Math. Res. Lett.
Volume 23, Number 6, 1645–1663, 2016

Ideals of regular functions of a quaternionic variable

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In this paper we prove that, for any $n \in \mathbb{N}$, the ideal generated by n slice regular functions $f_1, \ldots, f_n$ having no common zeroes coincides with the entire ring of slice regular functions. The proof required the study of the non-commutative syzygies of a vector of regular functions, that manifest a different character when compared with their complex counterparts.

1. Introduction

The theory of slice regular functions of a quaternionic variable (often simply called regular functions) has been introduced in [13], [14], and further developed in a series of papers, including in particular [3], where most of the recent developments are discussed. The full theory is presented in the monograph [12], while an extension of the theory to the case of real alternative algebras is discussed in [15], [16] and [17]. The theory of regular functions has been applied to the study of a non-commutative functional calculus, (see for example the monograph [6] and the references therein) and to address the problem of the construction and classification of orthogonal complex structures in open subsets of the space $\mathbb{H}$ of quaternions (see [10]). In many cases, the results one obtains in the theory of regular functions are inspired by complex analysis, though they often require essential modifications, due to the different nature of zeroes and singularities of regular functions. Examples of this latter kind of results include those on power and Laurent series expansions, and can be found in the monograph [12]. Some recent results of geometric theory of regular functions, not included in this monograph, appear in [5], [8], [11].

2010 Mathematics Subject Classification: 30G35.
Key words and phrases: Functions of a quaternionic variable, Ideals of regular functions.

In this paper we study the ideals in the (non-commutative) ring of regular functions, and we prove an analogue of a classical result for one (and several) complex variables, namely the fact that if a family of holomorphic functions has no common zeroes, then it generates the entire ring of holomorphic functions. In her doctoral dissertation [21], the author proved that this was the case for regular functions as well (in fact, she showed that this was true for bounded regular functions, an analogue of the corona theorem), under the strong hypothesis that not only the functions could not have common zeroes, but also that the functions could not have zeroes on the same spheres.

Here we show that such a request is not necessary, at least for the case of regular functions (we do not consider the bounded case), by employing some delicate local properties of such functions. We show how to reduce the study of the problem to the case of holomorphic functions, and we then use the coherence of the sheaf of holomorphic functions to show that the local solution to the problem extends to a global one.

As for the state of the art in the study of the corona problem in different contexts, we refer the reader to the recent, and rather exhaustive, volume [9], whose first chapter presents a short history of the problem itself. The papers [1], [22], [24], contain significative descriptions of ideals of holomorphic functions, in connection with their maximality. Sheaves of slice regular functions are introduced in [7].

2. Preliminary Results

Let $\mathbb{H}$ denote the non commutative real algebra of quaternions with standard basis $\{1, i, j, k\}$. The elements of the basis satisfy the multiplication rules

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik,$$

which, if we set 1 as the neutral element, extend by distributivity to all $q = x_0 + x_1 i + x_2 j + x_3 k$ in $\mathbb{H}$. Every element of this form is composed by the *real* part $\mathrm{Re}(q) = x_0$ and the *imaginary* part $\mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$. The *conjugate* of $q \in \mathbb{H}$ is then $\bar{q} = \mathrm{Re}(q) - \mathrm{Im}(q)$ and its *modulus* is defined as $|q|^2 = q\bar{q}$. We can therefore calculate the multiplicative inverse of each $q \neq 0$ as $q^{-1} = \frac{\bar{q}}{|q|^2}$. Notice that for all non real $q \in \mathbb{H}$, the quantity $\frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}$ is an imaginary unit, that is a quaternion whose square equals -1. Then we can express every $q \in \mathbb{H}$ as $q = x + yI$, where x, y are real (if $q \in \mathbb{R}$, then $y = 0$) and I is an element of the unit 2-dimensional sphere of purely imaginary quaternions,

$$\mathbb{S} = \{q \in \mathbb{H} \mid q^2 = -1\}.$$

In the sequel, for every $I \in \mathbb{S}$ we will denote by L_I the plane $\mathbb{R} + \mathbb{R}I$, isomorphic to $\mathbb{C}$ and, if Ω is a subset of $\mathbb{H}$, by Ω_I the intersection $\Omega \cap L_I$. As explained in [12], the natural domains of definition for slice regular functions are the symmetric slice domains. These domains actually play the role played by domains of holomorphy in the complex case:

Definition 2.1. Let Ω be a domain in $\mathbb{H}$ that intersects the real axis. Then:

1) Ω is called a *slice domain* if, for all $I \in \mathbb{S}$, the intersection Ω_I with the complex plane L_I is a domain of L_I;
2) Ω is called a *symmetric domain* if for all $x, y \in \mathbb{R}$, $x + yI \in \Omega$ implies $x + y\mathbb{S} \subset \Omega$.

We can now recall the definition of slice regularity. From now on, Ω will always be a symmetric slice domain in $\mathbb{H}$, unless differently stated.

Definition 2.2. A function $f : \Omega \to \mathbb{H}$ is said to be *(slice) regular* if, for all $I \in \mathbb{S}$, its restriction f_I to Ω_I has continuous partial derivatives and is *holomorphic*, i.e., it satisfies

$$\overline{\partial}_I f(x + yI) := \frac{1}{2}\Big(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\Big) f_I(x + yI) = 0$$

for all $x + yI \in \Omega_I$.

A basic result in the theory of regular functions, that relates slice regularity and classical holomorphy, is the following, [12, 14]:

Lemma 2.3 (Splitting Lemma). *If f is a regular function on Ω, then for every $I \in \mathbb{S}$ and for every $J \in \mathbb{S}$, J orthogonal to I, there exist two holomorphic functions $F, G : \Omega_I \to L_I$, such that for every $z = x + yI \in \Omega_I$, it holds*

$$f_I(z) = F(z) + G(z)J.$$

One of the first consequences of the previous result is the following version of the Identity Principle, [14]:

Theorem 2.4 (Identity Principle). *Let f be a regular function on Ω. Denote by Z_f the zero set of f, $Z_f = \{q \in \Omega \mid f(q) = 0\}$. If there exists $I \in \mathbb{S}$*

such that $\Omega_I \cap Z_f$ *has an accumulation point in* Ω_I*, then* f *vanishes identically on* Ω.

In the sequel we will use an important extension result (see [2], [3]) that we will present in the following special formulation:

Lemma 2.5 (Extension Lemma). *Let* Ω *be a symmetric slice domain and choose* $I \in \mathbb{S}$. *If* $f_I : \Omega_I \to \mathbb{H}$ *is holomorphic, then setting*

$$f(x+yJ) = \frac{1}{2}[f_I(x+yI) + f_I(x-yI)] + J\frac{I}{2}[f_I(x-yI) - f_I(x+yI)]$$

extends f_I *to a regular function* $f : \Omega \to \mathbb{H}$. *Moreover* f *is the unique extension and it is denoted by* $\mathrm{ext}(f_I)$.

The product of two regular functions is not, in general, regular. To guarantee the regularity we have to use a different multiplication operation, the $*$-product. From now on, if F is a holomorphic function, we will use the notation:

$$\hat{F}(z) := \overline{F(\bar{z})}.$$

Definition 2.6. Let f, g be regular functions on a symmetric slice domain Ω. Choose $I, J \in \mathbb{S}$ with $I \perp J$ and let F, G, H, K be holomorphic functions from Ω_I to L_I such that $f_I = F + GJ, g_I = H + KJ$. Consider the holomorphic function defined on Ω_I by

$$f_I * g_I(z) = \left[F(z)H(z) - G(z)\hat{K}(z)\right] + \left[F(z)K(z) + G(z)\hat{H}(z)\right] J. \tag{1}$$

Its regular extension $\mathrm{ext}(f_I * g_I)$ is called the *regular product* (or $*$*-product*) of f and g and it is denoted by $f * g$.

Notice that the $*$-product is associative but generally is not commutative. Its connection with the usual pointwise product is stated by the following result.

Proposition 2.7. *Let* $f(q)$ *and* $g(q)$ *be regular functions on* Ω. *Then, for all* $q \in \Omega$,

$$f * g(q) = \begin{cases} f(q)g(f(q)^{-1}qf(q)) & \text{if} \quad f(q) \neq 0 \\ 0 & \text{if} \quad f(q) = 0 \end{cases} \tag{2}$$

Corollary 2.8. *If f, g are regular functions on a symmetric slice domain Ω and $q \in \Omega$, then $f * g(q) = 0$ if and only if $f(q) = 0$ or $f(q) \neq 0$ and $g(f(q)^{-1}qf(q)) = 0$.*

To illustrate the natural meaning of the $*$-product of two regular functions, we consider two quaternionic power series, $\sum_{n=0}^{\infty} q^n a_n$ and $\sum_{n=0}^{\infty} q^n b_n$, both centered at zero and with radius of convergence $R > 0$. These power series define two regular functions $f(q) = \sum_{n=0}^{\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{\infty} q^n b_n$ on the open ball $B(0, R) \subseteq \mathbb{H}$ centered at 0 and with radius R (see e.g. [12]). Now, (polynomials and) power series with coefficients in a non commutative ring are classically endowed with the *Cauchy product*, that even in the non commutative case is still defined as

$$(3) \qquad \left(\sum_{n=0}^{\infty} q^n a_n\right) \cdot \left(\sum_{n=0}^{\infty} q^n b_n\right) = \sum_{n=0}^{\infty} q^n c_n \quad \text{with} \quad c_n = \sum_{m=0}^{n} a_m b_{n-m}$$

so that the sequence of coefficients $\{c_n\}$ is obtained by the convolution of the sequences $\{a_n\}$ and $\{b_n\}$. It turns out that

Proposition 2.9. *The $*$-product of the regular functions $f(q)$ and $g(q)$ coincides with the Cauchy product of their power series expansions, i.e.*

$$f * g(q) = \left(\sum_{n=0}^{\infty} q^n a_n\right) \cdot \left(\sum_{n=0}^{\infty} q^n b_n\right),$$

on $B(0, R)$.

Proof. Let us consider the coefficients $\{a_n\}, \{b_n\}, \{c_n\}$ of the power series appearing in Equation (3). Choose I, J in $\mathbb{S}$ with $I \perp J$ and write

$$a_n = \alpha_n + \beta_n J \quad \text{and} \quad b_n = \gamma_n + \delta_n J$$

for suitable $\alpha_n, \beta_n, \gamma_n, \delta_n$ in L_I. A direct computation shows that the splitting of c_n is

$$c_n = \sum_{m=0}^{n} (\alpha_m \gamma_{n-m} - \beta_m \bar{\delta}_{n-m}) + \sum_{m=0}^{n} (\alpha_m \delta_{n-m} + \beta_m \bar{\gamma}_{n-m}) J$$

and a comparison with equation (1) leads to the conclusion of the proof. □

It is immediate, and useful for the sequel, to notice that if $\{a_n\}$ are all real numbers, then we have

$$\begin{aligned} f * g(q) &= \left(\sum_{n=0}^{\infty} q^n a_n\right) \cdot \left(\sum_{n=0}^{\infty} q^n b_n\right) = fg(q) = gf(q) \\ &= \left(\sum_{n=0}^{\infty} q^n b_n\right) \cdot \left(\sum_{n=0}^{\infty} q^n a_n\right) = g * f(q). \end{aligned}$$

on the whole domain of convergence $B(0, R)$ of the power series, i.e. the $*$-product and the pointwise product coincide (and are commutative). Hence power series with real coefficients define, on their domains of convergence, regular functions that behave nicely with respect to the $*$-product; these functions are called *slice preserving* regular functions, since, for all $I \in \mathbb{S}$, they map subsets of L_I into L_I.

The following operations are naturally defined in order to study the zero set of regular functions.

Definition 2.10. Let f be a regular function on a symmetric slice domain Ω. Choose $I, J \in \mathbb{S}$ with $I \perp J$ and let F, G be holomorphic functions from Ω_I to L_I such that $f_I = F + GJ$. If f_I^c is the holomorphic function defined on Ω_I by

$$f_I^c(z) = \hat{F}(z) - G(z)J. \tag{4}$$

Then the *regular conjugate* of f is the regular function defined on Ω by $f^c = \mathrm{ext}(f_I^c)$, and the *symmetrization* of f is the regular function defined on Ω by $f^s = f * f^c = f^c * f$.

If the regular function $f : \Omega \to \mathbb{H}$ is such that $f_I(z) = F(z) + G(z)J$, with $F, G : \Omega_I \to L_I$ holomorphic functions, then it is easy to see that (see, e.g., [12])

$$f_I^s = f_I * f_I^c = f_I^c * f_I = F(z)\hat{F}(z) + G(z)\hat{G}(z). \tag{5}$$

Hence $f^s(\Omega_I) \subseteq L_I$ for every $I \in \mathbb{S}$, i.e., f^s is slice preserving. Moreover if g is a regular function on Ω, then

$$(f * g)^c = g^c * f^c \quad \text{and} \quad (f * g)^s = f^s g^s = g^s f^s. \tag{6}$$

Zeroes of regular functions have a nice geometric property:

Theorem 2.11. *Let f be a regular function on a symmetric slice domain Ω. If f does not vanish identically, then its zero set consists of isolated points or isolated 2-spheres of the form $x + y\mathbb{S}$ with $x, y \in \mathbb{R}$, $y \neq 0$.*

Notice that $f(q)^{-1}qf(q)$ belongs to the same sphere $x + y\mathbb{S}$ as q. Hence each zero of $f * g$ in $x + y\mathbb{S}$ corresponds to a zero of f or to a zero of g in the same sphere.

Lemma 2.12. *Let f be a regular function on a symmetric slice domain Ω and let f^s be its symmetrization. Then for each $S = x + y\mathbb{S} \subset \Omega$ either f^s vanishes identically on S or it has no zeroes in S.*

The *regular reciprocal* f^{-*} of a regular function f defined on a symmetric slice domain Ω can now be defined in $\Omega \setminus Z_{f^s}$ as

$$f^{-*} = (f^s)^{-1} f^c, \tag{7}$$

where Z_{f^s} denotes the zero set of the symmetrization f^s.

Remark 2.13. If f is a regular function defined on a slice symmetric domain of $\mathbb{H}$, then its regular reciprocal $f^{-*} = (f^s)^{-1} f^c$ has a sphere of poles at Z_{f^s} and is a semiregular function in the sense of [23].

3. Ideals generated by two regular functions

In this section we will prove that if f_1 and f_2 are two regular functions with no common zeroes on a symmetric slice domain Ω, then they generate the entire ring of regular functions on Ω, i.e. there are two regular functions h_1 and h_2 on Ω such that $f_1 * h_1 + f_2 * h_2 = 1$.

We begin by proving a local version of this result for holomorphic functions (in the sense of Definition 2.2), following the approach used in the case of several complex variables.

Theorem 3.1. *Let f_1, f_2 be two functions, regular in a symmetric slice domain Ω without common zeroes. Then, for any $I \in \mathbb{S}$, the equation*

$$f_1 * h_1 + f_2 * h_2 = 1. \tag{8}$$

restricted to Ω_I has local holomorphic solutions h_1, h_2 at any point of Ω_I.

Proof. By the Splitting Lemma, for any $I \in \mathbb{S}$, we can represent, for $\ell = 1, 2$, the functions f_ℓ via functions holomorphic in Ω_I as

$$f_\ell(z) = f_{\ell|I}(z) = F_\ell(z) + G_\ell(z)J,$$

where $J \in \mathbb{S}$ is orthogonal to I. Similarly, the functions h_ℓ that we are looking for can be written as

$$h_\ell(z) = h_{\ell|I}(z) = H_\ell(z) + K_\ell(z)J,$$

for suitable holomorphic functions H_ℓ and K_ℓ. Using (1), it is immediate to see that (8) can be rewritten as a system of two equations for holomorphic functions in L_I, namely, omitting the variable z,

$$\begin{cases} F_1H_1 - G_1\hat{K}_1 + F_2H_2 - G_2\hat{K}_2 = 1 \\ F_1K_1 + G_1\hat{H}_1 + F_2K_2 + G_2\hat{H}_2 = 0. \end{cases} \tag{9}$$

Since f_1 and f_2 do not have common zeroes in $\Omega_I \subset \Omega$, the same holds true for F_1, G_1, F_2, G_2. Hence, a classical one complex variable result implies that there exist H_1, K_1, H_2, K_2, holomorphic in Ω_I, which define a solution of the first equation of (9). In general, the functions H_1, K_1, H_2, K_2 will not define a solution of system (9). However, one can modify the solution to the first equation by adding an element of the syzygies of (F_1, G_1, F_2, G_2) and try to solve the system. Since the latter functions have no common zeroes on Ω_I, their syzygies (see, e.g., [4]) are generated by the columns of the following matrix

$$A = \begin{pmatrix} G_1 & F_2 & G_2 & 0 & 0 & 0 \\ -F_1 & 0 & 0 & F_2 & G_2 & 0 \\ 0 & -F_1 & 0 & -G_1 & 0 & G_2 \\ 0 & 0 & -F_1 & 0 & -G_1 & -F_2 \end{pmatrix}.$$

Hence the general solution to the first equation of (9) is given by

$$\begin{pmatrix} H_1 + \hat{\beta}_1 G_1 + \hat{\beta}_2 F_2 + \hat{\beta}_3 G_2 \\ -\hat{K}_1 - \hat{\beta}_1 F_1 + \hat{\beta}_4 F_2 + \hat{\beta}_5 G_2 \\ H_2 - \hat{\beta}_2 F_1 - \hat{\beta}_4 G_1 + \hat{\beta}_6 G_2 \\ -\hat{K}_2 - \hat{\beta}_3 F_1 - \hat{\beta}_5 G_1 - \hat{\beta}_6 F_2 \end{pmatrix} \tag{10}$$

where $\beta_1, \ldots, \beta_6$ are arbitrary holomorphic functions in Ω_I. Consider now the matrix B of holomorphic functions defined by

$$B = \begin{pmatrix} \hat{F}_1 & 0 & 0 & -\hat{F}_2 & -\hat{G}_2 & 0 \\ \hat{G}_1 & \hat{F}_2 & \hat{G}_2 & 0 & 0 & 0 \\ 0 & 0 & \hat{F}_1 & 0 & \hat{G}_1 & \hat{F}_2 \\ 0 & -\hat{F}_1 & 0 & -\hat{G}_1 & 0 & \hat{G}_2 \end{pmatrix}. \tag{11}$$

In order to obtain a solution of (9) we now need to request that the vector

$$\begin{pmatrix} K_1 + \beta_1 \hat{F}_1 - \beta_4 \hat{F}_2 - \beta_5 \hat{G}_2 \\ \hat{H}_1 + \beta_1 \hat{G}_1 + \beta_2 \hat{F}_2 + \beta_3 \hat{G}_2 \\ K_2 + \beta_3 \hat{F}_1 + \beta_5 \hat{G}_1 + \beta_6 \hat{F}_2 \\ \hat{H}_2 - \beta_2 \hat{F}_1 - \beta_4 \hat{G}_1 + \beta_6 \hat{G}_2 \end{pmatrix} \tag{12}$$

belongs to the syzygies of (F_1, G_1, F_2, G_2). That is, setting $H = {}^t(K_1, \hat{H}_1, K_2, \hat{H}_2)$, we need to find $\beta = {}^t(\beta_1, \ldots, \beta_6)$ and $\alpha = {}^t(\alpha_1, \ldots, \alpha_6)$ vectors of holomorphic functions such that

$$H + B\beta = A\alpha,$$

namely such that

$$\begin{pmatrix} A, & -B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = H. \tag{13}$$

Our next goal is to establish that the rank of the (4×12)-matrix $(A, -B)$ equals 4 on the entire Ω_I. Since F_1, G_1, F_2, G_2 have no common zeroes, it is easy to prove that both A and B have rank 3 at each point $z \in \Omega_I$. Consider for instance A and denote by $A^1, \ldots, A^6$ its columns. If $F_1(z) \neq 0$, then $\{A^1, A^2, A^3\}$ is a maximal subset of linearly independent columns on a neighborhood of z. If $F_1(z) = 0$ and $G_1(z) \neq 0$, we can take as a maximal subset of linearly independent columns $\{A^1, A^4, A^5\}$. If both $F_1(z)$ and $G_1(z)$ vanish, we proceed analogously considering F_2 and G_2. The rank of $(A, -B)$ is not maximum at a point $z \in \Omega_I$ if and only if all columns of B are linear combinations of columns of A, which is equivalent to the condition that all columns of B belong to the syzygies of (F_1, G_1, F_2, G_2). Hence the rank of $(A, -B)$ is 3 where (in Ω_I) the following six equations are simultaneously satisfied:

$$F_1\hat{F}_1 + G_1\hat{G}_1 = 0 \tag{14}$$
$$F_1\hat{F}_2 + G_2\hat{G}_1 = 0 \tag{15}$$
$$F_1\hat{G}_2 - F_2\hat{G}_1 = 0 \tag{16}$$
$$G_1\hat{F}_2 - G_2\hat{F}_1 = 0 \tag{17}$$
$$F_2\hat{F}_1 + G_1\hat{G}_2 = 0 \tag{18}$$
$$F_2\hat{F}_2 + G_2\hat{G}_2 = 0 \tag{19}$$

Equations (14) and (19) can be written in Ω_I as the quaternionic equations $f_1^s(z) = 0$ and $f_2^s(z) = 0$. We will now investigate the meaning of the other terms. Using (1) and the fact that Ω_I is symmetric (i.e. if it contains z then it contains $\bar{z}$ as well), we get

$$(f_1^c * f_2)_I(z) = (F_2(z)\hat{F}_1(z) + G_1(z)\hat{G}_2(z)) - (G_1(z)\hat{F}_2(z) - G_2(z)\hat{F}_1(z))J$$

$$(f_2^c * f_1)_I(z) = (F_1(z)\hat{F}_2(z) + G_2(z)\hat{G}_1(z)) + (G_1(z)\hat{F}_2(z) - G_2(z)\hat{F}_1(z))J$$

$$(f_1^c * f_2)_I(\bar{z}) = \overline{(F_1(z)\hat{F}_2(z) + G_2(z)\hat{G}_1(z))} + \overline{(F_1(z)\hat{G}_2(z) - F_2(z)\hat{G}_1(z))}J$$

$$(f_2^c * f_1)_I(\bar{z}) = \overline{(F_2(z)\hat{F}_1(z) + G_1(z)\hat{G}_2(z))} + (F_1(z)\hat{G}_2(z) - F_2(z)\hat{G}_1(z))J.$$

Hence if the matrix $(A, -B)$ has rank 3 at $z \in \Omega_I$, then equations (15)–(18) imply that $(f_1^c * f_2)_I(z) = (f_2^c * f_1)_I(z) = (f_1^c * f_2)_I(\bar{z}) = (f_2^c * f_1)_I(\bar{z}) = 0$. Consequently if $(A, -B)$ has rank 3 at $z \in U$, then we have

$$f_1^s(z) = 0 \tag{20}$$
$$f_1^c * f_2(z) = 0 \tag{21}$$
$$f_2^c * f_1(z) = 0 \tag{22}$$
$$f_1^c * f_2(\bar{z}) = 0 \tag{23}$$
$$f_2^c * f_1(\bar{z}) = 0 \tag{24}$$
$$f_2^s(z) = 0 \tag{25}$$

Let $z = x + yI$. From equations (20) and (25) we obtain that both f_1 and f_2 have a (non common and hence non spherical) zero in the sphere $x + y\mathbb{S}$. Equation (20) can be written as

$$f_1^c * f_1(z) = 0$$

which, by Proposition 2.7 leads to two possibilities:

(a) $f_1^c(z) = 0$ or

(b) $f_1^c(z) \neq 0$ and $f_1((f_1^c(z))^{-1} z f_1^c(z)) = 0$.

In case (a), we have that $f_1^c(\bar{z}) \neq 0$, since $x + y\mathbb{S}$ is not a spherical zero of (f_1 and hence of) f_1^c. Thanks to Proposition 2.7, equation (23) becomes

$$f_1^c(\bar{z}) f_2((f_1^c(\bar{z}))^{-1} \bar{z} f_1^c(\bar{z})) = 0,$$

which implies that

$$f_2((f_1^c(\bar{z}))^{-1} \bar{z} f_1^c(\bar{z})) = 0. \tag{26}$$

Moreover (20) yields that $x + y\mathbb{S}$ is a spherical zero of f_1^s, and hence that

$$0 = f_1^s(\bar{z}) = f_1^c(\bar{z}) f_1((f_1^c(\bar{z}))^{-1} \bar{z} f_1^c(\bar{z})),$$

leading to

$$f_1((f_1^c(\bar{z}))^{-1} \bar{z} f_1^c(\bar{z})) = 0. \tag{27}$$

The hypothesis that f_1 and f_2 have no common zeroes together with (26) and (27) gives us a contradiction.
In case (b), again thanks to Proposition 2.7, equation (21)

$$f_1^c(z) f_2((f_1^c(z))^{-1} z f_1^c(z)) = 0$$

yields that f_2 vanishes at $(f_1^c(z))^{-1} z f_1^c(z)$ which is a zero of f_1. Again a contradiction. In conclusion, equations (20)–(25) (and hence equations (14)–(19)) are never simultaneously satisfied, which implies that the matrix $(A, -B)$ has rank 4 at all points of Ω_I. Therefore, using the classical Rouché - Capelli method it is now possible to find a local holomorphic solution (α, β) of system (13) in the neighborhood of each point $z \in \Omega_I$. This gives us a local holomorphic solution of system (9) and hence of equation (8). □

To find a global solution of (8) on Ω_I we will apply results from the theory of analytic sheaves. More precisely we will use the following consequence of Cartan Theorem B, see [19].

Theorem 3.2. *Let $D \subseteq \mathbb{C}^n$ be a pseudoconvex domain, and let $(\mathcal{F}, D)$ be a coherent analytic sheaf. Suppose that there exist finitely many global sections $s_1, \dots s_k \in \Gamma(D, \mathcal{F})$ such that $(s_1)_z, \dots, (s_k)_z$ generate the stalk $\mathcal{F}_z$ over each $z \in D$. Then for any global section $g \in \Gamma(D, \mathcal{F})$, there exist $g_1, \dots, g_k \in \Gamma(D, \mathcal{O})$ holomorphic functions on D such that $g = s_1 g_1 + \dots + s_k g_k$.*

In our setting the sheaf $(\mathcal{F}, D)$ will be the coherent sheaf $(\mathcal{O}^4, \Omega_I)$ of 4-tuples of germs of holomorphic functions on Ω_I.

Theorem 3.3. *Let f_1, f_2 be regular functions on a symmetric slice domain $\Omega \subseteq \mathbb{H}$, with no common zeroes in Ω. Then there exist h_1 and h_2 regular functions on Ω such that*

$$f_1 * h_1 + f_2 * h_2 = 1 \tag{28}$$

on Ω.

Proof. Fix $I \in \mathbb{S}$ and, with the notation of the proof of Theorem 3.1, consider the linear system

$$\begin{pmatrix} A, & -B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = H \tag{29}$$

associated with equation (28) restricted to Ω_I. In the language of analytic sheaves, the proof of Theorem 3.1 read as follows: consider the coherent analytic sheaf $(\mathcal{O}^4, \Omega_I)$ of 4-tuples of germs of holomorphic functions on the pseudoconvex domain $\Omega_I \subseteq L_I \simeq \mathbb{C}$. The fact that the matrix $(A, -B)$ appearing in equation (29) has rank 4 at all point $z \in \Omega_I$ means that the twelve columns $\{A^1, \dots, A^6, B^1, \dots, B^6\}$ generate the stalk $\mathcal{O}^4_z$ of $(\mathcal{O}^4, \Omega_I)$ at any $z \in \Omega_I$. Theorem 3.2 implies then that for any 4-tuple $k \in \Gamma(\Omega_I, \mathcal{O}^4)$ of holomorphic functions on Ω_I, there exist twelve holomorphic functions $g_1, \dots, g_{12} \in \Gamma(\Omega_I, \mathcal{O})$ such that $k = g_1 A^1 + \cdots + g_6 A^6 + g_7 B^1 + \cdots + g_{12} B^6$. In particular, setting $k = H$ we obtain a global solution of (29) and therefore a global solution h_1, h_2 of equation (28) on Ω_I. To conclude, applying the Extension Lemma 2.5, we uniquely extend the functions h_1, h_2 to Ω as regular functions that satisfy

$$f_1 * h_1 + f_2 * h_2 = 1$$

everywhere on Ω. □

4. Ideals of regular functions

In this section we show how the proof of Theorem 3.3 can be extended to the case of $n (\geq 2)$ regular functions with no common zeroes.

Lemma 4.1. *Let $f_1, \ldots, f_n$ be n regular functions in a slice symmetric domain Ω without common zeroes. Then for any $I \in \mathbb{S}$ if $f_\ell = F_\ell + G_\ell J$ is the splitting of f_ℓ on Ω_I, for $\ell = 1, \ldots, n$, then:*

1) *the rank of the $\left(2n \times \binom{2n}{2}\right)$-matrix A whose columns are the standard generators of the syzygies of the vector $(F_1, G_1, \ldots, F_n, G_n)$ equals $2n - 1$ on Ω_I;*
2) *the rank of the $\left(2n \times \binom{2n}{2}\right)$-matrix B whose columns are the standard generators of the syzygies of the vector $(-\hat{G}_1, \hat{F}_1, \ldots, -\hat{G}_n, \hat{F}_n)$ equals $2n - 1$ on Ω_I;*
3) *the rank of the $\left(2n \times 2\binom{2n}{2}\right)$-matrix $(A, -B)$ equals $2n$ on Ω_I.*

Proof. Since $f_1, \ldots, f_n$ do not have common zeroes in $\Omega_I \subseteq \Omega$, the same condition is satisfied by $F_1, G_1, \ldots, F_n, G_n$. Reasoning as we did in the $n = 2$ case, if $F_1(z) \neq 0$, we can reorder the columns of A in such a way that all the elements in the subdiagonal are nonzero multiples of F_1 and all entries underneath the subdiagonal vanish. If $F_1(z) = 0$ and $G_1(z) \neq 0$, we can reorder (rows and columns) so that the subdiagonal is composed by nonzero multiples of G_1 and all the elements underneath vanish. The process can be iterated up to G_n. Moreover the matrix $\left(A^{2n}, A^{2n+1}, \ldots, A^{\binom{2n}{2}}\right)$ has a row of zeroes. This guarantees that A has rank $2n - 1$ on Ω_I. The same argument apply to B since $\hat{F}_1, \hat{G}_1, \ldots, \hat{F}_n, \hat{G}_n$ do not have common zeroes in Ω_I.

To prove the third assertion, we will proceed by contradiction. Suppose that the rank of $(A, -B)$ equals $2n - 1$ at $z \in \Omega_I$. Then each column of $-B$ is a linear combination of the columns of A, i.e. it belongs to the syzygies of $(F_1, G_1, \ldots, F_n, G_n)$. By taking the scalar product of each column of B by $(F_1, G_1, \ldots, F_n, G_n)$, we get $\binom{2n}{2}$ equations that, as in the case $n = 2$, lead to

$$\begin{cases} f_\sigma^s = 0 \\ f_\gamma^c * f_\delta(z) = 0 \\ f_\gamma^c * f_\delta(\bar{z}) = 0 \end{cases} \tag{30}$$

for any $\sigma, \gamma, \delta \in \{1, \ldots, n\}$, $\gamma \neq \delta$. As for $n = 2$, equations of the first type in system (30) imply that $f_1, \ldots, f_n$ all have a (not common and not spherical) zero on the 2-sphere generated by z. Following the lines of the proof of Theorem 3.1 it is possible to prove that the hypothesis that $f_1, \ldots, f_n$ do not have common zeroes leads to a contradiction. □

The previous lemma allows us to prove the following local result, using the same arguments of the case $n = 2$.

Theorem 4.2. *Let $f_1, \dots, f_n$ be n functions, regular on a symmetric slice domain Ω without common zeroes. Then for any $I \in \mathbb{S}$ the equation*

$$f_1 * h_1 + \cdots + f_n * h_n = 1. \tag{31}$$

restricted to Ω_I has local holomorphic solutions $h_1, \dots, h_n$ in the neighborhood of any point of Ω_I.

As in the proof of Theorem 3.3, the consequence of Cartan Theorem B stated in Theorem 3.2 lead us to find a global solution of equation (31) on Ω_I. The Extension Lemma 2.5 provides a global regular solution on Ω.

Theorem 4.3. *Let $f_1, \dots, f_n$ be regular functions on a symmetric slice domain $\Omega \subseteq \mathbb{H}$, with no common zeroes in Ω. Then there exist $h_1, \dots, h_n$ regular functions on Ω such that*

$$f_1 * h_1 + \cdots + f_n * h_n = 1$$

on Ω.

5. Syzygies of regular functions

We conclude the paper with a short description of the syzygies of regular functions. Let us begin by studying the structure of the sheaf of local syzygies of n regular functions.

Theorem 5.1. *Let $f_1, \dots, f_n$ be n regular functions on a symmetric slice domain Ω, with no common zeroes. For any $I \in \mathbb{S}$, and any $J \in \mathbb{S}, J \perp I$, let $f_\ell = F_\ell + G_\ell J$ $(\ell = 1, \dots, n)$ for suitable holomorphic functions F_ℓ, G_ℓ. If $(\mathcal{K}, \Omega_I)$ is the sheaf of germs of holomorphic solutions of the system*

$$\begin{cases} F_1 H_1 - G_1 \hat{K}_1 + \cdots + F_n H_n - G_n \hat{K}_n = 0 \\ F_1 K_1 + G_1 \hat{H}_1 + \cdots + F_n K_n + G_n \hat{H}_n = 0. \end{cases} \tag{32}$$

associated with

$$f_1 * h_1 + \cdots + f_n * h_n = 0 \tag{33}$$

restricted to Ω_I, *then*

$$(\mathcal{K}, \Omega_I) \cong (\mathcal{O}^{4n^2-4n}, \Omega_I)/(\mathcal{O}^{4n^2-6n+2}, \Omega_I).$$

Proof. Using the same notation of Lemma 4.1, the sheaf $(\mathcal{K}, \Omega_I)$ corresponds to the sheaf of germs of local solutions of the system of $2n$ equations in $2\binom{2n}{2}$ unknowns

$$\begin{pmatrix} A, & -B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \tag{34}$$

Lemma 4.1 yields that we can express locally $2n$ unknowns as holomorphic functions in terms of $2\binom{2n}{2} - 2n = 4n^2 - 4n$ germs of holomorphic functions. We therefore obtain a surjective map

$$\varphi : (\mathcal{O}^{4n^2-4n}, \Omega_I) \to (\mathcal{K}, \Omega_I).$$

The germ in $(\mathcal{O}^{4n^2-4n}, \Omega_I)$ associated with the vector ${}^t(\alpha, \beta)$, solution of (34), belongs to $\ker \varphi$ if and only if

$$A\alpha = B\beta = 0,$$

which, recalling that the rank of A and B equals $2n-1$, implies that the kernel of φ is isomorphic to $(\mathcal{O}^{4n^2-6n+2}, \Omega_I)$. Hence we conclude that $(\mathcal{K}, \Omega_I)$ is isomorphic to $(\mathcal{O}^{4n^2-4n}, \Omega_I)/(\mathcal{O}^{4n^2-6n+2}, \Omega_I)$. $\square$

In the complex case, if $f_1, \ldots, f_n$ are holomorphic functions of one complex variable with no common zeroes, then their syzygies are generated by $\binom{n}{2}$ vectors of holomorphic functions which can be constructed as follows: let e_ℓ, $\ell = 1, \ldots, n$, be the standard basis of $\mathbb{R}^n$. The generators of the syzygies are then

$$f_r e_t - f_t e_r = (0, \ldots, 0, -f_t, 0, \ldots, 0, f_r, 0, \ldots, 0)$$

for $1 \leq r < t \leq n$, a fact which we have repeatedly used in the previous section. It is therefore natural to ask if a similar situation occurs for regular functions without common zeroes. Since the $*$-multiplication is not commutative, the immediate analogue of these syzygies does not work in this context. Natural syzygies would on the other hand be the vectors

$$\begin{aligned}\mathrm{syz}(r,t) &:= (f_t^c * f_r^s)e_t - (f_r^c * f_t^s)e_r \\ &= (0,\dots,0,-f_r^c * f_t^s,0,\dots,0,f_t^c * f_r^s,0,\dots,0)\end{aligned}$$

for $1 \le r < t \le n$. In fact, Formula (6) implies that (see Definition 2.10),

$$f_r * (-f_r^c * f_t^s) + f_t * (f_t^c * f_r^s) = 0$$

for all $1 \le r < t \le n$. For $n \ge 2$, as in the case of holomorphic functions, there are $\binom{n}{2}$ syzygies, though Theorem 5.1 immediately implies the following proposition.

Proposition 5.2. *Let $f_1,\dots,f_n$ be regular functions on a slice symmetric domain Ω of $\mathbb{H}$ with no common zeroes. Then their syzygies are locally generated by $n-1$ vectors of regular functions.*

To understand this phenomenon, we note that for any three indices $1 \le p < r < t \le n$, we have

$$\mathrm{syz}(r,t) * f_p^s = \mathrm{syz}(p,t) * f_r^s - \mathrm{syz}(p,r) * f_t^s. \tag{35}$$

Let us fix a sphere $S = x + y\mathbb{S} \subseteq \Omega$. If one of the functions f_p, f_r, f_t never vanishes on S, assume f_p, then (35) immediately shows that $\mathrm{syz}(r,t)$ is a combination with regular coefficients of $\mathrm{syz}(p,t)$ and $\mathrm{syz}(p,r)$

$$\mathrm{syz}(r,t) = \mathrm{syz}(p,t) * f_r^s * (f_p^s)^{-1} - \mathrm{syz}(p,r) * f_t^s * (f_p^s)^{-1}. \tag{36}$$

If all f_p, f_r, f_t have a zero on S, without loss of generality, we can assume that f_p has the lesser order (for the notion of order of a zero see, e.g., [12]). Then, again, (36) can be used to represent $\mathrm{syz}(r,t)$ locally.

Remark 5.3. It therefore appears that the reason why we can reduce to $n-1$ the number of syzygies is a consequence of Remark 2.13, namely the fact that a (isolated, non real) zero of a regular function f generates a sphere of zeroes for f^s and a sphere of poles for its reciprocal f^{-*}.

Acknowledgements

The first two authors acknowledge the support of G.N.S.A.G.A. of INdAM and of Italian MIUR (Research Projects: PRIN "Varietà reali e complesse:

geometria, topologia e analisi armonica", FIRB "Geometria differenziale e teoria geometrica delle funzioni", SIR "Analytic aspects in complex and hypercomplex geometry"). They express their gratitude to Chapman University, where a portion of this work was carried out.

References

[1] A. Brudnyi, *Banach-valued holomorphic functions on the maximal ideal space of H^∞*, Invent. Math. **193** (2013), 187–227.

[2] F. Colombo, G. Gentili, and I. Sabadini, *A Cauchy kernel for slice regular functions*, Ann. Global Anal. Geom. **37** (2010), 361–378.

[3] F. Colombo, G. Gentili, I. Sabadini, and D. C. Struppa, *Extension results for slice regular functions of a quaternionic variable*, Adv. Math. **222** (2009), 1793–1808.

[4] F. Colombo, I. Sabadini, F. Sommen, and D. C. Struppa, *Analysis of Dirac Systems and Computational Algebra*, Birkhäuser, 2004.

[5] F. Colombo, I. Sabadini, and D. C. Struppa, *The Runge theorem for slice hyperholomorphic functions*, Proc. Amer. Math. Soc. **139** (2011), 1787–1803.

[6] F. Colombo, I. Sabadini, and D. C. Struppa, *Noncommutative Functional Calculus (Theory and Applications of Slice-Hyperholomorphic Functions)*, Progress in Mathematics series **289**, Birkhäuser Basel, (2011).

[7] F. Colombo, I. Sabadini, and D. C. Struppa, *Sheaves of slice regular functions*, Math. Nachr. **285** (2012), 949–958.

[8] C. Della Rocchetta, G. Gentili, and G. Sarfatti, *A Bloch-Landau theorem for slice regular functions*, in Advances in Hypercomplex Analysis, ed. by G. Gentili, I. Sabadini, M. V. Shapiro, F. Sommen, D. C. Struppa, Springer INdAM Series, Springer, Milan, 2013, pp. 55–74.

[9] R. G. Douglas, S. G. Krantz, E. T. Sawyer, S. Treil, and B. D. Wick (eds.), *The Corona Problem: Connections Between Operator Theory, Function Theory, and Geometry*, Fields Institute Communications **72**, Springer New York Heidelberg Dordrecht London, (2014).

[10] G. Gentili, S. Salamon, and C. Stoppato, *Twistor transforms of quaternionic functions and orthogonal complex structures*, J. Eur. Math. Soc. **16** (2014), 2323–2353.

[11] G. Gentili and G. Sarfatti, *Landau-Toeplitz theorems for slice regular functions*, Pacific J. Math. **265** (2013), 381–404.

[12] G. Gentili, C. Stoppato, and D. C. Struppa, *Regular Functions of a Quaternionic Variable*, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg, 2013.

[13] G. Gentili and D. C. Struppa, *A new approach to Cullen-regular functions of a quaternionic variable*, C. R. Math. Acad. Sci. Paris **342** (2006), 741–744.

[14] G. Gentili and D. C. Struppa, *A new theory of regular functions of a quaternionic variable*, Adv. Math. **216** (2007), 279–301.

[15] R. Ghiloni and A. Perotti, *A new approach to slice regularity on real algebras*, in: Hypercomplex Analysis and its Applications, I. Sabadini and F. Sommen eds., Trends in Mathematics, Birkhäuser, Basel (2011), 109–124.

[16] R. Ghiloni and A. Perotti, *Slice regular functions on real alternative algebras*, Adv. Math. **226** (2011), 1662–1691.

[17] R. Ghiloni and A. Perotti, *Zeros of regular functions of quaternionic and octonionic variable: a division lemma and the cam-shaft effect*, Ann. Math. Pura Appl. (4) **190** (2011), 539–551.

[18] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, reprint of the 1965 original, AMS Chelsea Publishing, AMS Providence, RI, 2009.

[19] S. G. Krantz, *Function Theory of Several Complex Variables*, second edition, Wadsworth & Brooks, Pacific Grove, CA, 1992.

[20] B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Translations of Mathematical Monographs, Vol. 5, American Mathematical Society, Providence, RI, 1964.

[21] G. Sarfatti, *Elements of function theory in the unit ball of quaternions*, Ph.D. Thesis, Università di Firenze, `http://www.students.math.unifi.it/users/sarfatti/PhDGiulia.pdf`, 2013.

[22] A. Sasane and S. Treil, *Estimates in corona theorems for some subalgebras of* H^∞, Ark. Mat. **45** (2007), 351–380.

[23] C. Stoppato, *Singularities of slice regular functions*, Math. Nachr. **285** (2012), 1274–1293.

[24] S. Treil, *The problem of ideals of H^∞: beyond the exponent* 3/2, J. Funct. Anal. **253** (2007), 220–240.

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Received August 16, 2013