Explicit models for some stable categories of maximal Cohen-Macaulay modules

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We give concrete DG-descriptions of certain stable categories of maximal Cohen-Macaulay modules. This makes in possible to describe the latter as generalized cluster categories in some cases.

1. Introduction

Throughout k is a field. Let (R, m) be a local complete noetherian Gorenstein k-algebra of Krull dimension n with an isolated singularity and with $R/m \cong k$. Then it is well-known that the stable category of Cohen-Macaulay R-modules $\underline{\mathrm{MCM}}(R)$ is an $n - 1$-Calabi-Yau category. This category is also sometimes called the singularity category of R.

Assume given a Cohen-Macaulay R-module N such that $\Lambda = \mathrm{End}_R(R \oplus N)$ has finite global dimension and is a Cohen-Macaulay R-module. Thus Λ is a non-commutative crepant resolution of R in the sense of [16]. Under these conditions Iyama has shown that N is an $n - 1$-cluster tilting object in $\underline{\mathrm{MCM}}(R)$ [6, Thm. 5.2.1] and it is a natural question if $\underline{\mathrm{MCM}}(R)$ may be obtained by one of the standard constructions of CY-categories with cluster tilting object [1, 5].

Assume for simplicity that N contains no repeated summands and no summands isomorphic to R. Let $(S_i)_{i=0,\ldots,l}$ be the simple Λ-modules with S_0 corresponding to the summand R of $R \oplus N$. Let e_0 the idempotent given by the projection $R \oplus N \to R$. Put $l = \Lambda/\mathrm{rad}\,\Lambda$ and $l^0 = le_0$.

In this note we observe the following (Lemma 4.1, Theorem 5.1.1, Remark 5.1.2)

Theorem 1.1. *Let $(T_lV, d) \to \Lambda$ be a finite minimal model for Λ (see §4). Put $\Gamma = T_lV/T_lV e_0 T_lV$. Then one has*

$$\underline{\mathrm{MCM}}(R) \cong \mathrm{Perf}(\Gamma)/\langle (S_i)_{i \neq 0} \rangle$$

and furthermore Γ has the following properties:

1) *Γ has finite dimensional cohomology in each degree;*

2) *Γ is concentrated in degrees ≤ 0;*

3) *$H^0(\Gamma) = \Lambda/\Lambda e_0 \Lambda$;*

4) *As a graded algebra Γ is of the form $T_{l^0}V^0$ for $V^0 = (1 - e_0)V(1 - e_0)$.*

The miminal model (T_lV, d) always exists for abstract reasons (see Lemma 4.1 below) but to use Theorem 1.1 effectively one must be able to describe it. There are at least two cases where this is easy:

- Λ is 3-Calabi-Yau. It then follows from [14] that the minimal model of Λ is derived from a super potential.

- Λ is Koszul.

We obtain in particular (see Proposition 5.2.1 below)

Proposition 1.2. *Assume that Λ is 3-Calabi Yau. Then $\underline{\mathrm{MCM}}(R)$ is the generalized cluster category associated to a quiver with super potential as introduced by Claire Amiot in [1].*

The Koszul case is well taylored to quotient singularities. In §6.4 we consider the case where $R = k[[x_1, \ldots, x_n]]^G$ for $G \cong \mathbb{Z}/m\mathbb{Z}$ acting linearly (Proposition 6.5.1) with weights $(\xi^{a_i})_i$ for ξ a primitive m'th root of unity and $0 \leq a_i \leq m - 1$, $\gcd(a_i, m) = 1$, $\sum_i a_i \cong 0 \bmod m$. We given an explicit description of Γ in this case.

As an application (Proposition 6.6.1 below) of this explicit description of Γ we obtain a new proof of the following result from [2, 7]

Proposition 1.3. *Assume $\sum_i a_i = m$. Let P be the quiver with vertices $[m - 1] \overset{def}{=} \{1, \ldots, m - 1\}$ and for each $j \in [m - 1]$ and $i \in [n]$ such that $j + a_i \in [m - 1]$ an arrow $x_i : j \to j + a_i$ with relations $x_i x_{i'} = x_{i'} x_i$. Let C be the path algebra of P modulo these relations. Let $\Theta_C = \mathrm{RHom}_{C^e}(C, C \otimes$*

C) *be the inverse dualizing complex of C [10, 15]. Then there is a quasi-isomorphism of DG-algebras*

$$\Gamma \cong T_C(\Theta_C[n-1])$$

In particular $\underline{\mathrm{MCM}}(R)$ *is a generalized* $n-1$-*cluster category.*

2. Acknowledgement

As said Proposition 1.3 was proved independently in [2] (see also [7]). Some time ago the authors had been looking at explicit DG-models for certain stable categories of maximal Cohen-Macaulay models (based on Theorem 1.1). We knew that Proposition 1.3 was true in the 3-dimensional case (as then it is a consequence of Proposition 1.2) and expected a suitable generalization to hold in higher dimension as well. However we became distracted by other interests.

In May 2010 the authors learned about the precise statement of Proposition 1.3 from a very interesting lecture by Osamu Iyama at the OberWolfach meeting "Interactions between Algebraic Geometry and Noncommutative Algebra" (see [7]). In that lecture the result was derived from a very general statement about graded Calabi-Yau algebras of Gorenstein parameter 1.

We expected the result to be a consequence of Theorem 1.1 as well and this is indeed the case although the derivation is a little more involved than we anticipated. Nonetheless since the methods in [2] appear to be quite different from ours we decided it would be useful to write down our own proof.

We are very grateful to Dong Yang for his careful reading of the first version of this manuscript. His comments have helped us to improve the exposition. Dong Yang has also informed us that he and Martin Kalck have a different approach to some of our results based on "recollement".

3. Notation and conventions

Throughout l is a separable algebra over the ground field k. If we refer to an "algebra" Γ then we mean an l-algebra. In other words Γ is implicitly equipped with a ring homomorphism $l \to \Gamma$. Unadorned tensor products are over k.

4. Preliminaries

In this paper we deal with two settings. The "complete" case and the "graded" case. This mainly affects our definition of a tensor algebra. Let V be an l-bimodule and denote by $T_l^\circ V$ the ordinary tensor algebra of V. In the complete case $T_l V$ is the completion of $T_l^\circ V$ at the twosided ideal $(T_l^\circ V)_+$ generated by V. If V carries an additional $\mathbb{Z}$-grading (in a homological sense) then $T_l V$ refers to the graded completion of $T_l^\circ V$ at $(T_l^\circ V)_+$. In the graded case V will be equipped with an *additional* (non-homological) grading which we sometimes refer to as the "Adams grading" (following [13]). In this case $T_l V$ is simply $T_l^\circ V$ equipped with the extended Adams grading.

- In the complete case we deal with (DG-)algebras which can be written as of $T_l V / I$ with V a finitely generated (graded) l-bimodule and I a (graded) closed ideal in $T_l V$.

- In the graded case we deal with graded (DG-)algebras which can be written as of $T_l V / I$ with V a finitely generated (bi)graded l-bimodule and I a (bi)graded ideal. Unless otherwise specified the Adams grading will be left bounded in each homological degree.

Below we will use finite minimal models for some algebras. For us a finite minimal model of an algebra Λ will be a DG-algebra of the form $(T_l V, d)$ for a finitely generated graded l-bimodule V living in degrees ≤ 0 such that dV is in the twosided ideal generated by $V \otimes_l V$, together with a quasi-isomorphism $T_l V \to \Lambda$ (with Λ being viewed as a DG-algebra concentrated in degree zero).

Lemma 4.1. *Let Λ be either complete of the from $l + \operatorname{rad} \Lambda$ or graded of the form $\Lambda = l + \Lambda_1 + \cdots$. Assume $\dim \sum_i \operatorname{Tor}_i^\Lambda(l, l) < \infty$. Then Λ has a finite minimal model with*

$$
(4.1) \qquad V = \bigoplus_{i \geq 1} \operatorname{Tor}_i^\Lambda(l, l)[i - 1]
$$

Conversely if Λ has a finite minimal model $T_l V$ then V is given by the formula (4.1).

Proof. This follows from the bar-cobar formalism. In the graded case it has been proved explicitly in [13] (see §1.1 in loc. cit.). One checks that the proof also goes through in the complete case (see e.g. [14, Prop. A.5.4]). $\qquad \square$

For our examples we only need the case where Λ is a Koszul algebra or a completion thereof. In that case it is trivial to construct the minimal model directly (see Proposition 6.1.1 and Remark 6.1.2 below).

5. Stable categories of Cohen-Macaulay modules

In this section we show how to obtain explicit models for some stable categories of Cohen-Macaulay modules. In order to have somewhat concise statements we intentionally do not work in the greatest possible generality.

5.1. General results

In this section we assume that (R, m) is a complete local noetherian Gorenstein k-algebra with an isolated singularity (by complete local we mean either literally complete local or else $\mathbb{N}$-graded local and $R/m = k$). Let $(M_i)_{i=1,\dots,l}$ be indecomposable finitely generated maximal Cohen-Macaulay R-modules which are pairwise non-isomorphic and also not isomorphic to R. Put $M_0 = R$, $M = \bigoplus_i M_i$ and $\Lambda = \operatorname{End}_R(M)$. Assume that Λ has finite global dimension so that Λ is a non-commutative crepant resolution of R in the sense of [16].

Let $e_i \in \Lambda$ be the idempotent given by the projection $M \to M_i$ and put $P_i = \Lambda e_i$. Let $S_i = P_i / \operatorname{rad} P_i$ be the corresponding simple Λ-module. Put $l = \sum_i k e_i$.

Theorem 5.1.1. *Let $T_l V$ be a finite minimal model of Λ (cfr Lemma 4.1). Let $\underline{\mathrm{MCM}}(R)$ be the stable category of maximal Cohen-Macaulay R-modules (graded if we are in the graded context). Put $\Gamma = T_l V / T_l V e_0 T_l V$.*

1) *The $(S_i)_{i\neq 0}$ are perfect Γ-modules and furthermore there is an exact equivalence*

$$\underline{\mathrm{MCM}}(R) = \operatorname{Perf}(\Gamma)/\langle (S_i)_{i\neq 0}\rangle$$

2) *The DG-algebra Γ has finite dimensional cohomology in each degree. Furthermore $H^0(\Gamma) = \Lambda/\Lambda e_0 \Lambda$.*

Remark 5.1.2. Put $V^0 = (1 - e_0)V(1 - e_0)$ and $l^0 = l/le_0$. Then it is not hard to see that

$$T_l V / T_l V e_0 T_l V \cong T_{l^0} V^0$$

Proof of Theorem 5.1.1. We first prove (1). According to Lemma 5.1.3 below we have an equivalence

$$D_f^b(\Lambda)/\langle(S_i)_{i\neq 0}\rangle \cong D_f^b(R) :$$

which sends P_0 to R.

Hence using Buchweitz's theorem [3] we get

$$\underline{\mathrm{MCM}}(R) = D_f^b(R)/\langle R\rangle = \left(D_f^b(\Lambda)/\langle(S_i)_{i\neq 0}\rangle\right)/\langle P_0\rangle$$
$$= D_f^b(\Lambda)/\langle P_0, (S_i)_{i\neq 0}\rangle = \left(D_f^b(\Lambda)/\langle P_0\rangle\right)/\langle(S_i)_{i\neq 0}\rangle$$

To obtain our conclusion it is now sufficient to prove

$$D_f^b(\Lambda)/\langle P_0\rangle = \mathrm{Perf}(T_lV/T_lV\,e_0T_lV)$$

By hypotheses Λ has finite global dimension and hence $D_f^b(\Lambda) = \mathrm{Perf}(\Lambda) = \mathrm{Perf}(T_lV)$. We may now conclude as in [10, Lemma 7.2] (and proof).

Since $H^i(\Gamma) = \mathrm{Hom}_{\mathrm{Perf}(\Gamma)}(\Gamma, \Gamma[i])$ and $\mathrm{Perf}(\Gamma) \cong D_f^b(\Lambda)$ the second statement is an immediate consequence of Proposition 5.1.4 below.   □

Lemma 5.1.3. *The functor*

(5.1) $$\Xi : D_f^b(\Lambda) \to D_f^b(R) : N \mapsto e_0N$$

induces an equivalence

$$D_f^b(\Lambda)/\langle(S_i)_{i\neq 0}\rangle \cong D_f^b(R) :$$

Proof. Let $U \in D_f^b(R)$. Since P_0 is a vector bundle on $\mathrm{Spec}\,R - \{m\}$ we have that $P_0 \otimes_R U$ has finite dimensional cohomology.

Let N be such that for $n \geq N$ we have $H^{-n}(U) = 0$. We claim that for $n \geq N$ we have that $H^{-n}(P_0 \overset{L}{\otimes}_R U)$ is an extension of $(S_i)_{i\neq 0}$, i.e. $e_0 H^{-n}(P_0 \overset{L}{\otimes}_R U) = 0$. Indeed

(5.2) $$e_0 H^{-n}(P_0 \overset{L}{\otimes}_R U) = H^{-n}(e_0\Lambda e_0 \overset{L}{\otimes}_R U) \cong H^{-n}(R \overset{L}{\otimes}_R U)$$
$$= H^{-n}(U) = 0$$

Define $\Phi(U) = \tau_{\geq -N}(P_0 \overset{L}{\otimes}_R U)$. Then $\overline{\Phi(U)}$ is a well defined object of $D_f^b(\Lambda)/\langle(S_i)_{i\neq 0}\rangle$. We claim that $\overline{\Phi(-)}$ yields a quasi-inverse to (5.1).

If $U \in D_f^b(R)$ then the computation (5.2) shows that $\Xi\overline{\Phi}(U) = U$. Conversely assume $V \in D_f^b(\Lambda)$. Then $\overline{\Phi}\Xi(V) = \tau_{\geq -N}(P_0 \overset{L}{\otimes}_R e_0 V)$. Let C be the cone of $P_0 \overset{L}{\otimes}_R e_0 V = \Lambda e_0 \overset{L}{\otimes}_R e_0 V \to V$. We see $e_0 C = 0$. In other words the cohomology of C is given by extensions of $(S_i)_{i \neq 0}$. Furthermore by our choice of N we have $e_0 H^{-n}(V) = H^{-n}(e_0 V) = 0$ for $n \geq N$ and hence $H^{-n}(V)$ is an also an extension of $(S_i)_{i \neq 0}$ for such n. Thus working modulo $\langle (S_i)_{i \neq 0} \rangle$ we have

$$\tau_{\geq -N}(P_0 \overset{L}{\otimes}_R e_0 V) = \tau_{\geq -N} V = V$$

which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 5.1.4. *The category $D_f^b(\Lambda)/\langle P_0 \rangle$ is Hom-finite and in addition*

$$(5.3) \qquad\qquad \mathrm{Hom}_{D_f^b(\Lambda)/\langle P_0 \rangle}(\Lambda, \Lambda) = \Lambda/\Lambda e_0 \Lambda$$

Proof. Since Λ has finite global dimension to prove Hom-finiteness it is sufficient to prove for any $M \in D_f^b(\Lambda)$:

$$(5.4) \qquad\qquad \dim \mathrm{Hom}_{D_f^b(\Lambda)/\langle P_0 \rangle}(\Lambda, M) < \infty$$

If $N \in \mathrm{mod}(\Lambda)$ then there is a map

$$(5.5) \qquad\qquad\qquad\qquad \phi : P_0^n \to N$$

such that $\mathrm{coker}\, \phi \in \langle (S_i)_{i \neq 0} \rangle$. Using ϕ we can "resolve" any $M \in D_f^b(\Lambda)$ with a complex $P \in \langle P_0 \rangle$ such that $\mathrm{cone}(P \to M)$ modulo $\langle (S_i)_{i \neq 0} \rangle$ is an object M_1 in $\mathrm{mod}(\Lambda)[n]$ for $n \gg 0$. Hence to prove (5.4) we have reduced ourselves to the cases $M \in \mathrm{mod}(\Lambda)[n]$ for $n \gg 0$ or $M = S_i[p]$ for $i \neq 0$ and $p \in \mathbb{Z}$.

To deal with the case $M = S_i[p]$ let N be an extension of $(S_i)_{i \neq 0}$ in $\mathrm{mod}(A)$. Then $\mathrm{Hom}^i_{D_f^b(\Lambda)}(P_0, N) = 0$ from which we easily deduce
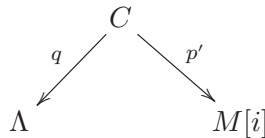
$$(5.6) \qquad\qquad \mathrm{Hom}^i_{D_f^b(\Lambda)/\langle P_0 \rangle}(\Lambda, N) = \begin{cases} N & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus this case is clear.

To deal with the other case it is sufficient to show the following statement. Assume $M \in \mathrm{mod}(\Lambda)$ then

$$(5.7) \qquad \mathrm{Hom}^i_{D^b_f(\Lambda)/\langle P_0 \rangle}(\Lambda, M) = 0$$

for $i > 0$. Let p be a map $\Lambda \to M[i]$ in $D^b_f(\Lambda)/\langle P_0 \rangle$ with $i > 0$. Then p is represented in $D^b_f(\Lambda)$ by a diagram of the following kind

$$
\begin{array}{ccc}
 & C & \\
 {}^{q}\swarrow & & \searrow{}^{p'} \\
\Lambda & & M[i]
\end{array}
$$

such that $P \stackrel{\mathrm{def}}{=} \mathrm{cone}\, q \in \langle P_0 \rangle$. It is easy to see that we may assume that this diagram is an actual diagram of complexes and that in addition C is a finite complex of finitely generated projectives. Then it follows that we may also assume that P is a finite complex of finite direct sums of P_0.

The composition $P[-1] \to C \xrightarrow{p'} M[i]$ factors through $\sigma_{\leq -i}(P[-1])$ where $\sigma_{\leq -i}$ denotes "naive" truncation. Hence $\sigma_{\leq -i}(P[-1]) \in \langle P_0 \rangle$.

We then obtain a morphism of distinguished triangles in $D^b_f(\Lambda)$

$$
\begin{array}{ccccc}
P[-1] & \longrightarrow & C & \xrightarrow{q} & \Lambda \\
\downarrow & & \downarrow{}^{p'} & & \vdots\downarrow \\
\sigma_{\leq -i}(P[-1]) & \longrightarrow & M[i] & \longrightarrow & Z
\end{array}
$$

where Z is the cone of the lower leftmost map. We see that Z has no cohomology in degree zero and hence $\mathrm{Hom}_{D^b_f(\Lambda)}(\Lambda, Z) = 0$. Thus p' factors through $\sigma_{\leq i}(P[-1])$. So p (which is the image of p' in $D^b_f(\Lambda)/\langle P_0 \rangle$) is the zero map. This finishes the proof of (5.7) for $i > 0$.

It follows also from (5.7) that $\mathrm{Hom}_{D^b(\Lambda)/\langle P_0 \rangle}(\Lambda, -)$ is a right exact functor on $\mathrm{mod}(\Lambda)$. To compute $\mathrm{Hom}_{D^b(\Lambda)/\langle P_0 \rangle}(\Lambda, M)$ let $\bar{M} = \mathrm{coker}\, \phi$ (cfr (5.5)). Applying (5.7) together with (5.6) to the right exact sequence

$$P_0^n \to M \to \bar{M} \to 0$$

we obtain

$$\mathrm{Hom}_{D^b(\Lambda)/\langle P_0 \rangle}(\Lambda, M) = \bar{M}$$

Applying this identity with $M = \Lambda$ we immediately obtain (5.3).

5.2. Application to Claire Amiot's generalized cluster categories

Let $(Q, w) = (Q_0, Q_1, w)$ be a quiver with super potential (thus w is an element of the completed path algebra $\widehat{kQ}$). The Ginzburg algebra $\Gamma(Q, w)$ is the DG-algebra $(k\widetilde{Q}, d)$ where $\widetilde{Q}$ is the graded quiver with vertices Q_0 and arrows

- The original arrows a in Q_1 (degree 0);

- Opposite arrows a^* for $a \in Q_1$ (degree -1);

- Loops c_i at vertices $i \in Q_0$ (degree -2). We put $c = \sum_i c_i$.

The differential is

$$da = 0 \qquad a \in Q_1$$
$$da^* = \frac{\partial w}{\partial a} \qquad a \in Q_1$$
$$dc = \sum_{a \in Q_1} [a^*, a]$$

Let $\widehat{\Gamma}(Q, w)$ be the graded completion of $\Gamma(Q, w)$ at path length. The co-homology in degree zero of $\widehat{\Gamma}(Q, w)$ is called the Jacobian algebra of (Q, w) and is denoted by $\mathcal{P}(Q, w)$. Concretely one has

$$\mathcal{P}(Q, w) = \widehat{kQ} \Big/ \Big\langle \left(\frac{\partial w}{\partial a} \right)_{a \in Q_1} \Big\rangle$$

Put

$$\mathcal{C}_{Q,w} = \mathrm{Perf}(\widehat{\Gamma}(Q, w)) / \langle (S_i)_{i \in Q_0} \rangle$$

where S_i is the simple $k\widetilde{Q}$ representation corresponding to the vertex i (this is perfect because of [11, §2.14]). In [1, Thm. 3.6] Claire Amiot shows that if $\mathcal{P}(Q, w)$ is finite dimensional then $\mathcal{C}_{Q,w}$ is a 2-CY category with cluster tilting object $\widehat{\Gamma}(Q, w)$. The endomorphism ring of this cluster tilting object is $\mathcal{P}(Q, w)$.

Assume now that R is a complete local 3-dimensional Gorenstein domain with residue field k with an isolated singularity. Assume that N is a Cohen-Macaulay R-module whose summands are not isomorphic to R and pairwise non-isomorphic such that in addition the following property holds:

- For $\Lambda = \mathrm{End}_R(R \oplus N)$ one has $\Lambda \cong H^*(\widehat{\Gamma}(Q, w))$ for some finite quiver and potential w.

This condition is equivalent to the 3-Calabi-Yau property for Λ (see [4, 10, 14]).

Let 0 be the vertex of Q corresponding to the R-summand of $R \oplus N$. Let (Q^0, w^0) be obtained from (Q, w) by deleting all edges in Q which are adjacent to 0 and all paths in w which pass through 0. By Theorem 5.1.1 one obtains

Proposition 5.2.1. *One has an exact equivalence of triangulated categories*

$$\underline{\mathrm{MCM}}(R) \cong \mathcal{C}_{Q^0, w^0}$$

Example 5.2.2. Assume that $R = k[[t, x, y, z]]/(xy - zt)$. Then R has a non-commutative crepant resolution $\mathrm{End}_R(R \oplus I)$ with $I = (x, z)$. The corresponding quiver is
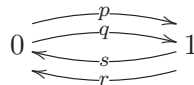
with super potential

$$w = psqr - prqs$$

We see that Q^0 consists of the single vertex 1 and $w^0 = 0$. Hence

$$\widehat{\Gamma}(Q^0, w^0) = k[c]$$

with $\deg c = -2$. Using the definitions in [9] it follows that $\underline{\mathrm{MCM}}(R)$ is the cluster category associated to the single vertex/no loops quiver; which is simply the category of $\mathbb{Z}/2\mathbb{Z}$ graded finite dimensional vector spaces.

6. Minimal models for Koszul algebras

The first few sections contain no new material. Their main purpose is to make explicit some formulas. Throughout we follow a version of the Sweedler convention. An element a of a tensor product $X \otimes Y$ is written as $a' \otimes a''$ (thus we suppress the summation sign) and a similar convention for longer tensor products.

6.1. The general case

Below $A = T_l^\circ V/(R)$ is a finitely generated quadratic algebra (thus V is a finitely generated l-bimodule and $R \subset V \otimes_l V$). We put

$$J_n = \bigcap_{i=0}^{n-1} V^{\otimes i-1} \otimes R \otimes V^{\otimes n-1-i}$$

and in particular $J_1 = V$, $J_2 = R$.

We note that A is naturally $\mathbb{N}$-graded by giving V degree 1 (as usual this is referred to as the Adams grading).

The canonical map

$$J_n \mapsto J_i \otimes_l J_{n-i}$$

is denoted by δ_i (or $\delta_{i,n-i}$ for clarity). It has degree zero for the Adams grading. Following our convention we write.

$$\delta_i(a) = \delta_i(a)' \otimes \delta_i(a)''$$

Put

$$\tilde{V} = \bigoplus_{n>0} J_n[n-1]$$

and

$$\tilde{A} = T_l \tilde{V}$$

For $a \in J_n$ considered as an element of degree $n-1$ in $\tilde{A}$ we put

$$da = (-1)^{i-1} \sum_i \delta_i(a)' \otimes \delta_i(a)''$$

It is easy to see that $d^2 = 0$.

Proposition 6.1.1. *Assume that A is Koszul. Then the map $q : \tilde{A} \to A$ induced from the projection map $\tilde{V} \to J_1 = V$ is a quasi-isomorphism.*

Proof. This follows in the standard way from the bar-cobar formalism (see Lemma 4.1). For the benefit of the reader we give a direct proof.

By definition the fact that A is Koszul means that the following complex of graded left A-modules is exact

$$(6.1) \qquad \cdots \to A \otimes_l J_2 \to A \otimes_l J_1 \to A \to l \to 0$$

where the differential is given by

$$d : A \otimes_l J_n \to A \otimes_l J_{n-1} : a \otimes b \mapsto a\delta_{1,n-1}(b)' \otimes \delta_{1,n-1}(b)''$$

Put

$$M = (T_l\tilde{V})_+ \overset{\text{def}}{=} \bigoplus_{n \geq 1} \tilde{V}^{\otimes_l n} \subset T_l\tilde{V}$$

We consider M as a left $\tilde{A}$ sub DG-module of $\tilde{A}$. As left graded $\tilde{A}$-module we have

$$M = \tilde{A} \otimes_l (J_1 \oplus J_2[1] \oplus J_3[2] \oplus \cdots)$$

Let $\tilde{C}$ be the cone of the inclusion map $i : M \to \tilde{A}$. As graded $\tilde{A}$-modules we have

$$\tilde{C} = \tilde{A} \otimes_l (l \oplus J_1[1] \oplus J_2[2] \oplus \cdots)$$

As $\operatorname{coker} i = l$ the obvious map $\tilde{C} \to l$ is a a quasi-isomorphism.

Put $C = A \otimes_{\tilde{A}} \tilde{C}$. Then one checks that C is precisely the complex (6.1) (without the right most l). Thus $C \to l$ is a quasi-isomorphism as well and hence so is the canonical map $\tilde{C} \to C$.

We now equip $\tilde{C}$ with an ascending filtration of sub-DG-$\tilde{A}$-modules as follows: $F_0\tilde{C} = \tilde{A}$, $F_1\tilde{C} = \tilde{A} \otimes (k \oplus J_1[1])$, ... We equip C with the similar filtration. The canonical map $\tilde{C} \to C$ is a map of filtered DG-$\tilde{A}$-modules.

Assume we have shown that $\tilde{A} \to A$ is a quasi-isomorphism in Adams degree $\leq n$. Then $(\tilde{C}/F_0\tilde{C})_{n+1} \to (C/F_0C)_{n+1}$ is a quasi-isomorphism. Given that $\tilde{C}_{n+1} \to C_{n+1}$ is a quasi-isomorphism we deduce that $F_0\tilde{C}_{n+1} = \tilde{A}_{n+1} \to A_{n+1} = (F_0C)_{n+1}$ is also a quasi-isomorphism. $\qquad \square$

Remark 6.1.2. It is easy to see that a suitable analogue of Proposition 6.1.1 holds for the completed rings $(T_l^\circ V/(R))\hat{\ }$. Indeed this simply means that the directs sums involved in the grading become direct products. Taking direct products of vector spaces is an extremely well behaved functor so it does not break anything. We will rely on this fact below without further elaboration.

6.2. Polynomial rings

We recall the familiar finite minimal model for polynomial rings. Assume $A = k[x_1, \ldots, x_n]$. If we put $V = \sum_i kx_i$ then $J_n = \wedge^n V \subset V^{\otimes n}$. Write $x_S = \wedge_{i \in S} x_i$ for $S \subset [n]$, $[n] = \{1, \ldots, n\}$, $S \neq \emptyset$. In the wedge product $\wedge_{i \in S} x_i$ we assume that the indices of the variables are in ascending order.

Specializing §6.1 to this situation we obtain that $\tilde{A}$ is equal to $k\langle (x_S)_{S \neq \emptyset} \rangle$ with differential

$$(6.2) \qquad dx_S = \sum_{S = A \coprod B, A \neq \emptyset, B \neq \emptyset} (-1)^{|A|-1} \epsilon_{A,B} x_A x_B$$

where $\epsilon_{A,B}$ is the sign defined by

$$\wedge_{i \in S} x_i = \epsilon_{A,B} \left(\wedge_{i \in A} x_i \right) \wedge \left(\wedge_{i \in B} x_i \right)$$

6.3. Crossed products

We now assume that k has characteristic zero. Assume that $\Lambda = A \# G$ is a crossed product with $A = k[x_1, \ldots, x_n]$ and G a finite group acting linearly on A. Since the construction of the finite minimal model of A is functorial (it is an application of the general construction in §6.1) we obtain a finite minimal model for Λ of the form

$$(6.3) \qquad \tilde{\Lambda} = \left(k\langle (x_S)_S \rangle \# G, d \right)$$

(with $l = kG$). The differential is kG-linear and on the variables $(x_S)_{S \neq \emptyset}$ it is given by the formula (6.2).

6.4. Crossed products for cyclic groups

In this section we specialize to the case where G is the cyclic group $\mathbb{Z}/m\mathbb{Z}$. We now assume that k is algebraically closed of characteristic zero so that we may assume that G acts diagonally on $V = kx_1 + \cdots + kx_n$. Fix a primitive m'th root of unity ξ. We write χ_i for the character of G

$$\chi_i(\bar{a}) = \xi^{ai}$$

and we let e_i be the corresponding primitive idempotent

$$e_i = \frac{1}{m} \sum_{a=0}^{m-1} \chi_i(\bar{a}) \bar{a} = \frac{1}{m} \sum_{a=0}^{m-1} \xi^{ai} \bar{a}$$

so that we have

$$kG = \sum_{i=0}^{m-1} ke_i$$

We assume that G acts on the variable x_i by the character χ_{a_i} for some $0 \leq a_i \leq m-1$. Thus in $\Lambda = A \# G$ with $A = k[x_1, \ldots, x_n]$ we have the relation

$$\bar{a} \cdot x_i = \xi^{a a_i} x_i \cdot \bar{a}$$

which implies

$$e_j \cdot x_i = x_i \cdot e_{j+a_i}$$

(from now we tacitly reduce indices modulo m). Specializing (6.3) we obtain a finite minimal model $\tilde{\Lambda}$ for Λ which is freely generated over l by the variables $(x_S)_S$ subject to the relations

$$e_j \cdot x_S = x_S \cdot e_{j+d(S)}$$

where

$$d(S) = \sum_{i \in S} a_i$$

and with differential as in (6.2).

6.5. McKay quiver description

We reformulate the results of the previous section in quiver language. Let Q be the quiver with vertices $\{0, \ldots, m-1\}$ (taken modulo m) and arrows $x_{j,i,j+a_i}$ for $i = 1, \ldots, n$, $j = 0, \ldots, m-1$ starting at j and ending at $j + a_i$. Then Λ is a quotient of the path algebra of Q where the arrow $x_{i,j,j+a_i}$ is sent to $e_j x_i = x_i e_{j+a_i} = e_j x_i e_{j+a_i} \in \Lambda$. The relation $x_k x_l = x_l x_k$ in Λ leads to the relation

(6.4) $$x_{j,k,j+a_k} \cdot x_{j+a_k,l,j+a_k+a_l} = x_{j,l,j+a_l} \cdot x_{j+a_l,k,j+a_k+a_l}$$

on Q. The path algebra of Q module all such relations is precisely Λ.

To describe the algebra $\tilde{\Lambda}$ we introduce the graded quiver $\tilde{Q}$ with the same vertices as Q and with arrows $x_{j,S,j+d(S)}$ of homological degree $-|S| + 1$

going from j to $j + d(S)$. The differential on $k\tilde{Q}$ is given by
(6.5)
$$dx_{j,S,j+d(S)} = \sum_{S=A\coprod B, A\neq\emptyset, B\neq\emptyset} (-1)^{|A|-1}\epsilon_{A,B}x_{j,A,j+d(A)} \cdot x_{j+d(A),B,j+d(S)}.$$

Let $\tilde{Q}^0$ be the quiver obtained from $\tilde{Q}$ by dropping all arrows adjacent to 0. Thus the arrows in $\tilde{Q}^0$ are of the form $x_{j,S,j+d(S)}$ with $j \neq 0$, $j + d(S) \neq 0$. Specializing Theorem 5.1.1 to the current situation we obtain the following result which we will formulate in the complete setting for variety. Note that by Remark 6.1.2 this makes little difference.

Proposition 6.5.1. *Let $A = k[x_1, \ldots, x_n]$ and assume that the cyclic group $G = \mathbb{Z}/m\mathbb{Z}$ acts linearly on A with weights $(\xi^{a_1}, \ldots, \xi^{a_n})$ satisfying the additional properties $\sum_i a_i \cong 0 \mod 0$ and $\gcd(a_i, m) = 1$. Then*

$$\underline{\mathrm{MCM}}(\hat{A}^G) \cong \mathrm{Perf}(k\tilde{Q}^0, d)/\langle (S_i)_{i=1,\ldots,m-1}\rangle$$

where the differential is given by (6.5) *(taking into account that arrows adjacent to the vertex 0 should be suppressed on the righthand side). The DG-algebra $(k\tilde{Q}^0, d)$ has finite dimensional cohomology and*

$$H^0(k\tilde{Q}^0, d) \cong \hat{\Lambda}/\hat{\Lambda}e_0\hat{\Lambda}.$$

Proof. This is a straight translation of Theorem 5.1.1. The conditions on the numbers $(a_i)_i$ are to insure that $\hat{A}^G$ is Gorenstein and has an isolated singularity. $\square$

Example 6.5.2. We discuss an example that occurred in [8, 12]. Assume that $G = \mathbb{Z}/2\mathbb{Z}$ acts with weights $(-1, -1, -1, -1)$ on $k[x_1, x_2, x_3, x_4]$. The resulting McKay quiver looks like
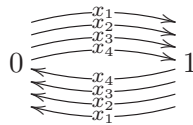
with relations

$$x_i x_j = x_j x_i$$

The quiver $\tilde{Q}^0$ contains a single vertex 1 and loops x_{ij}, $i < j$ and x_{1234} respectively of degree -1 and -3. The differential is given by

$$dx_{1234} = -[x_{12}, x_{34}] - [x_{23}, x_{14}] + [x_{24}, x_{13}]$$

where as usual $[-, -]$ denotes the graded commutator.

6.6. The minimal case

In this section we impose the additional condition.

$$(6.6) \qquad \sum_i a_i = m$$

It was proved in [2] that if condition (6.6) holds then $\underline{\mathrm{MCM}}(\hat{A}^G)$ has a description as a higher cluster category. In this section we indicate how to prove this starting from Proposition 6.5.1.

We order the vertices of $\tilde{Q}^0$ by their label. We we say that an arrow $x_{j,S,j+d(S)}$ is ascending if $j + d(S) > j$ (recall that the sum $j + d(S)$ is reduced mod m). Otherwise we say that the arrow is descending.

Let P be the quiver with the same vertices as $\tilde{Q}^0$ but only with ascending arrows of the form $x_{j,i,j+a_i}$ subject to relations

$$x_{j,k,j+a_k} \cdot x_{j+a_k,l,j+a_k+a_l} = x_{j,l,j+a_l} \cdot x_{j+a_l,k,j+a_k+a_l}$$

Let C be the resulting path algebra. As in [10, 15] we define the inverse dualizing complex of C as

$$\Phi_C = \mathrm{RHom}_{C^e}(C, C \otimes C)$$

Proposition 6.6.1. *There is a quasi-isomorphism of DG-algebras*

$$(6.7) \qquad (k\tilde{Q}^0, d) \cong T_C(\Theta_C[n-1])$$

In particular $\underline{\mathrm{MCM}}(\hat{A}^G)$ is a generalized $n-1$-cluster category.

The fact that $\underline{\mathrm{MCM}}(\hat{A}^G)$ has an $n-1$-cluster tilting object has already been established by Iyama [6, Thm. 5.2.1]. So we only need to be concerned with proving (6.7). The rest of this section will be devoted to this.

Let $\tilde{P}$ be the quiver with the same vertices as $\tilde{Q}^0$ but only with ascending arrows. Let D be the $k\tilde{P}$ bimodule generated by the descending arrows. Then we have

$$k\tilde{Q}^0 = T_{k\tilde{P}}D$$

Furthermore if we look at the formula (6.5) we see that both $k\tilde{P}$ and D are closed under the differential d (it is here that (6.6) is used).

Hence we need to prove two things

1) The natural map $k\tilde{P} \to C$ is a quasi-isomorphism.

2) $D \cong \Phi_{k\tilde{P}}[n-1]$

The first statement follows from the following lemma

Lemma 6.6.2. *The algebra C is Koszul over $l^0 \stackrel{\text{def}}{=} l/le_0$. Furthermore $k\tilde{P}$ is the minimal model for C constructed in §6.1.*

Proof. We give A an additional grading by putting $\deg x_i = a_i$. It is easy to see that the minimal resolution of S_i for C is obtained by resolving k as a graded A-module for the Adams grading and then truncating in degrees $\leq m - i$ for the additional grading. It is clear that this resolution is linear for the Adams grading.

Given the explicit form of the relations of C one verifies immediately that $k\tilde{P}$ is indeed the finite minimal form constructed in §6.1.   □

Now we prove the second statement. We first discuss some generalities on bimodules and differentials.

Let E be a DG-algebra and assume that P,Q are graded DG-E-bimodules. We write $P \otimes_{E^w} Q$ for the quotient of $P \otimes_E Q/[E, P \otimes_E Q]$. The dual of an E-bimodule Q is by definition $Q^* = \operatorname{Hom}_{E^e}(Q, {}_E E \otimes E_E)$. This is again a DG-$E$-bimodule through the surviving inner bimodule structure on $E \otimes E$.

If $\omega \in (P \otimes_{E^e} Q)_n$ then we obtain an induced map of degree n

$$\omega^\circ : P^* \to Q : \phi \mapsto (-1)^{|\phi(\omega')'|(|\omega''|+|\phi(\omega')''|)}\phi(\omega')''\omega''\phi(\omega')'$$

We say that ω is non-degenerate if P, Q are finitely generated projective E-bimodules and ω° induces an isomorphism $P^* \cong Q[n]$. If $d\omega = 0$ then ω° is a morphism of DG-bimodules.

As E-bimodule E is quasi-isomorphic to the cone of

$$0 \to \Omega_{E/l} \xrightarrow{\sigma} E \otimes_l E \to 0$$

where $\Omega_{E/l}$ is the bimodule of non-commutative differentials. As a graded E-bimodule $\Omega_{E/l}$ is generated by elements Db subject to the standard relations.[1] One has $\sigma(Db) = b \otimes 1 - 1 \otimes b$. The operator D has degree zero, whence $d(Db) = D(db)$.

---

[1]Using db would lead to confusion with the differential on E

We denote the cone of σ by $\tilde{\Omega}_{E/l}$. If $b \in E$ then Db considered as an element of degree $|b| - 1$ of $\tilde{\Omega}_{E/l}$ is written as $\tilde{D}b$.

Hence $\tilde{\Omega}_{E/l}$ is generated by $\tilde{D}b$ and an l-central element g of degree zero (this represents the generator $1 \otimes 1$ of $E \otimes_l E$). The differential on $\tilde{\Omega}_{E/l}$ is given by

$$
\begin{aligned}
dg &= 0 \\
d(\tilde{D}b) &= -\tilde{D}(db) + [b, g]
\end{aligned}
$$
(6.8)

We compute $\tilde{\Omega}_{k\tilde{P}/l^0}$. For $j < j + d(S)$ write $\tilde{x}_{j,d(S),j+d(S)} = \tilde{D}x_{j,S,j+d(S)}$. Thus $|\tilde{x}_{j,d(S),j+d(S)}| = -|S|$. Furthermore we introduce loops $\tilde{x}_{j,\emptyset,j}$ at j of degree zero which correspond to $-ge_j$. Then $\tilde{\Omega}_{k\tilde{P}/l^0}$ is the free bimodule with generators $\tilde{x}_{j,d(S),j+d(S)}$ for $j \leq j + d(S)$, $S \subsetneq [n]$ with $[n] = \{1, \ldots, n\}$. A simple computation using the formulas (6.8) yields

$$
(6.9) \quad d(\tilde{x}_{j,S,j+d(S)}) = \sum_{S = A \coprod B, B \neq \emptyset} (-1)^{|A|} \epsilon_{A,B} \tilde{x}_{j,A,j+d(A)} \cdot x_{j+d(A),B,j+d(S)}
$$
$$
- \sum_{S = A \coprod B, A \neq \emptyset} \epsilon_{A,B} x_{j,A,j+d(A)} \cdot \tilde{x}_{j+d(A),B,j+d(S)}
$$

We define an element $\omega \in \tilde{\Omega}_{k\tilde{P}/l^0} \otimes_{k\tilde{P}^e} D$ by the following formula

$$
\omega = \sum_{S \subsetneq [n]), j \leq j+d(S)} (-1)^{|S|-1} \epsilon_{S,S^c} \tilde{x}_{j,S,j+d(S)} \otimes x_{j+d(S),S^c,j}
$$

One verifies

1) $|\omega| = -n + 1$.

2) $d\omega = 0$.

3) ω is non-degenerate.

It follows that ω defines an isomorphism $\omega^\circ : \Theta_{k\tilde{P}} = \tilde{\Omega}^*_{k\tilde{P}/l^0} \to D[-n+1]$. Hence the proof is complete.

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