A pathology of asymptotic multiplicity in the relative setting

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We point out an example of a projective family $\pi : X \to S$, a sub-variety $V \subset X$, and a π-pseudoeffective divisor D on X for which the asymptotic multiplicity $\sigma_V(D; X/S)$ is infinite. This shows that the divisorial Zariski decomposition is not always defined for pseudoeffective divisors in the relative setting.

1. Introduction

We work throughout over $\mathbb{C}$. Suppose that X is a smooth projective variety and D is a pseudoeffective $\mathbb{R}$-divisor on X. The asymptotic multiplicity of D along a subvariety $V \subset X$, studied by Nakayama [11] and Ein-Lazarsfeld-Mustaţă-Nakamaye-Popa [5], has proved to be a fundamental tool in understanding the properties of the divisor D. For big divisors D, the definition of the asymptotic multiplicity is straightforward: roughly, one considers the linear series $|mD|$ for larger and larger values of m, and takes $\sigma_V(D) = \lim_{m \to \infty} \frac{1}{m} \operatorname{mult}_V |mD|$, where the multiplicity of a linear series along a subvariety is defined to be the multiplicity of a general member.

Complications arise, however, in carrying out this construction for divisors D which are pseudoeffective but not big, i.e. for divisors on the boundary of the pseudoeffective cone $\overline{\operatorname{Eff}}(X) \subset N^1(X)$. Nakayama realized that $\sigma_V(D)$ can be extended to a lower semicontinuous function on $\overline{\operatorname{Eff}}(X)$ by setting

$$\sigma_V(D) = \lim_{\epsilon \to 0} \sigma_V(D + \epsilon A),$$

where A is a fixed ample divisor. In some applications (e.g. in the construction of Zariski decompositions), it is important to know that the limit in question takes a finite value. While it is clear that the quantity on the right is nondecreasing as ϵ is made smaller, it might *a priori* be unbounded in the limit. That this does not happen in the non-relative setting was observed by Nakayama.

Our aim in this note is to demonstrate by example that when asymptotic multiplicity invariants are considered in the greater generality of divisors on a projective family $\pi : X \to S$, this finiteness need not hold: for a π-pseudoeffective divisor, the limit defining $\sigma_V(D; X/S)$ can indeed be infinite. This answers a question of Nakayama [11, pg. 33]. The example itself is familiar, a divisor on the versal deformation space of a fiber of Kodaira type I_2, which has been considered in related contexts by Reid [12, 6.8] and Kawamata [8, Example 3.8(2)],[9, Example 9].

Theorem 1. *There exists a projective family $\pi : X \to S$, a π-pseudoeffective divisor D, and a subvariety $V \subset X$ for which $\sigma_V(D; X/S)$ is infinite.*

An important use of asymptotic multiplicity invariants is in the construction of the divisorial Zariski decomposition, a higher-dimensional analog of the usual Zariski decomposition on surfaces. The example here shows that trouble arises if one generalizes this construction to pseudoeffective classes in the relative setting: after passing to a blow-up on which the valuation corresponding to V is divisorial, we obtain an example in which the decomposition is not defined.

Corollary 2. *Let $\pi : X \to S$ be as in Theorem 1. If $f : W \to X$ is the blow-up along V with exceptional divisor E, then $\tilde{D} = f^*D$ has $\sigma_E(\tilde{D}; W/S) = \infty$ and $N_\sigma(\tilde{D}; W/S)$ is not defined.*

Moreover, the divisor $\tilde{D}$ does not admit any Zariski decomposition in a very strong sense:

Corollary 3. *There does not exist a birational model $g : Z \to W$ for which $g^*\tilde{D}$ admits a decomposition $g^*\tilde{D} = P + N$ with P a $g \circ (f \circ \pi)$-movable divisor and N effective.*

In the next section, we recall the basic definitions and properties of the invariants $\sigma_V(D; X/S)$ and $N_\sigma(D; X/S)$ appearing in Theorem 1 and Corollary 2, before establishing the main claims in Section 3. In Section 4, we describe a more general setting for making computations in a similar spirit.

2. Preliminaries

Suppose that $\pi : X \to S$ is a projective, surjective morphism with connected fibers, with X and S smooth (hereafter, a *nice family*). We will find it convenient to allow the base S to be an analytic surface germ, following [7]. The proofs of the results in this section hold either when S is a quasiprojective variety or an analytic germ. In Section 3, it will be convenient to make computations with the base a germ. However, the germ we consider is algebraizable, and it follows that the same pathology occurs when the base is extended to be an affine scheme; this is discussed in Remark 1.

Two divisors D and D' on X are said to be numerically equivalent over S, or π-numerically equivalent, if $D \cdot C = D' \cdot C$ for any curve C that is contracted by π; write $D \equiv_\pi D'$ for the relation of numerical equivalence over S, and $N^1(X/S)$ for the vector space of $\mathbb{R}$-divisors on X, modulo this equivalence.

The familiar cones of positive divisors on a projective variety all have analogs in the relative setting: a divisor D on X is said to be

1) π-ample if D_s is ample on every fiber $X_s = \pi^{-1}(s)$;

2) π-strongly movable if the support of the cokernel of $f^* f_* \mathcal{O}_X(D) \to \mathcal{O}_X(D)$ has codimension at least 2;

3) π-big if the restriction of D to the generic fiber is big.

Corresponding to these classes of divisors are cones inside $N^1(X/S)$:

$$\mathrm{Amp}(X/S) \subseteq \mathrm{Mov}(X/S) \subseteq \mathrm{Big}(X/S).$$

The cones $\mathrm{Amp}(X/S)$ and $\mathrm{Big}(X/S)$ are both open. We will also consider the closures of these three cones inside $N^1(X/S)$, which are, respectively:

- $\mathrm{Nef}(X/S)$. A divisor is π-nef if D_s is nef on every fiber X_s (i.e. if $D \cdot C \geq 0$ for every curve C contracted by π);

- $\overline{\mathrm{Mov}}(X/S)$, the cone of π-movable divisors;

- $\overline{\mathrm{Eff}}(X/S)$, the cone of π-pseudoeffective divisors. A divisor D is called π-pseudoeffective if the restriction of D to the generic fiber is pseudo-effective.

The meaning of "movable" is unfortunately not entirely uniform in the literature, and we stress that here an $\mathbb{R}$-divisor is called π-movable if it lies in the closed cone $\overline{\mathrm{Mov}}(X/S)$; this is sometimes called π-nef in codimension

1. We note too that the cone $\overline{\mathrm{Eff}}(X/S)$ is not necessarily a strictly convex cone, in that it might contain an entire line through the origin; this contrasts with the familiar case when S is a point. For example, if D restricts to 0 on a general fiber of π, then D and $-D$ are both π-pseudoeffective.

For simplicity, we will assume that the base space S is either affine or an analytic germ. This is not really necessary, but the invariants under consideration can be computed when the base is projective simply by restricting to the preimage of a suitable affine open set; we refer to [11, §3.2] for details. The existence of a π-ample divisor A on X is automatic in this setting. If D is a π-big divisor, then $f_*\mathcal{O}_X(mD) \neq 0$ for sufficiently large and divisible m, and so $H^0(X, \mathcal{O}_X(mD)) = f_*(\mathcal{O}_X(mD))$ is nonzero as well. Hence in these settings, any π-big class has an effective representative.

Definition 1. Suppose that X is smooth. Given an irreducible subvariety $V \subset X$ and a π-big $\mathbb{R}$-divisor D, set

$$\sigma_V(D; X/S) = \inf_{\substack{D' \equiv_\pi D \\ D' \geq 0}} \mathrm{mult}_V(D').$$

Since D is π-big, there exists an effective $\mathbb{R}$-divisor D' that is π-numerically equivalent to D, and this infimum is taken over a nonempty set.

When D is a big integral divisor, a sequence D'_m of effective $\mathbb{R}$-divisors with multiplicities converging to the infimum can be found by taking $D'_m \in \frac{1}{m}|mD|$, where we choose a general element of the linear system $|mD|$.

We next extend the definition of the asymptotic multiplicity from π-big divisors to π-pseudoeffective divisors.

Definition 2. Given a π-pseudoeffective $\mathbb{R}$-divisor D, set

$$\sigma_V(D; X/S) = \lim_{\epsilon \to 0} \sigma_V(D + \epsilon A; X/S).$$

This is evidently a nondecreasing function as ϵ approaches 0, but it might have infinite limit. To show that it has a finite limit, it suffices to bound $\sigma_V(D + \epsilon A; X/S)$ above, independent of ϵ. Nakayama gives several conditions under which this can be achieved.

Theorem 4 ([11], Lemmas 2.1.2, 3.2.6). *If any of the following holds, then the asymptotic multiplicity $\sigma_V(D; X/S)$ is finite.*

 1) $S = \mathrm{Spec}\,\mathbb{C}$ *is a point;*

2) D is numerically equivalent over S to an effective $\mathbb{R}$-divisor Δ;

3) $\operatorname{codim} \pi(V) < 2$.

We recall the proof in case (1), perhaps the most important in practice. Case (2) is immediate from the definition, and we refer to [11] for (3). Assume for a moment that $V \subset X$ is an irreducible divisor; that this implies the general statement will follow from Theorem 5(2) below.

Proof of (1). For any ϵ, $(D + \epsilon A) - \sigma_V(D + \epsilon A)V$ is pseudoeffective, and so

$$((D + \epsilon A) - \sigma_V(D + \epsilon A)V) \cdot A^{n-1} \geq 0.$$

As long as $\epsilon < 1$ it follows that

$$\sigma_V(D + \epsilon A) \leq \frac{(D + \epsilon A) \cdot A^{n-1}}{V \cdot A^{n-1}} \leq \frac{(D + A) \cdot A^{n-1}}{V \cdot A^{n-1}}$$

is bounded above as ϵ decreases to 0. $\qquad\square$

This argument relies in a crucial way on the properness of X to carry out intersection theory, and is not applicable in the relative setting in general.

Proposition 5 ([11], Lemmas 2.1.4, 2.2.2, 2.1.7). *Suppose that $\pi : X \to S$ is a nice family and $V \subset X$ is an irreducible subvariety.*

1) *If F is any π-pseudoeffective divisor on X, then*

$$\lim_{\epsilon \to 0} \sigma_V(D + \epsilon F; X/S) = \sigma_V(D; X/S).$$

2) *Let $f : W \to X$ be the normalized blow-up of X along V, and let E be the irreducible exceptional divisor dominating V. Then $\sigma_E(f^*D; W/S) = \sigma_V(D; X/S)$.*

3) *The number of prime divisors Γ for which $\sigma_\Gamma(D; X/S) > 0$ is finite.*

The first of these shows that Definition 1 is independent of the choice of π-ample divisor A, while the second completes the proof of Theorem 4(1) in the case that V has codimension greater than 1.

Definition 3 ([11], [4]). Suppose that $\pi : X \to S$ is a nice family and that D is a π-pseudoeffective divisor such that $\sigma_\Gamma(D; X/S)$ is finite for every

prime divisor Γ. Then set

$$N_\sigma(D; X/S) = \sum_\Gamma \sigma_\Gamma(D; X/S)\, \Gamma,$$
$$P_\sigma(D; X/S) = D - N_\sigma(D; X/S).$$

It follows from Proposition 5(3) that there are only finitely many nonzero terms in the sum defining $N_\sigma(D; X/S)$.

We refer to $N_\sigma(D; X/S)$ as the negative part of the divisorial Zariski decomposition, and $P_\sigma(D; X/S)$ as the positive part. The negative part is a rigid, effective divisor. The positive part might not be nef, but it lies in the cone $\overline{\mathrm{Mov}}(X/S)$ of π-movable divisors. Corollary 2 shows that without the finiteness hypothesis on $\sigma_\Gamma(D; X/S)$, the definition is not always applicable in the relative setting.

In the non-relative setting, the divisorial Zariski decomposition is defined for any pseudoeffective class D, but it lacks certain useful properties of the classical Zariski decomposition in dimension 2: most importantly, the positive part $P_\sigma(D)$ is not nef in general. In many cases one may construct a birational model $f: W \to X$ on which $P_\sigma(f^*D)$ is actually nef, even if $P_\sigma(D)$ is not. However, a basic example of Nakayama shows that even this is not always possible [11, Theorem 5.2.6].

Another variant on Zariski decomposition in higher dimensions, the weak Zariski decomposition of Birkar, imposes fewer conditions and so exists for a larger class of divisors, including that of Nakayama's example. In this decomposition, the positive part P is allowed to be any relatively nef divisor, not necessarily the positive part $P_\sigma(f^*D)$ of the divisorial Zariski decomposition.

Definition 4 ([1]). Suppose that $\pi: X \to S$ is a nice family and that D is a pseudoeffective divisor on X. We say that D admits a *weak Zariski decomposition over S* if there exists a birational map $f: Y \to X$ and a decomposition $f^*D = P + N$, where P is $(f \circ \pi)$-nef and N is effective.

This condition is fairly unrestrictive, but there nevertheless exist pseudoeffective $\mathbb{R}$-divisors on smooth threefolds which do not admit a weak Zariski decomposition [10]. Corollary 3 asserts that the divisor $\tilde{D}$ provides another such example. Indeed, $\tilde{D}$ admits no Zariski decomposition in a still stronger sense: even after pulling back to a higher model, it cannot be decomposed as the sum of an effective divisor and a relatively movable divisor. The example

is qualitatively rather different from that of [10]: there, a certain pseudoeffective divisor D_λ has negative intersection with infinitely many curves; here, there is a single curve on which D is negative, but the multiplicity of D along this curve is infinite.

3. Main example

The claimed pathology follows from a few calculations on an example that has been studied by Reid [12, 6.8] and Kawamata [8, Example 3.8(2)]. Let $\pi : X \to S$ be the versal deformation space of a fiber of Kodaira type I_2. The base S is smooth, 2-dimensional germ. The fiber over the central point $0 \in S$ consists of two smooth rational curves C_1 and C_2, meeting transversely at two points p_1 and p_2. Let $C = \pi^{-1}(0)$ be the union of these two curves.

There are two divisors $\Gamma_1, \Gamma_2 \subset S$ corresponding to the smoothings of the two nodes of C. The fiber of π over a general point of Γ_i is a nodal rational curve, while the fiber over a general point of S is a smooth curve of genus 1.
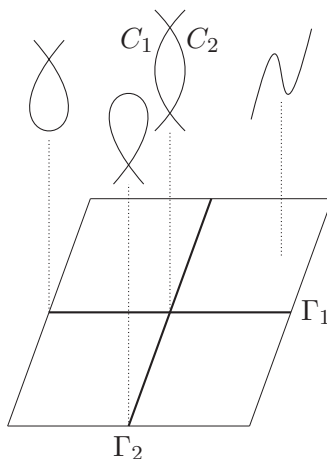

Figure 1: The family $\pi : X \to S$.

Remark 1. The computations that follow will give an example in which some $\sigma_V(D; X/S)$ is infinite, in the case where S is an analytic germ. The calculations rely on the fact that $\pi : X \to S$ is a versal deformation space.

However, the local analytic results imply that the same pathological behavior occurs even when the base S is an affine surface. Indeed, we will see in Lemma 7 below that there is a projective family $\bar{\pi} : \bar{X} \to \bar{S}$ where $\bar{S}$ is an affine surface, such that the restriction of $\bar{\pi}$ to the germ at a point $0 \in \bar{S}$ coincides with the map $\pi : X \to S$.

If $\bar{G}$ is a $\bar{\pi}$-big divisor, with restriction G to the germ, then $\sigma_{C_1}(\bar{G}; \bar{X}/\bar{S}) \geq \sigma_{C_1}(G; X/S)$: indeed, if $\bar{G}'$ is an effective divisor on $\bar{X}$ which is $\bar{\pi}$-numerically equivalent to $\bar{G}$, its restriction to the central germ is an effective divisor on X which is π-numerically equivalent to G. Thus the infimum defining $\sigma_{C_1}(\bar{G}; \bar{X}/\bar{S})$ in is taken over a subset of the infimum defining $\sigma_{C_1}(G; X/S)$ in Definition 1, giving the claimed inequality. It follows that in the limit at the pseudoeffective boundary, $\sigma_{\bar{C}_1}(\bar{D}; \bar{X}/\bar{S}) \geq \sigma_{C_1}(D; X/S)$, and it must be that $\sigma_{\bar{C}_1}(\bar{D}; \bar{X}/\bar{S})$ is infinite as well. The claims about Zariski decomposition follow in the same way.

Lemma 6. *The normal bundle $N_{C_i/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. Suppose for instance that $i = 1$. There is an exact sequence

$$0 \longrightarrow N_{C_1/X} \longrightarrow (N_{C/X})|_{C_1} \longrightarrow T_{C_2, p_1} \oplus T_{C_2, p_2} \longrightarrow 0$$

with the property that a first-order deformation of C, determined by a section $s \in H^0(C, N_{C/X})$, smooths the node at p_i if and only if s has nonzero image in T_{C_2, p_i} [6, Lemma 2.6]. The sheaf in the middle is the trivial $\mathcal{O}_C \oplus \mathcal{O}_C$. In one direction p_1 is smoothed, and in another p_2 is, so the map sends $(1, 0)$ to $(1, 0)$ and $(0, 1)$ to $(0, 1)$ with respect to the direct sum decompositions. It follows that the kernel is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. $\square$

Lemma 6 shows in particular that K_X has intersection 0 with both C_1 and C_2 and so is relatively numerically trivial.

Lemma 7. *There exists a flop $\tau : X \dashrightarrow X^+/S$ with flopping curve C_1. Let $C_1^+ \subset X^+$ be the flopped curve, and $C_2' \subset X^+$ be the strict transform of C_2. There exists an isomorphism $\sigma : X^+ \to X/S$ which sends C_1^+ to C_1 and C_2' to C_2. Furthermore, there exists an involutive automorphism $\imath : X \to X/S$ which exchanges the two curves C_1 and C_2.*

Proof. The arguments here are due to Kawamata [8, Example 3.8(2)]. We make some aspects of the proof explicit by working with local defining equations given by Reid [12]. In what follows, we use the notation $\bar{\cdot}$ to denote objects on a family $\bar{\pi} : \bar{X} \to \bar{S}$ over an affine base, while objects with no bar will be the restrictions to a certain germ.

Let $\bar{S} = \mathbb{A}^2$, with coordinates t_1 and t_2. Fix two distinct complex numbers a_1 and a_2 and define $\bar{X}_0 \subset (\mathbb{A}^1 \times \mathbb{A}^1) \times \bar{S}$ by the equation

$$x_1^2 = ((x_2 - a_1)^2 - t_1)((x_2 - a_2)^2 - t_2).$$

The closure $\bar{X} \subset (\mathbb{P}^1 \times \mathbb{P}^1) \times \bar{S}$ is smooth, and the second projection $\bar{\pi} : \bar{X} \to \bar{S}$ is proper. The fiber of $\bar{\pi}$ over a general point (t_1, t_2) is a smooth curve of genus 1. If exactly one of t_1 and t_2 is zero, the fiber is nodal, while if $t_1 = t_2 = 0$, the fiber is given by $x_1^2 = (x_2 - a_1)^2(x_2 - a_2)^2$. This central fiber has two components, the rational curves C_1 defined by $x_1 = -(x_2 - a_1)(x_2 - a_2)$ and C_2 defined by $x_1 = (x_2 - a_1)(x_2 - a_2)$. The restriction of $\bar{\pi} : \bar{X} \to \bar{S}$ to the germ at $(0, 0) \in \bar{S}$ is the versal deformation space $\pi : X \to S$ considered above. The involution $\imath : \bar{X} \to \bar{X}/\bar{S}$ defined by $\imath(x_1, x_2) = (-x_1, x_2)$ exchanges the two components of the central fiber.

There is a section $\bar{\sigma} : \bar{S} \to \bar{X}$ given by

$$x_2(t_1, t_2) = \frac{a_1 + a_2}{2} - \frac{t_1 - t_2}{2(a_1 - a_2)},$$
$$x_1(t_1, t_2) = (x_2(t_1, t_2) - a_1)^2 - t_1.$$

This has $\bar{\sigma}(0, 0) = \left(\frac{(a_2 - a_1)^2}{4}, \frac{1}{2}(a_1 + a_2) \right)$, which lies on C_1 and is disjoint from C_2.

Let $\bar{\Sigma}_1$ be the divisor $\sigma(\bar{S})$. Since $\bar{\Sigma}_1 \cdot C_1 = 1$ and $\bar{\Sigma}_1 \cdot C_2 = 0$, the curves C_1 and C_2 have distinct classes in $N_1(\bar{X}/\bar{S})$. Since all other fibers of $\bar{\pi}$ are irreducible, it must be that $N^1(\bar{X}/\bar{S})$ is has dimension 2. The divisor $2\imath_*(\bar{\Sigma}_1) - \bar{\Sigma}_1$ has positive degree on general fibers, and so is $\bar{\pi}$-big. Since $\bar{S}$ is affine, there is an effective divisor $\bar{\Delta}$ representing this class. For sufficiently small ϵ, the pair $(\bar{X}, \epsilon\bar{\Delta})$ is klt. Since $\bar{\Delta} \cdot C_1 < 0$, there exists a $(K_{\bar{X}/\bar{S}} + \epsilon\bar{\Delta})$-flip $\tau : \bar{X} \dashrightarrow \bar{X}^+$, which is a $K_{\bar{X}/\bar{S}}$-flop. The map $\bar{\pi}^+ : \bar{X}^+ \to \bar{S}$ is a minimal model of $\bar{X}^+$.

By Lemma 6, the map τ is the flop of a rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. It follows by looking at a resolution of τ (discussed in the proof of Lemma 8 below) that the map $\tau|_{\bar{S}}$ simply blows up the point $\bar{\Sigma}_1 \cap C_1$ and that the strict transform of $\bar{\Sigma}_1$ on $\bar{X}^+$ is smooth. This strict transform contains the curve C_1^+, and satisfies $\tau_*\bar{\Sigma}_1 \cdot C_2' = 2$.

Now, the varieties $\bar{X}^+$ and $\bar{S}$ are smooth and $\bar{\pi}^+ : \bar{X}^+ \to \bar{S}$ has all fibers 1-dimensional, so $\bar{\pi}^+$ is flat. Since $\pi : X \to S$ is a versal deformation space, there exists an isomorphism $\beta : X^+ \to X$ over some automorphism of S. However, β might not be defined over the identity map on S. The divisor $\Sigma_2 = \beta_*(\tau_*(\Sigma_1))$ is a smooth divisor on X, containing C_1, and meeting C_2

at two points. There is a translation on the smooth fibers of π sending Σ_1 to Σ_2, which defines a birational self-map $\gamma : X \dashrightarrow X$ over the identity on S. The map $\pi \circ \gamma : X \to S$ must be isomorphic to some minimal model of X over S, and indeed must be to isomorphic to $\pi^+ : X^+ \to S$ since the strict transforms of Σ_1 under γ and τ have the same numerical classes. It follows that there exists an isomorphism $\sigma : X^+ \to X$ over the identity of S. Replacing σ with $\sigma \circ \imath$ if necessary, we may assume that $\sigma(C_1^+) = C_1$ and $\sigma(C_2') = C_2$, as required. $\qquad\square$

Each of the maps $\sigma \circ \tau$ and $\imath$ is a birational involution of X over S, but we will soon see that the composition $\phi = (\sigma \circ \tau) \circ \imath$ is of infinite order. Since $\imath(C_2) = C_1$, the effect of repeatedly applying ϕ is to flop C_1, then C_2, then C_1 again, and so on. We will denote by $\phi_* D$ the strict transform of a divisor D under a birational map ϕ, and use the same notation for the induced map on numerical groups when confusion seems unlikely.
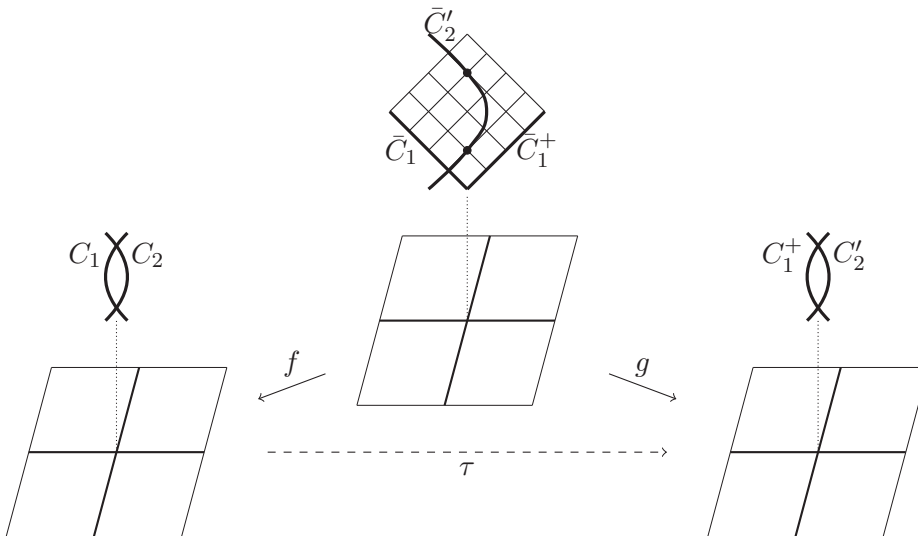

Figure 2: Resolution of the flop τ.

To an effective divisor D on X, associate the 4-tuples

$$v_D = (D \cdot C_1, D \cdot C_2, \mathrm{mult}_{C_1}(D), \mathrm{mult}_{C_2}(D)),$$
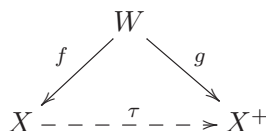$$\sigma_D = (D \cdot C_1, D \cdot C_2, \sigma_{C_1}(D; X/S), \sigma_{C_2}(D; X/S)).$$

Lemma 8. *Suppose that D is a divisor on X, and let $\tilde{D}$ denote the strict transform of D under the flop $\tau : X \dashrightarrow X^+$. Then*

1) $\tilde{D} \cdot C_1^+ = -D \cdot C_1,$

2) $\tilde{D} \cdot C_2' = D \cdot C_2 + 2(D \cdot C_1),$

3) $\mathrm{mult}_{C_1^+}(\tilde{D}) = \mathrm{mult}_{C_1}(D) + D \cdot C_1,$

4) $\mathrm{mult}_{C_2'}(\tilde{D}) = \mathrm{mult}_{C_2}(D).$

In matrix form, we have $v_{\phi_* D} = M v_D$ and $\sigma_{\phi_* D} = M \sigma_D$ where

$$
M = \begin{pmatrix}
2 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
$$

Proof. Let W be the graph of the flop τ:

Since τ is the flop of a rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by Lemma 6, the map f is simply the blow-up of X along C_1, while g is the blow-up of X^+ along C_1^+. There is a single f-exceptional divisor E on W, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and has normal bundle of bidegree $(-1, -1)$. Let $\bar{C}_1$ be a ruling of E contracted by g, so that f sends $\bar{C}_1$ isomorphically to C_1. Similarly, let $\bar{C}_1^+$ be a a ruling of E contracted by f, so that g maps $\bar{C}_1^+$ isomorphically onto C_1^+. Lastly, let $\bar{C}_2'$ be the strict transform of C_2 on W, a curve which meets E transversely at 2 points. This resolution is illustrated in Figure 2.

Then write

$$
f^* D + aE = g^* \tilde{D},
$$

for some constant a. Taking the intersection of both sides with $\bar{C}_1$ yields $D \cdot C_1 + a(E \cdot \bar{C}_1) = 0$. Since $E \cdot \bar{C}_1 = -1$, we obtain $a = D \cdot C_1$. Intersecting with $\bar{C}_1^+$, we have $-a = \tilde{D} \cdot C_1^+$. Similarly, intersecting with $\bar{C}_2'$, we have $D \cdot C_2 + a(E \cdot \bar{C}_2') = \tilde{D} \cdot C_2'$, and since $E \cdot \bar{C}_2' = 2$, we have (2). It is clear that $\mathrm{mult}_{C_2'}(\tilde{D}) = \mathrm{mult}_{C_2}(D)$, since τ is an isomorphism at the generic point of C_2. Finally,

$$
\mathrm{mult}_{C_1^+}(\tilde{D}) = \mathrm{mult}_E(g^* \tilde{D}) = \mathrm{mult}_E(f^* D) + a = \mathrm{mult}_{C_1}(D) + a.
$$

These calculations immediately yield $v_{\phi_* D} = M v_D$, since the second map $\imath$ exchanges the two curves C_1 and C_2. Write D_m for a general divisor linearly equivalent to mD, and then

$$\sigma_{C_1}(\phi_* D) = \lim_{m\to\infty} \frac{1}{m} \mathrm{mult}_{C_1}(\phi_* D_m) = \lim_{m\to\infty} \frac{1}{m} (\mathrm{mult}_{C_1} D_m + D_m \cdot C_1)$$
$$= \left(\lim_{m\to\infty} \frac{1}{m} \mathrm{mult}_{C_1} D_m\right) + D \cdot C_1 = \sigma_{C_1}(D) + a.$$

$\square$

We are now in position to make the main computation.

Theorem 9. *Let $\pi : X \to S$ be the versal deformation space of a singular fiber of Kodaira type I_2, and let C_1 be a component of the central fiber. There exists a divisor D on the boundary of the cone $\overline{\mathrm{Eff}}(X/S)$ with $\sigma_{C_1}(D; X/S) = \infty$.*

Proof. Fix a π-ample effective $\mathbb{Q}$-divisor $H = H_0$ on X with $H \cdot C_1 = H \cdot C_2 = 1$. Then $\sigma_{C_i}(H) = 0$ for each i. For example, we might take $H = \bar{\Sigma}_1 + \imath_*(\bar{\Sigma}_1)$ in the notation of Lemma 7.

Let $H_n = \phi_*^n(H)$ be the strict transform of H on X under n applications of ϕ. Using the last part of Lemma 8, and the Jordan decomposition of M, which has a 3×3 block associated to the eigenvalue 1, we compute $\sigma_{H_n} = (2n+1, -2n+1, n(n-1)/2, n(n+1)/2)$:

n	$H_n \cdot C_1$	$H_n \cdot C_2$	$\sigma_{C_1}(H_n)$	$\sigma_{C_2}(H_n)$
0	1	1	0	0
1	3	-1	0	1
2	5	-3	1	3
3	7	-5	3	6
		$\cdots$		
n	$2n+1$	$-2n+1$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}$

The key feature of the example is that while $H_n \cdot C_1$ grows linearly in n, the multiplicity $\sigma_{C_1}(H_n)$ grows quadratically. Let D be the divisor class on the boundary of $\overline{\mathrm{Eff}}(X/S)$ with $D \cdot C_1 = 1$ and $D \cdot C_2 = -1$. Since C_1 and C_2 span $N_1(X/S)$, we see that

$$H_n \equiv_\pi (2n)D + H_0.$$

It follows that $\frac{1}{2n}H_n \equiv_\pi D + \frac{1}{2n}H_0$ is a sequence of divisors converging to D, whose multiplicities along the curves is known. By Definition 1, we

compute

$$\sigma_{C_1}(D; X/S) = \lim_{n \to \infty} \sigma_{C_1}\left(D + \frac{1}{2n}H_0\right)$$

$$= \lim_{n \to \infty} \frac{1}{2n}\sigma_{C_1}H_n = \lim_{n \to \infty} \frac{n-1}{4} = \infty.$$

$\square$

Note that $\operatorname{codim} \pi(C_1) = 2$, so there is no contradiction with Theorem 4(3).

Corollary 10. *If $f : W \to X$ is the blow-up along C_1 with exceptional divisor E, then $\tilde{D} = f^*D$ has $\sigma_E(\tilde{D}; W/S) = \infty$ and $N_\sigma(\tilde{D}; W/S)$ contains the divisor E with infinite coefficient. In particular, there does not exist a birational model $g : Z \to W$ for which $g^*\tilde{D}$ admits a decomposition $g^*\tilde{D} = P + N$ with P a $g \circ (f \circ \pi)$-movable divisor and N effective.*

Proof. By Theorem 5(2), if $f : W \to X$ is the blow-up along C_1, with exceptional divisor E, we have $\sigma_E(f^*D; W/S) = \infty$. Now, suppose that $g : Y \to W$ is any birational map, and that $g^*f^*D = P + N$, where P is a $(g \circ f \circ \pi)$-movable divisor and N is effective. Let $\tilde{E}$ denote the strict transform of E on Y. Then

$$\sigma_{\tilde{E}}(g^*f^*D; Y/S) \leq \sigma_{\tilde{E}}(P; Y/S) + \sigma_{\tilde{E}}(N; Y/S) = \sigma_{\tilde{E}}(N; Y/S).$$

The last of these is finite since N is effective, while the first is infinite, a contradiction. This completes the proof. $\square$
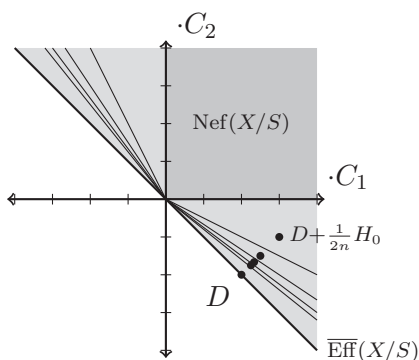

Figure 3: Chambers in $N^1(X/S)$.

4. A general set-up

The key feature that made possible the computation of the preceding example is that if the four numbers $D \cdot C_i$ and $\sigma_{C_i}(D)$ are all known, then the same four invariants can be computed for the strict transform of D under ϕ using Lemma 8. In this section, we give an explanation for this, and describe how to make analogous computations in a more general setting.

Suppose that X is normal and $\mathbb{Q}$-factorial and that $\phi : X \dashrightarrow X$ is a pseudoautomorphism over S (i.e. a birational map for which neither ϕ nor ϕ^{-1} contracts any divisors). We will say that a birational morphism $f : Y \to X$ from a normal $\mathbb{Q}$-factorial variety Y is a *small lift* of ϕ if the induced map $\psi : Y \dashrightarrow Y$ is also a pseudoautomorphism.

$$
\begin{array}{ccc}
Y & \dashrightarrow{\psi} & Y \\
\downarrow{f} & & \downarrow{f} \\
X & \dashrightarrow{\phi} & X
\end{array}
$$

Observe that if $f : Y \to X$ is a small lift, then the map ψ must permute the exceptional divisors of f.

Example 11. Suppose that $\phi : X \dashrightarrow X$ is a pseudoautomorphism and x is a point not contained in $\mathrm{indet}\,\phi$. The blow-up $f : \mathrm{Bl}_x X \to X$ is a small lift of ϕ if and only if x is a fixed point of ϕ. If x is not a fixed point, then the induced map $\psi : Y \dashrightarrow Y$ contracts the exceptional divisor E, while if x is fixed, then $\psi|_E : E \to E$ is an automorphism.

The more interesting examples are those in which f contracts a divisor lying over $\mathrm{indet}\,\phi$.

Example 12. Next we construct a small lift of the map $\phi : X \dashrightarrow X/S$ from Section 3. Let $f : W \to X$ be the blow-up along C_1 as before, with exceptional divisor E_1, and let $h : Y \to W$ be the blow-up along $\bar{C}_2'$, with exceptional divisor E_2. The two exceptional divisors E_1 and E_2 are swapped by the induced map $\psi : Y \dashrightarrow Y$, and $h \circ f$ is a small lift.

$$
\begin{array}{ccc}
Y & \xdashrightarrow{\psi} & Y \\
\ \ {\scriptstyle h\circ f}\downarrow\ \searrow{\scriptstyle h} & & \downarrow{\scriptstyle h\circ f} \\
 & W & \\
\ \ \swarrow{\scriptstyle f}\ \ \ \searrow{\scriptstyle g} & & \\
X & \xdashrightarrow{\phi} & X
\end{array}
$$

The curves C_1 and C_2 could have been blown up in the opposite order, yielding a different small lift $f' : Y' \to X$. This makes no real difference: the threefolds Y and Y' differ only by flops, and strict transform induces an identification $N^1(Y) \xrightarrow{\sim} N^1(Y')$ with respect to which the maps ψ_* and ψ'_* coincide.

If $f : Y \to X$ is a small lift, it follows from the negativity lemma [2, Lemma 3.6.2] and the $\mathbb{Q}$-factoriality of X that there is a decomposition $N^1(Y) = f^* N^1(X) \oplus V_E$, where $V_E = \bigoplus_i \mathbb{R} \cdot [E_i]$. If D is a divisor class on X, it is not necessarily true that $f^* \phi_* D = \psi_* f^* D$. However, the difference $f^* \phi_* D - \psi_* f^* D$ is an f-exceptional divisor, since

$$f_*(f^* \phi_* D - \psi_* f^* D) = \phi_* D - f_* \psi_* f^* D$$
$$= \phi_* D - \phi_* f_* f^* D = \phi_* D - \phi_* D = 0.$$

Define $K : N^1(X) \to V_E$ by $K = f^* \phi_* - \psi_* f^*$. The next lemma characterizes the action of the strict transform $\psi_* : N^1(Y) \to N^1(Y)$ with respect to this decomposition.

Lemma 13. *Suppose that X and Y are $\mathbb{Q}$-factorial and $f : Y \to X$ is a small lift of a pseudoautomorphism $\phi : X \dashrightarrow X$. With respect to the decomposition $N^1(Y) \cong f^* N^1(X) \oplus V_E$, ψ_* is given in block form as*

$$\psi_* = \left(\begin{array}{c|c} \phi_* & 0 \\ \hline -K & P \end{array} \right),$$

where P is the permutation matrix for the action of ψ_* on the E_i. The eigenvalues of ψ_* are the union of those of ϕ_* and those of P, which are roots of unity. Its eigenvectors are

1) *$f^* v_i - (\lambda_i I - P)^{-1} K v_i$, where v_i are the eigenvectors of ϕ_*, with eigenvalues λ_i;*

2) *E_i, the exceptional divisors of f, with eigenvalues that are roots of unity.*

Proof. For a divisor D on X, $\psi_* f^* D = f^* \phi_* D - KD$, while the exceptional divisors E_i are simply permuted by ψ; this gives the block form of the map. The eigenvectors follow from elementary linear algebra. $\qquad\square$

Note that if ψ_* fixes the exceptional divisors, which can always be arranged by replacing ϕ and ψ by suitable iterates, the permutation matrix P is the identity.

A rational map $\phi : X \dashrightarrow Y$ between $\mathbb{Q}$-factorial varieties is said to be D-non-negative for an $\mathbb{R}$-divisor D if on some common resolution $p : W \to X$, $q : W \to Y$, we have $p^*D + E = q^*(\phi_*D)$, where E is an effective q-exceptional divisor. If $\phi : X \dashrightarrow X$ is a pseudoautomorphism with a small lift $f : Y \to X$, then we may consider a resolution of the form
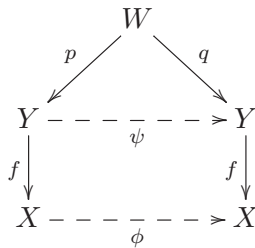
If D is a divisor on X for which ϕ is D-non-negative, then we have $p^*f^*D + E = q^*f^*\phi_*D$ with $E \geq 0$. Pushing forward both sides by q, this gives

$$q_*p^*f^*D + q_*E = f^*\phi_*D$$
$$\psi_*f^*D + E' = f^*\phi_*D,$$

where E' is an effective f-exceptional divisor. In particular, $KD = f^*\phi_*D - \psi_*f^*D = E'$ is effective.

Next we observe that if the divisorial Zariski decomposition of f^*D is known for some divisor D, the decomposition of $f^*\phi_*D$ can often be computed. Although in earlier sections we assumed that X was smooth, in what follows X need only be normal and $\mathbb{Q}$-factorial, as we will only consider asymptotic multiplicities along divisors, rather than higher-dimensional subvarieties (which might lie entirely in the singular locus of X). We assume for simplicity that ψ fixes each of the f-exceptional divisors E_i; this can always be arranged by replacing ϕ by a suitable iterate. This assumption implies that the permutation matrix P is the identity, and that $\psi_*(KD) = KD$ since KD is f-exceptional.

Lemma 14. *Suppose $\phi : X \dashrightarrow X$ is a pseudoautomorphism over S, and that D is a class in $N^1(X/S)$ with $\sigma_\Gamma(D; X/S)$ finite for all divisors Γ. Then $N_\sigma(\phi_*D; X/S) = \phi_*N_\sigma(D; X/S)$ and $P_\sigma(\phi_*D; X/S) = \phi_*P_\sigma(D; X/S)$. If ϕ is D-non-negative and $N_\sigma(f^*D; Y/S)$, then $P_\sigma(f^*\phi_*D; Y/S) = \psi_*P_\sigma(f^*D; Y/S)$.*

Proof. Since ϕ neither contracts nor extracts any divisors, for any prime divisor E we have $\sigma_E(D; X/S) = \sigma_{\phi_* E}(\phi_* D; X/S)$. The claim for $N_\sigma(\phi_* D; X/S)$ follows, and that for $P_\sigma(\phi_* D; X/S)$ is immediate.

Now, by the D-non-negativity hypothesis on ϕ, KD is an effective f-exceptional divisor. By [11, Lemma 3.5.1], if E is an effective f-exceptional divisor, we have $N_\sigma(f^* D + E) = N_\sigma(f^* D) + E$. This means that

$$N_\sigma(f^* \phi_* D) = N_\sigma(\psi_* f^* D + KD) = N_\sigma(\psi_*(f^* D + KD))$$
$$= \psi_* N_\sigma(f^* D + KD) = \psi_* N_\sigma(f^* D) + \psi_* KD$$
$$= \psi_* N_\sigma(f^* D) + KD.$$

We have made use of the fact that E is effective by the non-negativity hypothesis on D. It is now simple to compute the positive part of the decomposition:

$$P_\sigma(f^* \phi_* D) = f^* \phi_* D - N_\sigma(f^* \phi_* D) = f^* \phi_* D - \psi_* N_\sigma(f^* D) - KD$$
$$= \psi_* f^* D - N_\sigma(\psi_* f^* D) = P_\sigma(\psi_* f^* D) = \psi_* P_\sigma(f^* D).$$

$\square$

Remark 2. The example of Section 3 can be interpreted as an instance of the calculations in this section. A small lift of the map ϕ is constructed in Example 12. Let F_1, F_2 be a basis for $N^1(X/S)$ dual to C_1 and C_2. A basis for $N^1(Y/S)$ is given by the four classes $(h \circ f)^* F_1$, $(h \circ f)^* F_2$, E_1, and E_2. The vector v_D gives the coefficients for the class of the strict transform of D on Y with respect to this above basis. Lemma 8 is nothing more than the calculation of the induced map ψ_* of Lemma 13. The final calculation in Theorem 9 can then be carried out as a repeated application of Lemma 14.

Suppose now that $S = \operatorname{Spec} \mathbb{C}$ and $\phi : X \dashrightarrow X$ is a pseudoautomorphism whose action $\phi_* : N^1(X) \to N^1(X)$ has a unique largest eigenvalue $\lambda > 1$, and that $f : Y \to X$ is a small lift of ϕ. Since $S = \operatorname{Spec} \mathbb{C}$, the pseudoeffective cone is strictly convex, and by a version of the Perron-Frobenius theorem [3] there exists a λ-eigenvector D_ϕ which is contained in the pseudoeffective cone. Lemma 13 then provides the existence of a λ-eigenvector D_ψ for ψ_*.

We are then able to compute the Zariski decomposition of the divisor $f^* D_\phi$ using Lemma 14.

Corollary 15. *Let D_ϕ be the dominant eigenvector of $\phi_* : N^1(X) \to N^1(X)$, and D_ψ be the dominant eigenvector of $\psi_* : N^1(Y) \to N^1(Y)$. If D is ϕ-non-negative, then $P_\sigma(f^* D_\phi) = D_\psi$.*

Proof. If D is any pseudoeffective divisor on X, then for every n we have

$$P_\sigma(f^*(\lambda^{-n}\phi_*^n D)) = \lambda^{-n}\psi_*^n P_\sigma(f^*D).$$

Take $D = D_\phi + D_{\phi^{-1}}$, so that the above reduces to

$$P_\sigma(f^*(D_\phi + \lambda^{-2n}D_{\phi^{-1}})) = \lambda^{-n}\psi_*^n P_\sigma(f^*D).$$

The left hand side converges to $P_\sigma(f^*D_\phi)$ by Proposition 5(1). With a suitable choice of scaling, the right hand side converges to D_ψ. $\qquad\square$

Acknowledgements

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