Log rationally connected surfaces

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In this paper, combining the works of Miyanishi-Tsunoda [MT84a, MT84b] and Keel-McKernan [KM99], we prove the log Castelnuovo's rationality criterion for smooth quasiprojective surfaces over complex numbers.

1. Introduction

Iitaka [Iit77] proposes a program generalizing the classical theory of complex projective surfaces to open algebraic surfaces using the language of pairs. The classification theory of open algebraic surfaces according to the log Kodaira dimension has been carried out by works of Iitaka [Iit81, Iit82], Kawamata [Kaw79], Miyanishi-Tsunoda [MT84a, MT84b] and Keel-McKernan [KM99].

Open algebraic surfaces with log Kodaira dimension $-\infty$ are particularly interesting. Basic examples are *log ruled (log uniruled)* log surface pairs, i.e., the interior contains a Zariski open subset isomorphic to (dominated by) $\mathbb{A}^1 \times T_0$, where $\dim T_0 = 1$. Keel and McKernan [KM99] prove the following log Enriques' ruledness criterion.

Theorem 1.1 ([KM99]). *A log smooth log surface pair (X, D) is log uniruled if and only if $\kappa(X, D) = -\infty$.*

A *log rational curve* on a log pair (X, D) is a rational curve $f : \mathbb{P}^1 \to X$ which meets D at most once. Inspired by the theory of log rational curves developed by Chen and the author [CZ14a, CZ14b], we define that a log pair (X, D) is *log rationally connected* if there exists a log rational curve passing through a general pair of points.

In this paper, combining the works of Miyanishi-Tsunoda [MT84a], [MT84b] and Keel-McKernan [KM99], we obtain the following numerical criterion for log rationally connected log surface pairs, which generalizes the classical Castelnuovo's rationality criterion.

Theorem 1.2 (Log Castelnuovo's criterion). *The following statements are equivalent for a log smooth log surface pair* (X, D):

1) (X, D) *is log rationally connected;*

2) $H^0(X, (\Omega^1_X(\log D))^{\otimes m}) = 0$, *for any* $m \geq 1$;

3) $\kappa(X, D) = -\infty$ *and* $H^0(X, S^{12}\Omega^1_X(\log D)) = 0$.

By [CZ14a, Prop. 2.7], log rational connectedness implies there exists a log rational curve $f : \mathbb{P}^1 \to X$ such that $f^*\Omega^1_X(\log D)$ is the sum of line bundles of negative degree. Hence every section of $(\Omega^1_X(\log D))^{\otimes m}$ vanishes along $f(\mathbb{P}^1)$. Since such curves cover a dense open subset of X, (1) implies (2). It is obvious that (2) implies (3). For (3) $\Rightarrow$ (1), the idea is to run the log minimal model program for log surface pairs. The end products are classified by Miyanishi-Tsunoda [MT84a, MT84b] and Keel-McKernan [KM99]. And many of them are log rationally connected with the exception of the log-ruled case. The log ruled case is proved in Proposition 3.2. In the proof, we discover, somewhat unexpectedly, a close relationship between the non-existence of pluri log one forms and the solutions for the strong approximation problem of the affine line over function fields of complex curves.

Remark 1.3.

1) Log rationally connected log pairs are expected to be the natural geometric context for the strong approximation problem over function fields of curves, c.f., [HT08, CZ16, CZ15].

2) By [Iit81, p.91], the precise copy of Castelnuovo's rationality criterion fails. There exists a log surface pair (X, D) with $q(X, D) = 0$, $h^0(X, \mathcal{O}_X(2(K_X + D))) = 0$ but $h^0(X, \mathcal{O}_X(4(K_X + D))) = 1$. In fact, a more careful analysis of our proof will show that the condition (3) in Theorem 1.2 is sharp.

3) By [KM99, Cor. 7.9], log rational connectedness implies that the fundamental group of the interior is finite. However, this topological restriction plus negative log Kodaira dimension will not characterize log rational connectedness. See Propositon 4.1.

4) The log smooth log Fano surface pair $(\mathbb{P}^2, \{xy = 0\})$ has nonvanishing log irregularity and infinite fundamental group of the interior. In particular, it is not log rationally connected. So the log analogue of the

theorem of Campana [Cam92] and Kollár-Miyaoka-Mori [KMM92] fails even for surface pairs.

Notation 1.4. In this paper, we work with varieties and log pairs over complex numbers $\mathbb{C}$. A log pair (X, D) means a variety X with a reduced Weil divisor D. Let U be its interior $X - D$. We say that (X, D) is *log smooth* if X is smooth and D is a simple normal crossing divisor. A log pair is projective if the ambient variety is projective.

For a log smooth pair (X, D), we use $\kappa(X, D)$ to denote the logarithmic Kodaira dimension and $q(X, D)$ to denote the logarithmic irregularity, i.e., $q(X, D) = h^0(X, \Omega^1_X(\log D))$. They only depend on the interior of the pair.

2. Log minimal model program

Let (X, D) be a projective log smooth surface pair with $\kappa(X, D) = -\infty$. By [KM98, Theorem 3.47], we run the log minimal model program on this pair

$$(X, D) = (X_0, D_0) \rightarrow (X_1, D_1) \rightarrow \cdots \rightarrow (X_k, D_k) = (X^*, D^*),$$

such that:

1) each step is a divisorial contraction;

2) the log Kodaira dimension remains the same, i.e., $\kappa(X_i, D_i) = -\infty$;

3) the end product (X^*, D^*) is either
   a) log ruled, or
   b) a log del Pezzo surface of Picard number one, i.e., $\rho(X^*) = 1$.

Lemma 2.1. *If (X^*, D^*) is log ruled (log rationally connected), so is (X, D).*

Proof. At each step of the log MMP, we contract a curve representing a $(K_X + D)$-negative extremal ray. Such curve could be contracted to either a boundary point or an interior point. Thus after deleting $U := X - D$ with a pure codimension one subset, it is isomorphic to a dense open V^* of $U^* := X^* - D^*$ such that $\dim(U^* - V^*) = 0$. The lemma follows from log ruledness (log rational connectedness) remains after deleting finitely many points. $\square$

When (X, D) is a log del Pezzo surface pair, we recall the following two fundamental theorems established by Miyanishi-Tsunoda [MT84a, MT84b] and Keel-McKernan [KM99].

Theorem 2.2. *[Miy01, Ch.2, Thm 5.1.2] Let $(X, D \neq \emptyset)$ be a log del Pezzo surface pair of Picard number one with nonempty boundary. If (X, D) is not log ruled, then the interior $U := X - D$ is a Platonic $\mathbb{A}^1_*$-fiber space, i.e., U is isomorphic to $\mathbb{A}^2 - (0)$ modulo a noncyclic small finite subgroup of $GL(2, \mathbb{C})$. In particular, (X, D) is log rationally connected.*

Theorem 2.3. *[KM99, Theorem 1.6] Let $(X, D = \emptyset)$ be a log del Pezzo surface pair with empty boundary. Then the smooth locus of X^{sm} is rationally connected.*

Proof of Theorem 1.2. If (X^*, D^*) a log del Pezzo surface pair of Picard number one and not log ruled, the theorem follows from Theorem 2.2, 2.3 and Lemma 2.1. If (X^*, D^*) is log ruled, then by Lemma 2.1 again, (X, D) is log ruled. This case will be proved in Proposition 3.2. □

3. Log ruled case

Lemma 3.1. *Let (X, D) a projective log smooth surface pair. If (X, D) is log ruled and let $f : X \dashrightarrow T$ be a rational map to a smooth projective curve T, then there exists a birational morphism $(X', D') \to (X, D)$ extending f and $f|_{X'-D'}$ is proper onto the image $X - D$.*

Proof. To resolve the indeterminacy, we need to take a sequence of blow ups. For each intermediate blow up $g_i : X_{i+1} \to (X_i, D_i)$, we take $D_{i+1} := g_i^{-1}(D_i)$. Then it is clear that $g_i|_{X_{i+1}-D_{i+1}}$ is proper onto the image $X_i - D_i$. □

Proposition 3.2. *If (X, D) is log ruled and $H^0(X, S^{12}\Omega^1_X(\log D)) = 0$, then (X, D) is log rationally connected.*

Proof. By [Iit82, Prop. 11.3] and Lemma 3.1, we may assume that the rational map $(X, D) \to T_0$ extends to a proper flat morphism $f : (X, D) \to T$, where T is a smooth projective curve containing T_0. The log ruling on the generic fiber implies that $f(U)$ is nonempty open in T. Let S be the complement of $f(U)$ in T. By [Miy01, 2.1.17], we have a morphism of pairs

$$f : (X, D) \to (T, S),$$

which induces an injection

$$H^0(T, \Omega^1_T(\log S)) \to H^0(X, \Omega^1_X(\log D)).$$

Thus, by assumption, $q(T, S) = 0$. The classification of log curves implies that the pair (T, S) is isomorphic to either $(\mathbb{P}^1, \emptyset)$ or $(\mathbb{P}^1, \{\infty\})$.

Notation 3.3. Let $p_1, \cdots, p_k$ be the points on $T - S$ whose inverse image $f|_U^{-1}(p_i)$ contains no reduced component. Let d_i be the minimal multiplicity of all irreducible components of $f|_U^{-1}(p_i)$. We may assume that $d_1 \leq \cdots \leq d_k$.

Lemma 3.4. *If the base (T, S) is isomorphic to $(\mathbb{P}^1, \{\infty\})$ and*

$$H^0(X, S^2\Omega^1_X(\log D)) = 0,$$

then $k \leq 1$.

Proof. If $k \geq 2$, we may assume that $p_1 = 0$ and $p_2 = 1$. Let z be the coordinate on $\mathbb{P}^1 - \{\infty\}$. Since any irreducible component E of $f|_U^{-1}(p_i)$ has the form $(t^e = 0)$ where $e \geq 2$, the pullback of the tensor

$$f^*\left(\frac{dz}{z} \otimes \frac{dz}{z-1}\right) = e\frac{dt}{t} \otimes t^{e-1}dt = et^{e-2}dt \otimes dt$$

is regular on E. Thus, we have

$$0 \neq f^*\left(\frac{dz}{z} \otimes \frac{dz}{z-1}\right) \in H^0(X, S^2\Omega^1_X(\log D)),$$

which contradicts the assumption. $\square$

Lemma 3.5. *If the base (T, S) is isomorphic to $(\mathbb{P}^1, \emptyset)$ and*

$$H^0(X, S^{12}\Omega^1_X(\log D)) = 0,$$

then we have the following:

   1) $k = 0$, i.e., each fiber contains a reduced component;

   2) $k = 1$ and $d_1 = n \geq 2$;

   3) $k = 2$ and $d_1, d_2 \geq 2$;

   4) $k = 3$ and (d_1, d_2, d_3) are one of the following triples:
   a) $(2, 2, n \geq 2)$;
   b) $(2, 3, 3)$;
   c) $(2, 3, 4)$;
   d) $(2, 3, 5)$.

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Proof. If $k \geq 4$, let $p_1 = 0, p_2 = 1, p_3 = \infty, p_4 = c$. By the same argument as in Lemma 3.4, the pullback

$$f^* \left(\frac{dz}{z} \otimes \frac{dz}{(z-1)(z-c)} \right)$$

gives a nonzero element in $H^0(X, S^2\Omega^1_X(\log D))$.

Now we may assume that $k = 3$ and $p_1 = 0, p_2 = 1, p_3 = \infty$. Consider the pullback of one forms and we obtain the following table.

order at	0	1	∞
$f^*(\frac{dz}{z})$	-1	$d_2 - 1$	-1
$f^*(\frac{dz}{z-1})$	$d_1 - 1$	-1	-1
$f^*(\frac{dz}{z(z-1)})$	-1	-1	$d_3 - 1$

With the same argument as in Lemma 3.4, we obtain the following:

- If $d_1 \geq 3$, then

$$0 \neq f^* \left(\frac{dz}{z} \otimes \frac{dz}{z-1} \otimes \frac{dz}{z(z-1)} \right) \in H^0(X, S^3\Omega^1_X(\log D)).$$

- If $d_1 = 2, d_2 \geq 4$, then

$$0 \neq f^* \left(\frac{dz}{z} \otimes \left(\frac{dz}{z-1} \right)^{\otimes 2} \otimes \frac{dz}{z(z-1)} \right) \in H^0(X, S^4\Omega^1_X(\log D)).$$

- If $d_1 = 2, d_2 = 3, d_3 \geq 6$, then

$$0 \neq f^* \left(\left(\frac{dz}{z} \right)^{\otimes 3} \otimes (\frac{dz}{z-1})^{\otimes 2} \otimes \frac{dz}{z(z-1)} \right) \in H^0(X, S^6\Omega^1_X(\log D)).$$

All above cases contradict with $H^0(X, S^{12}\Omega^1_X(\log D)) = 0$. Therefore the lemma is proved. $\qquad \square$

Lemma 3.6. *With the same notations as above, assume that*

$$H^0(X, S^{12}\Omega^1_X(\log D)) = 0.$$

Then there exists a finite cover $g : \mathbb{P}^1 \to \mathbb{P}^1$ such that after taking the base change

$$
\begin{array}{ccc}
(X', D') & \xrightarrow{\ g'\ } & (X, D) \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
(T' = \mathbb{P}^1, S' = g^{-1}(S)) & \xrightarrow{\ g\ } & (T, S),
\end{array}
$$

the fiber $U'_t = X'_t - D'_t$ contains a reduced component for every $t \in T' - S'$.

Proof. When $k \leq 2$, we can choose g as a cyclic cover $z \mapsto z^n$. Now by Lemma 3.4 and 3.5, the only cases left are when the base is $(\mathbb{P}^1, \emptyset)$ and $k = 3$. On the other hand, every such case corresponds to a finite noncyclic subgroup

$$\Gamma \leq \operatorname{Aut}(\mathbb{P}^1) = PGL(2, \mathbb{C}) \cong SO(3; \mathbb{R})$$

as in [Art91, Chapter 5, Theorem 9.1]. The natural quotient map

$$g : \mathbb{P}^1 \to \mathbb{P}^1/\Gamma \cong \mathbb{P}^1$$

totally ramifies over three points on the target, say, p_1, p_2, p_3, such that the ramification order of every preimage of p_i is d_i, for $i = 1, 2, 3$. The lemma follows by taking base change via the map $g : \mathbb{P}^1 \to \mathbb{P}^1/\Gamma$. $\qquad\square$

Now let us return to the proof of Proposition 3.2. Let U' be the interior of (X', D'). By construction, f' is a morphism of log pairs. And every fiber of $f'|_{U'}$ contains a reduced component, i.e., there exists local integral section over any point in $T - S$.

Note that f' gives an integral model of $\mathbb{P}^1$ or $\mathbb{A}^1$ over the function field $\mathbb{C}(\mathbb{P}^1)$. Since strong approximation holds for $\mathbb{P}^1$ or $\mathbb{A}^1$ over the function field of curves [Ros02, Theorem 6.13], a general pair of points in U' can be connected by an integral section, which is either a rational curve or an $\mathbb{A}^1$-curve. The log rational connectedness of U follows because U' maps surjectively to U. $\qquad\square$

4. An example

Proposition 4.1. *There exists an log ruled surface (X, D) such that*

1) *$\pi_1(X - D)$ is trivial;*

2) *$q(X, D) = 0$;*

3) *$\kappa(X, D) = -\infty$;*

4) (X, D) is not log rationally connected.

Construction 4.2. Take $U = \mathbb{A}^1 \times \mathbb{A}^1$ with the natural projection π to the first factor $T = \mathbb{A}^1$. Pick two points p_1, p_2 on T. For each $i = 1, 2$, remove $\pi^{-1}(p_i)$ from U and glue back a disjoint union of a triple $\mathbb{A}^1$ and a double $\mathbb{A}^1$. We obtain a new surface $q : U' \to T$. We can achieve this by embedding U into $\mathbb{P}^1 \times \mathbb{P}^1$, taking further blow ups and deleting some extra divisors. Take any log smooth model (X, D) with the interior U'.

Proof of Proposition 4.1. First $q : U' \to T$ gives an $\mathbb{A}^1$-ruling over $\mathbb{A}^1$, in particular, we have $\kappa(X, D) = -\infty$. The proof of Lemma 3.4 implies that (X, D) is not log rationally connected. For the fundamental group of U', an easy calculation via Van Kampen's theorem shows that it is simply connected. Finally by the construction of log Albanese variety [Iit77] and the rationality of X, we have

$$\text{rank } H^0(\Omega^1_X(\log D)) = \text{rank } H_1(U', \mathbb{Z})^{free}.$$

The later is trivial because $\pi_1(U') = 0$. Therefore we have $q(X, D) = 0$. □

Acknowledgments

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