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Operator ideals in Tate objects

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Tate's central extension originates from 1968 and has since found many applications to curves. In the 80s Beilinson found an n-dimensional generalization: cubically decomposed algebras, based on ideals of bounded and discrete operators in ind-pro limits of vector spaces. Kato and Beilinson independently defined '(n-)Tate categories' whose objects are formal iterated ind-pro limits in general exact categories. We show that the endomorphism algebras of such objects often carry a cubically decomposed structure, and thus a (higher) Tate central extension. Even better, under very strong assumptions on the base category, the n-Tate category turns out to be just a category of projective modules over this type of algebra.

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In his 1980 paper "Residues and adèles" [4] A. A. Beilinson introduced the following algebraic structure, without giving it a name:

Definition 1. A *Beilinson n-fold cubical algebra* is

- an associative k-algebra A;
- two-sided ideals I_i^+, I_i^- such that $I_i^+ + I_i^- = A$ for $i = 1, \dots, n$;
- call $I_{tr} := \bigcap_{i=1,\dots,n} I_i^+ \cap I_i^-$ the *trace-class* operators of A.

In his 1987 paper "How to glue perverse sheaves" [5] he introduced the exact category $1\text{-}\mathsf{Tate}^{el}_{\aleph_0}\mathcal{C}$ for any given exact category $\mathcal{C}$. It was suggestively denoted by $\underleftrightarrow{\lim}\,\mathcal{C}$ in loc. cit. We shall recall its definition in §1.

Although these two papers do not cite each other, some ideas in them can be viewed as two sides of the same coin. In the present paper we establish a rigorous connection between them. In fact, the main idea is that the latter category — under a number of assumptions — are just the projective modules over the former type of algebras. But this really requires some assumptions — in general it is quite far from the truth.

Define $n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C}) := \mathsf{Tate}^{el}_{\aleph_0}(\,(n-1)\text{-}\mathsf{Tate}_{\aleph_0}(\mathcal{C})\,)$ and $n\text{-}\mathsf{Tate}_{\aleph_0}(\mathcal{C})$ as the idempotent completion of $n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$. We write $P_f(R)$ for the category of finitely generated projective right R-modules.

Theorem 1. *Let $\mathcal{C}$ be an idempotent complete split exact category.*

1) *For every object $X \in n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$ its endomorphism algebra canonically carries the structure of a Beilinson n-fold cubical algebra.*

2) *If there is a countable family of objects $\{S_i\}$ in $\mathcal{C}$ such that every object in $\mathcal{C}$ is a direct summand of some countable direct sum of objects from $\{S_i\}$, then there exists (non-canonically) an object $X \in n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$ such that*

$$n\text{-}\mathsf{Tate}_{\aleph_0}(\mathcal{C}) \xrightarrow{\sim} P_f(R) \quad \textit{with} \quad R := \mathrm{End}(X)$$

is an exact equivalence of exact categories.

3) *Under this equivalence, the ideals $I_i^{\pm}$ correspond to certain categorical ideals, which can be defined even if $\mathcal{C}$ is not split exact.*

See Theorem 15 and Theorem 17 in the paper for details. In other words: In some sense the approaches of the 1980 paper and the 1987 paper are essentially equivalent. If $\mathcal{C}$ is not split exact, the ideals I_i^+, I_i^- still exist, see Theorem 13 in the text. However, the property $I_i^+ + I_i^- = A$ can fail to hold; Example 7 will give a counter-example.

V. G. Drinfeld has also introduced a category fitting into the same context, his notion of "Tate R-modules" for a given ring R [23]. We call it $\mathsf{Tate}^{Dr}(R)$ and give the definition later. In loc. cit. these appear without a restriction on the cardinality. However, if we restrict to countable cardinality, then Theorem 1 also implies:

Theorem 2. *Let R be a commutative ring. Then there is an exact equivalence of categories*

$$\mathsf{Tate}^{Dr}_{\aleph_0}(R) \xrightarrow{\sim} P_f(E),$$

where E is the Beilinson 1-fold cubical algebra

$$E := \mathrm{End}_{\mathsf{Tate}^{Dr}_{\aleph_0}(R)}\left(R((t)) \right),$$

where "$R((t))$" is understood to be the 'Tate R-module à la Drinfeld' denoted by $R((t))$ in Drinfeld's paper [23].

See Theorem 18 in the paper. This also reveals a certain additional structure on endomorphisms of Drinfeld's Tate R-modules, which appears not to have been studied so far at all.

Beilinson has originally considered the category $1\text{-}\mathsf{Tate}^{el}_{\aleph_0}\mathcal{C}$, i.e. without idempotent completion. Our previous paper [17, §3.2.7] shows:

Theorem 3. *The category $1\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$ can fail to be idempotent complete. In particular, one cannot improve Theorem 1 to*

$$n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C}) \xrightarrow{\sim} P_f(R),$$

i.e. without the idempotent completion, regardless which ring R is taken.

This follows simply since $P_f(-)$ is always an idempotent complete category. For some constructions the categories $n\text{-}\mathsf{Tate}_{\aleph_0}(\mathcal{C})$ are too small since the admissible Ind- and Pro-limits are only allowed to be taken over *countable* diagrams. This happens for example when writing down the adèles of a curve over an uncountable base field as a 1-Tate object. In our previous paper [17] we have therefore constructed categories $n\text{-}\mathsf{Tate}_{\kappa}(\mathcal{C})$, constraining the size of limits by a general infinite cardinal κ. Examples due to J. Šťovíček and J. Trlifaj [17, Appendix B] demonstrate the following

Theorem 4. *Even if $\mathcal{C}$ is split exact and idempotent complete, the category $1\text{-}\mathsf{Tate}_{\kappa}(\mathcal{C})$ for $\kappa > \aleph_0$ can fail to be split exact. In particular, one cannot improve Theorem 1 to general cardinalities κ.*

A key application of our results are to adèles of schemes, as introduced by A. N. Parshin and Beilinson [47], [4]. A detailed account was given by A. Huber [36]. We state the next result in the language of these papers, but the reader will also find the necessary notation and background explained in the main body of the present text:

Theorem 5. *Let k be a field and X/k a finite type scheme of pure dimension n. Let $\mathcal{F}$ be any quasi-coherent sheaf and $\triangle \subseteq S(X)_n$ a subset.*

1) *Then the Beilinson-Parshin adèles $A(\triangle, \mathcal{F})$ can be viewed as an elementary n-Tate object in finite-dimensional k-vector spaces, i.e. so that*
$$A(\triangle, \mathcal{F}) \in n\text{-}\mathsf{Tate}^{el}(\mathsf{Vect}_f)\,.$$
2) *The ring $\operatorname{End}(A(\triangle, \mathcal{O}_X))$ carries the structure of an n-fold cubical Beilinson algebra as in Definition 1.*
3) *If $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ is a singleton and $\operatorname{codim}_X \overline{\{\eta_i\}} = i$, there is a canonical isomorphism $\operatorname{End}_{n\text{-}\mathsf{Tate}^{el}}(A(\triangle, \mathcal{O}_X)) \cong E_{\triangle}^{\mathrm{Beil}}$, where $E_{\triangle}^{\mathrm{Beil}}$ denotes Beilinson's original cubical algebra from [4, §3, "E_{Δ}"] (defined without Tate categories).*

See Theorem 22 in the paper — in a way this result is the counterpart of a recent result of Yekutieli [56, Theorem 0.4], who uses topologies instead of Tate objects however. Theorem 5 does not follow from Theorem 1 since adèles with very few exceptions hinge on forming uncountably infinite limits. Trying to generalize (1), one may also view the adèles as n-Tate objects over other categories, e.g. finite abelian groups if k gets replaced by the integers $\mathbf{Z}$, or coherent sheaves with zero-dimensional support. However, for these variations parts (2) and (3) of the theorem would be false. We refer the reader to §10 for counter-examples

Historically, J. Tate's paper [53] introduced the first example of a Beilinson n-fold cubical algebra, but only for the case $n = 1$. He developed a formalism of traces for his trace-class operators, lifting the trace of finite-dimensional vector spaces. We can generalize this to exact categories:

An *exact trace* is a natural notion of a formalism of traces for a general exact category, see §8 for details.

Theorem 6. *Suppose $\mathcal{C}$ is an idempotent complete exact category and $\mathrm{tr}_{(-)}$ an exact trace on $\mathcal{C}$ with values in an abelian group Q. Then for every object $X \in n\text{-}\mathsf{Tate}(\mathcal{C})$ and $I_{tr} := I_{tr}(\mathrm{End}(X))$ its trace-class operators, there is a canonically defined trace*

$$\tau_X : I_{tr}/[I_{tr}, I_{tr}] \to Q,$$

such that for a short exact sequence $A \hookrightarrow B \twoheadrightarrow B/A$ and $f \in I_{tr}(B)$ so that $f\mid_A$ factors over A, we have

$$\tau_B(f) = \tau_A(f\mid_A) + \tau_{B/A}(\overline{f}).$$

If $X \in \mathcal{C}$, this trace agrees with the given trace, $\tau_X = \mathrm{tr}_X$.

See Prop. 19 for the full statement, which is more detailed and gives a unique characterization of τ in terms of the input trace. We also get:

Theorem 7. *Let $\mathcal{C}$ be an idempotent complete exact category. Then for every trace-class morphism $\varphi \in I_{tr}(X, X)$ some sufficiently high power $\varphi^{\circ r}$ (or a sufficiently long word made from several such morphisms) will factor through an object in $\mathcal{C}$.*

This generalizes a property which Tate had baptized 'finite-potent' and which plays a key rôle in his construction of the trace.

Tate and Beilinson used the n-fold cubical algebras to produce (higher) central extensions. The classical example is Tate's central extension, which encodes the residue of a rational 1-form. Ultimately, these constructions can be translated into Lie (and Hochschild) homology classes. Under mild assumptions, we can construct these classes also for the endomorphism algebras of n-Tate objects.

Theorem 8. *Let k be a field. Suppose $\mathcal{C}$ is a k-linear abelian category with a k-valued exact trace. For every n-sliced object*[1] *$X \in n\text{-}\mathsf{Tate}(\mathcal{C})$, the*

[1]See the main body of the text for definitions.

endomorphism algebra $E := \mathrm{End}(X)$ *is a Beilinson* n*-fold cubical algebra and*

1) *its Lie algebra* $\mathfrak{g}_X := E_{Lie}$ *carries a canonical Beilinson–Tate Lie cohomology class,*

$$\phi_{\mathrm{Beil}} \in H^{n+1}_{\mathrm{Lie}}(\mathfrak{g}_X, k);$$

2) *as well as canonical Hochschild and cyclic homology functionals*

$$\phi_{HH} : HH_n(E) \to k \quad resp. \quad \phi_{HC} : HC_n(E) \to k.$$

See Theorem 21 for details. For $n = 1$ the class ϕ_{Beil} just happens to define a central extension as a Lie algebra. Of course, the classical examples are all special cases of this construction. We provide some examples in §9.

Tate categories and Beilinson cubical algebras have already found quite diverse applications. Ranging from residue symbols in [53], [4], glueing sheaves [5], over models for infinite-dimensional vector bundles [23][2], to higher local compactness and Fourier theory [40], [44], [45], e.g. for the representation theory of algebraic groups over higher local fields [39], [29], [30], [31], [18], [41].

Quite recently, B. Hennion has introduced Tate categories for stable ∞-categories [34], [33]. It would be interesting to study the counterparts of our results in this context. Higher local fields can be regarded as n-Tate objects and in [43] D. V. Osipov has already related adèles to categories similar to n-Tate categories and studied endomorphism rings in this context. In a quite different direction, A. Yekutieli [54], [56] develops the use of semi-topological algebraic structures to describe adèles. These also give rise to an n-fold cubical algebra, but in a different way based on picking coefficient fields for the individual layers of the involved higher local fields. The relation to his approach is explained in [16].

1. Tate categories

For every exact category $\mathcal{C}$ one can form the corresponding categories of admissible Ind-objects $\mathsf{Ind}^a\mathcal{C}$ or admissible Pro-objects $\mathsf{Pro}^a\mathcal{C}$, perhaps with

[2]Drinfeld's paper proposes several notions, one of them being Tate R-modules, which are closest to the subject of this paper. The other are flat Mittag-Leffler bundles, which are however also related via admissible Ind-objects of projectives. See [24], [17, Appendix].

some conditions on the allowed cardinality of diagrams, denoted by a subscript as in $\mathsf{Ind}^a_{\aleph_0}$ or more generally Ind^a_κ for κ an infinite cardinal. See [50], [17] for definitions and basic properties. Enlarging $\mathcal{C}$ in both of these ways, we arrive at the commutative square of inclusion functors

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathsf{Ind}^a\mathcal{C} \arrow[d] \\ \mathsf{Pro}^a\mathcal{C} \arrow[r] & \mathsf{Ind}^a\mathsf{Pro}^a\mathcal{C}. \end{tikzcd}$$

Definition 2. Let $\mathcal{C}$ be an exact category.

1) The category $\mathsf{Tate}^{el}\mathcal{C}$ is the smallest extension-closed full sub-category of the category $\mathsf{Ind}^a\mathsf{Pro}^a\mathcal{C}$ which contains both $\mathsf{Ind}^a\mathcal{C}$ and $\mathsf{Pro}^a\mathcal{C}$.
2) $\mathsf{Tate}(\mathcal{C})$ denotes the idempotent completion of $\mathsf{Tate}^{el}\mathcal{C}$.
3) Define n-$\mathsf{Tate}^{el}(\mathcal{C}) := \mathsf{Tate}^{el}\left((n-1)\text{-}\mathsf{Tate}(\mathcal{C}) \right)$ and n-$\mathsf{Tate}(\mathcal{C}) := n$-$\mathsf{Tate}^{el}(\mathcal{C})^{ic}$ as its idempotent completion.

All of these categories come with a natural exact category structure so that all basic tools of homological algebra are available, they have derived categories, K-theory, etc... Versions of the Tate category were first introduced by K. Kato in 1980 in an IHES preprint, published only much later [40], and independently by A. Beilinson under the suggestive name $\varprojlim\!\!\!\!\!\!\longleftrightarrow \mathcal{C}$ [5]. The equivalence of these two approaches was established by L. Previdi. There is also a slightly different variant due to V. Drinfeld [23]. We refer the reader to [50], [17] for extensive discussions of these categories and comparison results. The 'C_n categories' of D. V. Osipov [43] are based on similar ideas. The definition of Tate categories which we give here is due to [17].

Example 1 (Kapranov). If $\mathcal{C}$ is the abelian category of finite-dimensional k-vector spaces, $\mathsf{Tate}^{el}\mathcal{C}$ is equivalent to the exact category of locally linearly compact topological k-vector spaces.

Example 2. If R is a commutative ring and $P_f(R)$ the exact category of finitely generated projective R-modules, $\mathsf{Tate}_{\aleph_0}P_f(R)$ is equivalent to the category of countably generated "Tate R-modules" in the sense of Drinfeld [23]. Without the restriction on countable generation, the latter is in general only a full sub-category of the former. Both are proven in [17, Thm. 5.30].

Example 3. We refer the reader to the works of Kato [40], Kapranov [39, Appendix] and Previdi [50] for a discussion of Tate categories for non-additive categories. These will not appear in the present paper.

Every object in the category $\mathsf{Tate}^{el}\mathcal{C}$ comes with a notion of 'lattices'.

Definition 3. Let $X \in \mathsf{Tate}^{el}\mathcal{C}$ be an object. We call a sub-object[3] $L \hookrightarrow X$ a *lattice* (or *Tate lattice* if we wish to contrast the notion to other concepts of lattices) if $L \in \mathsf{Pro}^a\mathcal{C}$ and $X/L \in \mathsf{Ind}^a\mathcal{C}$. The set of all lattices in X is the *Sato Grassmannian* $Gr(X)$.

There are two basic properties which control most of the behaviour of this concept of lattices: If $L' \hookrightarrow L \hookrightarrow X$ are two lattices in X, then L/L' lies in the category $\mathcal{C}$ [17, Proposition 6.6] . Furthermore, if $\mathcal{C}$ is idempotent complete, any two lattices have a common sub-lattice and a common over-lattice [17, Theorem 6.7].

2. The motivating classical example

The following algebraic structure was introduced by Beilinson [4] for the purpose of generalizing Tate's 1968 construction of the one-dimensional residue symbol [53] to higher dimensions. The constructions in loc. cit. produce a kind of generalized residue symbol for any such algebraic structure. The importance of this structure extends far beyond just the residue symbol. In a way, it axiomatizes essential algebraic features of the endomorphism algebra of a well-behaved n-Tate object. Before addressing this, let us recall the definition:

Definition 4. [4, §1] Let k be a field. An *(n-fold) cubically decomposed algebra* over k is the datum $(A, (I_i^{\pm}), \tau)$:

- an associative k-algebra A;
- two-sided ideals I_i^+, I_i^- such that $I_i^+ + I_i^- = A$ for $i = 1, \ldots, n$;
- writing $I_i^0 := I_i^+ \cap I_i^-$ and $I_{tr} := I_1^0 \cap \cdots \cap I_n^0$, a k-linear map (called *trace*)

$$\tau : I_{tr}/[I_{tr}, A] \to k.$$

[3] In this paper, the symbols $\hookrightarrow$ and $\twoheadrightarrow$ denote admissible monics and epics with respect to the exact structure of a category. Moreover, a sub-object always refers to an admissible sub-object in the sense that the inclusion is an admissible monic.

In our applications A will usually be unital and τ very close to a classical notion of trace, but technically A could be a non-unital algebra and τ the zero map.

Next, let us recall the original key example for this structure, coming straight from geometry. Suppose X/k is a reduced scheme of finite type and pure dimension n. We shall use the same notation as in [4]. Notably, $S(X)_\bullet$ denotes the simplicial set of flags of points (i.e. $S(X)_n = \{(\eta_0 > \cdots > \eta_n)\}$ with $\eta_i \in X$ and $x > y$ means that $\overline{\{x\}} \ni y$). Further, given $\triangle \subseteq S(X)_n$ we write ${}_{\eta_0}\triangle := \{(\eta_1 > \cdots > \eta_n) \mid (\eta_0 > \cdots > \eta_n) \in \triangle\}$. Finally, $A(\triangle, \mathcal{M})$ denotes the *Beilinson-Parshin adèles* for $\triangle \subseteq S(X)_n$. This means that for any *coherent* sheaf $\mathcal{M}$ we define

$$A(\triangle, \mathcal{M}) := \prod_{\eta\in\triangle} \varprojlim_i (\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta^i) \qquad \text{(in the case } n = 0\text{)}$$

$$A(\triangle, \mathcal{M}) := \prod_{\eta\in X} \varprojlim_i A({}_\eta\triangle\,, \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta^i) \qquad \text{(in the case } n \geq 1\text{)}$$

and for a *quasi-coherent* sheaf $\mathcal{M}$ we define $A(\triangle, \mathcal{M}) := \underrightarrow{\operatorname{colim}}_{\mathcal{M}'} A(\triangle, \mathcal{M}')$ and the colimit is taken over the category of coherent sub-sheaves of $\mathcal{M}$ with inclusions as morphisms. These colimits and limits are usually taken in the bi-complete category of $\mathcal{O}_X$-module sheaves. We follow this viewpoint here as well, at least for the moment. Later, in §10 we shall address a novel perspective using Tate categories instead.

Definition 5 (Beilinson [4]). Let $\triangle = \{(\eta_0 > \cdots > \eta_i)\} \in S(X)_i$ be given and M a finitely generated $\mathcal{O}_{\eta_0}$-module. Then a *(Beilinson) lattice* in M is a finitely generated $\mathcal{O}_{\eta_1}$-module $L \subseteq M$ such that $\mathcal{O}_{\eta_0} \cdot L = M$. Now and later on, we shall use the abbreviation

$$M_\triangle := A(\triangle, M)$$

for M a quasi-coherent sheaf on X.

Whenever we are given a $\triangle$ as above, define $\triangle' := \{(\eta_1 > \cdots > \eta_n)\}$, removing the initial entry.

Definition 6 (Beilinson [4]). Let M_1 and M_2 be finitely generated $\mathcal{O}_{\eta_0}$-modules.

1) Let $\mathrm{Hom}_{\varnothing}(M_1, M_2) := \mathrm{Hom}_k(M_1, M_2)$ be the set of all k-linear maps. Then we define[4]

$$\mathrm{Hom}_{\triangle}(M_1, M_2) \subseteq \mathrm{Hom}_k(M_{1\triangle}, M_{2\triangle})$$

to be the sub-k-module of all $f \in \mathrm{Hom}_k(M_{1\triangle}, M_{2\triangle})$ such that for all (Beilinson) lattices $L_1 \hookrightarrow M_1, L_2 \hookrightarrow M_2$ there exist (Beilinson) lattices $L_1' \hookrightarrow M_1, L_2' \hookrightarrow M_2$ with

$$L_1' \hookrightarrow L_1, \quad L_2 \hookrightarrow L_2', \quad f(L'_{1\triangle'}) \hookrightarrow L_{2\triangle'}, \quad f(L_{1\triangle'}) \hookrightarrow L'_{2\triangle'}$$

and for all such L_1, L_1', L_2, L_2' the induced k-linear map

$$\overline{f} : (L_1/L_1')_{\triangle'} \to (L_2'/L_2)_{\triangle'}$$

lies in $\mathrm{Hom}_{\triangle'}(L_1/L_1', L_2'/L_2)$.

2) Let $I^{+}_{1\triangle}(M_1, M_2)$ be those morphisms $f \in \mathrm{Hom}_{\triangle}(M_1, M_2)$ such that there exists a lattice $L \hookrightarrow M_2$ with $f(M_{1\triangle}) \hookrightarrow L_{\triangle'}$.

3) Dually, $I^{-}_{1\triangle}(M_1, M_2)$ is formed of those such that there exists a lattice $L \hookrightarrow M_1$ with $f(L_{\triangle'}) = 0$.

4) For $i \geq 2$ we let $I^{+}_{i\triangle}(M_1, M_2)$ be those $f \in \mathrm{Hom}_{\triangle}(M_1, M_2)$ such that for all lattices L_1, L_1', L_2, L_2' as in part (1) the condition

$$\overline{f} \in I^{+}_{(i-1)\triangle'}(L_1/L_1', L_2'/L_2)$$

holds. Analogously, we define $I^{-}_{i\triangle}(M_1, M_2)$ to consist of those which satisfy the condition $\overline{f} \in I^{-}_{(i-1)\triangle'}(L_1/L_1', L_2'/L_2)$ instead.

With these definitions in place we are ready to formulate the principal source of algebras as in Definition 1:

Theorem 9 (Beilinson, [4, §3]). *Suppose X/k is a reduced finite type scheme of pure dimension n. Let $\eta_0 > \cdots > \eta_n \in S\,(X)_n$ be a flag with* $\mathrm{codim}_X \overline{\{\eta_i\}} = i$. *Then*

$$E^{\mathrm{Beil}}_{\triangle} := \mathrm{Hom}_{\triangle}(\mathcal{O}_{\eta_0}, \mathcal{O}_{\eta_0}) \quad \subseteq \mathrm{End}_k(\mathcal{O}_{X\triangle}, \mathcal{O}_{X\triangle})$$

is a unital associative sub-algebra. Define $I^{\pm}_{i\triangle} \subseteq E^{\mathrm{Beil}}_{\triangle}$ by $I^{\pm}_{i\triangle}(\mathcal{O}_{\eta_0}, \mathcal{O}_{\eta_0})$ for $1 \leq i \leq n$. Then $(E^{\mathrm{Beil}}_{\triangle}, (I^{\pm}_{i\triangle}), \mathrm{tr})$ is an n-fold cubically decomposed algebra.

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[4]Below, the slightly shortened notation $M_{1\triangle}$ resp. $M_{2\triangle}$ refers to $(M_1)_{\triangle}$ resp. $(M_2)_{\triangle}$.

Here "tr" refers to Tate's trace for finite-potent morphisms, see [53] for the definition. In particular, $E_{\triangle}^{\mathrm{Beil}}$ is an example of the algebras in Definition 1.

There are a number of other examples leading to cubically decomposed algebras:

Example 4 (Yekutieli). Every topological higher local field (TLF) [54], [55] carries Yekutieli's canonical cubically decomposed algebra structure [56, Thm. 0.4]. If the base field k is perfect, one can show that the adèles decompose as a kind of restricted product of TLFs. Yekutieli's cubically decomposed algebra then turns out to be isomorphic to Beilinson's. See [16] for details.

Example 5. Higher infinite matrix algebras also carry a cubically decomposed structure. This is probably the simplest non-trivial example [11].

So how can we connect Beilinson's Theorem with the category $\mathsf{Tate}^{el}\mathcal{C}$?

3. Operator ideals in Tate categories

First of all, we will show that the condition of Definition 6, part (1), naturally comes up in the context of lattices of Tate objects. This requires some preparation. We need to establish some features which would be entirely obvious if we dealt with k-vector spaces and the notion of lattices from Definition 5.

It was shown in [17] that Pro-objects are left filtering in $\mathsf{Tate}^{el}(\mathcal{C})$ and Ind-objects right filtering. The following result strengthens these two facts:

Proposition 10. *Let $\mathcal{C}$ be an exact category.*

1) *Every morphism $Y \xrightarrow{f} X$ in $\mathsf{Tate}^{el}(\mathcal{C})$ with $Y \in \mathsf{Pro}^{a}(\mathcal{C})$ can be factored as $Y \overset{\tilde{f}}{\to} L \hookrightarrow X$ with L a lattice in X.*
2) *Every morphism $X \xrightarrow{g} Y$ in $\mathsf{Tate}^{el}(\mathcal{C})$ with $Y \in \mathsf{Ind}^{a}(\mathcal{C})$ can be factored as $X \twoheadrightarrow X/L \overset{\tilde{g}}{\to} Y$ with L a lattice in X.*

Proof. A complete proof is given in [14, Proposition 2.7]. □

Lemma 1. *Suppose $\mathcal{C}$ is an exact category. Let $X, X' \in \mathsf{Tate}^{el}\mathcal{C}$ and $\varphi \in$ Hom(X, X').*

1) *For every lattice* $L \hookrightarrow X$ *there exists a lattice* $L' \hookrightarrow X'$ *admitting a factorization as depicted below on the left.*
2) *For every lattice* $L' \hookrightarrow X'$ *there exists a lattice* $L \hookrightarrow X$ *admitting a factorization as depicted on the right.*

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, dotted, "\varphi'"'] & X \arrow[d, "\varphi"] \\ L' \arrow[r, hook, dotted] & X' \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, hook, dotted] \arrow[d, dotted, "\varphi'"'] & X \arrow[d, "\varphi"] \\ L' \arrow[r, hook] & X' \end{tikzcd}$$

Proof. This immediately follows from Prop. 10. (1) Here $L \to X'$ is a morphism from a Pro-object, so it factors through a lattice L' of X'. (2) Here $X \to X'/L'$ is a morphism to an Ind-object, so it factors $X \twoheadrightarrow X/L \to X'/L'$ for some lattice L. □

Lemma 2 (Cartesian sandwich). *Suppose* $\mathcal{C}$ *is idempotent complete. Let* $X_1, X_2 \in \mathsf{Tate}^{el}(\mathcal{C})$ *and* $L \hookrightarrow X_1 \oplus X_2$ *a lattice. Then there exist lattices* $L_i' \subseteq L_i$ *of* X_i *so that*

$$L_1' \oplus L_2' \subseteq L \subseteq L_1 \oplus L_2.$$

Proof. Consider the composed morphism $L \hookrightarrow X_1 \oplus X_2 \twoheadrightarrow X_i$. By Prop. 10 this factors through a lattice of X_i, say $L_i \hookrightarrow X_i$. Now we already have

$$\begin{tikzcd} L \arrow[d] \arrow[dr, hook] & \\ L_1 \oplus L_2 \arrow[r, hook] & X_1 \oplus X_2 \end{tikzcd}$$

By [17, Lemma 6.9] the downward arrow is an admissible monic (this crucially makes use of the assumption that $\mathcal{C}$ is idempotent complete). Dually, consider the composition $X_i \hookrightarrow X_1 \oplus X_2 \twoheadrightarrow (X_1 \oplus X_2)/L$. Since Ind-objects are right filtering [17, Prop. 5.10 (2)], there must be a lattice L_i' so that the map factors as $X_i \twoheadrightarrow X_i/L_i' \to (X_1 \oplus X_2)/L$. Thus, the composition $L_1' \oplus L_2' \hookrightarrow X_1 \oplus X_2 \twoheadrightarrow (X_1 \oplus X_2)/L$ is zero and by the universal property of kernels we get a canonical morphism $L_1' \oplus L_2' \to L$. Again by [17, Lemma 6.9] the corresponding morphism of the lattice quotients must be an admissible epic, so that this is an admissible monic. □

Definition 7. Let $\mathcal{C}$ be an exact category. For objects $X, X' \in \mathsf{Tate}^{el}\mathcal{C}$ call a morphism $\varphi : X \to X'$

1) *bounded* if there exists a lattice $L' \subseteq X'$ so that φ factors as $X \to L' \hookrightarrow X'$;
2) *discrete* if there exists a lattice $L \subseteq X$ so that $L \hookrightarrow X \xrightarrow{\varphi} X'$ is the zero morphism;
3) *finite* if it is both bounded and discrete.

Denote by $I^+(X, X')$, $I^-(X, X')$ and $I^0(X, X')$ the subsets of $\operatorname{Hom}(X, X')$ of bounded, discrete and finite morphisms respectively.

Lemma 3. *Suppose $\mathcal{C}$ is idempotent complete and $s \in \{+, -, 0\}$. Then for arbitrary objects X, X', X'' the following are true:*

1) $I^s(X, X')$ *is a subgroup with respect to addition.*
2) $I^s(X, X')$ *is a categorical ideal, i.e. the composition of any morphism with a morphism in I^s lies in I^s. Thus, the composition of morphisms factors as*

$$I^s\left(X', X''\right) \otimes \operatorname{Hom}\left(X, X'\right) \longrightarrow I^s\left(X, X''\right)$$
$$\operatorname{Hom}\left(X', X''\right) \otimes I^s\left(X, X'\right) \longrightarrow I^s\left(X, X''\right).$$

3) *In the ring* $\operatorname{End}(X)$ *the subgroup* $I^s(X, X)$ *is a two-sided ideal.*
4) *Every morphism in $I^0(X, X')$ factors through a morphism from an Ind- to a Pro-object. Every product of at least two morphisms in I^0 factors through an object in $\mathcal{C}$.*

Example 6. Let $\mathcal{C} := \mathsf{Vect}_f$ be the category of finite-dimensional k-vector spaces. Define $X := k[t] \oplus k[[t]]$ and an endomorphism

$$\varphi : k[t] \oplus k[[t]] \longrightarrow k[t] \oplus k[[t]], \qquad (a, b) \longmapsto (0, a).$$

It is easy to see that $\varphi \in I^0(X, X)$, but φ does not factor through an object in $\mathcal{C}$. This shows that Lemma 3 (4) cannot be strengthened in this alluring fashion. Note that this phenomenon is already present in Tate's original work [53]. It is the reason why he has to work with 'finite-potent' morphisms rather than finite rank ones.

Proof. (1) Suppose $\varphi_1, \varphi_2 \in I^+(X, X')$ are given and they factor as

$$\varphi_1 : X \to L'_1 \hookrightarrow X' \quad \text{and} \quad \varphi_2 : X \to L'_2 \hookrightarrow X'.$$

Then by the directedness of the Sato Grassmannian [17, Thm. 6.7] there exists a lattice L'_3 so that $L'_i \subseteq L'_3$ for $i = 1, 2$ and thus without loss of generality we may assume $L'_i = L'_3$ in the above factorizations, so the claim is clear. The same works for $I^-(X, X')$ by taking a common sub-lattice. By $I^0(X, X') = I^+(X, X') \cap I^-(X, X')$ it also works for I^0. (2) For $\varphi \in I^+$ and ψ arbitrary it is trivial that $\varphi \circ \psi$ factors through a lattice, namely the same as φ does. In the reverse direction $\psi \circ \varphi$ we have a factorization as depicted on the left in

$$\begin{tikzcd} & L' \arrow[d, hook] & \\ X \arrow[ur] \arrow[r, "\varphi"'] & X' \arrow[r, "\psi"'] & X'' \end{tikzcd} \qquad \begin{tikzcd} & L' \arrow[d, hook] \arrow[r, dotted, "\psi|_{L'}"] & L'' \arrow[d, hook, dotted] \\ X \arrow[ur] \arrow[r, "\varphi"'] & X' \arrow[r, "\psi"'] & X'' \end{tikzcd}$$

since $\varphi \in I^+$. Then by Lemma 1 (1) there exists a lattice L'' in X'' so that we get a further factorization as depicted above on the right. Thus, we have a factorization $X \to L'' \hookrightarrow X''$ with L'' a lattice, so $\psi \circ \varphi$ is also bounded. For $\varphi \in I^-$ and ψ arbitrary, the proof is analogous. This time $\psi \circ \varphi$ trivially sends a lattice to zero, namely the same one as φ and the reverse direction $\varphi \circ \psi$ requires an argument, very analogous to the above one for I^+: Let L' be a lattice which is sent to zero by φ as shown left in

$$\begin{tikzcd} & L' \arrow[d, hook] \arrow[dr, "0"] & \\ X \arrow[r, "\psi"'] & X' \arrow[r, "\varphi"'] & X'' \end{tikzcd} \qquad \begin{tikzcd} L \arrow[d, hook, dotted] \arrow[r, dotted, "\psi|_L"] & L' \arrow[d, hook] \arrow[dr, "0"] & \\ X \arrow[r, "\psi"'] & X' \arrow[r, "\varphi"'] & X'' \end{tikzcd}$$

According to Lemma 1 (2) there exists a lattice L in X so that we can complete the diagram as depicted on the right. Hence $\varphi \circ \psi$ sends a lattice to zero. (3) is immediate from (2). For (4) let $X \xrightarrow{\varphi} X'$ be a morphism in $I^0(X, X')$. By boundedness we find a factorization through a lattice L' so that we get the diagram on the left

$$\begin{tikzcd} & L' \arrow[d, hook] \\ X \arrow[ur] \arrow[r, "\varphi"'] & X' \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, dotted] \arrow[d, hook] & L' \arrow[d, hook] \\ X \arrow[ur] \arrow[r, "\varphi"'] & X' \end{tikzcd}$$

and by discreteness a lattice L so that $L \to X \to X'$ is zero, as depicted on the right. Since $L' \hookrightarrow X'$ is a monomorphism, the induced upper horizontal arrow must be the zero map itself. By the universal property of kernels this yields a factorization

$$\begin{tikzcd} L \arrow[d, hook] \arrow[dr, "0"] & \\ X \arrow[r, "\varphi"] \arrow[d, two heads] & L' \\ X/L \arrow[ur] & \end{tikzcd}$$

Thus, we have obtained a factorization $X \twoheadrightarrow X/L \to L' \hookrightarrow X'$ as desired. If we compose any two morphisms, we may equivalently look at the composition of these factorizations,

$$X \twoheadrightarrow X/L \to L' \hookrightarrow X' \twoheadrightarrow X'/L'' \to L'' \hookrightarrow X''',$$

then $L' \to X'/L''$ is a morphism from a Pro-object to an Ind-object. Since Pro-objects are left filtering in $\mathsf{Tate}^{el}\mathcal{C}$, this factors through a Pro-sub-object of X'/L'', which is therefore also an Ind-object. However, by [17, Prop. 5.9] an object can only be simultaneously an Ind- and Pro-object if it actually lies in $\mathcal{C}$. $\square$

Definition 8. Let X be an elementary Tate object. If there exists a lattice $i : L \hookrightarrow X$ which admits a splitting $s : X \to L$ so that $si = \mathrm{id}_L$, we call X *sliced.*

Remark 4. Such a splitting must be an admissible epic. To see this, define $g : X \to X$ by $g := \mathrm{id}_X - is$. Then $gi = \mathrm{id}_X i - i\,\mathrm{id}_L = 0$ and by the universal property of cokernels, we deduce that g factors as $X \twoheadrightarrow X/L \xrightarrow{\tilde{g}} X$. If $q :$

$X \twoheadrightarrow X/L$ denotes the quotient map, a direct computation verifies that the maps in

$$X/L \underset{\tilde{g}}{\overset{q}{\leftrightarrows}} X \underset{s}{\overset{i}{\leftrightarrows}} L$$

produce a split exact sequence, with the given splittings. By [19, Lemma 2.7] this belongs to the exact structure and thus s is an admissible epic.

The following result is the categorical analogue of the decomposition used by J. Tate in his original article [53, Prop. 1].

Proposition 11. *Let $\mathcal{C}$ be an exact category.*

1) *If X is a sliced elementary Tate object,* $\mathrm{End}\,(X) = I^{+}(X,X) + I^{-}(X,X)$. *More generally, there is a short exact sequence of abelian groups*

$$0 \longrightarrow I^{0}(X,X) \longrightarrow I^{+}(X,X) \oplus I^{-}(X,X) \longrightarrow \mathrm{End}(X) \longrightarrow 0.$$

2) *If $\mathcal{C}$ is split exact and idempotent complete, every elementary Tate object in $\mathsf{Tate}^{el}_{\aleph_0}\mathcal{C}$ is sliced. In particular, each* $\mathrm{End}(X)$ *is a unital one-fold cubical algebra in the sense of Definition 1.*

Proof. (1) Define $P^{+} := is : X \to L \hookrightarrow X$. This is clearly a bounded morphism. Define $P^{-} := \mathrm{id}_X - P^{+}$. We find that after precomposing by $L \hookrightarrow X$ the two morphisms $X \overset{\mathrm{id}}{\to} X$, $X \overset{P^{+}}{\to} X$ agree, thus $P^{-}\mid_{L} = 0$, i.e. P^{-} is a discrete morphism. Finally, $\mathrm{id}_X = P^{+} + P^{-}$ by construction. Thus, any morphism $\varphi \in \mathrm{End}\,(X)$ can be written as $\varphi = P^{+}\varphi + P^{-}\varphi$ and since bounded and discrete morphisms form ideals, by Lemma 3, $P^{+}\varphi$ is bounded and $P^{-}\varphi$ discrete. (2) By [17, Prop. 5.23] the split exactness of $\mathcal{C}$ implies that $\mathsf{Tate}^{el}_{\aleph_0}\mathcal{C}$ is also split exact. Hence, we can pick any lattice (always exists), obtain $L \hookrightarrow X$, and the split exactness enforces the existence of a splitting. $\square$

Example 7. Let p be a prime number and $\mathcal{C}$ the abelian category of finite abelian p-groups. We shall show that $\mathsf{Tate}^{el}\mathcal{C}$ contains both sliced and non-sliced objects. Specifically, both

$$\text{“}\mathbf{F}_p((t))\text{”} = \underset{i}{\underrightarrow{\mathrm{colim}}}\, \underset{j}{\underleftarrow{\lim}}\, t^{-i}\mathbf{F}_p[t]/t^{j} \quad \text{and} \quad \text{“}\mathbf{Q}_p\text{”} = \underset{i}{\underrightarrow{\mathrm{colim}}}\, \underset{j}{\underleftarrow{\lim}}\, p^{-i}\mathbf{Z}/p^{j}$$

are elementary Tate objects.

1) The former is sliced via $\mathbf{F}_p((t)) \simeq \mathbf{F}_p[[t]] \oplus t^{-1}\mathbf{F}_p[t^{-1}]$ while the latter is not. To see this, note that $\mathsf{Ind}^a\mathcal{C}$ is equivalent to the category of p-primary torsion abelian groups. Hence, if there exists a splitting $\mathbf{Q}_p \simeq I \oplus P$ with $I \in \mathsf{Ind}^a\mathcal{C}$, we must have $I = 0$ since $\mathbf{Q}_p$ has no non-trivial torsion elements at all. Thus, $\mathbf{Q}_p \in \mathsf{Pro}^a\mathcal{C}$, forcing that $\mathbf{Q}_p/\mathbf{Z}_p \in \mathcal{C}$, but this is clearly absurd.

2) Proposition 11 fails for $\mathbf{Q}_p$. Suppose not. Then there exist $p \in I^+(\mathbf{Q}_p, \mathbf{Q}_p)$ and $q \in I^-(\mathbf{Q}_p, \mathbf{Q}_p)$ so that $\mathrm{id} = p + q$. Then pq and qp lie in $I^0(\mathbf{Q}_p, \mathbf{Q}_p)$, so by Lemma 3 they both factor through a morphism from an Ind- to a Pro-object. So they factor through torsion elements. Thus, $pq = qp = 0$. As a result, $p = p(p+q) = p^2$ and analogously for q. So these must be idempotents. Thus, in the idempotent completion we get a direct sum splitting $\mathbf{Q}_p \simeq \operatorname{im} p \oplus \operatorname{im} q$. Since q kills a lattice, say L, the map $\mathbf{Q}_p \overset{q}{\twoheadrightarrow} \operatorname{im} q$ descends to $\mathbf{Q}_p/L \twoheadrightarrow \operatorname{im} q$, forcing $\operatorname{im} q$ to be an Ind-object and therefore zero. Again, we obtain $\mathbf{Q}_p \in \mathsf{Pro}^a\mathcal{C}$, which is absurd.

We recall that in an additive category a morphism p is an epic if for any composition

$$X \xrightarrow{p} Y \xrightarrow{f} Z$$

which is zero, f must already have been zero. Now suppose we want to find a definition for 'locally epic'. Then lattices take over the rôle of a basis of open neighbourhoods of the neutral element. Hence, it makes sense to use the definition of epic morphisms, but restrict both the assumption as well as the conclusion to lattices. This leads to the following concept.

Definition 9. Let $p : X \to Y$ be a morphism of elementary Tate objects.

1) We call p *submersive* if for any morphism f and lattice $L \hookrightarrow X$ so that the diagonal arrow in

$$\begin{tikzcd}
L' \arrow[r, hook, dotted] & Y \arrow[r, "f"] & Z \\
L \arrow[r, hook] \arrow[urr, dashed, "0"] & X \arrow[u, "p"] &
\end{tikzcd}$$

is zero, there exists a lattice $L' \hookrightarrow Y$ (drawn with a dotted arrow) so that $L' \hookrightarrow Y \to Z$ is zero.

(Slogan: "vanishing on a lattice can be pushed forward")

2) Symmetrically, call p *immersive* if for any morphism f and lattice $L \hookrightarrow X$ so that the diagonal arrow in

$$\begin{tikzcd} Y/L' & Y \arrow[l, two heads, dotted] \arrow[d, "p"] & & Z \arrow[ll, "f"'] \arrow[dlll, dashed, "0"] \\ X/L & X \arrow[l, two heads] \end{tikzcd}$$

is zero, there exists a lattice $L' \hookrightarrow Y$ (whose quotient is drawn with a dotted arrow) so that $Z \to Y \twoheadrightarrow Y/L'$ is zero.

(Slogan: "vanishing modulo a lattice can be pulled back")

These two definitions are almost dual. One transforms one into the other by going to the opposite category and interchanging Ind- and Pro-objects.

Lemma 5. *Let $\mathcal{C}$ be an idempotent complete exact category.*

1) *Every admissible monic $p : Y \hookrightarrow X$ is immersive.*
2) *Every admissible epic $p : X \twoheadrightarrow Y$ is submersive.*

For example for an arbitrary lattice in an elementary Tate object, the inclusion is immersive and the respective quotient morphism submersive:

$$L \xhookrightarrow{\text{immersive}} X \xrightarrow{\text{submersive}} X/L$$

We shall show in Example 9 that the lemma can fail if we remove the word 'admissible'.

For a morphism $f : X \to Y$ in $\mathsf{Tate}^{el}(\mathcal{C})$, and $L \hookrightarrow X$ a lattice, the notation $f(L) = 0$ is shorthand for the statement that the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

commutes. As a first step towards the proof of the lemma, say for $p : X \twoheadrightarrow Y$ epic, we observe that for $Y \in \mathsf{Ind}^a(\mathcal{C})$ the statement is automatically true, since then we have that $0 \hookrightarrow Y$ is a lattice, and we certainly have $g(0) = 0$.

The general case relies on the following lemma.

Lemma 6. *Let $\mathcal{C}$ be an idempotent complete exact category. Let $g : M \to N$ be a morphism of admissible Pro-objects M, $N \in \mathsf{Pro}^a(\mathcal{C})$, which is sent to the zero morphism by the exact functor $\mathsf{Pro}^a(\mathcal{C}) \to \mathsf{Pro}^a(\mathcal{C})/\mathcal{C}$. Then there exists a commutative triangle*

$$\begin{tikzcd} & U \arrow[dr] & \\ M \arrow[ur, two heads, "g"] \arrow[rr] & & N, \end{tikzcd}$$

where $U \in \mathcal{C}$, and g is an admissible epic in $\mathsf{Pro}^a(\mathcal{C})$.

Proof. We use that $\mathcal{C} \subset \mathsf{Pro}^a(\mathcal{C})$ is right s-filtering [17, Prop. 4.2. (2)]. By an observation of Bühler this implies that the class Σ_m of admissible monomorphisms in $\mathsf{Pro}^a(\mathcal{C})$ with cokernel in $\mathcal{C}$ satisfies a calculus of right fractions, see [17, Prop. 2.19] for the broader context. Moreover, we also know from Bühler that $\mathsf{Pro}^a(\mathcal{C})[\Sigma_m^{-1}] \cong \mathsf{Pro}^a(\mathcal{C})/\mathcal{C}$, [17, Prop. 2.19]. Since $g : M \to N$ and $0 : M \to N$ induce the same map in $\mathsf{Pro}^a(\mathcal{C})/\mathcal{C}$, we see that there exists a commutative diagram

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr, "g"] & \\ M & M' \arrow[l] \arrow[r] \arrow[u, "h"] \arrow[d] & N \\ & M, \arrow[ul] \arrow[ur, "0"'] & \end{tikzcd}$$

where $h \colon M' \hookrightarrow M$ is an admissible monic with cokernel $Q \in \mathcal{C}$. The commutativity of the diagram implies that the horizontal arrow $M' \to N$ is zero. Therefore, we obtain by the universal property of cokernels a factorization

$$\begin{tikzcd} & Q \arrow[dr] & \\ M \arrow[ur, two heads] \arrow[rr, "g"] & & N \end{tikzcd}$$

as required to conclude the proof of the assertion. □

We are now ready to prove that admissible epimorphisms are submersive.

Proof of Lemma 5. We shall only treat the case of an admissible epic, and leave the necessary modifications for the monic case to the reader. By

Lemma 1 we have a commutative diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ M \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ N \arrow[r] & Z \end{tikzcd}$$

where the horizontal arrows are inclusions of lattices. We also know that the inclusion $\mathsf{Ind}^a(\mathcal{C}) \hookrightarrow \mathsf{Tate}^{el}(\mathcal{C})$ is right s-filtering [15, Cor. 2.3], and the quotient category is equivalent to $\mathsf{Pro}^a(\mathcal{C})/\mathcal{C}$ [17, Prop. 5.34]. Inclusions of lattices are sent to isomorphisms in $\mathsf{Tate}^{el}(\mathcal{C})/\mathsf{Ind}^a(\mathcal{C})$. Hence, we obtain that the composition $f \circ p$ is sent to 0 in $\mathsf{Tate}^{el}(\mathcal{C})/\mathsf{Ind}^a(\mathcal{C})$. However, exact functors send admissible epimorphisms to admissible epimorphisms; and every admissible epimorphism is an epimorphism in the categorical sense. The relation $f \circ p = 0 \circ p$ in $\mathsf{Tate}^{el}(\mathcal{C})/\mathsf{Ind}^a(\mathcal{C})$ implies now that $f = 0$ in $\mathsf{Tate}^{el}(\mathcal{C})/\mathsf{Ind}^a(\mathcal{C})$.

We have shown above that the morphism $M \to N$ in $\mathsf{Pro}^a(\mathcal{C})$ is sent to 0 in $\mathsf{Pro}^a(\mathcal{C})/\mathcal{C}$. By Lemma 6, this yields a factorization $M \twoheadrightarrow Q \to N$ with $Q \in \mathcal{C}$. Let L' be the kernel of the admissible epimorphism $M \twoheadrightarrow Q$. By construction $L' \subset Y$ is a lattice, and $f(L') = 0$. This concludes the proof. $\square$

Example 8. A submersive morphism does not need to be an epimorphism. For example, for $\mathcal{C} := \mathsf{Vect}_f$ the zero morphism $k[[t]] \xrightarrow{0} k[t]$ is a submersion. This makes sense topologically since we would think of $k[t]$ as having zero-dimensional tangent spaces. It is however also a finite immersion, which appears rather strange from the point of view of topological intuition.

Example 9. Let us construct a (non-admissible!) monomorphism which is not immersive. Let $\mathcal{C} := \mathsf{Vect}_f$ be the category of finite-dimensional k-vector spaces. We have a morphism

$$\varphi : k[t] \longrightarrow k((t)),$$

the obvious inclusion. This morphism is monic, but it is not an admissible monic since otherwise a Pro-object would have an Ind-object as a sub-object. We claim that this morphism is not immersive. Suppose it is. Take $Z \to Y$ to be the identity $\mathrm{id}_{k[t]}$ and $L := k[[t]]$, which is clearly a lattice in $k((t))$. The immersion property now implies that $k[t]$ must be a lattice in itself. In particular, it must be a Pro-object, which is absurd.

Lemma 7. *Submersive morphisms have the following properties:*

1) *Isomorphisms are submersive.*
2) *The composition of submersive morphisms is submersive.*

Proof. (1) is trivial, just transport the lattice along the isomorphism. (2) Let p, q be composable submersive morphisms. Let f be an arbitrary morphism and L a lattice that gets sent to zero by $(f \circ q) \circ p$, i.e. the lower diagonal arrow in

$$\begin{tikzcd}
L'' \arrow[r, hook, dotted] & Z \arrow[r, "f"] & W \\
L' \arrow[r, hook, dotted] \arrow[urr, dashed, "0"] & Y \arrow[u, "q"] & \\
L \arrow[r, hook] \arrow[uurr, dashed, "0"] & X \arrow[u, "p"] &
\end{tikzcd}$$

The submersiveness of p (for the morphism $f \circ q$) guarantees the existence of a lattice L' so that $f \circ q$ sends it to zero. Now the submersiveness of q yields the existence of a lattice L'' which is being sent to zero by f. But this is all we had to show. $\square$

Lemma 8. *Immersive morphisms have the following properties:*

1) *Isomorphisms are immersive.*
2) *The composition of immersive morphisms is immersive.*

Proof. The proof is essentially dual to the proof of Lemma 7, just reverse the direction of all arrows. $\square$

Lemma 9 ([17, Prop. 5.23]). *If $\mathcal{C}$ is idempotent complete and split exact,* $\mathsf{Tate}^{el}_{\aleph_0}\mathcal{C}$ *is split exact.*

Lemma 10. *Suppose we are given one of the squares*

$$\begin{tikzcd} X_1 \arrow[r, "\text{discrete}"] \arrow[d, "\text{submersive}"'] & X_2 \\ Y_1 \arrow[r, "f"'] & Y_2 \arrow[u, hook] \end{tikzcd} \qquad \textit{resp.} \qquad \begin{tikzcd} X_1 \arrow[r, "\text{bounded}"] \arrow[d, two heads] & X_2 \\ Y_1 \arrow[r, "g"'] & Y_2 \arrow[u, "\text{immersive}"'] \end{tikzcd}$$

Then f is discrete (resp. g bounded).

For this statement to be true the monic (resp. epic) would need not be admissible.

Proof. If $X_1 \to X_2$ is discrete, there is a lattice L so that the upper row in

$$\begin{tikzcd} L \arrow[r, hook] & X_1 \arrow[r, "\text{discrete}"] \arrow[d, "\text{submersive}"'] & X_2 \\ & Y_1 \arrow[r, "g"'] & Y_2 \arrow[u, hook] \end{tikzcd}$$

is the zero morphism. Since the right-hand side upward arrow is a monomorphism, it follows that $L \hookrightarrow X_1 \to Y_1 \to Y_2$ must already be zero. Now being submersive implies that there is a lattice L' in Y_1 so that $L' \hookrightarrow Y_1 \xrightarrow{g} Y_2$ is zero. Hence, g is discrete. The argument for the other square is dual. $\square$

We collect a few more useful properties.

Lemma 11. *Suppose $\mathcal{C}$ is idempotent complete.*

1) *If $p : X \to Y$ is submersive, either no lattice $L \hookrightarrow X$ is sent to zero, or Y is an Ind-object.*
2) *Submersive discrete morphisms are precisely the morphisms $X \to Y$ with Y an Ind-object.*
3) *If $p : X \to Y$ is immersive, either it does not factor through any lattice in Y, or X is a Pro-object.*
4) *Immersive bounded morphisms are precisely the morphisms $X \to Y$ with X a Pro-object.*

Proof. (1) If a lattice L exists that p sends to zero, being submersive gives a lattice L' in

$$\begin{tikzcd} L' \arrow[r, hook, dotted] & Y \arrow[rr, "1"] & & Y \\ L \arrow[r, hook] \arrow[urrr, dashed, "0"] & X \arrow[u, "p"] & & \end{tikzcd}$$

so that $L' \hookrightarrow Y$ is the zero map. So the zero object is a lattice, which forces Y to be an Ind-object. (2) if $p : X \to Y$ is discrete, a lattice is sent to zero, so just use (1). Conversely, if Y is an Ind-object, by Prop. 10 the morphism p factors through a lattice quotient $p : X \twoheadrightarrow X/L \to Y$. In particular p sends L to zero and so p is discrete. As Y is an Ind-object, the zero object is a lattice, so p is clearly submersive. (3) and (4) are dual. $\square$

Finally, we can show that the boundedness of a morphism is preserved under passing to sub-objects or quotients, and analogously for discreteness and finiteness.

Proposition 12. *Suppose $\mathcal{C}$ is idempotent complete. Let $\varphi : X \to X$ be a bounded (resp. discrete, finite) morphism and $Y \hookrightarrow X$ an admissible monic such that $\varphi\mid_Y$ factors over Y, i.e.*

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, hook] & Y \arrow[d, hook] \\ X \arrow[r, "\varphi"] & X \end{tikzcd} \tag{3.1}$$

Then

1) *the restriction $\varphi\mid_Y : Y \to Y$ is also bounded (resp. discrete, finite), and*
2) *the quotient map $\overline{\varphi} : X/Y \to X/Y$ is also bounded (resp. discrete, finite).*

If φ is discrete and $L' \hookrightarrow L \hookrightarrow X$ are lattices so that φ factors as

$$\varphi : L/L' \longrightarrow L/L',$$

then there exist lattices $L_1' \hookrightarrow L_1 \hookrightarrow Y$ *and* $L_2' \hookrightarrow L_2 \hookrightarrow X/Y$ *so that* $\varphi\mid_Y$ *and* $\overline{\varphi}$ *factor as*

$$\begin{aligned} \varphi\mid_Y : L_1/L_1' &\longrightarrow L_1/L_1' \\ \overline{\varphi} : L_2/L_2' &\longrightarrow L_2/L_2' \end{aligned}$$

and

$$L_1/L_1' \hookrightarrow L/L' \twoheadrightarrow L_2/L_2'$$

is short exact.

Proof. (1, Bounded) As $\varphi : X \to X$ is bounded, it factors over a lattice, say L. Thus, $Y \hookrightarrow X \xrightarrow{\varphi} X$ factors over L in the target, but by the commutativity of Diagram 3.1 this means that $Y \xrightarrow{\varphi} Y \hookrightarrow X$ factors over L in the target. Hence, we get the diagram

$$\begin{tikzcd} Y/L' & Y \arrow[l, two heads, dotted] \arrow[d, hook, "p"'] & Y \arrow[l, "\varphi"'] \arrow[dll, dashed, "0"] \\ X/L & X \arrow[l, two heads] \end{tikzcd}$$

By Lemma 5 the admissible monic p is immersive. Thus, a lattice L' as in the above diagram exists, showing that $\varphi\mid_L$ is bounded.

(1, Discrete) This is simpler. As $\varphi : X \to X$ is discrete, there exists a lattice $L \hookrightarrow X$ so that $L \hookrightarrow X \xrightarrow{\varphi} X$ is zero. By Lemma 1 there exists a lattice $L' \hookrightarrow Y$ such that under $Y \hookrightarrow X$ it maps to $L \hookrightarrow X$, and then the composition

$$L' \to L \to X \to X$$

is zero. Thus, by commutativity $L' \hookrightarrow Y \to Y \hookrightarrow X$ is zero, and by the defining property of monics, the composition $L' \hookrightarrow Y \to Y$ must already be zero. Since L' is a lattice, it follows that $\varphi\mid_L$ is discrete.

(1, Finite) Just combine both statements.

(2, Bounded) Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, two heads] & X \arrow[d, two heads] \\ X/Y \arrow[r, "\overline{\varphi}"] & X/Y. \end{tikzcd}$$

As φ is bounded, there exists a lattice $L \hookrightarrow X$ so that $\varphi : X \to L \hookrightarrow X$. By Lemma 1 there exists a lattice $L' \hookrightarrow X/Y$ so that $X \twoheadrightarrow X/Y$ restricted to L factors over $L' \hookrightarrow X/Y$. In other words, $X \stackrel{\varphi}{\to} X \twoheadrightarrow X/Y \twoheadrightarrow (X/Y)/L'$ is zero. By the commutativity of the diagram,

$$X \twoheadrightarrow X/Y \stackrel{\overline{\varphi}}{\to} X/Y \twoheadrightarrow (X/Y)/L'$$

must be zero as well. Since the first morphism is an epic, we deduce that $X/Y \stackrel{\overline{\varphi}}{\to} X/Y \twoheadrightarrow (X/Y)/L'$ is already the zero map. By the universal property of cokernels, this means that there is a factorization $\overline{\varphi} : X/Y \to L' \hookrightarrow X/Y$, i.e. $\overline{\varphi}$ is bounded.

(2, Discrete) As φ is discrete, there exists a lattice $L \hookrightarrow X$ such that $L \hookrightarrow X \stackrel{\varphi}{\to} X$ is zero. Thus, we obtain that the diagonal morphism in

$$\begin{tikzcd} L' \arrow[r, hook, dotted] & X/Y \arrow[rr, "\overline{\varphi}"] & & X/Y \\ L \arrow[r, hook] \arrow[urrr, dashed, "0"] & X \arrow[u, two heads, "p"] & & \end{tikzcd}$$

is zero. Following Lemma 5 the admissible epic p is submersive, i.e. there exists a lattice $L' \hookrightarrow X/Y$ such that $L' \hookrightarrow X/Y \stackrel{\overline{\varphi}}{\to} X/Y$ is zero. But this just means that $\overline{\varphi}$ is discrete, too.

(2, Finite) Just combine the last two statements.

(Lattices) Finally, combine the above constructions for a discrete morphism and lattices $L' \hookrightarrow L \hookrightarrow X$ such that φ factors as

$$\varphi : L/L' \longrightarrow L/L'.$$

We see that they construct lattices $L_1', L_1 \hookrightarrow Y$ so that $L_1 \hookrightarrow L$ and $L_1' \hookrightarrow L'$ under $Y \hookrightarrow X$; without loss of generality use the (co-)directedness of the Sato Grassmannian [17, Thm. 6.7] to achieve that $L_1' \hookrightarrow L_1$ holds, by replacing L_1 by a common over-lattice of the two constructed lattices if necessary. Proceed similarly for the quotient X/Y. $\square$

4. General Tate objects

In this section we extend the previous definitions to non-elementary Tate objects.

Let $\mathcal{C}$ be an exact category. We recall that its idempotent completion $\mathcal{C}^{ic}$ is the category whose objects are pairs (X, p) with $X \in \mathcal{C}$ and $p : X \to X$ with $p^2 = p$ an idempotent. Its morphisms are

$$\begin{aligned} \text{(4.1)} \quad \operatorname{Hom}_{\mathcal{C}^{ic}}((X,p),(Y,q)) &= \{f \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \mid qfp = f\} \\ &= \{f \mid \exists g \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \text{ so that } f = qgp\}\,. \end{aligned}$$

We refer to [19, §6] or [49, Ch. II] for a detailed construction and basic properties of the idempotent completion. Recall that $\mathsf{Tate}(\mathcal{C}) := (\mathsf{Tate}^{el}\mathcal{C})^{ic}$. We will now define all basic types of morphisms between general Tate objects by simply requiring that the morphism of the underlying elementary Tate objects has the relevant property. This is, by the way, the same mechanism which is employed to equip $\mathcal{C}^{ic}$ with an exact structure: A kernel-cokernel sequence in $\mathcal{C}^{ic}$ is called exact iff it is a direct summand of an exact sequence in $\mathcal{C}$. For example, if p is an idempotent of X,

$$(X,p) \to X \to (X, 1-p) \tag{4.2}$$

is a direct summand of

$$X \overset{(p\ 1-p)}{\hookrightarrow} X \oplus X \overset{\binom{1-p}{p}}{\twoheadrightarrow} X.$$

Hence, Sequence 4.2 is actually a short exact sequence in $\mathcal{C}^{ic}$. In particular, $(X,p) \hookrightarrow X$ is an admissible monic and, since $1-p$ is also an idempotent, it also follows that $X \twoheadrightarrow (X,p)$ is an admissible epic. See [19, Prop. 6.13] for more on this. In particular, admissible monics and epics in $\mathcal{C}^{ic}$ are represented by admissible monics and epics in $\mathcal{C}$.

Lemma 12. *Let $\mathcal{C}$ be a split exact category and $\mathcal{C}^{ic}$ its idempotent completion. Then $\mathcal{C}^{ic}$ is also split exact.*

Proof. Suppose $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{C}^{ic}$. Then by definition [19, §6, cf. Prop. 6.13] it arises as a direct summand of an exact sequence in $\mathcal{C}$, viewed as a sequence in $\mathcal{C}^{ic}$. Thus, there is an exact sequence in $\mathcal{C}$ so that

$$0 \longrightarrow A \oplus A' \xrightarrow{i} B \oplus B' \xrightarrow{j} C \oplus C' \longrightarrow 0$$

is exact in $\mathcal{C}$. Since $\mathcal{C}$ is split exact, there exists a left splitting $\pi : B \oplus B' \to A \oplus A'$ so that $\pi i = 1$. It is now easy to check that $B \to B \oplus B' \overset{\pi}{\to} A \oplus A' \to A$, where the outer arrows are the inclusion and projection from the direct summands, is a left splitting of the original exact sequence. $\square$

Definition 10. For objects $X, X' \in \mathsf{Tate}(\mathcal{C})$ we say that $\varphi : X \to X'$ is

1) bounded,
2) discrete,
3) finite,
4) immersive,
5) submersive,

if, when unwinding the definition of idempotent completion, we have $X = (Y, p)$ and $X' = (Y', q)$ and $\varphi : Y \to Y'$ (so that $q\varphi p = \varphi$) is a morphism of elementary Tate objects so that φ has the named property.

Lemma 13. *Let $\mathcal{C}$ be an idempotent complete exact category. In* $\mathsf{Tate}(\mathcal{C})$

1) *Lemma 3 remains valid, i.e. bounded, discrete and finite morphisms form categorical ideals,*
2) *Lemma 5 remains valid, i.e. admissible monics (resp. epics) are immersive (resp. submersive) as before.*
3) *Lemma 7 remains valid, i.e. submersions behave as before.*
4) *Lemma 8 remains valid, i.e. immersions behave as before.*
5) *Lemma 9 remains valid, i.e. if $\mathcal{C}$ is split exact,* $\mathsf{Tate}_{\aleph_0}\mathcal{C}$ *is split exact.*
6) *Lemma 10 remains valid, i.e. given*

$$
\begin{tikzcd}
X_1 \arrow[r, "\text{discrete}"] \arrow[d, "\text{submersive}"'] & X_2 \\
Y_1 \arrow[r, "f"'] & Y_2 \arrow[u, hook]
\end{tikzcd}
\qquad \textit{resp.} \qquad
\begin{tikzcd}
X_1 \arrow[r, "\text{bounded}"] \arrow[d, two heads] & X_2 \\
Y_1 \arrow[r, "g"'] & Y_2 \arrow[u, "\text{immersive}"']
\end{tikzcd}
$$

f is discrete (resp. g bounded).

Proof. Nothing really happens. We give some details nonetheless: (1) The ideal property follows from the corresponding property for elementary Tate objects since $\mathrm{Hom}_{\mathcal{C}^{ic}}(X, X')$ is a subgroup of $\mathrm{Hom}_{\mathcal{C}}(Y, Y')$ for $X =: (Y, p)$ and $X' =: (Y', q)$, Equation 4.1. (2), (3), (4) similar. (5) Use Lemma 12 for split exactness. For (6) note that we have such squares in $\mathsf{Tate}(\mathcal{C})$ only if they come from a square of elementary Tate objects with morphisms with the same properties, so Lemma 10 applies to this square, implying that f is discrete in $\mathsf{Tate}^{el}\mathcal{C}$ and then so is in $\mathsf{Tate}(\mathcal{C})$. Analogously for g. $\square$

Remark 14. Clearly our approach is based on drawing parallels to similar concepts in functional analysis. For example our notion of bounded morphisms is not too remote from the concept of a compact operator. The same remark applies to trace-class operators. The idea to look at higher local fields, i.e. special cases of n-Tate objects over vector spaces, from a functional analytic perspective has already been pursued in the work of A. Cámara [20] and [21].

5. Cubical structure

In Beilinson's definition, that is Definition 6, an interesting continuity condition appears. One looks at all k-linear maps "such that for all lattices $L_1 \hookrightarrow M_1, L_2 \hookrightarrow M_2$ there exist lattices $L_1' \hookrightarrow M_1, L_2' \hookrightarrow M_2$ such that

$$L_1' \hookrightarrow L_1, \quad L_2 \hookrightarrow L_2', \quad f(L'_{1\triangle'}) \hookrightarrow L_{2\triangle'}, \quad f(L_{1\triangle'}) \hookrightarrow L'_{2\triangle'}$$

holds."

In order to relate this to Tate objects, we first need to show that the very definition of morphisms of Tate objects implies this kind of behaviour automatically. This is not entirely obvious from the outset due to the rather different style of definition of lattices:

Lemma 15. *Suppose $\mathcal{C}$ is an idempotent complete exact category and $X_1, X_2 \in \mathsf{Tate}^{el}\mathcal{C}$. Let $f \in \mathrm{Hom}\,(X_1, X_2)$ be an arbitrary morphism. For all lattices $L_1 \hookrightarrow X_1, L_2 \hookrightarrow X_2$ there exist lattices L_1', L_2' and a* double lattice factorization

$$\begin{tikzcd} X_1 \arrow[r, "f"] & X_2 \\ L_1 \arrow[u, hook] \arrow[r, dotted, "\exists"] & L_2' \arrow[u, hook] \\ L_1' \arrow[u, hook] \arrow[r, dotted, "\exists"] & L_2 \arrow[u, hook] \end{tikzcd} \tag{5.1}$$

and for all such L_1, L_1', L_2, L_2' we get an induced morphism

$$\overline{f} : L_1/L_1' \to L_2'/L_2 \quad \text{in} \quad \mathrm{Hom}_{\mathcal{C}}\left(L_1/L_1', L_2'/L_2\right)$$

in the category $\mathcal{C}$.

We keep the notation $\overline{f}$ for later use.

Proof. From the assumptions we just get the diagram depicted on the left in:

$$\begin{tikzcd} X_1 \arrow[r, "f"] & X_2 \\ L_1 \arrow[u, hook] & \\ & L_2 \arrow[uu, hook] \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "f"] & X_2 \\ L_1 \arrow[u, hook] \arrow[r, dotted, "\exists"] & L_2' \arrow[u, hook] \\ & L_2 \arrow[u, hook] \end{tikzcd}$$

By Lemma 1 (1) the restriction $f\,|_{L_1}$ factors through some lattice of X_2, say $\tilde{L}_2$. By the directedness of the Sato Grassmannian [17, Thm. 6.7] we can find a common over-lattice of both $\tilde{L}_2$ and L_2, call it L_2', so that we arrive at the diagram on the right. By Lemma 1 (2) there exists some lattice $\tilde{L}_1$ of X_1 so that $f\,|_{\tilde{L}_1}$ factors through L_2. By the codirectedness of the Sato Grassmannian [17, Thm. 6.7] we can find a common sub-lattice of both L_1 and $\tilde{L}_1$, call it L_1', so that we arrive at the Diagram 5.1. Finally, this induces a canonical morphism $\overline{f} : L_1/L_1' \longrightarrow L_2'/L_2$ and by [17, Prop. 6.6] quotients of nested lattices lie in the base category, i.e. both source and target of $\overline{f}$ lie in the sub-category $\mathcal{C}$. $\square$

Later, we will need to understand how the composition of morphisms leads to the composition of such induced morphisms $\overline{f}$. In order to do this, we need to be able to find intermediate double lattice factorizations. The best we can hope for in this direction is the following existence result:

Lemma 16. *Suppose $\mathcal{C}$ is idempotent complete. Let $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$ be arbitrary morphisms between elementary Tate objects. Then for every double lattice factorization as in Diagram 5.1 for the composite $g \circ f$ we can find lattices $\tilde{L}_1$ in X_1, L_2, L_2' in X_2 and $\tilde{L}_3$ in X_3 so that*

$$\begin{tikzcd} L_1/\tilde{L}_1 \arrow[r, "\overline{f}"] \arrow[d, two heads] & L_2/L_2' \arrow[r, "\overline{g}"] & \tilde{L}_3/L_3 \\ L_1/L_1' \arrow[rr, "\overline{g\circ f}"] & & L_3'/L_3 \arrow[u, hook] \end{tikzcd} \tag{5.2}$$

commutes.

Proof. For the beginning, let f, g be arbitrary morphisms. Suppose we are given a double lattice factorization for $g \circ f$, i.e.

$$\begin{tikzcd} X_1 \arrow[r, "g\circ f"] & X_3 \\ L_1 \arrow[u, hook] \arrow[r] & L_3' \arrow[u, hook] \\ L_1' \arrow[u, hook] \arrow[r] & L_3. \arrow[u, hook] \end{tikzcd}$$

In general there is no reason why it should be possible to factor the two lower horizontal arrows over lattices in X_2. Thus, we first need to refine a given factorization. Using Lemma 1 (1) there exists a lattice L_2 in X_2, and (using the Lemma again) a lattice $\tilde{L}_3$ so that the diagram depicted below on the left commutes:

$$
\begin{tikzcd}
X_1 \arrow[r, "f"] & X_2 \arrow[r, "g"] & X_3 \\
 & L_2 \arrow[u, hook, dotted] \arrow[r, dotted] & \tilde{L}_3 \arrow[u, hook, dotted] \\
L_1 \arrow[uu, hook] \arrow[ur, dotted] \arrow[rr] & & L_3' \arrow[u, hook] \\
L_1' \arrow[u, hook] \arrow[rr] & & L_3 \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \arrow[r, "f"] & X_2 \arrow[r, "g"] & X_3 \\
 & L_2 \arrow[u, hook] \arrow[r] & \tilde{L}_3 \arrow[u, hook] \\
L_1 \arrow[uu, hook] \arrow[ur] \arrow[rr] & & L_3' \arrow[u, hook] \\
L_1' \arrow[u, hook] \arrow[rr] & & L_3 \arrow[u, hook] \\
\tilde{L}_1 \arrow[u, hook, dotted] \arrow[r, dotted] & L_2' \arrow[uuu, hook, dotted] \arrow[ur, dotted] &
\end{tikzcd}
$$

Here we may have without loss of generality replaced $\tilde{L}_3$ in the diagram by a common over-lattice of L_3' and $\tilde{L}_3$ so that the diagram still commutes (use directedness of the Sato Grassmannian). Now consider the bottom horizontal arrow in this diagram. Analogous to the previous refinement, using Lemma 1 (2) we find a lattice L_2' in X_2 which (after possibly replacing L_2' by a common sub-lattice with L_2) fits in the diagram depicted above on the right. Repeating this step again for $\tilde{L}_1$ yields the full diagram on the right. Taking quotients we get Diagram 5.2. $\square$

The following definition is a fairly precise imitation (even regarding the naming of the variables) of the continuity condition employed by Beilinson in his adèle paper, compare with Definition 6, or see the original paper [4].

Definition 11. Suppose $\mathcal{C}$ is idempotent complete. Let $X_1, X_2 \in n\text{-}\mathsf{Tate}^{el}\mathcal{C}$ be elementary n-Tate objects.

1) Let $I_1^s(X_1, X_2)$ for $s \in \{+, -, 0\}$ denote the bounded, discrete and finite morphisms in $\mathrm{Hom}(X_1, X_2)$ respectively, exactly as in Definition 7.

2) For $i = 2, \ldots, n$ let $I_i^s(X_1, X_2)$ denote the morphisms $f \in \mathrm{Hom}(X_1, X_2)$ such that for all lattices L_1, L_1', L_2, L_2' and double lattice factorizations as in Diagram 5.1 we have
$$\overline{f} \in I^s_{(i-1)}(L_1/L_1', L_2'/L_2).$$

3) We define
$$I_{tr}(X_1, X_2) := \bigcap_{i=1,\ldots,n} I_i^0(X_1, X_2),$$
its elements will be called *trace-class* morphisms.

As in §4 this immediately implies a reasonable definition for general (i.e. non-elementary) Tate objects:

Definition 12. If (X_1, p_1) and (X_2, p_2) are general Tate objects, define $I_i^s(X_1, X_2)$ to consist of those morphisms $f : (X_1, p_1) \to (X_2, p_2)$ such that the underlying morphism of elementary Tate objects $X_1 \to X_2$ lies in $I_i^s(X_1, X_2)$ in the above sense.

Theorem 13. *Suppose $\mathcal{C}$ is idempotent complete and $X, X', X'' \in n\text{-}\mathsf{Tate}^{el}\mathcal{C}$ or $n\text{-}\mathsf{Tate}(\mathcal{C})$.*

1) *The $I_i^s(-,-)$ for $i = 1, \ldots, n$ are categorical ideals. This means that the composition of morphisms factors as*

$$I_i^s\left(X', X''\right) \otimes \mathrm{Hom}\left(X, X'\right) \longrightarrow I_i^s\left(X, X''\right)$$
$$\mathrm{Hom}\left(X', X''\right) \otimes I_i^s\left(X, X'\right) \longrightarrow I_i^s\left(X, X''\right).$$

2) *In the ring* $\mathrm{End}(X)$ *the $I_i^s(X, X)$ are two-sided ideals.*
3) *Every composition of $\geq 2^n$ morphisms from $I_{tr}(-,-)$ factors through an object in $\mathcal{C}$. For words in $< 2^n$ letters this is in general false.*

The following argument is very close in spirit to the handling of lattices by A. Yekutieli in [56]. However, we encounter a number of additional technical issues because of the less concrete notion of lattice we work with. There is also a similar study in the case of vector spaces and n-local fields by D. V. Osipov [43].

Proof. (1) We only show this for elementary Tate objects since the general case follows directly along the same lines as the proofs in §4. We will reduce this to the case of a single Tate category, notably Lemma 3. Let f, g be composable morphisms as depicted in the top row of the diagram below. In order to prove that $g \circ f$ lies in I_i^s (for some s and i), the condition to check reduces to proving a property $g \circ f$ induced to a certain iterated subquotient of lattices. The lattice subquotients arise from an inductive choice. More precisely: Starting with $m := 1$, consider any double lattice factorization of the composition as in diagram

$$
\begin{tikzcd}
X_1 \arrow[r, "f"] & X_2 \arrow[r, "g"] & X_3 \\
L_1 \arrow[u, hook] \arrow[rr] & & L'_3 \arrow[u, hook] \\
L'_1 \arrow[u, hook] \arrow[rr] & & L_3 \arrow[u, hook]
\end{tikzcd}
\tag{5.3}
$$

with X_1, X_2, X_3 being elementary $(n-m+1)$-Tate objects. By Lemma 16 we can construct a commutative diagram

$$
\begin{tikzcd}
L_1/\tilde{L}_1 \arrow[r, "\overline{f}"] \arrow[d, two heads, "\text{submersive}"] & L_2/L'_2 \arrow[r, "\overline{g}"] & \tilde{L}_3/L_3 \\
L_1/L'_1 \arrow[rr, "\overline{g \circ f}"] & & L'_3/L_3 \arrow[u, hook, "\text{immersive}"]
\end{tikzcd}
$$

where the left and right outer arrows are an admissible epic (resp. monic) and thus are submersive (resp. immersive) by Lemma 13. Now continue with a picking another double lattice factorization as in Equation 5.3, but this time with $m_{new} := m_{old} + 1$ and using the top row of the above diagram in place of $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$. Note that the objects in this new row are quotients of nested lattices, so by [17, Prop. 6.6] they are elementary $(n-m)$-Tate objects. Repeat this until we reach $m = i$. For the rest of the proof $\overline{g \circ f}$ will refer to the respective morphism coming from the last step in this inductive procedure, i.e. when $m = i$. In particular, $\overline{g \circ f}$ is a morphism between $(n-i)$-Tate objects and from now on the word lattice will only refer to lattices in such. No more interplay of lattices of varying Tate categories will be needed, let us also rename the entries of the above diagram into neutral terms

$$
\begin{tikzcd}
W_1 \arrow[r, "\overline{f}"] \arrow[d, two heads, "\text{submersive}"] & A \arrow[r, "\overline{g}"] & W_2 \\
Z_1 \arrow[rr, "\overline{g \circ f}"] & & Z_2 \arrow[u, hook, "\text{immersive}"]
\end{tikzcd}
$$

Now by assumption one of $\overline{f}$ or $\overline{g}$ lies in I^s, so by Lemma 3 the entire top row lies in I^s. Then by Lemma 10 it follows that the bottom row lies in I^s as well. (2) trivially follows from (1). For (3) first note that it suffices to show this for elementary n-Tate objects. Now we show the claim by induction on

n. For $n = 1$ Lemma 3 gives the claim. Hence, assume the case $n-1$ has been dealt with and suppose $f_j \in I_{tr}(-,-)$ for $j = 1, \dots, 2^n$ are given and composable so that $f_1 \circ \cdots \circ f_{2^n}$ makes sense. By a minimal variation of the argument for Lemma 3 there is a factorization of $f_j \circ f_{j+1}$ as

$$X_1 \twoheadrightarrow X_1/L \xrightarrow{\overline{f_j \circ f_{j+1}}} L' \hookrightarrow X_2, \tag{5.4}$$

where L is a lattice in X_1 and L' a lattice in X_2. Following the argument of Lemma 3 further, the composition of any two morphisms having a factorization as in Equation 5.4, factors through an object in $(n-1)\text{-}\mathsf{Tate}(\mathcal{C})$. Thus, for every second index there is a factorization $\overline{f_1 f_2}, \overline{f_3 f_4}, \overline{f_5 f_6}, \dots :$ $X_* \to C_* \to X_*$ with '$*$' replaced by suitable indices and with $C_j \in (n-1)$-$\mathsf{Tate}(\mathcal{C})$. Now if we compose these $2^n/2 = 2^{n-1}$ morphisms, by induction it factors over an object in $\mathcal{C}$. To see that one cannot do with less than 2^n morphisms, we ask the reader to adapt Example 6 accordingly. $\square$

Definition 13. Suppose we are given $A := \operatorname{End}(X)$ in the situation of Theorem 13. Pairwise commuting elements $P_i^+ \in A$ (with $i = 1, \dots, n$) such that the following conditions are met:

- $P_i^{+2} = P_i^+$.
- $P_i^+ A \subseteq I_i^+$.
- $P_i^- A \subseteq I_i^-$ (and we define $P_i^- := \mathbf{1}_A - P_i^+$)

will be called a *system of good idempotents*. We shall call an (elementary) n-Tate object *n-sliced* if $A = \operatorname{End}(X)$ admits a system of good idempotents.

A very explicit example for good idempotents will be given in Example 10.

Proposition 14. *For every n-sliced object $X \in n\text{-}\mathsf{Tate}^{el}(\mathcal{C})$ or $n\text{-}\mathsf{Tate}(\mathcal{C})$ we have*

$$I_i^+(X,X) + I_i^-(X,X) = \operatorname{End}(X) \tag{5.5}$$

for all $i = 1, \dots, n$. Moreover, $\operatorname{End}(X)$ *is a Beilinson n-fold cubical algebra as in Definiton 1.*

Proof. By Theorem 13 we have the necessary ideals $I_i^{\pm}$. In order to meet all axioms of Definiton 1, it suffices to prove Equation 5.5. However, this can be done using the idempotents, by an immediate generalization of the proof of Prop. 11. $\square$

Lemma 17. *If $X \in \mathcal{C}$ then every endomorphism is trace-class, i.e.*

$$I_{tr}(X) = \mathrm{End}_{n\text{-}\mathsf{Tate}(\mathcal{C})}(X) = \mathrm{End}_{\mathcal{C}}(X).$$

Proof. The first equality holds since *every* sub-object of $X \in \mathcal{C}$ is a lattice with respect to the top 1-Tate structure, $n\text{-}\mathsf{Tate}(\mathcal{C}) = \mathsf{Tate}(\,(n-1)\text{-}\mathsf{Tate}(\mathcal{C})\,)$. Then for all quotients N/N' of such lattices $N' \hookrightarrow N \hookrightarrow X$, we still have $N/N' \in \mathcal{C}$. Thus, inductively, every endomorphism is trace-class. Then second equality holds since the embeddings $\mathcal{C} \hookrightarrow \mathsf{Tate}(\mathcal{C})$ are all fully faithful. $\square$

6. The countable split exact case

With the previous results we have seriously approached arriving at a structure as in Definition 1. Suppose $\mathcal{C}$ is an idempotent complete exact category. Now suppose X is an elementary n-Tate object. We may present it as

$$X = \underset{\overrightarrow{L_1}}{\mathrm{colim}}\,\underset{\overleftarrow{L_1'}}{\lim}\,\frac{L_1}{L_1'}, \tag{6.1}$$

where $L_1' \hookrightarrow L_1 \hookrightarrow X$ are a nested pair of lattices. One could also write this as

$$X = \underset{\overrightarrow{L_1 \in Gr(X)}}{\mathrm{colim}}\;\underset{\overleftarrow{L_1' \in Gr(X),\, L_1' \subseteq L_1}}{\lim}\;\frac{L_1}{L_1'}, \tag{6.2}$$

where $Gr(X)$ denotes the Sato Grassmannian of all lattices in X.

Remark 18. Let us look at the situation of a general n-Tate object, without the cardinality hypothesis. We can always write X as an Ind-diagram (of Pro-objects) or Pro-diagram (of Ind-objects)

$$X = \underset{\overrightarrow{L_1}}{\mathrm{colim}}\, L_1, \qquad X = \underset{\overleftarrow{L_1}}{\lim}\,\frac{X}{L_1} \tag{6.3}$$

where L_1 runs over the partially ordered set of lattices in X. The first presentation follows trivially from [17] and would work for a general exact category $\mathcal{C}$; for the second one needs the dual viewpoint developed in [15, § Duality], requiring $\mathcal{C}$ to be idempotent complete. This asymmetry stems from [17] defining the Tate category as a sub-category of $\mathsf{Ind}^a\mathsf{Pro}^a\mathcal{C}$ so that some 'preferred viewpoint' is built into the theory. Working from the outset with the opposite category would shove the idempotent completeness assumption to the first presentation and remove it from the second. The presentation in Equation 6.2 can be obtained by first using the left-hand side presentation in Equation 6.3, and then employing the right-hand side presentation for each L_1 individually. The co-directedness of the Sato Grassmannian and [15, Corollary 2] imply that instead of $L_1 = \varprojlim_{L_1'} \frac{L_1}{L_1'}$ with L_1' running through lattices of the Pro-object L_1, we may alternatively run through the lattices L_1' of X which are contained in L_1.

By [17, Prop. 6.6] any such quotient L_1/L_1' is an $(n-1)$-Tate object. This observation generalizes to the case where X is a general n-Tate object by refining our presentation to

$$X = P \operatorname*{colim}_{\overrightarrow{L_1}} \varprojlim_{L_1'} \frac{L_1}{L_1'},$$

where P denotes an idempotent. By induction it follows that

$$X = P \operatorname*{colim}_{\overrightarrow{L_1}} \varprojlim_{L_1'} P_{L_1,L_1'} \operatorname*{colim}_{\overrightarrow{L_2}} \varprojlim_{L_2'} P_{L_1,L_1',L_2,L_2'} \cdots \operatorname*{colim}_{\overrightarrow{L_n}} \varprojlim_{L_n'} P_{(\dots)} \frac{L_n}{L_n'},$$

where $L_1' \hookrightarrow L_1$ are nested lattices of the elementary n-Tate object underlying X, $L_2' \hookrightarrow L_2$ are nested lattices of the elementary $(n-1)$-Tate object underlying L_1/L_1', $L_3' \hookrightarrow L_3$ are nested lattices of the elementary $(n-2)$-Tate object underlying L_2/L_2',..., and finally L_n/L_n' is an object of $\mathcal{C}$. The letters $P, P_{L_1,L_1'}, \dots$ denote idempotents cutting out the respective Tate objects.

The results of the last section on double lattice factorizations, notably Lemma 15, tell us that any morphism

$$f : X_1 \longrightarrow X_2$$

of n-Tate objects stems from a system of compatible morphisms $L_n/L_n' \to N_n/N_n'$ in the category $\mathcal{C}$ so that f is induced from assembling these morphisms into

$$\operatorname*{colim}_{\overrightarrow{L_1}} \varprojlim_{L_1'} \cdots \operatorname*{colim}_{\overrightarrow{L_n}} \varprojlim_{L_n'} \frac{L_n}{L_n'} \xrightarrow{f} \operatorname*{colim}_{\overrightarrow{N_1}} \varprojlim_{N_1'} \cdots \operatorname*{colim}_{\overrightarrow{N_n}} \varprojlim_{N_n'} \frac{N_n}{N_n'}.$$

If we take over from Lemma 15 the notation that the induced morphism of a double lattice factorization is $\overline{f}$, the morphisms $L_n/L'_n \to N_n/N'_n$ here are nothing but "n-fold overline f".

Theorem 15. *Suppose $\mathcal{C}$ is a split and idempotent complete exact category and $X \in n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$ or $n\text{-}\mathsf{Tate}_{\aleph_0}(\mathcal{C})$, i.e. X is a countable n-Tate object. Then X is n-sliced and* $\operatorname{End}(X)$ *carries the structure of an n-fold cubical Beilinson algebra in the sense of Definition 1.*

Proof. For a 1-Tate object this is literally Prop. 11. In general, by Prop. 14, it suffices to find a system of good idempotents. Proceed by induction in n. Write an elementary n-Tate object as

$$X = \underset{\overrightarrow{L_1}}{\operatorname{colim}}\,\underset{\overleftarrow{L'_1}}{\lim} \frac{L_1}{L'_1}. \tag{6.4}$$

As any quotient L_1/L'_1 is an $(n-1)$-Tate object and we assume our claim, i.e. the existence of a system of good idempotents, for $n-1$, the idempotents supply a direct sum decomposition

$$L_1/L'_1 = \bigoplus_{s_1,\ldots,s_{n-1}\in\{\pm\}} P_1^{s_1}\cdots P_{n-1}^{s_{n-1}}\left(L_1/L'_1\right).$$

If $L_1 \hookrightarrow L_2$ is a larger lattice in X and $L'_2 \hookrightarrow L'_1$ a smaller lattice, the split exactness allows one to find a direct sum decomposition

$$\frac{L_2}{L'_2} \simeq \frac{L_1}{L'_1} \oplus (\text{another } (n-1)\text{-Tate object}).$$

We can use the same idempotents $P_1^{\pm},\ldots,P_{n-1}^{\pm}$ to decompose the new summand. As our indexing categories are countable, we can exhaust X in this fashion to get a choice of $n-1$ good idempotents on all of X. Finally, on all of X, we get a further idempotent $P_n^{\pm}$, just by splitting the entire presentation of Equation 6.4 as

$$0 \longrightarrow \underset{\overleftarrow{L'_1}}{\lim} \frac{\tilde{L}}{L'_1} \longrightarrow \underset{\overrightarrow{L_1}}{\operatorname{colim}}\,\underset{\overleftarrow{L'_1}}{\lim} \frac{L_1}{L'_1} \longrightarrow \underset{\overrightarrow{L_1}}{\operatorname{colim}} \frac{L_1}{\tilde{L}} \longrightarrow 0,$$

where $\tilde{L}$ is some fixed lattice of X. This gives us a full system of n good idempotents and thus proves our claim. If X is a general Tate object, (X,p), use the idempotents $pP_i^{\pm}p$ of the underlying elementary Tate object X instead. $\square$

Open Problem. *What is the correct analogue of this theorem in the context of Hennion's Tate categories for stable ∞-categories? [34]*

We close this section by presenting an example where it is particularly easy to find a system of good idempotents.

Example 10. Let R be a ring, possibly non-commutative. Define the ring of formal Laurent series by $R((t)) := R[[t]][t^{-1}]$. Then $R((t_1))((t_2))\cdots((t_n))$ canonically has a representative in $n\text{-}\mathsf{Tate}^{el}(\mathsf{Mod}_R)$ via

$$\begin{aligned} X &:= \text{“}R((t_1))((t_2))\cdots((t_n))\text{”} \\ &= \underset{\overrightarrow{i_n}}{\operatorname{colim}}\underset{\overleftarrow{j_n}}{\lim}\cdots\underset{\overrightarrow{i_1}}{\operatorname{colim}}\underset{\overleftarrow{j_1}}{\lim}\frac{1}{t_1^{i_1}\cdots t_n^{i_n}}R[t_1,\ldots,t_n]/(t_1^{j_1},\ldots,t_n^{j_n}). \end{aligned}$$

When evaluating the colimits and limits in Mod_R, we get the usual R-module

$$R((t_1))((t_2))\cdots((t_n))$$

and with a little more work one obtains the ring structure on it. Since the (co)limits are taken over projective R-modules so that this object actually could be constructed on the left in

$$n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(P_f(R)) \longrightarrow n\text{-}\mathsf{Tate}^{el}(\mathsf{Mod}_R),$$

we deduce that the implications of Thm. 15 apply to this particular object. For $i = 1,\ldots,n$ define idempotents

$$P_i^+ \sum a_{m_1,\ldots,m_n} t_1^{m_1}\cdots t_n^{m_n} := \sum \delta_{m_i\geq 0} a_{m_1,\ldots,m_n} t_1^{m_1}\cdots t_n^{m_n}$$

for $a_{m_1,\ldots,m_n} \in R$ and $P_i^- := 1 - P_i^+$. This is a system of good idempotents for X in the sense of Definition 13. A description of the ideals $I^{\pm}_{(-)}$ is easy to give; the statements would be analogous to those in Lemma 27 in §10.

Remark 19. A different approach has been introduced by A. Yekutieli. He developed the concept of semi-topological rings in [54], [55]. If one prefers Yekutieli's semi-topological theory over n-Tate objects, an analogous construction is possible: If R is a semi-topological ring, e.g. with the discrete topology, Yekutieli shows that $R((t_1))((t_2))\cdots((t_n))$ also possesses a canonical structure as a semi-topological ring itself. He has established a result in the style of Theorem 15 in [56]. Whichever way one proceeds, one needs a replacement for classical topological concepts: For example, it is known that for $n \geq 2$ an n-local field like $k((t_1))\cdots((t_n))$ is *not* a topological field. A. N. Parshin and I. B. Fesenko have resolved this issue by using sequential topologies, cf. [26]. K. Kato's version of Tate categories was also introduced to address exactly this issue, we refer to the introduction of [40]. See [16] for a comparison of these different viewpoints.

7. Relation to projective modules

For a possibly non-commutative ring R we denote by $P_f(R)$ the category of finitely generated projective *right* R-modules. First, let us recall the following:

Theorem 16. *([17, Thm. 5.30]) Suppose R is a commutative ring.*

1) *Then $\mathsf{Tate}^{Dr}(R)$ admits a canonical fully faithful embedding as a subcategory of $\mathsf{Tate}(P_f(R))$.*
2) *When restricting to countable cardinality, this becomes an equivalence of categories,*

$$\mathsf{Tate}^{Dr}_{\aleph_0}(R) \xrightarrow{\sim} \mathsf{Tate}_{\aleph_0}(P_f(R))$$

We shall also need a result identifying split exact categories and projective module categories, a type of projective generator argument. It applies to a wide range of situations, so let us state it in this generality.

Lemma 20. *Let $\mathcal{C}$ be an idempotent complete, split exact category such that every object is a direct summand of some fixed object $S \in \mathcal{C}$. Then the functor*

$$\begin{aligned} \mathcal{C} &\longrightarrow P_f(\mathrm{End}_{\mathcal{C}}(S)) \\ Z &\longmapsto \mathrm{Hom}_{\mathcal{C}}(S, Z) \end{aligned} \tag{7.1}$$

is an exact equivalence of exact categories.

Proof. Firstly, define $E := \mathrm{End}_{\mathcal{C}}(S)$ and note that for any $Z \in \mathcal{C}$ the group $\mathrm{Hom}_{\mathcal{C}}(S, Z)$ becomes a right E-module by the composition of morphisms, i.e.

$$\mathrm{Hom}_{\mathcal{C}}(S, Z) \times \mathrm{Hom}_{\mathcal{C}}(S, S) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(S, Z).$$

This produces a functor $\mathcal{C} \to \mathsf{Mod}(E)$. Thus, for any objects $Z_1, Z_2 \in \mathcal{C}$ we get an induced map of homomorphism groups

$$\mathrm{Hom}_{\mathcal{C}}(Z_1, Z_2) \longrightarrow \mathrm{Hom}_{\mathsf{Mod}(E)}(\mathrm{Hom}_{\mathcal{C}}(S, Z_1), \mathrm{Hom}_{\mathcal{C}}(S, Z_2)). \tag{7.2}$$

For $Z_1 = Z_2 := S$ it is an isomorphism. It also sends idempotents to idempotents, so the object S and all its direct summands are sent to E and direct summands of it. However, by assumption any object of $\mathcal{C}$ is of this shape, so this gives an alternative description of the same functor, and therefore implies that the map in Equation 7.2 is an isomorphism for arbitrary Z_1, Z_2. So the functor is fully faithful. Moreover, we see that every object is sent to a direct summand of the free E-module E, so the image of the functor consists of finitely generated projective E-modules, which shows that the functor is well-defined. Conversely, every finitely generated projective E-module M is a direct summand of $E^{\oplus n}$ for some n. Since Equation 7.2 is also an isomorphism for $S^{\oplus n}$, the idempotent defining M comes from an idempotent of $S^{\oplus n}$. As $\mathcal{C}$ is idempotent complete, this idempotent possesses a kernel. This shows that the functor is essentially surjective. As a result, we have an equivalence of categories and since both are split exact, this equivalence is necessarily exact. In either case, the only short exact sequences are the split ones. This finishes the proof. $\square$

Remark 21. Let us quickly address the uniqueness of such a presentation. Suppose $S, S' \in \mathcal{C}$ are objects both satisfying the assumptions of the lemma. For example, $S' := S \oplus S$. Then the equivalences of categories are related by the functor

$$\begin{aligned} P_f(\mathrm{End}_{\mathcal{C}}(S)) &\longrightarrow P_f(\mathrm{End}_{\mathcal{C}}(S')) \\ M &\longmapsto M \otimes_{\mathrm{End}_{\mathcal{C}}(S)} \mathrm{Hom}_{\mathcal{C}}(S', S), \end{aligned}$$

which itself is an exact equivalence. Note that this is precisely the shape of a Morita equivalence: $\mathrm{Hom}_{\mathcal{C}}(S', S)$ is the Morita bimodule with the rings $\mathrm{End}_{\mathcal{C}}(S)$ and $\mathrm{End}_{\mathcal{C}}(S')$ acting from the left- and right respectively. Exchanging the rôles of S and S' yields the Morita bimodule for the reverse direction.

Suppose $\mathcal{C}$ is a split exact category. Moreover, suppose there is a collection $\{S_i\}_{i \in \mathbf{N}}$ of objects $S_i \in \mathcal{C}$ such that every object $X \in \mathcal{C}$ is a direct summand of some countable direct sum of these S_i. Then $\mathcal{C}' := \mathsf{Tate}_{\aleph_0}\mathcal{C}$ is idempotent complete and split exact by Lemma 13. Moreover, there is the canonical object

$$Y := \widehat{\prod_{\mathbf{N}} S} \oplus \widehat{\bigoplus_{\mathbf{N}} S}, \tag{7.3}$$

defined by

$$\widehat{\prod_{\mathbf{N}} S} := \prod_{\mathbf{N}} \prod_{i \in \mathbf{N}} S_i, \quad \text{and} \quad \widehat{\bigoplus_{\mathbf{N}} S} := \bigoplus_{\mathbf{N}} \bigoplus_{i \in \mathbf{N}} S_i,$$

viewed as a Pro^a-object (respectively Ind^a-object), and it follows from [17, Prop. 5.24] that every object $X \in \mathsf{Tate}^{el}_{\aleph_0}\mathcal{C}$ is a direct summand of Y. Then of course the same holds in the idempotent completion $\mathcal{C}'$. As a result, we have shown the assumptions of our argument, but for $\mathsf{Tate}_{\aleph_0}\mathcal{C}$ instead of $\mathcal{C}$ and for the family $\{S_i\}_{i \in \mathbf{N}}$ we can take the single object Y. We may now iterate this procedure to obtain the following.

Definition 14. Let $\mathcal{C}$ be a split exact category. For any object $X \in \mathcal{C}$ define

$$X((t)) := \prod_{\mathbf{N}} X \oplus \bigoplus_{\mathbf{N}} X \quad \in \mathsf{Tate}^{el}\mathcal{C}.$$

This is just a special case of Equation 7.3 in the case of a single object.

Definition 15. Let $\mathcal{C}$ be a split exact category and $\{S_i\}_{i \in \mathbf{N}}$ a collection of objects such that every $X \in \mathcal{C}$ is a direct summand of a countable direct sum of objects in $\{S_i\}$. Then we call

$$S := \left(\widehat{\prod_{\mathbf{N}} S} \oplus \widehat{\bigoplus_{\mathbf{N}} S} \right) ((t_2)) \cdots ((t_n))$$

a *standard object* for n-$\mathsf{Tate}_{\aleph_0}\mathcal{C}$.

Theorem 17. *Let $\mathcal{C}$ be a idempotent complete and split exact category with a countable collection $\{S_i\}$ of objects as in Definition 15.*

1) *Then every object $X \in n$-$\mathsf{Tate}_{\aleph_0}\mathcal{C}$ is a direct summand of a standard object.*
2) *There is an exact equivalence of exact categories*

$$n\text{-}\mathsf{Tate}_{\aleph_0}(\mathcal{C}) \xrightarrow{\sim} P_f(R)$$

for $R := \operatorname{End}_{n\text{-}\mathsf{Tate}_{\aleph_0}(\mathcal{C})}(S)$ and S any standard object.

3) *Under this equivalence, the ideals $I_i^{\pm}$ in R correspond to the categorical ideals of Definition 11.*

Proof. The first claim is just [17, Prop. 7.4] and the second is a direct consequence thanks to Lemma 20. Part (3) is obvious for the standard object and then use that every object is a direct summand of the latter. □

Let us now adapt this result to the case of 'Tate modules *à la Drinfeld*', we refer to [17, §5.4] for a definition and background information. This type of object has been introduced by Drinfeld in his paper [23] as a candidate for the local sections of a reasonable notion of infinite-dimensional vector bundles over a scheme.

Theorem 18. *Let R be a commutative ring. Then there is an exact equivalence of categories*

$$\mathsf{Tate}^{Dr}_{\aleph_0}(R) \xrightarrow{\sim} P_f(E),$$

where E is the Beilinson 1-fold cubical algebra

$$E := \mathrm{End}_{\mathsf{Tate}^{Dr}_{\aleph_0}(R)}\left(R((t))\right),$$

where "$R((t))$" is understood as the Tate module à la Drinfeld with this name in Drinfeld's paper [23].

Proof. We claim that we have exact equivalences of categories, namely

$$\mathsf{Tate}^{Dr}_{\aleph_0}(R) \xrightarrow{\sim} \mathsf{Tate}_{\aleph_0}(P_f(R)) \xrightarrow{\sim} P_f\left(E\right),$$

where the first equivalence stems from Theorem 16. The latter exists since $P_f(R)$ is an idempotent complete split exact category so that Theorem 17 is applicable with

$$E := \mathrm{End}_{\mathsf{Tate}_{\aleph_0}(P_f(R))}\left(R((t))\right).$$

To justify this, note that every finitely generated projective R-module is a direct summand of a finitely generated free module $R^{\oplus n}$ for n large enough, so $R((t))$, as in Definition 14, is indeed a standard object. However, now using the equivalence of Theorem 16 again, the full faithfulness yields an isomorphism of rings

$$E \cong \mathrm{End}_{\mathsf{Tate}^{Dr}_{\aleph_0}(R)}\left(R((t))\right),$$

where now $R((t))$ is to be understood as the (rather: one choice of a) Tate module à la Drinfeld corresponding to the Tate object with the same name. However, Drinfeld himself introduced the corresponding object already in his original paper [23, §3, especially Example 3.2.2], and in fact it is also called $R((t))$ in loc. cit. This establishes the claim. □

Open Problem. *It would seem interesting to study the analogous problem without the restriction to countable cardinality. This probably would lead to a very complicated picture. Kaplansky has shown that a projective module over a ring must necessarily be a direct sum of countably generated modules. Over the years it has become increasingly clear that this perhaps surprising appearance of questions of cardinality permeate the entire field [10]. See for example [35] or [24] for intricacies in the context of Drinfeld's ideas.*

8. Trace-class operators

Suppose A is an n-fold cubical algebra as in Definition 1. Then we call the intersection of ideals

$$I_{tr} := \bigcap_{i=1,\ldots,n} I_i^+ \cap I_i^-$$

the ideal of *trace-class* operators in A. Let us say a few things about the historical precursors of this concept: While the name is inspired from a vaguely related definition in functional analysis, the present format originates from Tate's 1968 article on residues for curves [53]. He considers a 1-fold cubical algebra of k-linear maps and wants to define a trace on I_{tr}, mimicking the usual trace. Sadly, the maps in his ideal I_{tr} need not have finite rank, so a priori it is not clear whether a notion of trace exists for them at all. Tate then follows the principle that any nilpotent map should have trace zero, no matter whether it has finite rank or not. From this he distills the concept of a 'finite-potent' map — a map for which some finite power has finite rank. Tate manages to develop a well-defined trace for such maps. Nonetheless, this trace has some fairly mysterious properties. Most notably it is not always linear, as was conjectured by Tate and later shown by F. Pablos Romo [46], see also [2], [51] for a fairly complete analysis of this issue. However, in all applications of Tate's trace one usually only needs it for trace-class operators, i.e. maps in I_{tr}, rather than all finite-potent maps. Restricted to I_{tr}, Tate's generalized trace becomes linear and very well-behaved. In this section we shall generalize this concept to Tate categories. Just as Tate's original work takes the classical finite rank trace as input, we shall also use a notion of trace on the input category $\mathcal{C}$ as the starting point for the construction:

Definition 16. Let $\mathcal{C}$ be an exact category. An *exact trace* on $\mathcal{C}$ with values in an abelian group Q is for each object $X \in \mathcal{C}$ a group homomorphism

$$\mathrm{tr}_X : \mathrm{End}_{\mathcal{C}}(X) \longrightarrow Q$$

so that the following properties hold:

1) *(Zero on commutators)* For $f, g \in \mathrm{End}_{\mathcal{C}}(X)$ we have $\mathrm{tr}_X(fg - gf) = 0$.
2) *(Additivity)* For a short exact sequence $A \hookrightarrow B \twoheadrightarrow B/A$ and $f \in \mathrm{End}_{\mathcal{C}}(B)$ so that $f\mid_A$ factors over A, we have

$$\mathrm{tr}_B(f) = \mathrm{tr}_A(f\mid_A) + \mathrm{tr}_{B/A}(f). \tag{8.1}$$

Example 11. For $\mathcal{C} := \mathsf{Vect}_f$ the usual trace of a k-linear endomorphism is an exact trace with values in the base field k.

Example 12 (Hattori-Stallings trace). Suppose R is any unital associative ring, not necessarily commutative. Let $P_f(R)$ be the category of finitely generated right R-modules. For any $X \in P_f(R)$ the *Hattori-Stallings trace* is the morphism

$$\begin{aligned} \mathrm{tr}_X : \mathrm{End}_R(X) &\longrightarrow R/[R,R] \\ X \otimes X^{\vee} &\longrightarrow R/[R,R] \\ x \otimes x^{\vee} &\longmapsto x^{\vee}(x). \end{aligned}$$

It is an exact trace. In fact, it is known to be universal on the category $P_f(R)$, i.e. any exact trace with values in an abelian group Q arises as the composition of the Hattori-Stallings trace with a morphism $R/[R,R] \to Q$. See [32], [52] for the original papers, [3, §1] for a review. If $R := k$ is a field, we recover the classical trace.

Example 13. In Tate's theory in [53] every nilpotent endomorphism has trace zero. This need not hold in our context — for entirely trivial reasons. We give an explicit counter-example nonetheless: Take $\mathcal{C} := P_f(\mathbf{Z}/2^{10})$. Define a trace $tr_{\mathbf{Z}/2^{10}} : \mathrm{End}(\mathbf{Z}/2^{10}) \to \mathbf{Z}/2^{10}$ as the identity. Since $\mathcal{C}$ is split exact, the axiom regarding additivity for exact sequences determines a unique continuation of $tr_{\mathbf{Z}/2^{10}}$ to the entire category $\mathcal{C}$. Clearly, multiplication with 2 is a nilpotent endomorphism of $\mathbf{Z}/2^{10}$, yet has trace 2.

Once such a trace is available, we can lift it to the trace-class operators of n-Tate objects:

Proposition 19. *Suppose $\mathcal{C}$ is an idempotent complete exact category and $\mathrm{tr}_{(-)}$ an exact trace with values in an abelian group Q. Then for every object*

$X \in n\text{-}\mathsf{Tate}(\mathcal{C})$ *there is a canonically defined morphism*

$$\tau_X : I_{tr} \to Q$$

and these morphisms are uniquely determined by the following properties:

1) *If* $X \in \mathcal{C}$ *then* $\tau_X(f) = \mathrm{tr}_X(f)$ *for all* $f \in \mathrm{End}(X)$.
2) *Suppose* $N' \hookrightarrow N \hookrightarrow X$ *are any lattices of* X *such that* $\varphi \in I_{tr}(X)$ *admits a lift*

$$\begin{tikzcd} & N \arrow[d, hook] \\ X \arrow[ur, dotted, "\varphi"] \arrow[r, "\varphi"] & X \end{tikzcd}$$

and for which $\varphi\mid_{N'}$ *is zero, and thus factors as* $N/N' \xrightarrow{\overline{\varphi}} N/N'$. *Then* $\tau_X(\varphi) := \tau_{N/N'}(\overline{\varphi})$. *This element is independent of the choice of* N, N'.

3) (Zero on commutators) *For* $f, g \in I_{tr}(X)$ *we have* $\tau_X(fg - gf) = 0$.
4) (Additivity) *For a short exact sequence* $A \hookrightarrow B \twoheadrightarrow B/A$ *and* $f \in I_{tr}(B)$ *so that* $f\mid_A$ *factors over* A, *we have*

$$\tau_B(f) = \tau_A(f\mid_A) + \tau_{B/A}(\overline{f}).$$

We automatically have $f\mid_A \in I_{tr}(A)$ *and* $\overline{f} \in I_{tr}(B/A)$.

The endomorphism group in (1) makes sense in view of Lemma 17.

Proof. (Step 1) The case $n = 0$ is trivial and directly reduces to the axioms of an exact trace. We deal with the case of an elementary 1-Tate object X first, i.e. assume $n = 1$. Suppose $X \xrightarrow{\varphi} X$ is any endomorphism in $\mathsf{Tate}^{el}(\mathcal{C})$. We call a diagram

$$\begin{tikzcd} N \arrow[r, dotted, "\overline{\varphi}"] \arrow[d, hook] & N \arrow[d, hook] \\ X \arrow[ur] \arrow[r, "\varphi"] & X \end{tikzcd}$$

a 1-*factorization* if $N \hookrightarrow X$ is a lattice, the diagram commutes, and $\overline{\varphi}$ factors over $\overline{\varphi} : N/N' \to N/N'$ with $N' \hookrightarrow N \hookrightarrow X$ a further (smaller) lattice. For any 1-factorization of φ we can define a preliminary trace by

$\tau(\varphi) := \mathrm{tr}_{N/N'}(\overline{\varphi}) \in Q$ for the simple reason that any quotient of lattices, e.g. N/N', must be an object in $\mathcal{C}$, [17, Prop. 6.6]. Next, we note that any *finite* morphism has a 1-factorization: Since φ is bounded, it factors as $X \to N \hookrightarrow X$. Now, restrict this to N. Since φ is also discrete, there exists some lattice V with $V \hookrightarrow X \to X$ being zero, so let N' be any common sub-lattice of V and N. Such exists because of the co-directedness of the Sato Grassmannian [17, Theorem 6.7]. Now N and N' satisfy all necessary criteria. Suppose we find a further 1-factorization with N' replaced by a smaller lattice N''. From $N'' \hookrightarrow N' \hookrightarrow N$ we get the short exact sequence

$$\frac{N'}{N''} \hookrightarrow \frac{N}{N''} \twoheadrightarrow \frac{N}{N'}.$$

Since $\overline{\varphi}$ already vanishes on N', the trace must be zero on the left-hand side term. The additivity axiom of the trace, Equation 8.1, implies that $\mathrm{tr}_{N/N''}(\overline{\varphi}) = \mathrm{tr}_{N/N'}(\overline{\varphi}) + 0$. Similarly, if we replace N by a larger lattice N^+, we get the short exact sequence

$$\frac{N}{N'} \hookrightarrow \frac{N^+}{N'} \twoheadrightarrow \frac{N^+}{N}$$

and as $\overline{\varphi}$ factors over N by assumption, the trace must be zero on the right-hand side term. Again, we get $\mathrm{tr}_{N^+/N'}(\overline{\varphi}) = \mathrm{tr}_{N/N'}(\overline{\varphi}) + 0$ by Equation 8.1. This shows that our preliminary definition of a trace is independent under replacing N' by a smaller lattice and N by a larger one. Since any two lattices have a common sub-lattice and over-lattice, [17, Theorem 6.7], and $\mathrm{tr}_{N/N'}(\overline{\varphi}) \in Q$ is unchanged under replacing lattices this way, we conclude that $\mathrm{tr}_{N/N'}(\overline{\varphi})$ is actually independent of N and N'. In the same way we can show that the trace is linear: Pick N_i, N_i' for $i = 1, 2$ and both morphisms in consideration and then verify the claim by replacing N_1, N_2 by a joint over-lattice, resp. sub-lattice for N_1', N_2'. For a general (not necessarily elementary) 1-Tate object, we proceed as in §4: The morphism φ is called finite if the underlying morphism of elementary Tate objects $(X, p) \to (X, p)$ is finite, so we can take the trace which we have just constructed.

Summarizing our findings, we have seen that axioms (1) and (2) actually dictate a well-defined construction of a group homomorphism $\tau_X : I_{tr} \to Q$. This implies the uniqueness and it remains to show that axioms (3) and (4) hold. Vanishing on $[I_{tr}, I_{tr}]$ is immediately clear since we find $\overline{\varphi \circ \varphi'} = \overline{\varphi} \circ \overline{\varphi'}$ for composable trace-class morphisms φ, φ', so the 1-factorization of a commutator can be expressed as the commutator of 1-factorizations. Now use the vanishing of exact traces on commutators. This proves (3). For (4), suppose

$$A \hookrightarrow B \twoheadrightarrow B/A \tag{8.2}$$

is a short exact sequence and $f \in I_{tr}(B)$ is such that $f\mid_A$ factors over A. In this situation, Prop. 12 guarantees that $f\mid_A \in I_{tr}(A)$ and $\overline{f} \in I_{tr}(B/A)$ are also trace-class. Moreover, if f factors over N/N', it supplies us with a short exact sequence

$$N_1/N_1' \hookrightarrow N/N' \twoheadrightarrow N_2/N_2',$$

where $N_1' \hookrightarrow N_1 \hookrightarrow A$ and $N_2' \hookrightarrow N_2 \hookrightarrow B/A$ are lattices. Of course these quotients are objects in $\mathcal{C}$, so the additivity of the exact trace tells us that

$$\mathrm{tr}_{N_1/N_1'}(f) = \mathrm{tr}_{N/N'}(f\mid_A) + \mathrm{tr}_{N_2'/N_2}(f),$$

but by (1) and (2) each of these traces is just a way to evaluate our trace $\tau_{(-)}$, and we get

$$\tau_B(f) = \tau_A(f\mid_A) + \tau_{B/A}(\overline{f}),$$

which is exactly what we wanted to show. This settles axiom (4).

(Step 2) For an elementary n-Tate object proceed exactly as above, but combined with an induction: Define a *n-factorization* just like a 1-factorization

$$\begin{tikzcd}
N \arrow[r, dotted, "\varphi"] \arrow[d, hook] & N \arrow[d, hook] \\
X \arrow[r, "\varphi"] \arrow[ur] & X
\end{tikzcd}$$

but with $\overline{\varphi} : N/N' \to N/N'$ an endomorphism of an $(n-1)$-Tate object. By the definition of trace-class operators, Definition 11, this is now again a trace-class operator for this $(n-1)$-Tate object:

$$\varphi \in I_{tr} = I_1^+ \cap I_1^- \cap \left(\bigcap\nolimits_{i=2,\dots,n} I_i^+ \cap I_i^- \right),$$

from which we deduce that $\overline{\varphi} \in I_{tr}(N/N')$, as $(n-1)$-Tate objects. Next, pick a $(n-1)$-factorization for $\overline{\varphi}$ and proceed this way until we get a 1-factorization. Define $\tau(\varphi)$ as before by $\tau(\varphi) := \mathrm{tr}_{N/N'}(\overline{\varphi})$ of this 1-factorization. As in the case of 1-factorizations, verify that for any j-factorization ($j = 1, \dots, n$), replacing lattices by over- resp. sub-lattices does not affect the value of $\mathrm{tr}_{N/N'}(\overline{\varphi})$: We prove this by induction starting from

$j = 1$. But this case has already been dealt with in Step 1. For $j \geq 2$ use the same argument as in Step 1, adapted as follows: In Step 1 we essentially only used the additivity property of the exact trace. Replace this by using the additivity of our τ, i.e. its axiom (4), of the previous induction step $j - 1$.

Now, by construction properties (1), (2) in our claim are satisfied. Property (3) follows easily as in Step 1. In order to show axiom (4), we can again just copy the proof in Step 1 since it only uses the additivity of the exact trace, which we can again replace by the additivity of our τ, axiom (4), of the previous induction step. $\square$

It would be very nice if one could prove the following result in greater generality.

Proposition 20 (Strong Commutator Vanishing). *Let $\mathcal{C}$ be an abelian category. Suppose $X \in n\text{-}\mathsf{Tate}(\mathcal{C})$ and $R := \operatorname{End}(X)$, $I_{tr} \subseteq R$ the trace-class ideal. Then*

$$\tau_X([I_{tr}, R]) = 0,$$

i.e. a commutator of a trace-class endomorphism with an arbitrary *endomorphism vanishes.*

Proof. Let $X \in n\text{-}\mathsf{Tate}(\mathcal{C})$ and $\varphi \in R$, $\varphi_0 \in I_{tr}(R)$. Since I_{tr} is a two-sided ideal, φ_0, $\varphi\varphi_0$ and $\varphi_0\varphi$ are all trace-class and thus we know that for each of them we can quotient $X \to X$ step-by-step through lattices as in Prop. 19 (2), going from n-Tate objects to 0-Tate objects while preserving the value of τ, so that we may assume from the outset that $X \in \mathcal{C}$. We can also find such lattices simultaneously for all three of them since in each step the directedness and co-directedness of the Sato Grassmannian [17, Theorem 6.7] assures us that we can take a common over- (respectively sub-)lattice of the lattices we find for each individual morphism. Now consider the commutative diagram

$$\begin{tikzcd}
\ker(\varphi_0) \arrow[r, "\varphi\varphi_0"] \arrow[d, hook] & \ker(\varphi_0) \arrow[d, hook] \\
X \arrow[r, "\varphi\varphi_0"] \arrow[d, two heads, "\varphi_0"'] & X \arrow[d, two heads, "\varphi_0"'] \\
\operatorname{im}(\varphi_0) \arrow[r, "\varphi_0\varphi"] & \operatorname{im}(\varphi_0)
\end{tikzcd}$$

The kernel and image exist since $\mathcal{C}$ is abelian. The top horizontal arrow is clearly the zero map so that $\tau_X(\varphi\varphi_0) = \tau_{\mathrm{im}(\varphi_0)}(\varphi_0\varphi)$ by the additivity axiom of the trace. Moreover, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{im}(\varphi_0) \arrow[r, "\varphi_0\varphi"] \arrow[d, hook] & \mathrm{im}(\varphi_0) \arrow[d, hook] \\
X \arrow[r, "\varphi_0\varphi"] \arrow[d, two heads] & X \arrow[d, two heads] \\
X/\mathrm{im}(\varphi_0) \arrow[r, "\varphi_0\varphi"] & X/\mathrm{im}(\varphi_0)
\end{tikzcd}$$

Again, by additivity we must have $\tau_X(\varphi_0\varphi) = \tau_{\mathrm{im}(\varphi_0)}(\varphi_0\varphi)$, since the bottom horizontal arrow is the zero map. $\square$

9. The Tate extension

Next, we recall a construction due to Beilinson [4], generalizing Tate's ingenious insight from [53] for $n = 1$:

Construction 1. *Let k be a field. For every n-fold cubically decomposed algebra $(A, (I_i^{\pm}), \tau)$ over k, as in Definition 4, there is a canonically defined Lie cohomology class*

$$\phi_{\mathrm{Beil}} \in H^{n+1}_{\mathrm{Lie}}(\mathfrak{g}, k),$$

where $\mathfrak{g} := A_{Lie}$ is the Lie algebra associated to A via the commutator $[x, y]_{\mathfrak{g}} := xy - yx$.

This cohomology class was introduced in [4]. An explicit formula and example computations can be found in [11], [12].

Example 14. For $n = 1$, Tate constructs "the original" cubically decomposed algebra in [53, Prop. 1] — this is the example which has started the entire subject in a way. It was independently found by many others, notably Kac–Peterson [38] or the Japanese school, cf. Date–Jimbo–Kashiwara–Miwa [22]. There are also the computations by Feigin and Tsygan [25], covering the other cohomological degrees as well. For a certain field K, Tate's paper

[53, Theorem 1] constructs a map, following the notation of *loc. cit.*,

$$\mathrm{res} : K \wedge K \longrightarrow k, \qquad \text{(the "abstract residue")}$$

which can be re-interpreted as $\phi_{\mathrm{Beil}} \in H^2_{\mathrm{Lie}}(K_{Lie}, k)$. It produces a map $\Omega^1_{K/k} \to k$, $f\mathrm{d}g \mapsto \mathrm{res}(f \wedge g)$, which agrees with the usual one-dimensional residue of a rational 1-form at a point. Going well beyond the viewpoint in [53, Prop. 1], one can regard the Lie 2-cocycle ϕ_{Beil} as defining a Lie algebra central extension

$$k \longrightarrow \widehat{\mathfrak{g}} \longrightarrow K.$$

The Lie algebra $\widehat{\mathfrak{g}}$ is an example of what is nowadays called *Tate's central extension*. In this case, $\widehat{\mathfrak{g}}$ is known as the *Heisenberg Lie algebra*. The theory is presented and used from this perspective for example in [8, §2.4], [6, §2.10, §2.13], [7, §2.7], [9], [28], etc… Applications abound.

Example 15. For $n = 2$, the cocycle $\phi_{\mathrm{Beil}} \in H^3_{\mathrm{Lie}}(\mathfrak{g}, k)$, applied to a doubly infinite matrix Lie algebra, has been studied in great detail by Frenkel and Zhu [28].

The construction of a trace for trace-class operators in the previous section allows us to define a higher Tate extension class for the Lie algebras underlying endomorphism algebras of suitable n-Tate objects. In [12] Beilinson's construction was lifted from Lie cohomology to Hochschild and cyclic homology. These generalize analogously to our present situation.

Theorem 21. *Let k be a field and let $\mathcal{C}$ be a k-linear abelian category with a k-valued exact trace. For every n-sliced object $X \in n\text{-}\mathsf{Tate}(\mathcal{C})$, the endomorphism algebra $E := \mathrm{End}(X)$ is a cubically decomposed algebra in the sense of Definition 4.*

1) *In particular, its Lie algebra $\mathfrak{g}_X := E_{Lie}$ carries a canonical Beilinson–Tate Lie cohomology class,*

$$\phi_{\mathrm{Beil}} \in H^{n+1}_{\mathrm{Lie}}(\mathfrak{g}_X, k)$$

via Construction 1. Alternatively, one may view this as a functional $\phi_{\mathrm{Beil}} : H^{\mathrm{Lie}}_{n+1}(\mathfrak{g}_X, k) \to k$.

2) *There is also a canonical Hochschild homology and cyclic homology functional*

$$\phi_{HH} : HH_n(E) \to k \quad \textit{resp.} \quad \phi_{HC} : HC_n(E) \to k.$$

The Hochschild and the Lie invariant are not completely independent of each other, cf. [12] for details on the interplay of these constructions.

Example 16. Recall that, by Theorem 15, if $\mathcal{C}$ is split exact and idempotent complete, every countable n-Tate object is automatically n-sliced. For example, if we consider the category $\mathcal{C} := \mathsf{Vect}_f$, then the above theorem applies to all objects in $n\text{-}\mathsf{Tate}_{\aleph_0}(\mathcal{C})$.

Proof. (1) By Prop. 14, the endomorphisms $E := \mathrm{End}(X)$ form a Beilinson cubical algebra, but so far without a trace formalism τ. Prop. 19 promotes the k-valued exact trace on $\mathcal{C}$ to a trace

$$\tau_X : I_{tr}/[I_{tr}, I_{tr}] \to k$$

for any n-Tate object X and $I_{tr} = I_{tr}(X, X)$ its trace-class operators. As $\mathcal{C}$ is abelian, Prop. 20, shows that this trace satisfies the stronger axioms of a cubically decomposed algebra. Finally, Construction 1 applies and provides us with ϕ_{Beil}; here we refer to [4] for the actual construction. (2) The construction of these maps just takes a cubically decomposed algebra as its input, so we can directly feed E into [12]. We leave the details regarding the existence of local units to the reader. $\square$

Example 17. If $n = 1$, this means that $\mathfrak{g}_X$ comes equipped with a canonical Lie central extension

$$0 \longrightarrow k \longrightarrow \widehat{\mathfrak{g}_X} \longrightarrow \mathfrak{g}_X \longrightarrow 0$$

and if $\mathcal{C} := \mathsf{Vect}_f$ and if we employ the usual trace, this produces most of the classical examples of Tate's central extension. For example, if $\mathfrak{g}$ is a simple Lie algebra, its loop Lie algebra

$$\mathfrak{g}((t)) := \varinjlim_i \varprojlim_j t^{-i}\mathfrak{g}[t]/t^j$$

can naturally be viewed as an object in $1\text{-}\mathsf{Tate}_{\aleph_0}\mathcal{C}$. The adjoint representation can be promoted to a Lie algebra embedding

$$\begin{aligned} \widetilde{\mathrm{ad}} : \mathfrak{g}((t)) &\hookrightarrow \mathfrak{E} := \mathrm{End}_{1\text{-}\mathsf{Tate}_{\aleph_0}\mathcal{C}}(\mathfrak{g}((t)))_{Lie} \\ x &\mapsto (y \mapsto [x, y]) \end{aligned}$$

(on the left-hand side view $\mathfrak{g}((t))$ as a plain Lie algebra and not as a 1-Tate object). The pullback of $\phi_{\mathrm{Beil}} \in H^2_{\mathrm{Lie}}(\mathfrak{E}, k)$ along $\widetilde{\mathrm{ad}}$ is the Kac-Moody cocycle. This mechanism defines a higher Lie cohomology class also for higher loop Lie algebras $\mathfrak{g}((t_1))\cdots((t_n))$, [28], [11].

Example 18. The classical residue for a rational 1-form on a curve can be obtained as follows: Let X/k be a smooth integral curve and $x \in X$ a closed point. Then the field of fractions of the completed local ring at x has a canonical structure as a 1-Tate object in $\mathcal{C} := \mathsf{Vect}_f$: To see this, observe that

$$\widehat{\mathcal{O}}_{X,x} = \varprojlim_i \mathcal{O}_{X,x}/\mathfrak{m}^i_{X,x}$$

is a Pro-object of finite-dimensional k-vector spaces. The field of fractions $\widehat{\mathcal{K}}_{X,x} := \operatorname{Frac}\widehat{\mathcal{O}}_{X,x}$ can be written as the colimit over all finitely generated $\widehat{\mathcal{O}}_{X,x}$-submodules of $\widehat{\mathcal{K}}_{X,x}$. Combining both presentations allows us to view $\widehat{\mathcal{K}}_{X,x} \in 1\text{-}\mathsf{Tate}_{\aleph_0}\mathcal{C}$. Of course, this is only a special case of the Parshin-Beilinson adèles, see §2. The Hochschild functional of Theorem 21 supplies us with a canonical map

$$\phi_{HH} : HH_1(\operatorname{End}_{1\text{-}\mathsf{Tate}_{\aleph_0}\mathcal{C}}(\widehat{\mathcal{K}}_{X,x})) \longrightarrow k.$$

Since the multiplication map $z \mapsto \alpha \cdot z$ for any $\alpha \in \widehat{\mathcal{K}}_{X,x}$ defines an endomorphism of this 1-Tate object, there is a canonical ring map from the rational function field $k(X)$ (or $\widehat{\mathcal{K}}_{X,x}$) to the above endomorphism algebra. Composing them, we get

$$HH_1(k(X)) \longrightarrow k$$

and the Hochschild-Kostant-Rosenberg isomorphism identifies the left-hand side with $\Omega^1_{k(X)/k}$. This map turns out to be the residue. This is the Hochschild analogue of Tate's construction of the residue. See [12] for details.

Note that Beilinson's paper [4] would have used Beilinson's cubically decomposed algebra, see Theorem 9, instead of using a Tate category. However, by our Theorem 5 these are isomorphic. Alternatively, one could also use Yekutieli's cubically decomposed algebra, see [56].

Remark 22. The structures produced by Theorem 21 can be viewed as "linearizations" of a more involved non-linear extension on the level of groups, resp. algebraic K-theory. See [13] for the non-linear version. For K-groups in low degrees, notably K_1 and K_2, this has been pioneered by [38] and [1]. See [48] for the analogue in topological K-theory.

10. Applications to adèles of schemes

We refer to [17, §7.2] for a detailed treatment of the relation between Parshin-Beilinson adèles and n-Tate objects. Let Ab (resp. Ab_f) be the category of all (resp. finite) abelian groups. Suppose X is a scheme, finite type of pure Krull dimension n over $\operatorname{Spec} R$ for R some Noetherian commutative ring. Moreover, let $\mathcal{F}$ be a quasi-coherent sheaf on X and we fix a subset $\triangle \subseteq S(X)_n$ of the flags of scheme points.

The treatment of [17, §7.2] views adèles as an elementary n-Tate object in coherent sheaves of X with zero-dimensional support, i.e. with a slight abuse of language, we could say

$$A(\triangle, \mathcal{F}) \in n\text{-}\mathsf{Tate}^{el}(\mathsf{Coh}_0 X)\ .$$

This gives an exact functor $\mathsf{QCoh}\,(X) \to n\text{-}\mathsf{Tate}^{el}(\mathsf{Coh}_0 X)$, $\mathcal{F} \mapsto A(\triangle, \mathcal{F})$. Of course, one might wish to distinguish between $A(\triangle, \mathcal{O}_X)$ as an n-Tate object of coherent sheaves, or as the $\mathcal{O}_X$-module sheaf one obtains by carrying out the respective limits and colimits in the bi-complete category of $\mathcal{O}_X$-module sheaves $A(\triangle, \mathcal{F}) \in \mathsf{Mod}(\mathcal{O}_X)$. However, this distinction will always be clear from the context.

Remark 23. The category of quasi-coherent $\mathcal{O}_X$-module sheaves $\mathsf{QCoh}\,(X)$ is also bi-complete, but carrying out the limits in this category instead would *not* form an exact functor $n\text{-}\mathsf{Tate}^{el}(\mathsf{Coh}_0 X) \to \mathsf{QCoh}\,(X)$, and furthermore the resulting objects do not appear to be particularly interesting. In fact, in $\mathsf{QCoh}\,(X)$ even taking countable infinite products $\prod_{\mathbf{Z}}$ is not an exact functor.

However, if $R = k$ is a field, we may alternatively take global sections, $\mathsf{Coh}_0 X \xrightarrow{\Gamma} \mathsf{Vect}_f$, and view the adèles as an n-Tate object in finite-dimensional k-vector spaces,

$$A(\triangle, \mathcal{F}) \in n\text{-}\mathsf{Tate}^{el}(\mathsf{Vect}_f)\ .$$

For applications in number theory it is interesting to look at schemes of finite type over $\operatorname{Spec} \mathbf{Z}$. Then the global section functor allows to formulate the adèles as an n-Tate object in finite abelian groups, that is

$$A(\triangle, \mathcal{F}) \in n\text{-}\mathsf{Tate}^{el}(\mathsf{Ab}_f)\ .$$

All these variations of the adèles provide a rich source of examples of higher Tate objects. We shall show:

Theorem 22. *Let k be a field and X/k a reduced finite type scheme of pure dimension n. For any quasi-coherent sheaf $\mathcal{F}$ and subset $\triangle \subseteq S(X)_n$ the Beilinson-Parshin adèles $A(\triangle, \mathcal{F})$, viewed as an elementary n-Tate object in finite-dimensional k-vector spaces, i.e. so that*

$$A(\triangle, \mathcal{F}) \in n\text{-}\mathsf{Tate}^{el}\left(\mathsf{Vect}_f\right),$$

is n-sliced (cf. Definition 13). In particular,

$$E_{\triangle}^{\mathrm{Tate}} := \operatorname{End}\left(A(\triangle, \mathcal{O}_X)\right)$$

carries the structure of an n-fold cubical Beilinson algebra (cf. Definition 1).

The claim of this theorem fails if we instead view the adèles as n-Tate objects over $\mathsf{Coh}_0(X)$ or Ab_f. These variations are usually not n-sliced. We defer the proof and begin with some negative examples:

Example 19. Suppose $X := \operatorname{Spec} \mathbf{Z}[t]$. We shall only consider singleton flags $\triangle = \{(\eta_0 > \eta_1 > \eta_2)\}$, defining objects in the category $2\text{-}\mathsf{Tate}^{el}(\mathsf{Ab}_f)$. We shall look at some examples modelled after 2-local fields of mixed characteristics $(0, 0, p)$ and $(0, p, p)$.

1) $A((0) > (t) > (p, t), \mathcal{O}_X)$ evaluates to what could be called $\mathbf{Q}_p((t))$. The objects $t^n \mathbf{Q}_p[[t]]$ for $n \in \mathbf{Z}$ are lattices and the relative quotients

$$\frac{t^n \mathbf{Q}_p[[t]]}{t^m \mathbf{Q}_p[[t]]} \simeq \mathbf{Q}_p^{\oplus(m-n)} \tag{10.1}$$

for $m \geq n$ lie in $1\text{-}\mathsf{Tate}^{el}(\mathsf{Ab}_f)$. Here the sub-objects $p^i \mathbf{Z}_p^{\oplus(n-m)}$ are examples of lattices, with respective quotients $\simeq (\mathbf{Z}/p^j\mathbf{Z})^{\oplus n-m} \in \mathsf{Ab}_f$. We have

$$I_i^{+}(X, X) + I_i^{-}(X, X) = \operatorname{End}(X) \tag{10.2}$$

for $i = 1$ by the presence of the splitting $\mathbf{Q}_p((t)) \twoheadrightarrow \mathbf{Q}_p[[t]]$ which chops off the principal part of the Laurent series. On the other hand, for $i = 2$ Equation 10.2 fails. It suffices to apply Example 7 to the lattice quotients appearing in Equation 10.1.

2) $A((0) > (p) > (p, t), \mathcal{O}_X)$ evaluates to something interesting. It is denoted by $\mathbf{Q}_p\{\{t\}\}$ in [27], and can be described explicitly as doubly

infinite $\mathbf{Q}_p$-valued sequences with boundedness conditions, namely

$$\mathbf{Q}_p\{\{t\}\} = \left\{ \sum_{i=-\infty}^{+\infty} a_i t^i \,\middle|\, \exists C \in \mathbf{R} : a_i \in \mathbf{Q}_p, |a_i|_p \leq C, \lim_{i \to -\infty} |a_i|_p = 0 \right\}.$$

It carries the structure of a 2-local field. The objects $L_n := p^n A((p) > (p,t), \mathcal{O}_X)$ for $n \in \mathbf{Z}$, which identify with

$$L_n = \left\{ \sum_{i=-\infty}^{+\infty} a_i t^i \,\middle|\, a_i \in \mathbf{Q}_p, |a_i|_p \leq p^{-n}, \lim_{i \to -\infty} |a_i|_p = 0 \right\},$$

are lattices and the relative quotients

$$\frac{p^n A((p) > (p,t), \mathcal{O}_X)}{p^m A((p) > (p,t), \mathcal{O}_X)} \simeq (\mathbf{Z}/p^{m-n}\mathbf{Z})((t))$$

for $m \geq n$ lie in $1\text{-}\mathsf{Tate}^{el}(\mathsf{Ab}_f)$. Here we are in the opposite situation. Equation 10.2 holds for $i = 2$, but fails for $i = 1$. The argument of Example 7 can be adapted to show the latter. For this note that $\mathbf{Q}_p\{\{t\}\}/L_n$ is a p-primary torsion group.

See for example [27, Ch. I] or [42] for a further discussion of higher local fields. These sources also explain the construction of $F\{\{t\}\}$ for F a general complete discrete valuation field. We leave it to the reader to formulate its Tate object structure in general. All these higher local fields arise as special cases of adèles of suitably chosen singleton flags.

Example 20. Suppose $X := \operatorname{Spec} k[t]$ and we view its adèles as an elementary 1-Tate object in $1\text{-}\mathsf{Tate}^{el}(\mathsf{Coh}_0 X)$. We show that it cannot be sliced. For simplicitly, let us look at $\triangle = \{(0) > (t)\}$ and $\triangle' = \{(t)\}$. Then

$$A(\triangle', \mathcal{O}_X) \hookrightarrow A(\triangle, \mathcal{O}_X) \twoheadrightarrow A(\triangle, \mathcal{O}_X)/A(\triangle', \mathcal{O}_X)$$

is a short exact sequence. Roughly speaking, it identifies with

$$k[[t]] \hookrightarrow k((t)) \twoheadrightarrow k((t))/k[[t]].$$

The Ind-object $k((t))/k[[t]] = \underset{\overrightarrow{i}}{\operatorname{colim}} \frac{1}{t^i} k[[t]]/k[[t]]$ is t-torsion. In particular, if there was a section to $k((t))$, the latter would have to possess non-trivial t-torsion elements. This is a contradiction. Quite differently, in $1\text{-}\mathsf{Tate}^{el}(\mathsf{Vect}_f)$ a section exists.

Proof of Theorem 22. This is not very difficult because we can produce the required idempotents explicitly. For the sake of simplicity let us write η^i to denote the i-th ideal power of the ideal sheaf of the reduced closed subscheme $\overline{\{\eta\}}$ for a given scheme point $\eta \in X$. Moreover, let us write "$\mathcal{O}\left\langle f^{-1}\right\rangle$" to denote coherent sub-sheaves of $\mathcal{O}_{X,\eta}$, indexed by f, so that the quasi-coherent sheaf $\mathcal{O}_{X,\eta}$ is presented as the $\mathcal{O}_X$-module colimit over them, i.e. as depicted on the left below:

$$\mathcal{O}_{X,\eta} = \underset{f\notin\eta}{\underrightarrow{\operatorname{colim}}}\, \mathcal{O}\left\langle f^{-1}\right\rangle. \qquad R_P = \underset{f\notin P}{\underrightarrow{\operatorname{colim}}}\, \frac{1}{f}R \subset R\left[\frac{1}{f}\right].$$

(This notation is supposed to be suggestive of the corresponding presentation if R is a ring and $P \in \operatorname{Spec} R$ a prime ideal, as depicted above on the right). We unwind the formation of adèles directly from the definition; but recall that by the limits and colimits we really mean the respective diagrams of n-Tate objects and do not refer to carrying out actual limits in the categories themselves, see [17, §7.2] for details on how this can be implemented explicitly. We arrive at

$$\begin{aligned}
A(\triangle,\mathcal{O}_X) &= \prod_{\eta_0\in X} \underset{i_0}{\underleftarrow{\lim}}\, A\left({}_{\eta_0}\triangle\,,\frac{\mathcal{O}_{X,\eta_0}}{\eta_0^{i_0}}\right)\\
&= \prod_{\eta_0\in X} \underset{i_0}{\underleftarrow{\lim}}\,\underset{f_0\notin\eta_0}{\underrightarrow{\operatorname{colim}}}\, A\left({}_{\eta_0}\triangle\,,\frac{\mathcal{O}\left\langle f_0^{-1}\right\rangle}{\eta_0^{i_0}}\right)\\
&= \prod_{\eta_0\in X} \underset{i_0}{\underleftarrow{\lim}}\,\underset{f_0\notin\eta_0}{\underrightarrow{\operatorname{colim}}}\,\prod_{\eta_1\in X} \underset{i_1}{\underleftarrow{\lim}}\, A\left({}_{\eta_1\eta_0}\triangle\,,\frac{\mathcal{O}\left\langle f_0^{-1}\right\rangle}{\eta_0^{i_0}} \underset{\mathcal{O}_X}{\otimes} \frac{\mathcal{O}_{X,\eta_1}}{\eta_1^{i_1}}\right)\\
&= \prod_{\eta_0\in X} \underset{i_0}{\underleftarrow{\lim}}\,\underbrace{\underset{f_0\notin\eta_0}{\underrightarrow{\operatorname{colim}}}\,\prod_{\eta_1\in X} \underset{i_1}{\underleftarrow{\lim}}}_{\mathsf{Tate}^{el}}\,\underset{f_1\notin\eta_1}{\underrightarrow{\operatorname{colim}}}\, A\left({}_{\eta_1\eta_0}\triangle\,,\frac{\mathcal{O}\left\langle f_0^{-1}\right\rangle}{\eta_0^{i_0}} \underset{\mathcal{O}_X}{\otimes} \frac{\mathcal{O}\left\langle f_1^{-1}\right\rangle}{\eta_1^{i_1}}\right)\\
&= \prod_{\eta_0\in X} \underset{i_0}{\underleftarrow{\lim}}\,\underbrace{\underset{f_0\notin\eta_0}{\underrightarrow{\operatorname{colim}}}\,\prod_{\eta_1\in X} \underset{i_1}{\underleftarrow{\lim}}}_{\mathsf{Tate}^{el}}\,\underbrace{\underset{f_1\notin\eta_1}{\underrightarrow{\operatorname{colim}}}\,\prod_{\eta_2\in X} \underset{i_2}{\underleftarrow{\lim}}}_{\mathsf{Tate}^{el}}\,\underset{f_2\notin\eta_2}{\underrightarrow{\operatorname{colim}}}\,(\cdots)\\
&\quad\text{and so forth}\ldots,
\end{aligned}$$

where we only run through those $\eta_0,\ldots,\eta_n$ such that $\eta_0 > \cdots > \eta_n \in \triangle$. The underbraces emphasize which parts of this expression are to be read as limits or colimits respectively, and how to group them to form Tate diagrams. We need to justify why the left-most limits, left of the initial underbrace, exist: Since our scheme is of finite type and η_0 runs through the irreducible components of X, the product over the η_0 is finite. Similarly, the ideal sheaf of each respective irreducible component is necessarily nilpotent so that for each η_0 the limit over i_0 is over an essentially finite diagram. Unwinding this further presents the adèle object as an elementary n-Tate object (in the sketch above only the first two outer-most Tate category iterations is visible). Our claim is proven if we can exhibit pairwise commuting idempotents P_j^+, $j = 1, \dots, n$, projecting this object onto the respective lattice, indexed by $f_{j-1} = 1$. This reduces to constructing sections

$$\frac{\mathcal{O}\left\langle f_0^{-1}\right\rangle}{\eta_0^{i_0}} \otimes \cdots \frac{\mathcal{O}\left\langle f_{j-1}^{-1}\right\rangle}{\eta_{j-1}^{i_{j-1}}} \cdots \otimes \frac{\mathcal{O}\left\langle f_{n-1}^{-1}\right\rangle}{\eta_{n-1}^{i_n}} \twoheadrightarrow \frac{\mathcal{O}\left\langle f_0^{-1}\right\rangle}{\eta_0^{i_0}} \otimes \cdots \frac{\mathcal{O}_X}{\eta_{j-1}^{i_{j-1}}} \cdots \otimes \frac{\mathcal{O}\left\langle f_{n-1}^{-1}\right\rangle}{\eta_{n-1}^{i_n}}$$

in the category of finite-dimensional k-vector spaces (since once these exist, they define straight morphisms between the respective Tate diagrams and therefore the desired idempotents in the category of n-Tate objects). However, the latter is obvious since the category of vector spaces is split exact. $\square$

Of course Theorem 22 provokes a question:

$$E_\triangle^{\mathrm{Beil}} = E_\triangle^{\mathrm{Tate}}\,?$$

Beilinson had already shown, see Theorem 9, that for flags $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ in a scheme X a cubical algebra $E_\triangle^{\mathrm{Beil}}$ can be formed from his notion of lattices. Its definition *hinges crucially* on geometric data of X simply because the underlying notion of lattice depends on X. On the other hand, we have just seen that $E_\triangle^{\mathrm{Tate}}$ is also a cubical algebra. It comes with its own notion of lattices, which now only depends on the structure as a Tate object. One can show that these two types of lattices are different, albeit very closely related to each other. We refer to [16, §5] for an explicit example illustrating this discrepancy.

However, the answer to our question is still affirmative:

Theorem 23. *Let k be a field. Suppose X/k is a reduced scheme of pure dimension n, and view the Beilinson-Parshin adèles $A(\triangle, \mathcal{O}_X)$, for $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ with $\mathrm{codim}_X \overline{\{\eta_i\}} = i$, as an n-Tate object in finite-dimensional k-vector spaces. Then there is a canonical isomorphism of*

Beilinson cubical algebras

$$E_{\triangle}^{\mathrm{Beil}} \cong E_{\triangle}^{\mathrm{Tate}}.$$

We shall split the proof into several lemmata. For notational clarity, let us temporarily introduce the following distinction:

Definition 17. Suppose M is a finitely generated $\mathcal{O}_{\eta_0}$-module and $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ a flag, $\triangle' := \{(\eta_1 > \cdots > \eta_n)\}$.

1) A *Beilinson lattice* is a lattice in the sense of Definition 5, i.e. a finitely generated $\mathcal{O}_{\eta_1}$-module $L \subseteq M$ such that $\mathcal{O}_{\eta_0} \cdot L = M$.
2) A *Tate lattice* is a lattice in the sense of Tate objects, i.e. a sub-object of the n-Tate object $M_{\triangle} := A(\triangle, M)$ which is a Pro-object with an Ind-quotient.

Lemma 24. *For $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ and M a finitely generated $\mathcal{O}_{\eta_0}$-module,*

1) *$M_{\triangle}$ is an elementary n-Tate object and*
2) *for every Beilinson lattice $L \subseteq M$ we have that $L_{\triangle'} \hookrightarrow M_{\triangle}$ is a Tate lattice,*
3) *for every Tate lattice T there exist Beilinson lattices $L_1 \subseteq L_2$ such that*

$$L_{1\triangle'} \subseteq T \subseteq L_{2\triangle'}.$$

Proof. (1) The statement about $M_{\triangle} = A(\eta_0 > \cdots > \eta_n, M)$ is clear from the discussion opening the section, i.e. essentially nothing but [17, §7.2].

(2) Unwinding Equation 10.4 for $L_{\triangle'} = A(\eta_1 > \cdots > \eta_n, L)$, we see that L is one of the Tate lattices in the outer-most colimit so that we clearly have, just by unwinding definitions, a canonical morphism

$$\begin{gathered} L_{\triangle'} \hookrightarrow M_{\triangle} \\ A(\triangle', L) \hookrightarrow A(\triangle, M) \\ \underset{\overleftarrow{L'_1}}{\lim} \underbrace{\underset{\overrightarrow{L_2}}{\mathrm{colim}} \cdots \underset{\overrightarrow{L_n}}{\mathrm{colim}} \underset{\overleftarrow{L'_n}}{\lim} \frac{L_n}{L'_n}}_{(n-1)\text{-Tate object}} \hookrightarrow \underset{\overrightarrow{L_1}}{\mathrm{colim}} \underset{\overleftarrow{L'_1}}{\lim} \underbrace{\underset{\overrightarrow{L_2}}{\mathrm{colim}} \cdots \underset{\overrightarrow{L_n}}{\mathrm{colim}} \underset{\overleftarrow{L'_n}}{\lim} \frac{L_n}{L'_n}}_{(n-1)\text{-Tate object}}, \end{gathered}$$

where on the left-hand side we have replaced the colimit over L_1 by the single value for $L_1 := L$ the lattice at hand. This is visibly a Pro-object with an Ind-quotient, thus a Tate lattice.

(3) Let $T \hookrightarrow M_{\triangle}$ be a Tate lattice. Since we may write $M_{\triangle}$ as $M_{\triangle} = \underrightarrow{\mathrm{colim}}_L L_{\triangle'}$ (by definition), i.e. as an Ind-diagram over Pro-objects, where L runs through all Beilinson lattices, the Pro-subobject T must factor through one of these $L_{\triangle'}$. If L_2 denotes one such index, i.e. the underlying Beilinson lattice, this means that $T \hookrightarrow L_{2\triangle'}$. The other direction is a little more complicated: Let L be any Beilinson lattice. Then the composition

$$L_{\triangle'} \hookrightarrow M_{\triangle} \twoheadrightarrow M_{\triangle}/T$$

is a morphism from a Pro-object to an Ind-object. Thus, it must factor through an $(n-1)$-Tate object C, i.e.

$$L_{\triangle'} \to P \hookrightarrow M_{\triangle}/T \tag{10.3}$$

(*Proof:* Since Pro-objects are left filtering in Tate objects by [17, Prop. 5.8], the composed arrow $L_{\triangle'} \to M_{\triangle}/T$ factors as $L_{\triangle'} \to P \hookrightarrow M_{\triangle}/T$, with P a Pro-object. But $M_{\triangle}/T$ is an Ind-object, so P must be an Ind-object, too. By [17, Prop. 5.9] it follows that P is an $(n-1)$-Tate object). The object $L_{\triangle'}$ can be presented as the Pro-diagram

$$L \mapsto (L/L')_{\triangle'} = L_{\triangle'}/L'_{\triangle'},$$

where $L' \subseteq L$ runs through all Beilinson sub-lattices, partially ordered by inclusion. The quotients $(L/L')_{\triangle'}$ are $(n-1)$-Tate objects, and since these are right filtering in Pro-objects over them, [17, Theorem 4.2 (2)], it follows that the arrow $L_{\triangle'} \to P$ factors through the projection to an object in the Pro-diagram

$$L_{\triangle'} \twoheadrightarrow (L/L')_{\triangle'} \to P$$

for a suitable $L' \subseteq L$. Thus, returning to Equation 10.3 the composition

$$L'_{\triangle'} \hookrightarrow L_{\triangle'} \to M_{\triangle}/T$$

is zero. Thus, $L'_{\triangle'} \hookrightarrow T$ follows from the universal property of kernels. □

Remark 25. The apparent asymmetry in the complexity of proving the existence of L_1 resp. L_2 in $L_{1\triangle'} \subseteq T \subseteq L_{2\triangle'}$ is caused by the fact that we view Tate objects as a sub-category of $\mathsf{Ind}^a\mathsf{Pro}^a(\mathcal{C})$. This is the place where this kind of asymmetry is built in.

Next, we observe that one can present the limits and colimits underlying the adèles in a particularly convenient format:

Lemma 26. *Suppose we are in the situation of the theorem.*

- *Then for any $j = 1, \dots, n$ the following describes the same object:*

$$\varinjlim_{L_1} \varprojlim_{L'_1} \cdots \varinjlim_{L_j} \varprojlim_{L'_j} A\left(\eta_{j+1} > \cdots > \eta_n, \frac{L_j}{L'_j}\right)$$

  where all the L_ℓ run increasingly through all finitely generated $\mathcal{O}_{\eta_\ell}$-submodules of $\frac{L_{\ell-1}}{L'_{\ell-1}}$ (if $\ell \geq 2$) or $\mathcal{O}_{\eta_0}$ (if $\ell = 1$); the $L'_\ell \subseteq L_\ell$ run decreasingly through all finitely generated $\mathcal{O}_{\eta_\ell}$-submodules of L_ℓ having full rank.

- *This statement holds true irrespective of whether we carry out the limits and colimits in the category of all k-vector spaces, or interpret it as an elementary j-Tate object with values in an elementary $(n-j)$-Tate object of finite-dimensional k-vector spaces.*

Proof. The immediate evaluation of $A(\eta_0 > \cdots > \eta_n, \mathcal{O}_X)$ straight from the definition unravels easily to become the case $j = 1$ in the statement. Inductively, one can transform the expression into its counterpart for $j + 1$. For details, cf. [16, Lemma 4.8]. This procedure works with both interpretations, verbatim. $\square$

We may read this lemma as a kind of induction step. For its final step, $j = n$, we arrive at

$$A(\eta_0 > \cdots > \eta_n, \mathcal{O}_X) = \varinjlim_{L_1} \varprojlim_{L'_1} \cdots \varinjlim_{L_n} \varprojlim_{L'_n} \frac{L_n}{L'_n}, \tag{10.4}$$

presenting the adèle object on the left-hand side entirely in terms of Beilinson lattices. This presentation bridges from the definition of the adèles in Beilinson's original paper [4] (or [36], [54], [37] for secondary sources) to the ideals in Beilinson's cubical algebra structure as given in Definition 1:

Lemma 27. *We keep the assumptions as in the theorem. Below, the 'roof symbol' $\widehat{(\cdots)}$ will denote omission:*

1) *Suppose M_1, M_2 are finitely generated $\mathcal{O}_{\eta_0}$-modules. Then a k-linear map $f \in \operatorname{Hom}_k(M_{1\triangle}, M_{2\triangle})$ lies in $\operatorname{Hom}_\triangle(M_1, M_2)$ if and only if it*

stems from a compatible system of k-linear morphisms

$$\frac{L_n}{L'_n} \longrightarrow \frac{N_n}{N'_n},$$

with $L'_n \subseteq L_n$ (in M_1) and $N'_n \subseteq N_n$ (in M_2) suitable Beilinson lattices, inducing a morphism in the limit/colimit

$$f : M_{1\triangle} \longrightarrow M_{2\triangle}$$
$$\varinjlim_{L_1}\varprojlim_{L'_1} \cdots \varinjlim_{L_n}\varprojlim_{L'_n} \frac{L_n}{L'_n} \longrightarrow \varinjlim_{N_1}\varprojlim_{N'_1} \cdots \varinjlim_{N_n}\varprojlim_{N'_n} \frac{N_n}{N'_n}.$$

2) *We remain in the situation of (1). We have $f \in I^+_{i\triangle}(M_1, M_2)$ if and only if f factors as*

$$\varinjlim_{L_1}\varprojlim_{L'_1} \cdots \varinjlim_{L_n}\varprojlim_{L'_n} \frac{L_n}{L'_n} \longrightarrow \varinjlim_{N_1}\varprojlim_{N'_1} \cdots \widehat{\varinjlim_{N_i}} \cdots \varinjlim_{N_n}\varprojlim_{N'_n} \frac{N_n}{N'_n},$$

i.e. instead of the colimit over all N_i we can take a fixed N_i (depending on $N_1, N'_1, \ldots, N_{i-1}, N'_{i-1}$). We have $f \in I^-_{i\triangle}(M_1, M_2)$ if and only if f factors as

$$\varinjlim_{L_1}\varprojlim_{L'_1} \cdots \widehat{\varprojlim_{L_i}} \cdots \varinjlim_{L_n}\varprojlim_{L'_n} \frac{L_n}{L'_n} \longrightarrow \varinjlim_{N_1}\varprojlim_{N'_1} \cdots \varinjlim_{N_n}\varprojlim_{N'_n} \frac{N_n}{N'_n},$$

i.e. instead of the limit over all L_i we can take a fixed L_i (depending on $L_1, L'_1, \ldots, L_{i-1}, L'_{i-1}$).

Proof. This follows rather directly from the definition. Firstly, unravel $A(\eta_0 > \cdots > \eta_n, \mathcal{O}_X)$ in terms of iterated limits and colimits of lattices as in Equation 10.4. But then ideal membership for $I^{\pm}_i$ is exactly the property to factor through a Beilinson lattice of the target, resp. a Beilinson lattice of the source. □

Proof of Thm. 23. For the sake of clarity, we shall denote a Beilinson lattice by the letter $\mathcal{L}$ in this proof, and Tate lattices by the letter L. We know that every Beilinson lattice gives rise to a Tate lattice, Lemma 24, and conversely by the same Lemma every possible Tate lattice L is sandwiched as $\mathcal{L}_{1\triangle} \subseteq L \subseteq \mathcal{L}_{2\triangle}$ between Beilinson lattices $\mathcal{L}_1, \mathcal{L}_2$. We now claim that

$$E_{\triangle}^{\mathrm{Beil}} \cong E_{\triangle}^{\mathrm{Tate}} \tag{10.5}$$

holds as sets. This is seen as follows: Given a k-linear map $f \in E_{\triangle}^{\mathrm{Beil}}$ the definition of the subgroup $\mathrm{Hom}_{\triangle}(\mathcal{O}_{\eta_0}, \mathcal{O}_{\eta_0}) \subseteq \mathrm{End}_k(\mathcal{O}_{X\triangle}, \mathcal{O}_{X\triangle})$ in Definition 6 guarantees that there exist factorizations

$$\overline{f} : (\mathcal{L}_1/\mathcal{L}_1')_{\triangle'} \to (\mathcal{L}_2'/\mathcal{L}_2)_{\triangle'}$$

over suitable Beilinson lattices $\mathcal{L}_1, \mathcal{L}_1', \mathcal{L}_2', \mathcal{L}_2$. By the exactness of the adèle functor $(-)_{\triangle'}$, this is nothing but $\overline{f} : \mathcal{L}_{1\triangle'}/\mathcal{L}'_{1\triangle'} \to \mathcal{L}'_{2\triangle'}/\mathcal{L}_{2\triangle'}$. Hence, we get a (straight) morphism of the explicit Tate diagrams arising from the presentation of Equation 10.4. In particular, this datum induces a morphism of n-Tate objects. Conversely, any morphism of n-Tate objects can be factored over lattice quotients in the desired shape by Lemma 15. This proves Equation 10.5 as an equality of sets because both maps are inverse to each other. However, it is easy to check that these maps are in fact group homomorphisms and also respect composition, so we get an isomorphism of associative algebras. Lemma 27 then establishes the equality of ideals $I_i^{\pm}$: Just unravel the ideal membership conditions and use that all Beilinson lattices give rise to Tate lattices, and conversely we have the sandwiching property. $\square$

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