h^0(L) - (4g-4).$$

When $LC = 0$, we can just apply a very similar argument as above to deduce that

$$PL \geq (2g-2)(h^0(L)-1) \geq 6h^0(L) - (2g-2) > 3h^0(L) - (4g-4).$$

When $LC = 2$, using the "at most two triples" method verbatim as before and by Remark 2.3 and Proposition 2.4, we have

$$h^0(L) \leq a_0r_0 + a_1r_1 = 2a_0 + a_1, \tag{3.4}$$

$$PL \geq a_0d_0 + a_1d_1 - d_0 \geq 6a_0 + 6a_1 - (2g-2). \tag{3.5}$$

Hence

$$PL \geq 3h^0(L) - (2g-2) > 3h^0(L) - (4g-4).$$

It remains to consider the case when $LC = 4$. In this case, since $r_0 = h^0(L|_C) = 3$, by Theorem 2.1 and Remark 2.2, we can get the following triples

$$\{(L_i, Z_i, a_i) : i = 0, \ldots, N\}$$

where $N \leq 2$. We first assume that we get three triples, i.e., $N = 2$. Then both $|L_1|_C|$ and $|L_2|_C|$ are base point free. Note that $r_1 = h^0(L_1|_C) = 2$ and $r_2 = h^0(L_2|_C) = 1$. It is easy to see that $L_1|_C = g_2^1$ and $L_2|_C = \mathcal{O}_C$. We divide the proof into two cases again.

[2]In fact, in this case, we only need to use the argument in Case 1.2.

Case 2.1: $a_0 \le a_2$. In this case, by Remark 2.3 and Proposition 2.4, we have

$$h^0(L) \le \sum_{i=0}^{2} a_i r_i = 3a_0 + 2a_1 + a_2, \tag{3.6}$$

$$PL \ge \sum_{i=0}^{2} a_i d_i - d_0 \ge 6(a_0 + a_1 + a_2) - (2g - 2). \tag{3.7}$$

Hence

$$PL + (4g - 4) > 6(a_0 + a_1 + a_2) \ge 3(3a_0 + 2a_1 + a_2) \ge 3h^0(L).$$

Case 2.2: $a_0 > a_2$. We follow the idea in Case 1.2 once more. By [15, Proposition 2.3], we have

$$L^2 \ge 2a_0 d_0' + \sum_{i=1}^{2} (d_{i-1}' + d_i') a_i - 2d_0' = 8a_0 + 6a_1 + 2a_2 - 8. \tag{3.8}$$

Here $d_i' = L_i C$ so that $d_0' = 4$, $d_1' = 2$, and $d_2' = 0$. $\square$

By a similar argument as in Case 1.2, we can get

$$L(P - L) \ge (a_0 - 1)(P - L)C = (a_0 - 1)(2g - 6) \ge 2a_0 - 2.$$

Hence

$$\begin{aligned} PL = L^2 + L(P - L) &\ge 10a_0 + 6a_1 + 2a_2 - 10 \\ &> 3(3a_0 + 2a_1 + a_2) - 10 > 3h^0(L) - (4g - 4). \end{aligned}$$

If we only get one triple or two, the proof here still applies after suitable modifications. We just list the modifications and leave the proof to the reader:

- If $N = 0$, set $a_1 = a_2 = d_1' = d_2' = 0$;
- If $N = 1$ and $L_1|_C = g_2^1$, set $a_2 = d_2' = 0$;
- If $N = 1$ and $L_1|_C = \mathcal{O}_C$, set $a_1 = d_1' = d_2' = 0$.

4. Linear series on varieties fibered by curves

In this section, we will generalize the results in Section 3 to arbitrary dimension.

Assumption 4.1. Throughout this section, we assume that:

- X is a projective, normal and minimal $\mathbb{Q}$-Gorenstein variety of general type, and $L \leq K_X$ is a nef Cartier divisor on X.
- $f : X \to Y$ is a fibration of genus $g \geq 2$ from X to a normal projective variety Y of dimension $n-1$ with connected fibers, and C is a general fiber.
- M is a Cartier divisor on Y such that $M^{n-1} > 0$ and $|M|$ is base point free. Write $B = f^*M$.
- Since $h^0(lK_X) > 0$ for $l \in \mathbb{N}$ sufficiently large, we write

$$K_X \sim_{\mathrm{lin}_{\mathbb{Q}}} \sum_{i=1}^{I_0} q_i H_i + V \tag{4.1}$$

  up to $\mathbb{Q}$-linear equivalence. Here $q_i \in \mathbb{Q}_{>0}$, each $H_i \geq 0$ is an irreducible, reduced and horizontal Cartier divisor, and $V \geq 0$ is the vertical part. Fix an integer $k > 0$ such that $(kB - K_X)|_{H_i}$ is pseudo-effective for each i and that $kB - V$ is pseudo-effective.[3]

We have the following numerical results.

Lemma 4.2. $K_X B^{n-1} > 0$.

Proof. By the assumption, we know that $B^{n-1} \sim_{\mathrm{num}} \deg_Y(M) \cdot C$ as one cycles on X, where $\deg_Y(M) = M^{n-1} > 0$. Hence

$$K_X B^{n-1} = \deg_Y(M)(K_X C) = (2g-2)M^{n-1} > 0.$$

The proof is finished. □

Lemma 4.3. *For any* $0 \leq j \leq n-2$*, we have*

$$K_X L^{n-1-j} B^j \leq 2k K_X L^{n-2-j} B^{j+1}.$$

[3]Such an integer $k > 0$ exists, since $f|_{H_i}$ is generically finite and thus $B|_{H_i}$ is nef and big for any i. Also, because V is vertical, it has to be contained in the pullback of certain divisor on Y. Hence, increasing k, we may assume that $kB - V$ is pseudo-effective.

Proof. Note that K_X, L, and B are all nef, and $(kB - K_X)|_{H_i}$, $kB - V$ are both pseudo-effective. It follows that

$$\begin{aligned} K_X L^{n-1-j} B^j &\leq K_X^2 L^{n-2-j} B^j \\ &= \sum_{i=1}^{I_0} q_i K_X L^{n-2-j} B^j H_i + K_X L^{n-2-j} B^j V \\ &\leq \sum_{i=1}^{I_0} k q_i L^{n-2-j} B^{j+1} H_i + k K_X L^{n-2-j} B^{j+1} \\ &\leq 2k K_X L^{n-2-j} B^{j+1}. \end{aligned}$$

This ends the proof. □

Theorem 4.4. *Under Assumption 4.1. Assume that C is hyperelliptic. Let $\pi : X' \to X$ be any birational morphism and $L' \leq \pi^* L$ be any nef Cartier divisor on X'.*

- *If $g \geq 3$, then*

$$h^0(L') - \frac{3(\pi^* K_X)(L')^{n-1}}{8(n-1)!} \leq n(2k+2)^{n-2} K_X B^{n-1}.$$

- *If $g \geq 4$, then*

$$h^0(L') - \frac{(\pi^* K_X)(L')^{n-1}}{3(n-1)!} \leq n(2k+2)^{n-2} K_X B^{n-1}.$$

Proof. The proofs of the two inequalities are almost identical. We will give the detailed proof of the first one and leave the other to the reader.

We prove this theorem by induction. When $n = 2$, i.e., X is a fibered surface, we know that $(\pi^* K_X)(\pi^* C) = K_X C = 2g - 2$. By Theorem 3.2, we have

$$h^0(L') - \frac{3}{8}(\pi^* K_X) L' \leq \frac{3}{8}(4g - 4).$$

Since $K_X B = (2g - 2) \deg M \geq 2g - 2$, we deduce that

$$h^0(L') - \frac{3}{8}(\pi^* K_X) L' \leq \frac{3}{4} K_X B < 2 K_X B.$$

This verifies the result for $n = 2$.

Now, suppose that the theorem holds when $\dim X \le n-1$ ($n \ge 3$). We first show that

$$h^0(L) - \frac{3K_X L^{n-1}}{8(n-1)!} \le \alpha_n K_X B^{n-1}, \tag{4.2}$$

where

$$\alpha_n = n(2k+2)^{n-2}.$$

Choose a general pencil in $|B|$ and denote by $\sigma : X_0 \to X$ the blow-up of the indeterminacies of this pencil. Then X_0 is a fibered variety over $\mathbb{P}^1$ whose general fiber F is isomorphic to the general member of this chosen pencil.

Let $L_0 = \sigma^* L$, $P_0 = \sigma^*(K_X + B)$, and $B_0 = \sigma^* B$. Applying Theorem 2.1 to X_0, L_0 and P_0, we obtain the following quadruples

$$\{(X_i, L_i, Z_i, a_i) : i = 0, \ldots, N\}$$

and P_i's which satisfy the conditions therein. Moreover, by Proposition 2.4, we have the following inequalities:

$$h^0(L) = h^0(L_0) \le \sum_{i=0}^{N} a_i r_i, \tag{4.3}$$

$$(K_X + B)L^{n-1} = P_0 L_0^{n-1} \ge (n-1)\sum_{i=0}^{N} a_i d_i - (n-1)d_0. \tag{4.4}$$

Here

$$r_i = h^0(L_i|_{F_i}), \quad d_i = (P_i|_{F_i})(L_i|_{F_i})^{n-2}.$$

In the following, we will compare r_i and d_i by induction.

First, we assume that $K_X L^{n-2} B > 0$. This implies that

$$d_0 = (P_0|_F)(L_0|_F)^{n-2} = (K_X + B)L^{n-2}B \ge K_X L^{n-2} B > 0.$$

Note that for each i, F_i has a fibered structure by curves of genus g induced by f. Moreover, by adjunction, $P_0|_F = K_F$. Hence by induction on the dimension, we can get the following numerical inequality for each i:

$$r_i - \frac{3d_i}{8(n-2)!} \le \alpha_{n-1} K_F (B|_F)^{n-2} = \alpha_{n-1} K_X B^{n-1}. \tag{4.5}$$

Since $d_0 > 0$ and $L \le K_X$, by Lemma 2.5 and Lemma 4.3, we obtain

$$\sum_{i=0}^{N} a_i - 1 \le \frac{P_0 L_0^{n-1}}{d_0} \le \frac{K_X L^{n-1} + L^{n-1}B}{K_X L^{n-2} B} \le \frac{K_X L^{n-1}}{K_X L^{n-2} B} + 1 \le 2k + 1,$$

i.e.,

$$\sum_{i=0}^{N} a_i \le 2k + 2. \tag{4.6}$$

Applying Lemma 4.3 repeatedly, we get

$$L^{n-1}B \le K_X L^{n-2} B \le (2k)^{n-2} K_X B^{n-1}, \tag{4.7}$$

$$L^{n-2}B^2 \le K_X L^{n-3} B^2 \le (2k)^{n-3} K_X B^{n-1}. \tag{4.8}$$

In particular, (4.7) and (4.8) imply that

$$d_0 = (P_0|_F)(L_0|_F)^{n-2} = (K_X + B)L^{n-2}B < 2 \cdot (2k)^{n-2} K_X B^{n-1}. \tag{4.9}$$

Combine (4.3)–(4.9) together. It follows that

$$\begin{aligned} h^0(L) - \frac{3K_X L^{n-1}}{8(n-1)!} &\le \sum_{i=0}^{N} \left(r_i - \frac{3d_i}{8(n-2)!} \right) a_i + \frac{3d_0}{8(n-2)!} + \frac{3L^{n-1}B}{8(n-1)!} \\ &< \alpha_{n-1} K_X B^{n-1} \sum_{i=0}^{N} a_i \\ &\quad + \left(\frac{3(2k)^{n-2}}{4(n-2)!} + \frac{3(2k)^{n-2}}{8(n-1)!} \right) K_X B^{n-1} \\ &\le (2k+2)\alpha_{n-1} K_X B^{n-1} + \frac{(2k)^{n-2}}{(n-2)!} K_X B^{n-1} \\ &< \alpha_n K_X B^{n-1}. \end{aligned}$$

In the last step, we use the fact that

$$(2k+2)\alpha_{n-1} + \frac{(2k)^{n-2}}{(n-2)!} \le (n-1)(2k+2)^{n-2} + (2k)^{n-2} < \alpha_n.$$

Second, we assume that $K_X L^{n-2} B = 0$. In this case, we have

$$h^0(L - B) = 0.$$

Otherwise, $L - B$ is effective. By Lemma 4.2, we would have

$$K_X L^{n-2} B = K_X L^{n-3}(L - B + B)B \geq K_X L^{n-3} B^2 \geq \cdots \geq K_X B^{n-1} > 0,$$

which is absurd. As a result, if we choose a general member $W \in |B|$, we get

$$h^0(L) \leq h^0(L|_W).$$

Since W itself is also fibered by curves of genus g and $L|_W \leq K_W$, by induction, we know that

$$h^0(L) \leq h^0(L|_W) \leq \frac{3K_W(L|_W)^{n-2}}{8(n-2)!} + \alpha_{n-1} K_W (B|_W)^{n-2}. \tag{4.10}$$

By adjunction, $K_W = (K_X + B)|_W$. It follows that

$$K_W (B|_W)^{n-2} = (K_X + B)B^{n-1} = K_X B^{n-1}. \tag{4.11}$$

Also, using Lemma 4.3 repeatedly, we have

$$\begin{aligned} K_W(L|_W)^{n-2} &= (K_X + B)L^{n-2}B = L^{n-2}B^2 \\ &\leq K_X L^{n-3} B^2 \leq (2k)^{n-3} K_X B^{n-1}. \end{aligned} \tag{4.12}$$

Combine (4.10)–(4.12) together, and we get

$$h^0(L) \leq \left(\frac{3 \cdot (2k)^{n-3}}{8(n-2)!} + \alpha_{n-1} \right) K_X B^{n-1} \leq \alpha_n K_X B^{n-1}.$$

This finishes the proof of (4.2).

Now, let $\pi : X' \to X$ be a birational morphism. Let $B' = \pi^* B$. We can check that $kB' - \pi^* V$ is still pseudo-effective. Moreover, $(kB' - \pi^* K_X)|_{\pi^* H_i}$ is also pseudo-effective for each i. Using the method in the proof of (4.2) verbatim, one can get

$$h^0(L') - \frac{3(\pi^* K_X)(L')^{n-1}}{8(n-1)!} \leq \alpha_n (\pi^* K_X)(B')^{n-1} = \alpha_n K_X B^{n-1}.$$

This finishes the whole proof. □

Theorem 4.5. *Under Assumption 4.1. Assume that there is no family of hyperelliptic curves on X. Let $\pi : X' \to X$ be a birational morphism. Then for any nef Cartier divisor $L' \le \pi^* L$, we have*

$$h^0(L') - \frac{(\pi^* K_X)(L')^{n-1}}{3(n-1)!} \le \frac{7}{3} n(2k+2)^{n-2} K_X B^{n-1}.$$

Proof. The induction method in Theorem 4.4 applies here verbatim. If $n = 2$, i.e., X is a fibered surface, then by Proposition 3.1,

$$h^0(L') - \frac{1}{3}(\pi^* K_X) L' \le \frac{7}{3}.$$

This verifies the theorem when $n = 2$.

With the above result, by the same induction method as in Theorem 4.4, we can get the conclusion. We leave the detailed proof to the reader. □

The direct application of the above theorems is when $X' = X$ and $L' = K_X$. We have the following summarized result.

Theorem 4.6. *Suppose that we are under Assumption 4.1.*

- *If C is hyperelliptic of genus $g \ge 3$, then*

$$h^0(K_X) - \frac{3K_X^n}{8(n-1)!} \le n(2k+2)^{n-2} K_X B^{n-1}.$$

- *If C is hyperelliptic of genus $g \ge 4$, then*

$$h^0(K_X) - \frac{K_X^n}{3(n-1)!} \le n(2k+2)^{n-2} K_X B^{n-1}.$$

- *If there is no family of hyperelliptic curves on X, then*

$$h^0(K_X) - \frac{K_X^n}{3(n-1)!} \le \frac{7}{3} n(2k+2)^{n-2} K_X B^{n-1}.$$

Proof. We just prove the first inequality. The others are almost the same.

Let $\pi : X' \to X$ be a birational modification such that the movable part of $\pi^* K_X$ is base point free, i.e., we can write

$$\pi^* K_X = L' + Z',$$

where $|L'|$ is base point free, $Z' \geq 0$ is a $\mathbb{Q}$-divisor, and $h^0(L') = h^0(K_X)$. Apply Theorem 4.4 to X' and L'. It follows that

$$h^0(K_X) - \frac{3(\pi^* K_X)(L')^{n-1}}{8(n-1)!} \leq n(2k+2)^{n-2} K_X B^{n-1}.$$

Note that

$$(\pi^* K_X)(L')^{n-1} \leq (\pi^* K_X)^n = K_X^n.$$

We can easily get the desired result. □

5. Proof of the main theorem

In this section, we prove the main theorem.

Theorem 5.1. *Let X be an n-dimensional normal, irregular minimal $\mathbb{Q}$-Gorenstein variety of general type. Assume X has one dimensional Albanese fiber.*

(1) *The general Albanese fiber is of genus 2 provided that*

$$K_X^n < \frac{8}{3}(n-1)!\chi(\omega_X).$$

(2) *The general Albanese fiber is hyperelliptic of genus 2 or 3 provided that*

$$K_X^n < 3(n-1)!\chi(\omega_X).$$

Proof. Our proof is based on the following three ingredients: (1) Theorem 4.6, (2) the covering and limiting method by Pardini [9], and (3) the generic vanishing theorem by Green and Lazarsfeld [6]. We give the proof by steps.

Step 1: Construction of covers and set up. Let A be the Albanese variety of X, and $\alpha = \mathrm{Alb}_X : X \to A$ be the Albanese map of X. For any $d \in \mathbb{Z}_{>0}$, denote by μ_d the multiplication map of A by d. Let $X_d = X \times_{\mu_d} A$.

Then we have the following commutative diagram:

$$\begin{array}{ccc} X_d & \xrightarrow{\phi_d} & X \\ {\scriptstyle \alpha_d}\downarrow & & \downarrow{\scriptstyle \alpha=\mathrm{Alb}_X} \\ A & \xrightarrow{\mu_d} & A \end{array}$$

Note that X_d is always irreducible, because the induced map $\alpha^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$ is injective, and in fact, an isomorphism here.

Fix a sufficiently ample Cartier divisor H on A. Let $B = \alpha^* H$ and $B_d = \alpha_d^* H$ for any $d \geq 1$. By [3, Chapter 2, Proposition 3.5], we have the following numerical equivalence:

$$d^2 B_d \sim_{\mathrm{num}} \phi_d^* B.$$

This implies that for any $1 \leq i \leq n$,

$$K_{X_d}^{n-i} B_d^i = (\phi_d^* K_X)^{n-i} \left(\frac{1}{d^2} \phi_d^* B \right)^i = d^{2m-2i} K_X^{n-i} B^i. \tag{5.1}$$

Here $m = h^1(\mathcal{O}_X) = \dim A$.

For each $d > 0$, let $X_d \xrightarrow{f_d} Y_d \xrightarrow{h_d} A$ be the Stein factorization of α_d. Let $M_d = h_d^* H$. Thus $B_d = f_d^* M_d$.

Step 2: Reduction to the case of positive genus. Note that to prove the theorem, we can assume that $\chi(\omega_X) > 0$. Since $\dim A = m$ and X has one dimensional Albanese fiber, we have $m \geq n - 1$.

Notice that ϕ_d is étale. Thus it follows that

$$d^{2m} \chi(\omega_X) = \chi(\omega_{X_d}) = \sum_{i=0}^{n} (-1)^i h^i(\omega_{X_d}) \leq h^0(K_{X_d}) + \sum_{i=2}^{n} (-1)^i h^i(\omega_{X_d}).$$

On the other hand, by [14, Theorem 4.1], which is in the spirit of the generic vanishing theorem of Green and Lazarsfeld [6], for each $i = 0, \ldots, n-2$, we have

$$\lim_{d \to \infty} \frac{h^i(\mathcal{O}_{X_d})}{d^{2m}} = 0.$$

By duality, it implies that $h^i(\omega_{X_d}) \sim o(d^{2m})$ for $i \geq 2$. Hence

$$h^0(K_{X_d}) \geq d^{2m} \chi(\omega_X) + o(d^{2m}). \tag{5.2}$$

In particular, we can assume that $h^0(K_{X_d}) > 0$ when d is large. Since it suffices to prove this theorem on any étale cover, we can assume that $h^0(K_X) > 0$.

Step 3: Pseudo-effectiveness. As in (4.1), we can write

$$K_X \sim_{\mathrm{lin}_{\mathbb{Q}}} \sum_{i=1}^{I_0} q_i H_i + V,$$

where $q_i \in \mathbb{Q}_{>0}$, each $H_i \geq 0$ is an irreducible, reduced and horizontal divisor, and $V \geq 0$ is the vertical part. We can also find an integer $k > 0$ such that $(kB - K_X)|_{H_i}$ is pseudo-effective for each i and that $kB - V$ is pseudo-effective.

Since ϕ_d is étale, we know that $K_{X_d} = \phi_d^* K_X$. Hence we can write

$$K_{X_d} \sim_{\mathrm{lin}_{\mathbb{Q}}} \sum_{i=1}^{I_0} q_i \phi_d^* H_i + \phi_d^* V = \sum_{i=1}^{I_0} q_i \left(\sum_{j=1}^{l_d} H_{i,j} \right) + \phi_d^* V.$$

Here each $H_{i,j}$ is an irreducible and reduced component of the divisor $\phi_d^* H_i$ on X_d.

By pulling back, we know that

$$(k\phi_d^* B - K_{X_d})|_{H_{i,j}} \quad \text{and} \quad k\phi_d^* B - \phi_d^* V$$

are pseudo-effective. Because $d^2 B_d \sim_{\mathrm{num}} \phi_d^* B$, we deduce that

$$(kd^2 B_d - K_{X_d})|_{H_{i,j}} \quad \text{and} \quad kd^2 B_d - \phi_d^* V$$

are pseudo-effective.

Step 4: Hyperelliptic case. If the general Albanese fiber of X is hyperelliptic of genus $g \geq 3$, so is the general fiber of f_d. By the first inequality in Theorem 4.6, we can get

$$h^0(K_{X_d}) - \frac{3K_{X_d}^n}{8(n-1)!} \leq n(2kd^2 + 2)^{n-2} K_{X_d} B_d^{n-1}.$$

Recall that by (5.1), we have

$$K_{X_d}^n = d^{2m} K_X^n, \qquad K_{X_d} B_d^{n-1} = d^{2m-2n+2} K_X B^{n-1}.$$

Combine this with (5.2), and it follows that

$$d^{2m}\chi(\omega_X) + o(d^{2m}) - \frac{3d^{2m} K_X^n}{8(n-1)!} \leq n(2kd^2 + 2)^{n-2} d^{2m-2n+2} K_X B^{n-1}.$$

Letting $d \to \infty$, we deduce that

$$K_X^n \geq \frac{8}{3}(n-1)!\chi(\omega_X).$$

By the same method and applying the second inequality in Theorem 4.6, we deduce that if the Albanese fiber is hyperelliptic of genus $g \geq 4$, then

$$K_X^n \geq 3(n-1)!\chi(\omega_X).$$

Step 5: Non-hyperelliptic case. Now assume that the general Albanese fiber of X is non-hyperelliptic. So is the general fiber of f_d.

Step 5.1. We first show that, up to étale covers, we have

$$h^0(K_X - B) > 0. \tag{5.3}$$

If this is true, then $\alpha^*|H|$ can be identified as a subspace of $|K_X|$. Note that H is sufficiently ample. Thus the map given by this (non-complete) linear system $\alpha^*|H|$ is identical to the Albanese map of X. It follows that the Albanese map of X factors through the canonical map of X.

Now let us prove (5.3). Choose a very general member $W \in |B_d|$. Since $h^0(K_{X_d}) > 0$, by adjunction, $h^0(K_W) > 0$. Hence we can consider the canonical map of W. We have the following diagram:

$$\begin{array}{ccc} W' & & \\ \sigma \big\downarrow & \overset{\psi}{\searrow} & \\ W & \underset{\phi_{K_W}}{\dashrightarrow} & \Sigma \end{array}$$

where $\Sigma \subseteq \mathbb{P}^{h^0(K_W)-1}$ is the canonical image of W, and σ is the blow-up of the indeterminacies of $|K_W|$. By Lemma 5.2, we have

$$h^0(K_{X_d}|_W) \leq h^0(K_W) \leq (\mathcal{O}_\Sigma(1))^{\dim \Sigma} + \dim \Sigma. \tag{5.4}$$

On the other hand, let $l \in \mathbb{Z}_{>0}$ be a positive integer such that lK_X is Cartier. By adjunction, lK_W is also Cartier. Note that K_W is nef and big. We have

$$\begin{aligned} K_W^{n-1} &= (\sigma^* K_W)^{n-1} \geq (\sigma^* K_W)^{n-1-\dim \Sigma} \left(\psi^* \mathcal{O}_\Sigma(1)\right)^{\dim \Sigma} \\ &\geq \frac{1}{l^{n-1-\dim \Sigma}} \left(\mathcal{O}_\Sigma(1)\right)^{\dim \Sigma}. \end{aligned} \tag{5.5}$$

Hence it follows from (5.1), (5.4) and (5.5) that

$$\begin{aligned} h^0(K_{X_d}|_W) &\leq l^{n-1-\dim\Sigma} K_W^{n-1} + \dim\Sigma \\ &\leq l^{n-1-\dim\Sigma}(K_{X_d}+B_d)^{n-1}B_d + n - 1 \sim o(d^{2m}). \end{aligned}$$

Therefore, up to étale covers, we can assume that

$$h^0(K_X) - h^0(K_X|_W) > 0$$

for a general member $W \in |B|$. By the following exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_X(K_X - B)) \longrightarrow H^0(\mathcal{O}_X(K_X)) \longrightarrow h^0(\mathcal{O}_W(K_X)),$$

we know that

$$h^0(K_X - B) > 0.$$

Note that in the meantime, we also obtain

$$h^0(K_{X_d} - B_d) > 0$$

for any $d > 0$, which implies that the map α_d factors through the canonical map of X_d. That is, we have the following commutative diagram:

$$\begin{array}{ccc} X_d & \overset{\phi_{K_{X_d}}}{\dashrightarrow} & \Gamma_d \\ {\scriptstyle f_d}\downarrow & \searrow^{\alpha_d} & \downarrow \\ Y_d & \underset{h_d}{\longrightarrow} & A \end{array}$$

where Γ_d is the canonical image of X_d.

Step 5.2. We claim that *there is no family of hyperelliptic curves on X_d*.

If the claim is not true, then there exists a hyperelliptic family on X_d. Furthermore, we have the following two possibilities: either this family is contracted by $\phi_{K_{X_d}}$ or not. If it is contracted by $\phi_{K_{X_d}}$, it would also be contracted by α_d, thus by f_d. This is impossible since the general fiber of f_d is non-hyperelliptic. The second case is still impossible. Because if this family is not contracted by $\phi_{K_{X_d}}$, then its image under $\phi_{K_{X_d}}$ is a family of $\mathbb{P}^1$ curves on Γ_d. Since $\alpha_d(X_d)$ can not have any $\mathbb{P}^1$ family, this hyperelliptic family has to be contracted by α_d, thus by f_d. It contradicts our assumption again.

Step 5.3. Since X_d has no family of hyperelliptic curves, by the third inequality in Theorem 4.6, we have

$$h^0(K_{X_d}) - \frac{K_{X_d}^n}{3(n-1)!} \leq \frac{7}{3}n(2kd^2+2)^{n-2}K_{X_d}B_d^{n-1}.$$

Use the method in Step 4 and let $d \to \infty$. It follows that

$$K_X^n \geq 3(n-1)!\chi(\omega_X).$$

The proof will be completed by showing the following lemma which is undoubtedly known to experts. □

Lemma 5.2. *Let L be a Cartier divisor on a projective variety X such that $|L|$ is base point free. Let $\phi_L : X \to \mathbb{P}^{h^0(L)-1}$ be the morphism induced by $|L|$. Denote $\Sigma = \phi_L(X)$ and $d = \dim \Sigma$. Then*

$$h^0(L) \leq (\mathcal{O}_\Sigma(1))^d + d.$$

Proof. Denote by W a very general hyperplane section of Σ. We have

$$0 \longrightarrow H^0(\mathcal{O}_\Sigma) \longrightarrow H^0(\mathcal{O}_\Sigma(1)) \longrightarrow H^0(\mathcal{O}_W(1)).$$

It gives

$$h^0(L) = h^0(\mathcal{O}_\Sigma(1)) \leq h^0(\mathcal{O}_W(1)) + 1.$$

Repeat this $d-1$ times to get

$$h^0(L) \leq h^0(\mathcal{O}_C(1)) + (d-1),$$

where C is the complete intersection of $d-1$ general hyperplane sections. In particular,

$$\deg \mathcal{O}_C(1) = (\mathcal{O}_\Sigma(1))^d.$$

Therefore,

$$h^0(\mathcal{O}_C(1)) \leq (\mathcal{O}_\Sigma(1))^d + 1.$$

This ends the proof. □

6. Final Remark

In [1, Remark 4.5], by using Xiao's method on the Harder-Narasimhan filtration, Barja shows the following:

Theorem (Barja). *Let X be an n-dimensional normal, irregular minimal Gorenstein variety of general type with Albanese fiber dimension one. Then*

$$K_X^n \geq 2(n-1)!\chi(\omega_X).$$

In fact, using the proof of Theorem 1.1, we can give an alternative proof of this result. Moreover, our proof is also valid for $\mathbb{Q}$-Gorenstein varieties. The induction method can be used here verbatim. We only need to show the corresponding result for surfaces, which is nothing but the following Noether type result:

Proposition 6.1. *Let S' be a minimal surface of general type, and $\sigma : S \to S'$ be a birational map. Then for any nef divisor $L \leq \sigma^* K_{S'}$ on S, we have*

$$(\sigma^* K_{S'})L \geq 2h^0(L) - 4.$$

Proof. See [8, Lemmas 2.2, 2.3]. □

We leave the proof to the reader.

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