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On angles determined by fractal subsets of the Euclidean space

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We prove that if the Hausdorff dimension of a compact subset of $\mathbb{R}^d$ is greater than $\frac{d+1}{2}$, then the set of angles determined by triples of points from this set has positive Lebesgue measure. This result improves on the the threshold $\min\{\frac{d}{2}+\frac{4}{3}, d-1\}$ obtained by A. Máthé ([15]). The result complements those of V. Harangi, T. Keleti, G. Kiss, P. Maga, A. Máthé, P. Mattila and B. Stenner in ([8]) and those of V. Harangi in ([7]). We also obtain new upper bounds for the number of times an angle can occur among N points in $\mathbb{R}^d$, $d \geq 4$, motivated by the results of Apfelbaum and Sharir ([1]) and Pach and Sharir ([16]). We then use this result to establish sharpness thresholds in the continuous setting.

1. Introduction

In this paper we study angles determined by subsets of the Euclidean space of a given Hausdorff dimension.

Definition 1.1. Given $E \subset \mathbb{R}^d$, $d \geq 2$, let $\theta(x,y,x)$ denote the interior angle of the triangle with vertices at x, y and z , at y, where $x,y,z \in E$. Define the angle set

$$\mathcal{A}(E) = \{\theta(x,y,z) : x,y,z \in E\}.$$

The question we ask is, how large does the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, need to be to ensure that the Lebesgue measure of $\mathcal{A}(E)$ is positive. Similarly, we would like to know if the angles are uniformly distributed in the sense that a small neighborhood of a given angle does not arise more often than is its share.

A. Máthé obtained the Hausdorff dimension threshold $\min\{\frac{d}{2}+\frac{4}{3}, d-1\}$ for this question in [15]. His result was based on a projection argument and the best known bounds, by Wolff and Erdogan, for the Falconer distance

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problem. He also considered a related question on what is the largest dimension of a compact set $E \subset \mathbb{R}^d$ such that it does not contain a particular angle. In [15] he for example establishes the existence of such a set E of dimension $\frac{d}{2}$ which does not contain the angle $\frac{\pi}{2}$. This result is sharp for d even. In his paper Máthé introduces a comprehensive set of techniques for tackling configuration questions of that type. See also the work of V. Harangi ([7]) for more results in this direction.

Our results are also motivated by a paper due to V. Harangi, T. Keleti, G. Kiss, P. Maga, A. Máthé, P. Mattila and B. Stenner ([8]) where it is proved that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$ is greater than $d-1$, then every angle $\theta \in [0, \pi]$ is in $\mathcal{A}(E)$. The authors also prove that if the Hausdorff dimension is greater than $\frac{d}{2}$, if d is even, and $\frac{d+1}{2}$ if d is odd, then the angle $\frac{\pi}{2}$ is in $\mathcal{A}(E)$. Furthermore, they demonstrate that the threshold $d-1$ is best possible for $\theta = \pi$.

In this paper we show that the Hausdorff dimensional threshold $\frac{d+1}{2}$ is sufficient to ensure that $\mathcal{A}(E)$ has positive Lebesgue measure. This will follow from the fact that in this regime, no angle is overrepresented.

Definition 1.2. Let $E \subset \mathbb{R}^d$, $d \geq 2$. We say that an angle $\alpha \in [0, \pi]$ is *equitably represented* in $\mathcal{A}(E)$ if for every Frostman measure μ supported on E and any $\epsilon > 0$,

$$\mu \times \mu \times \mu\{(x, y, z) : \alpha - \epsilon \leq \theta(x, y, z) \leq \alpha + \epsilon\} \lesssim \epsilon. \tag{1.1}$$

Here and throughout, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$.

Recall that a probability measure μ on a compact set $E \subset \mathbb{R}^d$ is a *Frostman measure* if, for any ball B_δ of radius δ,

$$\mu(B_\delta) \lesssim \delta^s, \tag{1.2}$$

where $s < dim_{\mathcal{H}}(E))$. For discussion and proof of the existence of such measures see, e.g., [14].

Our main result is the following.

Theorem 1.3. *Let E be a compact subset of $\mathbb{R}^d$ of Hausdorff dimension greater than $\frac{d+1}{2}$. Then every $\alpha \in (0, \pi)$ is equitably represented in $\mathcal{A}(E)$.*

Remark 1.4. We note that the implicit constant in (1.1) that we obtain depends only on the dimension d and on $I_s(\mu)$ for some $s \in (\frac{d+1}{2}, dim_{\mathcal{H}}(E))$, where

$$I_s(\mu) = \iint |x-y|^{-s} d\mu(x) d\mu(y).$$

Corollary 1.5. *Let E be a compact subset of $\mathbb{R}^d$ of Hausdorff dimension greater than $\frac{d+1}{2}$. Then the Lebesgue measure of $\mathcal{A}(E)$ is positive.*

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1.2. Sharpness of results

We show that a necessary lower bound on the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is $\frac{d}{2}$ in order to ensure that every angle is not overrepresented. Thus it is an open question whether our bound $\frac{d+1}{2}$ from the main theorem can be improved or whether an example can be found that shows the sharpness of our result.

Theorem 1.6. *For every $d \geq 2$ and $s \in (0, \frac{d}{2})$, there exists $E \subset \mathbb{R}^d$ of Hausdorff dimension s such that $\frac{\pi}{2}$ is not equitably represented in $\mathcal{A}(E)$.*

The main ingredient in the proof is the following generalization to $\mathbb{R}^d$ of a theorem by Apfelbaum and Sharir in [1], which they state in $\mathbb{R}^3$.

Theorem 1.7. *Let $P_n = \{1, \ldots, \lfloor n^{1/d} \rfloor\}^d$. Then the number of triplets $(p, q, r) \in P_n^3$ such that $\angle pqr = \pi/2$ is $\Omega(n^{3-\frac{2}{d}})$. Here and throughout, $X = \Omega(Y)$ with the controlling parameter n means that there exists $c > 0$ such that $X \geq cY$ with c independent of n.*

Remark 1.8. We do not know if a more efficient construction is possible. We also do not know if a version of Theorem 1.6 exists for angles other than $\frac{\pi}{2}$ in the same range of exponents. Similarly, it is not known if a version of Theorem 1.7 exists for other angles. See [1] for a detailed discussion of this issue.

We also have a necessity result with respect to the positive Lebesgue measure of $\mathcal{A}(E)$, the set of angles. Since E may be a subset of a line, we immediately see that in order to ensure that the Lebesgue measure of $\mathcal{A}(E)$ is positive, we must assume that the Hausdorff dimension of E is greater than one.

1.3. Applications to discrete geometry

The following results were obtained by Pach and Sharir, in [16], and Apfelbaum and Sharir, in [1]. In [16], it is shown that for a set of n points in $\mathbb{R}^2$, no angle can occur more than $cn^2 \log n$ times. Since there are about n^3 triples of points, this implies that there must be at least $c\frac{n}{\log n}$ distinct angles. In [1], it is shown that for a set of n points in $\mathbb{R}^3$, no angle can occur more than $cn^{\frac{7}{3}}$ times, which gives a lower bound of at least $cn^{\frac{2}{3}}$ distinct angles. They also show that for a set of n points in $\mathbb{R}^4$, no angle besides $\frac{\pi}{2}$ can occur more than $cn^{\frac{5}{2}}\beta(n)$ times, where $\beta(n)$ grows extremely slowly with respect to n. This means that there must be about $n^{\frac{1}{2}}(\beta(n))^{-1}$ distinct angles.

In dimensions four and higher, no results are currently available. In order to describe our main result in this direction, we need the following definition.

Definition 1.9. Let P be a set of n points contained in $[0,1]^d$, $d \geq 2$. Define the measure

$$d\mu_P^s(x) = n^{-1} \cdot n^{\frac{d}{s}} \cdot \sum_{p \in P} \chi_{B(p, n^{-\frac{1}{s}})}(x)dx,$$

where $B(p, n^{-\frac{1}{s}})$ denotes the Euclidean ball centered at p of radius $n^{-\frac{1}{s}}$ and $\chi_{B(p,n^{-\frac{1}{s}})}(x)$ is the characteristic function on that ball.

We say that P is <u>s-adaptable</u> if P is $n^{-\frac{1}{s}}$-separated and

$$I_s(\mu_P) = \iint |x-y|^{-s} d\mu_P^s(x) d\mu_P^s(y) < \infty.$$

This is equivalent to the statement

$$n^{-2} \sum_{p \neq p' \in P} |p - p'|^{-s} \lesssim 1.$$

To put it simply, s-adaptability means that a discrete point set P can be thickened into a set which is uniformly s-dimensional in the sense that its energy integral of order s is finite. Unfortunately, it is shown in [12] that there exist finite point sets which are not s-adaptable for certain ranges of the parameter s. However, many commonly used classes of discrete sets, such as homogeneous sets, studied, for example, by Solymosi and Vu, are indeed s-adaptable for $0 \leq s \leq d$. See [12] for a detailed description of these issues. Our main discrete geometric result is the following.

Theorem 1.10. *Let $P_1, P_2, P_3 \subset \mathbb{R}^d$, $\#P_i = N$, $i = 1,2,3$, $d \geq 2$, be s-adaptable sets for $s > \frac{d+1}{2}$. Further assume there exists a constant C such that $\min\{|x_i - x_j| : x_i \in P_i, x_j \in P_j, i \neq j\} \geq C > 0$, so in other words the sets P_1, P_2, P_3 are C separated. Then*

$$\#\{(x, y, z) \in P_1 \times P_2 \times P_3 : \theta(x, y, z) = \theta_0\} \lesssim N^{3-\frac{1}{s}}.$$

In dimensions two and three, these exponents are not as good as the aforementioned results of Apfelbaum and Sharir and Pach and Sharir. However, Theorem 1.10 gives non-trivial exponents in all dimensions. It should be noted however that our result is proved under an additional assumption that the sets P_1, P_2, P_3 are s-adaptable and separated by a fixed threshold.

This theorem implies that the number of distinct angles is at least $N^{\frac{1}{s}}$. That result extends to the case of looking at angles in an s-adaptable set P with N points. Using the pigeonhole theorem one can select three subsets with a number of points of the order of N each, that are separated by a constant distance. Applying the above theorem to those subsets then shows that the number of distinct angles in P must be at least $N^{\frac{1}{s}}$.

2. Proof of Corollary 1.5

Consider

$$\frac{1}{\epsilon}\mu \times \mu \times \mu \left\{ (x, y, z) : t - \epsilon \leq \frac{x - y}{|x - y|} \cdot \frac{y - z}{|y - z|} \leq t + \epsilon \right\},$$

where μ is a Frostman measure on E. Theorem 1.3 states precisely that

$$\mu \times \mu \times \mu \left\{ (x, y, z) : t - \epsilon \leq \frac{x - y}{|x - y|} \cdot \frac{y - z}{|y - z|} \leq t + \epsilon \right\} \lesssim \epsilon. \tag{2.1}$$

First pigeon hole the set E such that x, y, z can not lie on a single line. Then cover $\mathcal{A}(E)$ by $\cup_i (t_i - \epsilon_i, t_i + \epsilon_i)$. It follows that

$$\begin{aligned} 1 &= \mu \times \mu \times \mu\{E \times E \times E\} \\ &\leq \sum_i \mu \times \mu \times \mu \left\{ (x, y, z) : t_i - \epsilon_i \leq \frac{x - y}{|x - y|} \cdot \frac{y - z}{|y - z|} \leq t_i + \epsilon_i \right\} \\ &\lesssim \sum_i \epsilon_i, \end{aligned}$$

where the last inequality follows by (2.1). We conclude that $\sum_i \epsilon_i \gtrsim 1$, which implies that the Lebesgue measure of $\mathcal{A}(E)$ is positive. $\square$

3. Preliminary estimates needed for the proof of theorem 1.3

The key objects we need to study are generalized Radon transforms. Given $f : \mathbb{R}^d \to \mathbb{R}$, $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $t \in \mathbb{R}$, define

$$T_{\phi_t} f(x) := \int_{\{\phi(x,y)=t\}} f(y)\psi(x,y)d\sigma_{x,t}(y),$$

where $d\sigma_{x,t}$ is the Lebesgue measure on the set $\{y : \phi(x,y) = t\}$ and ψ is a smooth cut-off function. We further require

$$\nabla_x \phi(x,y) \quad \text{and} \quad \nabla_y \phi(x,y) \tag{3.1}$$

form two linearly independent vectors in $\mathbb{R}^d$ in a neighborhood of the sets

$$\{x : \phi(x,y) = t\} \quad \text{and} \quad \{y : \phi(x,y) = t\}, \tag{3.2}$$

respectively; this can be justified by details in the note of Phong and Stein [17] and is meant to provided an underlying smooth structure. We call T_{ϕ_t} the Radon transform associated to ϕ.

The following definition is stated in [17].

Definition 3.1. We say that $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies the *Phong-Stein rotational curvature condition* at t if

$$det \begin{pmatrix} 0 & \nabla_x \phi \\ -(\nabla_y \phi)^T & \frac{\partial^2 \phi}{dx_i dy_j} \end{pmatrix} \neq 0 \tag{3.3}$$

on the set $\{(x,y) : \phi(x,y) = t\}$.

Under this condition we have the following Lemma.

Lemma 3.2. *Suppose $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies the Phong-Stein rotational curvature condition. Then*

$$T_{\phi_t} : L^2(\mathbb{R}^d) \to L^2_{\frac{d-1}{2}}.$$

This follows, for example, from the main result in [17]. See also [10] and [18] for a thorough description of related estimates.

Remark 3.3. Here and throughout $L^2_\alpha(\mathbb{R}^d)$ denotes the Sobolev space with index α where

$$\|f\|^2_{L^2_\alpha(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |\xi|^{2\alpha} d\xi$$

and

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx,$$

the Fourier transform of $f : \mathbb{R}^d \to \mathbb{R}$.

Proposition 3.4. *Let μ be a Frostman measure on a compact set E and $T_{\phi_t} : L^2(\mathbb{R}^d) \to L^2_s(\mathbb{R}^d)$ with $d - s < \alpha < d$, for $\alpha = dim_{\mathcal{H}}(E)$, the Hausdorff dimension of E. Then*

$$\mu \times \mu\{(x, y) \in E \times E : t - \varepsilon \leq \phi(x, y) \leq t + \varepsilon\} \lesssim \varepsilon. \tag{3.4}$$

We prove this result in Section 6.

4. Proof of Theorem 1.3

For a fixed angle θ recall that if $x, y, z \in \mathbb{R}^d$ form the angle θ centered at y then by the law of cosines we have

$$|x - z|^2 = |x - y|^2 + |y - z|^2 - 2|x - y||y - z| \cos(\theta)$$

which can be rewritten as

$$(y - x) \cdot (y - z) - |x - y||y - z| \cos(\theta) = 0.$$

Define

$$\phi_\theta(x, y) = (y - x) \cdot (y - z) - |x - y||y - z| \cos(\theta)$$

where we have suppressed the dependence on z. We have the following key Lemma.

Lemma 4.1. *Suppose z is fixed. Then ϕ_θ satisfies the Phong-Stein rotational curvature condition at $t = 0$ as long as θ is not a multiple of π.*

By Lemma 3.2 we get that $T_{\phi_\theta} : L^2(\mathbb{R}^d) \to L^2_{\frac{d-1}{2}}$ and then by Proposition 3.4 we have that for all z and θ away from an angle that is a multiple of π we get

$$\mu \times \mu\{(x,y) \in E \times E : |\phi_\theta(x,y)| \le \varepsilon\} \lesssim \varepsilon.$$

Since μ is a probability measure we conclude as long as θ is not a multiple of π that

$$\mu \times \mu \times \mu\{(x,y,z) \in E^3 : |(y-x)\cdot(y-z) - |x-y||y-z|\cos(\theta)| \le \varepsilon\} \lesssim \varepsilon$$

which implies that θ is equitably represented in $\mathcal{A}(E)$.

Thus the proof of Theorem 1.3 has been reduced to proving Lemma 4.1 and this is where we now turn our attention. □

5. Proof of Lemma 4.1

Since we are showing that

$$\phi_\theta(x,y) = (y-x)\cdot(y-z) - |x-y||y-z|\cos(\theta)$$

satisfies the Phong-Stein rotational curvature condition at $t = 0$ we get to assume $\phi_\theta(x,y) = 0$. Without loss of generality we can assume that the pinned vector $z = 0$ and for ease of notation let's write $\beta = \cos(\theta)$. We then have to study

$$f(x,y) = (y-x)\cdot y - \beta|y-x||y|$$

where we can assume $f(x,y) = 0$. The last simplification is that we do a change of variables $u = y - x$, $v = y$ by defining

$$g(u,v) := u\cdot v - \beta|u||v|$$

and noting that $f(x,y) = g(y-x, y)$. We write out the calculations for the Phong-Stein rotational curvature condition in $\mathbb{R}^2$ as the higher dimensional cases follow similarly. First write $x = (x_1, x_2)$, $y = (y_1, y_2)$, $u = (u_1, u_2)$ and $v = (v_1, v_2)$ and then switch u and v to polar coordinates by writing $u = (r\cos(\omega), r\sin(\omega))$ and $v = (s\cos(\tau), s\sin(\tau))$. The determinant we have to calculate is

$$D = \begin{vmatrix} 0 & f_{x_1} & f_{x_2} \\ -f_{y_1} & f_{x_1y_1} & f_{x_2y_1} \\ -f_{y_2} & f_{x_1y_2} & f_{x_2y_2} \end{vmatrix}$$

where in polar coordinates we have

$$
\begin{aligned}
f_{x_1} &= -g_{u_1} = -s(\cos(\tau) - \beta\cos(\omega)) \\
f_{x_2} &= -g_{u_2} = -s(\sin(\tau) - \beta\sin(\omega)) \\
f_{y_1} &= g_{u_1} + g_{v_1} = s(\cos(\tau) - \beta\cos(\omega)) + r(\cos(\omega) - \beta\cos(\tau)) \\
f_{y_2} &= g_{u_2} + g_{v_2} = s(\sin(\tau) - \beta\sin(\omega)) + r(\sin(\omega) - \beta\sin(\tau)) \\
f_{x_1y_1} &= -g_{u_1u_1} - g_{u_1v_1} = \frac{\beta s}{r}\sin^2(\omega) - 1 + \beta\cos(\omega)\cos(\tau) \\
f_{x_2y_1} &= -g_{u_2u_1} - g_{u_2y_1} = -\frac{\beta s}{r}\cos(\omega)\sin(\omega) + \beta\sin(\omega)\cos(\tau) \\
f_{x_1y_2} &= -g_{u_1u_2} - g_{u_1v_2} = -\frac{\beta s}{r}\cos(\omega)\sin(\omega) + \beta\cos(\omega)\sin(\tau) \\
f_{x_2y_2} &= -g_{u_2u_2} - g_{u_2v_2} = \frac{\beta s}{r}\cos^2(\omega) - 1 + \beta\sin(\omega)\sin(\tau).
\end{aligned}
$$

Note that the condition $f(x, y) = 0$ is equivalent to $g(u, v) = 0$ which is equivalent to

$$\beta = \frac{u \cdot v}{|u||v|} = \cos(\omega - \tau).$$

To simplify our calculations of the determinant we can, without loss of generality, assume that $r = 1$. The result of the determinant is a degree 3 polynomial in s. We will now calculate the coefficients of that polynomial. For the constant coefficient note that if you set $s = 0$ in the matrix the top row is a zero row so the constant coefficient has to be 0.

To find the coefficient in front of s note that the middle and far right elements in the top row are both multiples of s so the only way we get s terms is if they are multiplied by terms in lower rows that are not multiples of s. This leads to the following coefficient

$$
\begin{aligned}
&\begin{vmatrix} 0 & \beta\cos(\omega) - \cos(\tau) & \beta\sin(\omega) - \sin(\tau) \\ \beta\cos(\tau) - \cos(\omega) & \beta\cos(\omega)\cos(\tau) - 1 & \beta\sin(\omega)\cos(\tau) \\ \beta\sin(\tau) - \sin(\omega) & \beta\cos(\omega)\sin(\tau) & \beta\sin(\omega)\sin(\tau) - 1 \end{vmatrix} \\
=\; & -(\beta\cos(\omega) - \cos(\tau))\left(-\beta\cos(\tau)\cos^2(\omega) - \beta\cos(\omega)\sin(\omega)\sin(\tau) + \cos(\omega)\right) \\
& + (\beta\sin(\omega) - \sin(\tau))\left(\beta\sin(\tau)\sin^2(\omega) + \beta\sin(\omega)\cos(\omega)\cos(\tau) - \sin(\omega)\right) \\
=\; & (\cos(\tau) - \beta\cos(\omega))(1 - \beta\cos(\tau - \omega))\cos(\omega) \\
& + (\beta\sin(\omega) - \sin(\tau))(\beta\cos(\tau - \omega) - 1)\sin(\omega) \\
=\; & (1 - \beta\cos(\tau - \omega))\left(\cos(\tau)\cos(\omega) - \beta\cos^2(\omega) - \beta\sin^2(\omega) + \sin(\omega)\sin(\tau)\right) \\
=\; & (1 - \beta\cos(\tau - \omega))(\cos(\tau - \omega) - \beta) \\
=\; & 0
\end{aligned}
$$

where in the last step we use that $\beta = \cos(\omega - \tau)$. This shows that the coefficient in front of s is also 0.

To find the coefficient in front of s^2 we go through all the possible ways that the non-zero elements in the top row get multiplied with exactly one s-term as we expand out the determinant. This leads to the following expression.

$$
\begin{aligned}
&(\cos(\tau) - \beta\cos(\omega))\times\\
&\Big(\beta\cos^2(\omega)(\beta\cos(\tau) - \cos(\omega)) + (\beta\cos(\omega) - \cos(\tau))(\beta(\sin(\omega)\sin(\tau) - 1)\\
&+ \beta\cos(\omega)\sin(\omega)(\beta\sin(\tau) - \sin(\omega)) + \beta\sin(\omega)\cos(\tau)(\sin(\tau) - \beta\sin(\omega))\Big)\\
&(\beta\sin(\omega) - \sin(\tau))\times\\
&\Big(\beta\cos(\omega)\sin(\tau)(\beta\cos(\omega) - \cos(\tau)) + \beta\cos(\omega)\sin(\omega)(\cos(\omega) - \beta\cos(\tau))\\
&+ \beta\sin^2(\omega)(\sin(\omega) - \beta\sin(\tau)) + (\sin(\tau) - \beta\sin(\omega))(\beta\cos(\omega)\cos(\tau) - 1)\Big)\\
&= (\cos(\tau) - \beta\cos(\omega))\Big(\beta^2\cos(2\omega - \tau) - 2\beta\cos(\theta) + \cos(\tau)\Big)\\
&+ (\beta\sin(\omega) - \sin(\tau))\Big(\beta^2\sin(\tau - 2\omega) + 2\beta\sin(\omega) - \sin(\tau)\Big)\\
&= -\beta^3(\cos(\omega)\cos(2\omega - \tau) + \sin(\omega)\sin(2\omega - \tau))\\
&+ \beta^2(\cos(\tau)\cos(2\omega - \tau) + 2\cos^2(\omega) + \sin(\tau)\sin(2\omega - \tau) + 2\sin(\omega))\\
&+ \beta(-2\cos(\omega)\cos(\tau) - \cos(\omega)\cos(\tau) - \sin(\omega)\sin(\tau) - 2\sin(\omega)\sin(\tau)) + 1\\
&= -\beta^3\cos(\omega - \tau) + \beta^2(\cos(2\omega - 2\tau) + 2) - 3\beta\cos(\omega - \tau) + 1\\
&= -\beta^3\cos(\omega - \tau) + \beta^2(2\cos^2(\omega - \tau) + 1) - 3\beta\cos(\omega - \tau) + 1
\end{aligned}
$$

Substituting in $\beta = \cos(\omega - \tau)$ yields

$$
\begin{aligned}
&= \cos^4(\omega - \tau) - 2\cos^2(\omega - \tau) + 1\\
&= (1 - \cos^2(\omega - \tau))^2\\
&= \sin^4(\omega - \tau)
\end{aligned}
$$

so we conclude that the coefficient in front of s^2 is $sin^4(\omega - \tau)$.

Finally to find the coefficient in front of s^3 we have to pick up a multiple of s from each row which leads to the following coefficient

$$\begin{aligned}
&\begin{vmatrix} 0 & \beta\cos(\omega)-\cos(\tau) & \beta\sin(\omega)-\sin(\tau) \\ \beta\cos(\omega)-\cos(\tau) & \beta\sin^2(\omega) & -\beta\cos(\omega)\sin(\omega) \\ \beta\sin(\omega)-\sin(\tau) & -\beta\cos(\omega)\sin(\omega) & \beta\cos^2(\omega) \end{vmatrix} \\
&= -(\beta\cos(\omega)-\cos(\tau))\big(\beta^2\cos(\omega)-\cos(\omega)\cos(\omega-\tau)\big) \\
&\quad + (\beta\sin(\omega)-\sin(\tau))\big(\beta\sin(\omega)\cos(\omega-\tau)-\beta^2\sin(\omega)\big) \\
&= \beta(0+0) \\
&= 0
\end{aligned}$$

so we conclude that the coefficient in front of s^3 is 0.

To conclude, we have computed the determinant D under the assumption $r=1$ and found

$$D = \sin^4(\omega-\tau)s^2.$$

This allows us to conclude that if the determinant is 0 then we must have that $\sin(\omega-\tau)=0$, which implies that $\beta=1$, which only happens if our original angle θ is a multiple of π. This concludes the proof. $\square$

6. Proof of Proposition 3.4

Take Schwartz the class functions $\eta_0(\xi)$ supported in the ball $\{|\xi|\le 4\}$ and $\eta(\xi)$ supported in the annulus

$$\{1<|\xi|<4\} \text{ with } \eta_j(\xi)=\eta(2^{-j}\xi) \text{ for } j\ge 1$$

with

$$\eta_0(\xi)+\sum_{j=1}^{\infty}\eta_j(\xi)=1.$$

Define the Littlewood-Paley piece of μ_j by the relation

$$\widehat{\mu}_j(\xi)=\widehat{\mu}(\xi)\eta_j(\xi).$$

Consider the left hand side of (3.4). This can be rewritten as

$$\sum_{j,k}\int\int_{\{t-\varepsilon\le\phi(x,y)\le t+\varepsilon\}}\psi(x,y)d\mu_j(x)\,d\mu_k(y)=\sum_{j,k}\langle\mu_j,T^{\varepsilon}\mu_k\rangle \tag{6.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\mathbb{R}^d)$ inner product and

$$T^{\varepsilon}\mu_k(x) = \int_{\{t-\varepsilon \le \phi(x,y) \le t+\varepsilon\}} \psi(x,y) d\mu_k(y)$$

$$= \int_{t-\varepsilon}^{t+\varepsilon} \int_{\phi(x,y)=r} \psi(x,y)\mu_k(y) d\sigma_{x,r}(y) dr, \tag{6.2}$$

where $d\sigma_{x,r}$ is the Lebesgue measure on the set $\{y : \phi(x,y) = r\}$. It should be noted that the innermost integral on the right side of (6.2) is just T_{ϕ_r} applied to μ_k. It follows that the right hand side of (6.1) becomes

$$\int_{t-\varepsilon}^{t+\epsilon} \sum_{j,k} \langle \mu_j, T_{\phi_r}(\mu_k) \rangle \, dr.$$

We will now use the mapping properties of T_{ϕ_r} and the fact that μ is a Frostman measure to prove $\langle \mu, T_{\phi_r} \mu \rangle$ is uniformly bounded in r over the the domain of integration. This, in turn, will prove our desired theorem.

We have

$$\langle \mu, T_{\phi_r}(\mu) \rangle = \sum_{j,k} \langle \mu_j, T_{\phi_r}(\mu_k) \rangle$$

$$= \sum_{|j-k| \le K} \langle \mu_j, T_{\phi_r}(\mu_k) \rangle + \sum_{|j-k| > K} \langle \mu_j, T_{\phi_r}(\mu_k) \rangle \tag{6.3}$$

for K large enough; the choice of K will be justified later. We will estimate each of the above sums separately. We will assume both $j \ge 0$ and $k \ge 0$; at the end of the proof, we will prove the estimate for negative values.

For the first sum in equation 6.3

$$\sum_{|j-k| \le K} \langle \mu_j, T_{\phi_r}(\mu_k) \rangle = \sum_{|j-k| \le K} \left\langle \widehat{\mu_j}, \widehat{T_{\phi_r}(\mu_k)} \right\rangle$$

$$\sim \sum_{|j-k| \le K} \left\langle \widehat{\mu_j}, \widehat{T_{\phi_r}(\mu_k)} \eta_j \right\rangle$$

$$\lesssim \sum_{|j-k| \le K} \|\mu_j\|_2 \, \|\widehat{T_{\phi_r}(\mu_k)} \eta_j\|_2$$

where we use the Cauchy-Schwartz inequality and $\eta_j \sim \eta_j^2$.

Our first observation is that since μ is a Frostman measure on a set of Hausdorff dimension α, we have that

$$\|\mu_j\|_2 \lesssim 2^{\frac{j(d-\alpha)}{2}}. \tag{6.4}$$

This we see by writing

$$\begin{aligned}\|\mu_j\|_2^2 &= \int |\widehat{\mu}(\xi)|^2 \eta(2^{-j}\xi) d\xi \\ &= \iiint e^{2\pi i(x-y)\cdot\xi} \eta(2^{-j}\xi) d\xi d\mu(x) d\mu(y) \\ &\quad 2^{dj} \iint \widehat{\eta}(2^j(x-y)) d\mu(x) d\mu(y)\end{aligned}$$

and then noting that the absolute value of this quantity is bounded, for every $N > 0$, by

$$\begin{aligned}&C_N 2^{dj} \iint (1 + 2^j|x-y|)^{-N} d\mu(x) d\mu(y) \\ &= C_N 2^{dj} \iint_{|x-y|\le 2^{-j}} (1 + 2^j|x-y|)^{-n} d\mu(x) d\mu(y) \\ &\quad + C_N 2^{dj} \sum_{l=0}^{\infty} \iint_{2^l \le 2^j|x-y| \le 2^{l+1}} (1 + 2^j|x-y|)^{-n} d\mu(x) d\mu(y) \\ &= I + II.\end{aligned}$$

Since μ is a Frostman measure

$$I \lesssim C_N 2^{dj} 2^{-j\alpha}.$$

Since μ is compactly supported, there exists $M > 0$ such that

$$II = C_N 2^{dj} \sum_{l=0}^{j+M} \iint_{2^l \le 2^j|x-y| \le 2^{l+1}} (1 + 2^j|x-y|)^{-n} d\mu(x) d\mu(y)$$

which again, since μ is a Frostman measure, is bounded by

$$C_N 2^{dj} \sum_{l=0}^{j+M} 2^{-ja} 2^{la} 2^{-lN} \lesssim C_N 2^{j(d-\alpha)}.$$

In conclusion $I + II \lesssim 2^{j(d-\alpha)}$ and equation 6.4 is established.

Our second observation is that

$$\|\widehat{T_{\phi_r}(\mu_k)}\eta_j\|_2 \lesssim 2^{-ks}2^{\frac{k(d-\alpha)}{2}} \tag{6.5}$$

by the mapping properties of the operator T_{ϕ_r} in the regime of $|j-k|<K$.

Combined, equations 6.4 and 6.5 allow us to estimate the first sum in equation 6.3 by

$$\sum_{|j-k|\le K} \langle \mu_j, T_{\phi_r}(\mu_k)\rangle \lesssim \sum_{|j-k|\le K} 2^{j\frac{d-\alpha}{2}} 2^{k\frac{d-\alpha}{2}} 2^{-ks} \lesssim 1$$

provided that $d-s<\alpha<d$.

In order to similarly obtain uniform bounds on the second sum in equation 6.3 we need the following lemma which is a variant of a calculation in [11]. It immediately yields

$$\sum_{|j-k|>K} \langle \mu_j, T_{\phi_r}(\mu_k)\rangle \lesssim \sum_{|j-k|>K} C_M 2^{-M\max\{j,k\}} \lesssim 1.$$

Lemma 6.1. *For any $M>2d+2$ there exists a constant $C_M>0$ such that for all indices j,k with $|j-k|>K$ with K large enough,*

$$\langle T_{\phi_r}(\mu_j), \mu_k\rangle \le C_M 2^{-M\max\{j,k\}}.$$

To prove the lemma, for simplicity, we replace T_{ϕ_r} by T and write

$$T\mu_k = \int_{\{y:\phi(x,y)=r\}} \psi(x,y)\mu_k(y)d\sigma_{x,r}(y),$$

where $d\sigma_{x,r}$ is the Lebesgue measure on the set $\{y:\phi(x,y)=r\}$. It follows from our upcoming arguments that as long as $t-\varepsilon\le r\le t+\varepsilon$, the estimates hold uniformly in r.

As ϕ satisfies the linear independence condition from (3.1) on a relatively open, bounded subset of $\{y:\phi(x,y)=t\}$ we can assume that $|\nabla_y\phi(x,y)|\approx 1$ on this set by making the support of ψ small enough. Next, we use an approximation argument on T by letting

$$T_n\mu_k(x) = n\int_{\mathbb{R}^d} \psi(x,y)\chi(n(\phi(x,y)-r))\mu_k(y)dy$$

where χ is a smooth cutoff supported near 0 and equal to 1 near 0. It is shown in [5] that

$$n\chi(n(\phi(x,y)-r))dy$$

converges to the measure that appears in T_{ϕ_r} as $n \to \infty$. Therefore, proving the estimate in the case where T_{ϕ_r} is replaced by T_n is sufficient by convergence theorems found in [4] which in turn shows the uniformity in r. We will drop the domains of integration in the upcoming calculations for brevity.

By Fourier inversion, we have

$$T_n\mu(x) = \int e^{iy\cdot\xi} e^{is\cdot(\phi(x,y)-r)} \psi(x,y)\widehat{\chi}(n^{-1}s)\widehat{\mu}(\xi)d\xi ds dy$$

and therefore

$$\widehat{T_n\mu}(\eta) = \int e^{-ix\cdot\eta} e^{iy\cdot\xi} e^{is\cdot(\phi(x,y)-r)} \psi(x,y)\widehat{\chi}(n^{-1}s)\widehat{\mu}(\xi)dxdydsd\xi. \tag{6.6}$$

Invoking the properties of the Fourier transform on L^2, we see that

$$\begin{aligned}\langle T_n\mu_j, \mu_k\rangle &= \langle \widehat{T_n\mu_j}, \widehat{\mu_k}\rangle \\ &= \int e^{-ix\cdot\eta} e^{iy\cdot\xi} e^{is\cdot(\phi(x,y)-r)} \psi(x,y)\widehat{\chi}(n^{-1}s)\widehat{\mu_j}(\xi)\widehat{\mu_k}(\eta)dxdydsd\xi d\eta \\ &= \int \widehat{\mu_j}(\xi)\widehat{\mu_k}(\eta)\widehat{\chi}(n^{-1}s)I_{jk}(\xi,\eta,s)d\eta d\xi ds \qquad (6.7)\end{aligned}$$

where

$$I_{jk}(\xi,\eta,s) = \psi_0(2^{-j}|\xi|)\psi_0(2^{-k}|\eta|)\int e^{is\cdot(\phi(x,y)-r)} e^{iy\cdot\xi} e^{-ix\cdot\eta}\psi(x,y)dxdy \tag{6.8}$$

and ψ_0 is smooth cutoff equal to 1 on $\{1 \le |z| \le 10\}$ and vanishing in the ball of radius $1/2$. The justification of such cutoffs comes from the support of $\widehat{\mu_j}(\xi)$ and $\widehat{\mu_k}(\eta)$ and again that $\eta_j \approx \eta_j^2$. We will show that

$$|I_{jk}(\xi,\eta,s)| \le C_M 2^{-M\max\{j,k\}} \tag{6.9}$$

when $|j-k| > K$ for a large enough K.

Computing the critical points of the phase function in (6.8), we see that

$$s\nabla_x\phi(x,y) = \eta \quad \text{and} \quad s\nabla_y\phi(x,y) = -\xi,$$

The compact support of ψ along with the linear independence from (3.1) implies that

$$|\nabla_x\phi(x,y)| \approx |\nabla_y\phi(x,y)| \approx 1.$$

More precisely, the upper bound follows from smoothness and compact support. The lower bound follows from the fact that a continuous non-negative function achieves its minimum on a compact set. This minimum is not zero because of the linear independence condition (3.1).

It follows that

$$|\xi| \approx |\eta| \tag{6.10}$$

when we are near critical points in (x, y). The support of the cutoffs ψ_0, when $|j-k| > K$, tell us that we are supported away from critical points in (x, y) since (6.10) no longer holds. This condition implies that for some h or h' in $\{1, 2, ..., d\}$,

$$s\frac{\partial \phi}{\partial x_h} - \eta_h \neq 0 \quad \text{or} \quad s\frac{\partial \phi}{\partial y_{h'}} + \xi_{h'} \neq 0.$$

Without loss of generality, assume the former holds and that $k > j$. It is immediate that $e^{-ix\cdot\eta}e^{is\cdot(\phi(x,y)-r)}$ is an eigenfunction of the differential operator

$$L = \frac{1}{i(s\frac{\partial \phi}{\partial x_h} - \eta_h)}\frac{\partial}{\partial x_h}.$$

We will integrate by parts in (6.8) using this operator. The expression that we get after performing this procedure $M > 2d + 2$ times is

$$I(\xi, \eta, s) \lesssim \sup_{x,y}\left|s\frac{\partial \phi}{\partial x_h} - \eta_h\right|^{-M}.$$

Now, suppose that we are in the region $\{|s| << |\eta|\}$ (i.e $|s| \leq c|\eta|$ with a sufficiently small constant $c > 0$). Since $|s\nabla_x\phi| \approx |s|$ it follows, after possibly changing our initial choice of h, that

$$\left|s\frac{\partial \phi}{\partial x_h} - \eta_h\right| \gtrsim \left|\left|s\frac{\partial \phi}{\partial x_h}\right| - |\eta|\right| \approx |\eta|.$$

Similarly, if $\{|s| >> |\eta|\}$ then

$$\left|s\frac{\partial \phi}{\partial x_h} - \eta_h\right| \gtrsim \left|\left|s\frac{\partial \phi}{\partial x_h}\right| - |\eta|\right| \approx |s|.$$

In either region,

$$|I_{jk}(\xi, \eta, s)| \lesssim \sup(|s|, |\eta|)^{-M} \lesssim 2^{-Mk}.$$

Considering (6.7), the integrand $\widehat{\chi}(n^{-1}s)I_{jk}(\xi,\eta,s)$ is integrable in s as the first term is at most 1 and I_{jk} is bounded above by $|s|^{-M}$. Performing the remaining integrations and keeping in mind the support properties of $\widehat{\mu_j}$ and $\widehat{\mu_k}$, it follows that

$$\sum_{|j-k|>K} \langle \mu_j, T_{\phi_r}(\mu_k)\rangle \lesssim \sum_{|j-k|>K} C_M 2^{-(M-2d)\max\{j,k\}} \lesssim 1.$$

This completes the proof of Lemma 6.1.

We are now ready to prove the proposition. Since both sums in (6.3) are bounded by 1, up to constants, this implies that the left hand side of (6.1) is bounded above by ε.

As mentioned, we have assumed throughout the proof that both $j > 0$ and $k > 0$. We now show that the estimate holds in the case that either j or k ranges in the negative integers. From (6.6) along with the definitions of both μ_j and μ_k, we immediately get

$$\langle T_n\mu_j, \mu_k\rangle = \langle \widehat{T_n\mu_j}, \widehat{\mu_k}\rangle \lesssim 2^{kd}2^{jd}.$$

This gives the necessary decay in the case that both $j < 0$ and $k < 0$. For the case that $j < 0$ and $k > L$, we use (6.7) to get decay in j and (6.9) to get decay in k. For the case that $k < 0$ and $j > L$, we use (6.7) to get decay in k and (6.9) to get decay in j.

7. Proof of Theorem 1.6

Let E_n denote the $n^{-\frac{1}{s}}$-neighborhood of

$$P_n = \frac{1}{n^{\frac{1}{d}}}\left(\mathbb{Z}^d \cap [0, n^{1/d}]^d\right)$$

where $0 < s < d/2$. It is known that the Hausdorff dimension of

$$E = \bigcap_{k=K}^{\infty} E_{2^{d\cdot 2^k}},$$

where K is a non-negative integer that we can choose, is s and furthermore that it is Ahlfors-David regular. See, for example, [2], [3]. See also [14] and [19] for a thorough description of the background material pertaining to fractal geometry and its connections with harmonic analysis.

Let μ_s be the s-dimensional Hausdorff measure on E and take $\epsilon = n^{-\frac{1}{s}}$. In order to show that $\frac{\pi}{2}$ is not equitably represented in $\mathcal{A}(E)$ it is sufficient to establish

$$(7.1) \quad \mu_s \times \mu_s \times \mu_s \left\{ (x,y,z) \in \mathbb{R}^{3d} : -n^{-\frac{1}{s}} \leq \frac{x-y}{|x-y|} \cdot \frac{y-z}{|y-z|} \leq n^{-\frac{1}{s}} \right\} \gtrsim n^{-\frac{1}{s}}.$$

The left hand side of 7.1 can be bounded below by

$$(7.2) \quad \mu_s \times \mu_s \times \mu_s \left\{ (x,y,z) \in E_n^3 : -n^{-\frac{1}{s}} \leq \frac{x-y}{|x-y|} \cdot \frac{y-z}{|y-z|} \leq n^{-\frac{1}{s}} \right\}$$

for all n where we impose the extra condition that the arms of the angles are bigger than a small fixed constant $0 < c < 1$. It then follows that if we have $(x_0, y_0, z_0) \in P_n^3$ such that $\theta(x_0, y_0, z_0) = \frac{\pi}{2}$ then

$$B\left(x_0, \frac{c^2}{8d} n^{-\frac{1}{s}}\right) \times B\left(y_0, \frac{c^2}{8d} n^{-\frac{1}{s}}\right) \times B\left(z_0, \frac{c^2}{8d} n^{-\frac{1}{s}}\right)$$
$$\subseteq \left\{ (x,y,z) \in E_n^3 : -n^{-\frac{1}{s}} \leq \frac{x-y}{|x-y|} \cdot \frac{y-z}{|y-z|} \leq n^{-\frac{1}{s}} \right\}.$$

We note that It is crucial here that the arms of the angles are required to be bigger than an absolute constant c. If the arms are allowed to be too small then tiny perturbations of the arms will radically change the angle. For the estimates we use the fact that $n^{-\frac{1}{s}} \ll n^{-\frac{1}{d}}$, given n large enough, since $0 < s < \frac{d}{2}$, and that the smallest distance between any two distinct points in P_n^3 is $n^{-\frac{1}{d}}$. This last observation also tells us that the sets

$$B(x_0, n^{-\frac{1}{s}}) \times B(y_0, n^{-\frac{1}{s}}) \times B(z_0, n^{-\frac{1}{s}})$$

are disjoint for any two different choices of $(x_0, y_0, z_0) \in P_n^3$. Those sets are also finitely many so we see that we can bound 7.2 below by

$$(7.3) \quad \left(\inf_{x_0 \in P_n} \mu_s(B(x_0, n^{-\frac{1}{s}})) \right)^3 \cdot \begin{array}{l} \text{the number of right angles} \\ \text{with arms longer than } cn^{\frac{1}{d}} \text{ in } \mathcal{A}(P_n). \end{array}$$

Note that a priori that x_0 need not be in in E. However since these inequalities hold for all n we can choose to use $n = 2^{d \cdot 2^K}$, where we have chosen K to be large enough. It is clear by our construction that $P_{2^{d\cdot 2^K}} \subseteq P_{2^{d\cdot 2^k}}$ for all $k \geq K$ and thus by our construction of E we have $P_{2^{d\cdot 2^K}} \subseteq E$. Thus we can guarantee that the x_0 above is in E.

Recall that since E is Ahlfors-David regular we know there exists a constant C such that for all $x \in E$ and all $0 < r \leq 1$ we have

$$C^{-1}r^s \leq \mu_s(B(x,r)) \leq Cr^s.$$

Since we can make sure that x_0 above is in E then this in particular means that we can bound 7.3 below by

$$n^{-3} \cdot \text{ the number of right angles with arms longer than } cn^{\frac{1}{d}} \text{ in } \mathcal{A}(P_n).$$

Using Theorem 1.7 (scaling does not change number of right angles) and in particular the note at the end of the proof of Theorem 1.7 the above is bounded below by

$$n^{-3} \cdot n^{3-\frac{2}{d}}$$

and since $s < \frac{d}{2}$ we have

$$n^{-3} \cdot n^{3-\frac{2}{d}} > n^{-\frac{1}{s}} = \epsilon.$$

This shows that 7.1 holds true. □

8. Proof of Theorem 1.7

This is a relatively straight forward generalization of the argument of Apfelbaum and Sharir in [1]. Assume for simplicity that n is a d-th power and a multiple of 5 so that all the quantities in the proof are integers. For a fixed d then this assumption does not change the order of magnitude of the lower bound.

Recall that we write $f = O(g)$ if there exists an x_0 such that $f(x) \lesssim g(x)$ for all $x > x_0$. We write $f = \Omega(g)$ if there exists an x_0 such that $f(x) \gtrsim g(x)$ for all $x > x_0$. Finally we write $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

Let $Q = \{\frac{2}{5}n^{1/d} + 1, \ldots, \frac{3}{5}n^{1/d}\}^d$ be the middle $\frac{1}{5}n^{1/d} \times \frac{1}{5}n^{1/d} \times \cdots \times \frac{1}{5}n^{1/d}$ portion of P. We have $|Q| = \frac{n}{5^d} = \Theta(n)$. For each pair of points in Q, the square of the distance between them is an integer of magnitude at most $\frac{d}{5^d}n^{2/d}$. Hence there are at most $\frac{d}{5^d}n^{2/d} = O(n^{2/d})$ distinct distances between the points of Q. For every point $x \in Q$ we take the spheres centered at x and containing at least one point $p \in Q$. There are $O(n^{2/d})$ such spheres. Do this for all points in Q and let S denote the resulting set of such spheres. We thus have $|S| = O(n^{1+\frac{2}{d}})$. By choosing Q to be small enough in P we are guaranteed that for every point $p \in P$ on a sphere $\sigma \in S$, the point on σ, antipodal to p, is also in P.

For each $\sigma \in S$, let $m_\sigma = |P \cap \sigma|$ denote the number of lattice points on σ. We observe that $\sum_{\sigma\in S} m_\sigma \geq 2\binom{|Q|}{2} = \Omega(n^2)$, since in the sum we count every pair $p, p' \in Q$ exactly twice - once with p at the center of the sphere and p' on the sphere itself, and once the other way around. In a similar manner $\sum_{\sigma\in S} m_\sigma \leq |Q| \cdot |P| = O(n^2)$, so this sum is $\Theta(n^2)$. Let $\sigma \in S$ be one of the spheres and let $p, q, r \in \sigma \cap P$ be three distinct points such that p and r are antipodal points of σ. Then $\angle pqr = \pi/2$. There are $m_\sigma/2$ choices of an antipodal pair $p, r \in \sigma \cap P$ and $m_\sigma - 2$ choices of a third point q. This yields $m_\sigma(m_\sigma - 2)/2$ right angles on σ. The lower bound on the number of right angles in P is obtained by summing over all the spheres of S. Note that each pair of points can be antipodal on at most one sphere, hence every angle is counted only once. This gives a lower bound of

$$\frac{1}{2}\sum_{\sigma\in S} m_\sigma(m_\sigma - 2) \geq \frac{1}{2|S|}\left(\sum_{\sigma\in S} m_\sigma\right)^2 - \sum_{\sigma\in S} m_\sigma = \frac{1}{2|S|}\Theta(n^4) - \Theta(n^2),$$

where we have used the Cauchy-Schwarz inequality. Substituting $|S| = O(n^{1+\frac{2}{d}})$ in the inequality gives $\Omega(n^{3-\frac{2}{d}})$ right angles determined by the points of P.

If the additional condition that the lengths of the arms of the right angles are at least of length $cn^{\frac{1}{d}}$ where c is a small constant then the entire construction still goes through with only a loss in the constant of the lower bound. □

9. Proof of Theorem 1.10

Since P_i is s-adaptable we can thicken it into a set E_i, which is uniformly s-dimensional for $i = 1, 2, 3$. Let $(x_0, y_0, z_0) \in P_1 \times P_2 \times P_3$ such that $\theta(x_0, y_0, z_0) = \theta_0$. Then straightforward estimates show that if you pick $\tilde{C} = \min\left\{1, \frac{C^2}{8d}\right\}$ then

$$B\left(x_0, \tilde{C}N^{-\frac{1}{s}}\right) \times B\left(y_0, \tilde{C}N^{-\frac{1}{s}}\right) \times B\left(z_0, \tilde{C}N^{-\frac{1}{s}}\right)$$

is contained in

$$\{(x, y, z) \in E_1 \times E_2 \times E_3 : \theta_0 - N^{-\frac{1}{s}} \leq \theta(x, y, z) \leq \theta_0 + N^{-\frac{1}{s}}\}.$$

Furthermore, since P_i is $N^{-\frac{1}{s}}$ separated for $i = 1, 2, 3$ then two such sets, for two different $(x_0, y_0, z_0) \in P_1 \times P_2 \times P_3$, are disjoint. Finally note that

$$\mu_P^s \left(B(x_0, \tilde{C}N^{-\frac{1}{s}}) \right) \leq N^{-1}.$$

Thus we can bound

$$\#\{(x, y, z) \in P_1 \times P_2 \times P_3 : \theta(x, y, z) = \theta_0\}$$

above by

$$N^3 \cdot \mu_P^s \times \mu_P^s \times \mu_P^s \Big\{ (x, y, z) \in E_1 \times E_2 \times E_3 : \\ \theta_0 - N^{-\frac{1}{s}} \leq \theta(x, y, z) \leq \theta_0 + N^{-\frac{1}{s}} \Big\}.$$

It is clear from the proof of Theorem 1.3 that it extends to the case where the points that form the angle come from distinct sets. Thus from that theorem we know that

$$\mu_P^s \times \mu_P^s \times \mu_P^s \Big\{ (x, y, z) \in E_1 \times E_2 \times E_3 : \\ \theta_0 - N^{-\frac{1}{s}} \leq \theta(x, y, z) \leq \theta_0 + N^{-\frac{1}{s}} \Big\} \lesssim N^{-\frac{1}{s}}.$$

Thus we have shown

$$\#\{(x, y, z) \in P_1 \times P_2 \times P_3 : \theta(x, y, z) = \theta_0\} \lesssim N^{3-\frac{1}{s}}$$

as claimed. □

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