Math. Res. Lett.
Volume 23, Number 5, 1405–1432, 2016

Quaternionic contact Einstein manifolds

Stefan Ivanov, Ivan Minchev, and Dimiter Vassilev

We show that a seven dimensional quaternionic contact Einstein manifold has constant qc-scalar curvature. In addition, we characterize qc-Einstein structures with certain flat vertical connection and develop their local structure equations. Finally, regular qc-Ricci flat structures are shown to fiber over hyper-Kähler manifolds.

1. Introduction

Following the work of Biquard [Biq1] quaternionic contact (qc) manifolds describe the Carnot-Carathéodory geometry on the conformal boundary at infinity of quaternionic Kähler manifolds. The qc geometry also became a crucial geometric tool in finding the extremals and the best constant in the L^2 Folland-Stein Sobolev-type embedding on the quaternionic Heisenberg groups [F2, FS], see [IMV, IMV2, IMV3]. An extensively studied class of quaternionic contact structures are provided by the 3-Sasakian manifolds. From the point of view of qc geometry, 3-Sasakian structures are qc manifolds whose torsion endomorphism of the Biquard connection vanishes. In turn, the latter property is equivalent to the qc structure being qc-Einstein, i.e., the trace-free part of the qc-Ricci tensor vanishes, see [IMV]. The qc-scalar curvature of a 3-Sasakian manifold is a non-zero constant. Conversely, it was shown in [IMV, IV2] that the Biquard torsion is the obstruction for a given qc structure to be locally 3-Sasakian provided the qc-scalar curvature $Scal$ is a non zero constant. Furthermore, as a consequence of the Bianchi identities, [IMV, Theorem 4.9] shows that the qc-scalar curvature of a qc-Einstein manifold of dimension at least eleven is constant while the seven dimensional case was left open.

The main purpose of this paper is to show that the qc-scalar curvature of a seven dimensional qc-Einstein manifold is constant, i.e., to prove the following

---

2010 Mathematics Subject Classification: 58G30, 53C17.
Key words and phrases: quaternionic contact structures, qc-conformal flatness, qc-conformal curvature, Einstein structures.

Theorem 1.1. *If M is a qc-Einstein qc manifold of dimension seven, then, the qc-scalar curvature is a constant, $Scal = const$.*

The proof of Theorem 1.1 makes use of the qc-conformal curvature tensor [IV1], which characterizes locally qc-conformally flat structures, a result of Kulkarni [Kul] on algebraic properties of curvature tensors in four dimensions, and an extension of [IMV, Theorem 1.21] which describes explicitly all qc-Einstein structures defined on open sets of the quaternionic Heisenberg group that are point-wise qc-conformal to the standard flat qc structure on the quaternionic Heisenberg groups. The main application of Theorem 1.1 is the removal of the a-priori assumption of constancy of the qc-scalar curvature in some previous papers concerning seven dimensional qc-Einstein manifolds, see for example Corollaries 3.2, 3.3 and 6.1.

The remaining parts of this paper are motivated by known properties of qc-Einstein manifolds with non-vanishing qc-scalar curvature, in that we prove corresponding results in the case of vanishing qc-scalar curvature. With this goal in mind and because of its independent interest, in Section 4 we define a connection on the canonical three dimensional vertical distribution of a qc manifold. We show that qc-Einstein spaces can be characterized by the flatness of this vertical connection. This allows us to write the structure equations of a qc-Einstein manifold in terms of the defining 1-forms, their exterior derivatives and the qc-scalar curvature, see Theorem 5.1. The latter extends the results of [IV2] and [IV3, Section 4.4.2] to the vanishing qc-scalar curvature case.

Recall that complete and regular 3-Sasakian and nS-spaces (called negative 3-Sasakian here) have canonical fibering with fiber $Sp(1)$ or $SO(3)$, and base a quaternionic Kähler manifold. This shows that if $S > 0$ (resp. $S < 0$), the qc-Einstein manifolds are "essentially" $SO(3)$ bundles over quaternionic Kähler manifolds with positive (resp. negative) scalar curvature. In Section 6 we show that in the "regular" case, similar to the non-zero qc-scalar curvature cases, a qc-Einstein manifold of zero scalar curvature fibers over a hyper-Kähler manifold, see Proposition 6.3.

We conclude the paper with a brief section where we show that every qc-Einstein manifold of non-zero scalar curvature carries two Einstein metrics. Note that the corresponding results concerning the 3-Sasakian case is well known, see [BGN]. In the negative qc-scalar curvature case both Einstein metrics are of signature $(4n, 3)$ of which the first is locally (negative) 3-Sasakian, while the second "squashed" metric is not 3-Sasakian, see Proposition 6.4.

Convention 1.2. Throughout the paper, unless explicitly stated otherwise, we will use the following conventions.

a) The triple (i, j, k) denotes any cyclic permutation of $(1, 2, 3)$ while s, t will denote any numbers from the set $\{1, 2, 3\}$, $s, t \in \{1, 2, 3\}$.

b) For a decomposition $TM = V \oplus H$ we let $[.]_V$ and $[.]_H$be the corresponding projections to V and H.

c) A, B, C, etc. will denote sections of the tangent bundle of M, i.e., $A, B, C \in TM$.

d) X, Y, Z, U will denote horizontal vector fields, i.e., $X, Y, Z, U \in H$.

e) ξ, ξ', ξ'' will denote vertical vector fields, i.e., $\xi, \xi', \xi'' \in V$.

f) $\{e_1, \dots, e_{4n}\}$ denotes an orthonormal basis of the horizontal space H;

g) The summation convention over repeated vectors from the basis $\{e_1, \dots, e_{4n}\}$ is used. For example, $k = P(e_b, e_a, e_a, e_b)$ means $k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b)$.

2. Preliminaries

It is well known that the sphere at infinity of a non-compact symmetric space M of rank one carries a natural Carnot-Carathéodory structure, see [M, P]. Quaternionic contact (qc) structures were introduced by O. Biquard [Biq1] and are modeled on the conformal boundary at infinity of the quaternionic hyperbolic space. Biquard showed that the infinite dimensional family [LeB91] of complete quaternionic-Kähler deformations of the quaternion hyperbolic metric have conformal infinities which provide an infinite dimensional family of examples of qc structures. Conversely, according to [Biq1] every real analytic qc structure on a manifold M of dimension at least eleven is the conformal infinity of a unique quaternionic-Kähler metric defined in a neighborhood of M. Furthermore, [Biq1] considered CR and qc structures as boundaries of infinity of Einstein metrics rather than only as boundaries at infinity of Kähler-Einstein and quaternionic-Kähler metrics, respectively. In fact, [Biq1] showed that in each of the three hyperbolic cases (complex, quaternionic, octoninoic) any small perturbation of the standard Carnot-Carathéodory structure on the boundary is the conformal infinity of an essentially unique Einstein metric on the unit ball, which is asymptotically symmetric.

We refer to [Biq1], [IMV] and [IV3] for a more detailed exposition of the definitions and properties of qc structures and the associated Biquard connection. Here, we recall briefly the relevant facts needed for this paper. A quaternionic contact (qc) manifold is a $4n+3$-dimensional manifold M with a codimension three distribution H equipped with an $Sp(n)Sp(1)$ structure locally defined by an $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$. Thus, $H = \cap_{s=1}^3 Ker\, \eta_s$ is equipped with a positive definite symmetric tensor g, called the horizontal metric, and a compatible rank-three bundle $\mathbb{Q}$ consisting of endomorphisms of H locally generated by three orthogonal almost complex structures I_s, $s = 1, 2, 3$, satisfying the unit quaternion relations: (i) $I_1 I_2 = -I_2 I_1 = I_3$, $I_1 I_2 I_3 = -id_{|_H}$; (ii) $g(I_s., I_s.) = g(., .)$; and (iii) the compatibility conditions $2g(I_s X, Y) = d\eta_s(X, Y)$, $X, Y \in H$ hold true.

The transformations preserving a given quaternionic contact structure η, i.e., $\bar\eta = \mu\Psi\eta$ for a positive smooth function μ and an $SO(3)$ matrix Ψ with smooth functions as entries are called *quaternionic contact conformal (qc-conformal) transformations*. If the function μ is constant $\bar\eta$ is called *qc-homothetic* to η and in the case $\mu \equiv 1$ we call $\bar\eta$ *qc-equivalent* to η. Notice that in the latter case, η and $\tilde\eta$ define the same qc structure. The qc-conformal curvature tensor W^{qc}, introduced in [IV1], is the obstruction for a qc structure to be locally qc-conformal to the standard 3-Sasakian structure on the $(4n+3)$-dimensional sphere [IV1, IV3].

Biquard showed that on a qc manifold of dimension at least eleven there is a unique connection ∇ with torsion T and a unique supplementary to H in TM subspace V, called the *vertical space*, such that the following conditions are satisfied: (i) ∇ preserves the decomposition $H \oplus V$ and the $Sp(n)Sp(1)$ structure on H, i.e., $\nabla g = 0, \nabla\sigma \in \Gamma(\mathbb{Q})$ for a section $\sigma \in \Gamma(\mathbb{Q})$, and its torsion on H is given by $T(X, Y) = -[X, Y]_{|V}$; (ii) for $\xi \in V$, the endomorphism $T_\xi = T(\xi, \cdot) : H \to H$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$; (iii) the connection on V is induced by the natural identification φ of V with the subspace $sp(1)$ of the endomorphisms of H, i.e., $\nabla\varphi = 0$. Furthermore, [Biq1] also described the supplementary distribution V, which is (locally) generated by the so called Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$ determined by

$$(1) \qquad \eta_s(\xi_t) = \delta_{st}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \quad (\xi_s \lrcorner d\eta_t)_{|H} = -(\xi_t \lrcorner d\eta_s)_{|H},$$

where $\lrcorner$ denotes the interior multiplication.

If the dimension of M is seven Duchemin showed in [D] that if we assume, in addition, the existence of Reeb vector fields as above, then the Biquard result holds. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying (1). We shall call ∇ *the Biquard connection.*

Notice that equations (1) are invariant under the natural $SO(3)$ action. Using the triple of Reeb vector fields we extend the horizontal metric g to a metric h on M by requiring $span\{\xi_1,\xi_2,\xi_3\}=V\perp H$ and $h(\xi_s,\xi_t)=\delta_{st}$,

$$(2)\qquad h|_H=g,\qquad h|_V=\eta_1\otimes\eta_1+\eta_2\otimes\eta_2+\eta_3\otimes\eta_3,\qquad h(\xi_s,X)=0.$$

The Riemannian metric h as well as the Biquard connection do not depend on the action of $SO(3)$ on V, but both change if η is multiplied by a conformal factor [IMV].

The fundamental 2-forms ω_s and the fundamental 4-form Ω of the quaternionic structure $\mathbb{Q}$ are defined, respectively, by

$$2\omega_{s|H}=d\eta_{s|H},\quad \xi\lrcorner\omega_s=0,\qquad \Omega=\omega_1\wedge\omega_1+\omega_2\wedge\omega_2+\omega_3\wedge\omega_3.$$

2.1. The torsion of the Biquard connection

It was shown in [Biq1] that the torsion T_ξ is completely trace-free, $tr\,T_\xi=tr\,T_\xi\circ I_s=0$. Decomposing the endomorphism $T_\xi\in(sp(n)+sp(1))^\perp$ into its symmetric part T^0_ξ and skew-symmetric part $b_\xi, T_\xi=T^0_\xi+b_\xi$, we have $T^0_{\xi_i}I_i=-I_iT^0_{\xi_i}$, $I_2(T^0_{\xi_2})^{+--}=I_1(T^0_{\xi_1})^{-+-}$, $I_3(T^0_{\xi_3})^{-+-}=I_2(T^0_{\xi_2})^{--+}$, $I_1(T^0_{\xi_1})^{--+}=I_3(T^0_{\xi_3})^{+--}$, where the upper script $+++$ denotes the component commuting with all three I_i, $+--$ indicates the component commuting with I_1 and anti-commuting with the other two, etc. Furthermore, the symmetric part T^0_ξ satisfies the identity

$$(3)\qquad g(T^0_\xi(X),Y)=\frac{1}{2}\mathcal{L}_\xi g(X,Y),\qquad \xi\in V,\quad X,Y\in H,$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ. The skew-symmetric part can be represented as $b_{\xi_i}=I_iu$, where u is a traceless symmetric (1,1)-tensor on H which commutes with I_1,I_2,I_3. Therefore we have $T_{\xi_i}=T^0_{\xi_i}+I_iu$. If $n=1$ then the tensor u vanishes identically, $u=0$, and the torsion is a symmetric tensor, $T_\xi=T^0_\xi$. Following [IMV] we define the $Sp(n)Sp(1)$ components T^0 and U of the torsion tensor by

$$T^0(X,Y)=g((T^0_{\xi_1}I_1+T^0_{\xi_2}I_2+T^0_{\xi_3}I_3)X,Y),\qquad U(X,Y)=-\,g(uX,Y).$$

Then, as shown in [IMV], both T^0 and U are trace-free, symmetric and invariant under qc homothetic transformations. Using the fixed horizontal metric g, we shall also denote by T^0 and U the corresponding endomorphisms of H, $g(T^0(X),Y) = T^0(X,Y)$ and $g(U(X),Y) = U(X,Y)$. The torsion of the Biquard connection ∇ is described by the formulas [Biq1] and [IMV]

$$
\begin{aligned}
T(X,Y) &= -[X,Y]_V = 2\sum_{s=1}^{3}\omega_s(X,Y)\xi_s,\\
T(\xi_s,X) &= \frac{1}{4}(I_sT^0 - T^0I_s)(X) + I_sU(X),\\
T(\xi_i,\xi_j) &= -S\xi_k - [\xi_i,\xi_j]_H,
\end{aligned}
\tag{4}
$$

where $[\xi_i,\xi_j]_H$ stands for the H-component of the commutator of the vector fields ξ_i, ξ_j and S is the normalized qc-scalar curvature defined below.

2.2. The curvature of the Biquard connection

We denote by $R = [\nabla,\nabla] - \nabla_{[,]}$ the curvature tensor of ∇ and by the same letter R the curvature $(0,4)$-tensor $R(A,B,C,D) = h(R_{A,B}C,D)$. The *qc-Ricci tensor*, the *qc-scalar curvature*, and the three *qc-Ricci 2-forms* are defined as follows, cf. Convention 1.2 for the used notation,

$$
\begin{aligned}
&Ric(A,B) = R(e_a,A,B,e_a), \qquad Scal = Ric(e_a,e_a),\\
&\rho_s(A,B) = \frac{1}{4n}R(A,B,e_a,I_se_a).
\end{aligned}
\tag{5}
$$

The *normalized qc-scalar curvature* S is defined by $8n(n+2)S = Scal$.

A fundamental fact, [IMV, Theorem 3.12], is that the torsion endomorphism determines the (horizontal) qc-Ricci tensor and the (horizontal) qc-Ricci forms of the Biquard connection,

$$
\begin{aligned}
Ric(X,Y) &= (2n+2)T^0(X,Y) + (4n+10)U(X,Y)\\
&\quad + 2(n+2)Sg(X,Y)\\
\rho_s(X,I_sY) &= -\frac{1}{2}\big[T^0(X,Y) + T^0(I_sX,I_sY)\big] - 2U(X,Y)\\
&\quad - Sg(X,Y).
\end{aligned}
\tag{6}
$$

We say that M is a *qc-Einstein manifold* if the horizontal Ricci tensor is proportional to the horizontal metric g,

$$
Ric(X,Y) = \frac{Scal}{4n}g(X,Y) = 2(n+2)Sg(X,Y),
\tag{7}
$$

which taking into account (6) is equivalent to $T^0 = U = 0$. Furthermore, by [IMV, Theorem 4.9] if $\dim(M) > 7$ then any qc-Einstein structure has a constant qc-scalar curvature. It was left as an open question whether a qc-Einsten manifold of dimension seven has constant qc-scalar curvature. The main result of the current paper Theorem 1.1 shows that this is indeed the case.

If the covariant derivatives with respect to ∇ of the endomorphisms I_s, the fundamental 2-forms ω_s, and the Reeb vector fields ξ_s are given by

$$\begin{aligned} &\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \quad \nabla \omega_i = -\alpha_j \otimes \omega_k + \alpha_k \otimes \omega_j, \\ &\nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j, \end{aligned} \tag{8}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the local connection 1-forms, then [Biq1] proved that $\alpha_i(X) = d\eta_k(\xi_j, X) = -d\eta_j(\xi_k, X) \quad \text{for all} \quad X \in H$. On the other hand, as shown in [IMV] the vertical and the $\mathfrak{sp}(1)$ parts of the curvature endomorphism $R(A,B)$ are related to the $\mathfrak{sp}(1)$-connection 1-forms α_s by

$$R(A, B, \xi_i, \xi_j) = 2\rho_k(A, B) = (d\alpha_k + \alpha_i \wedge \alpha_j)(A, B). \tag{9}$$

Finally, we have the following commutation relations [IMV]

$$\begin{aligned} &R(B, C, I_i X, Y) + R(B, C, X, I_i Y) \\ &= 2\big[-\rho_j(B, C)\omega_k(X, Y) + \rho_k(B, C)\omega_j(X, Y)\big]. \end{aligned} \tag{10}$$

In the next section we give the proof of our main result.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is achieved with the help of the following Lemma 3.1 where we calculate the curvature $R(Z, X, Y, V)$ at points where the horizontal gradient of the qc-scalar curvature does not vanish, $\nabla S \neq 0$. The proof of Theorem 1.1 proceeds by showing that on any open set where S is not locally constant M is locally qc-conformally flat. In fact, on any open set where $\nabla S \neq 0$ the qc-conformal curvature W^{qc} defined in [IV1] will be seen to vanish, hence by [IV1, Theorem 1.2] the qc manifold is locally qc-conformally flat. The final step involves a generalization of [IMV, Theorem 1.1], which follows from the proof of [IMV, Theorem 1.1], allowing the explicit description of all qc-Einstein structures defined on open sets of the quaternionic Heisenberg group that are point-wise qc-conformal to the standard flat qc structure on the quaternionic Heisenberg groups. It turns out that all such qc structures are of constant qc-scalar curvature, which allows the completion of the proof of Theorem 1.1.

Lemma 3.1. *On a seven dimensional qc-Einstein manifold we have the following formula for the horizontal curvature of the Biquard connection on any open set where the qc-scalar curvature is not constant,*

$$R(Z,X,Y,V) = 2S\big[g(Z,V)g(X,Y) - g(X,V)g(Z,Y)\big]. \tag{11}$$

Proof of Lemma 3.1. Our first goal is to show the next identity,

$$R(Z,X,Y,\nabla S) = 2S\big[dS(Z)g(X,Y) - dS(X)g(Z,Y)\big], \tag{12}$$

where ∇S is the horizontal gradient of S defined by $g(X,\nabla S) = dS(X)$. For this, recall the general formula proven in [IV1, Theorem 3.1, (3.6)],

$$\begin{aligned} R(\xi_i,\xi_j,X,Y) = {} & (\nabla_{\xi_i}U)(I_jX,Y) - (\nabla_{\xi_j}U)(I_iX,Y)\\ & - \frac{1}{4}\big[(\nabla_{\xi_i}T^0)(I_jX,Y) + (\nabla_{\xi_i}T^0)(X,I_jY)\big]\\ & + \frac{1}{4}\big[(\nabla_{\xi_j}T^0)(I_iX,Y) + (\nabla_{\xi_j}T^0)(X,I_iY)\big]\\ & - (\nabla_X\rho_k)(I_iY,\xi_i) - \frac{Scal}{8n(n+2)}T(\xi_k,X,Y)\\ & - T(\xi_j,X,e_a)T(\xi_i,e_a,Y) + T(\xi_j,e_a,Y)T(\xi_i,X,e_a) \end{aligned} \tag{13}$$

where the Ricci two forms are given by, cf. [IV1, Theorem 3.1],

$$\begin{aligned} & 6(2n+1)\rho_s(\xi_s,X)\\ = {} & (2n+1)X(S) - 2(\nabla_{e_a}U)(e_a,X)\\ & + \frac{1}{2}\big[(\nabla_{e_a}T^0)(e_a,X) - 3(\nabla_{e_a}T^0)(I_se_a,I_sX)\big],\\ & 6(2n+1)\rho_i(\xi_j,I_kX) = -6(2n+1)\rho_i(\xi_k,I_jX)\\ = {} & (2n-1)(2n+1)X(S)\\ & - \frac{1}{2}\big[(4n+1)(\nabla_{e_a}T^0)(e_a,X) + 3(\nabla_{e_a}T^0)(I_ie_a,I_iX)\big]\\ & - 4(n+1)(\nabla_{e_a}U)(e_a,X). \end{aligned} \tag{14}$$

In our case $T^0 = U = 0$, hence (13) takes the form

$$R(\xi_i,\xi_j,X,Y) = -(\nabla_X\rho_k)(I_iY,\xi_i). \tag{15}$$

Letting $n = 1$ and $T^0 = U = 0$ in (14) it follows $\rho_i(I_kY,\xi_j) = -\frac{1}{6}dS(Y)$, which after a cyclic permutation of ijk and a substitution of Y with I_kY

yields

$$\rho_k(I_iY,\xi_i) = -\frac{1}{6}dS(I_kY). \tag{16}$$

Taking the covariant derivative of (16) with respect to the Biquard connection and applying (8) we calculate

$$\begin{aligned}(\nabla_X\rho_k)(I_iY,\xi_i) &- \alpha_i(X)\rho_j(I_iY,\xi_i) + \alpha_j(X)\rho_i(I_iY,\xi_i)\\ &- \alpha_j(X)\rho_k(I_kY,\xi_i) + \alpha_k(X)\rho_k(I_jY,\xi_i)\\ &- \alpha_j(X)\rho_k(I_iY,\xi_k) + \alpha_k(X)\rho_k(I_iY,\xi_j)\\ = -\frac{1}{6}\nabla^2S(X,I_kY) &+ \frac{1}{6}\alpha_i(X)dS(I_jY) - \frac{1}{6}\alpha_j(X)dS(I_iY).\end{aligned} \tag{17}$$

Applying (14) with $n=1$ and $T^0=U=0$ we see that the terms involving the connection 1-forms cancel and (17) turns into

$$(\nabla_X\rho_k)(I_iY,\xi_i) = -\frac{1}{6}\nabla^2S(X,I_kY). \tag{18}$$

A substitution of (18) in (15) taking into account the skew-symmetry of $R(\xi_i,\xi_j,X,Y)$ with respect to X and Y allows us to conclude the following identity for the horizontal Hession of S

$$\nabla^2S(X,I_sY) + \nabla^2S(Y,I_sX) = 0. \tag{19}$$

The trace of (19) together with the Ricci identity yield

$$\begin{aligned}0 = 2\nabla^2S(e_a,I_ke_a) &= \nabla^2S(e_a,I_ke_a) - \nabla^2S(I_ke_a,e_a)\\ &= -2\sum_{s=1}^{3}\omega_s(e_a,I_ke_a)dS(\xi_s) = -8dS(\xi_k),\end{aligned}$$

i.e., we have

$$dS(\xi_s) = 0, \qquad \nabla^2S(\xi_s,\xi_t) = 0. \tag{20}$$

The equality (20) shows that S is constant along the vertical directions, $dS(\xi_s)=0$, hence, in view of (8), the second equation of (20) holds as well. In addition, we have $\nabla^2S(X,\xi_s) = XdS(\xi_s) - dS(\nabla_X\xi_s) = 0$ since ∇ preserves the vertical directions due to (8). Moreover, the Ricci identity

$$\nabla^2 S(\xi_s, X) - \nabla^2 S(X, \xi_s) = dS(T(\xi_s, X)) = 0$$

together with the above equality leads to

$$\nabla^2 S(\xi_s, X) = \nabla^2 S(X, \xi_s) = 0. \tag{21}$$

Next, we show that the horizontal Hessian of S is symmetric. Indeed, we have the identity

$$\begin{aligned} \nabla^2 S(X, Y) - \nabla^2 S(Y, X) &= d^2 S(X, Y) - dS(T(X, Y)) \\ &= -2\sum_{s=1}^{3} \omega_s(X, Y) dS(\xi_s) = 0 \end{aligned} \tag{22}$$

where we applied (20) to conclude the last equality. Now, (19) and (22) imply

$$\nabla^2 S(X, Y) - \nabla^2 S(I_s X, I_s Y) = 0 \tag{23}$$

which shows that the $[-1]$-component of the horizontal Hessian vanishes. Hence, the horizontal Hessian of S is proportional to the horizontal metric since $n = 1$, i.e.,

$$\nabla^2 S(X, Y) = \frac{\nabla^2 S(e_a, e_a)}{4} g(X, Y) = -\frac{\triangle S}{4} g(X, Y), \tag{24}$$

where $\triangle S = -\nabla^2 S(e_a, e_a)$ is the sub-Laplacian of S. We have the following Ricci identity of order three (see e.g. [IPV]

$$\begin{aligned} &\nabla^3 S(X, Y, Z) - \nabla^3 S(Y, X, Z) \\ &= -R(X, Y, Z, \nabla S) - 2\sum_{s=1}^{3} \omega_s(X, Y) \nabla^2 S(\xi_s, Z). \end{aligned} \tag{25}$$

Applying (21) we conclude from (25) that

$$\nabla^3 S(X, Y, Z) - \nabla^3 S(Y, X, Z) = -R(X, Y, Z, \nabla S). \tag{26}$$

Combining (26) and (24) we obtain the next expression for the curvature

$$R(Z, X, Y, \nabla S) = \frac{\nabla^3 S(X, e_a, e_a)}{4} g(Z, Y) - \frac{\nabla^3 S(Z, e_a, e_a)}{4} g(X, Y). \tag{27}$$

The trace of (27) together with the first equality of (6) computed for $n = 1, T^0 = 0$ and $U = 0$ yield

$$Ric(Z, \nabla S) = 6SdS(Z) = -\frac{3}{4}\nabla^3 S(Z, e_a, e_a).$$

Thus, we have

$$\nabla^3 S(Z, e_a, e_a) = -8SdS(Z). \tag{28}$$

Now, a substitution of (28) in (27) gives (12).

Turning to the general formula (11) we note that the horizontal curvature of the Biquard connection in the qc-Einstein case satisfies the identity

$$R(X, Y, Z, V) + R(Y, Z, X, V) + R(Z, X, Y, V) = 0. \tag{29}$$

This follows from the first Bianchi identity since $(\nabla T)(X, Y) = 0$ and $T(T(X, Y), Z) = \sum_{s=1}^{3} 2\omega_s(X, Y)T(\xi_s, Z) = 0$. Thus, the horizontal curvature has the algebraic properties of the Riemannian curvature, namely it is skew-symmetric with respect to the first and the last pairs and satisfies the Bianchi identity (29). Therefore it also has the fourth Riemannian curvature property,

$$R(X, Y, Z, V) = R(Z, V, X, Y). \tag{30}$$

The equalities (12) and (30) imply

$$\begin{aligned} 0 &= R(I_i\nabla S, I_j\nabla S, I_k\nabla S, \nabla S) = R(I_k\nabla S, \nabla S, I_i\nabla S, I_j\nabla S),\\ 0 &= R(I_i\nabla S, I_j\nabla S, I_j\nabla S, \nabla S) = R(I_j\nabla S, \nabla S, I_i\nabla S, I_j\nabla S). \end{aligned} \tag{31}$$

Moreover, using (10) and the second equality in (6) with $T^0 = U = 0$ we calculate

$$\begin{aligned} &R(I_j\nabla S, I_i\nabla S, I_i\nabla S, I_k\nabla S) - R(I_j\nabla S, I_i\nabla S, \nabla S, I_j\nabla S)\\ &= -2\rho_j(I_j\nabla S, I_i\nabla S)\omega_k(\nabla S, I_k\nabla S) + 2\rho_k(I_j\nabla S, I_i\nabla S)\omega_j(\nabla S, I_k\nabla S)\\ &= 0 \end{aligned} \tag{32}$$

The second equality of (31) together with (32) yields

$$R(I_j\nabla S, I_i\nabla S, I_i\nabla S, I_k\nabla S) = 0. \tag{33}$$

Finally, (12), (31), (32), (33) together with (10) and (6) imply for any $s \neq t$ the identities

$$R(I_s\nabla S, I_t\nabla S, I_t\nabla S, I_s\nabla S) = R(I_s\nabla S, \nabla S, \nabla S, I_s\nabla S) = 2S|\nabla S|^4. \tag{34}$$

In a neighborhood of any point where $\nabla S \neq 0$ the quadruple $\left\{\frac{\nabla S}{|\nabla S|}, \frac{I_1\nabla S}{|\nabla S|}, \frac{I_2\nabla S}{|\nabla S|}, \frac{I_3\nabla S}{|\nabla S|}\right\}$ is an orthonormal basis of H, hence after a small calculation taking into account (31), (33) and (34), we see that for any orthonormal basis $\{Z, X, Y, V\}$ of H we have

$$R(Z, X, Y, V) = 0, \qquad R(Z, X, Z, V) - R(Y, X, Y, V) = 0, \tag{35}$$

where the second equation follows from the first using the orthogonal basis $\{Z + Y, X, Z - Y, V\}$. For the "sectional curvature" $K(Z, X) = R(Z, X, Z, X)$ we have then the identities

$$\begin{aligned}
&K(Z, X) + K(Y, V) - K(Z, V) - K(Y, X)\\
&= R(Z, X, Z, X) + R(Y, V, Y, V) - R(Z, V, Z, V) - R(Y, X, Y, X)\\
&= R(Y, X, Y, X) + R(Y, X, Y, V) - R(Y, V, Y, X) + R(Y, V, Y, V)\\
&\quad - R(Z, X, Z, X) - R(Z, X, Z, V) + R(Z, V, Z, X) - R(Z, V, Z, V)\\
&= R(Z, X + V, Z, X - V) - R(Y, X + V, Y, X - V) = 0
\end{aligned}$$

using (30) in the second equality and (35) in the last equality. Now, [Kul, Theorem 3], shows that the Riemannian conformal tensor of the horizontal curvature R vanishes. In view of $Ric = 6S \cdot g$, we conclude that the curvature restricted to the horizontal space is given by (11) which proves the lemma. □

Proof of Theorem 1.1. Let M be a qc-Einstein manifold of dimension seven with a local $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ defining the given qc structure. Suppose the qc-scalar curvature is not a locally constant function. We shall reach a contradiction by showing that M is locally qc-conformally flat, which will be shown to imply that the qc-scalar curvature is locally constant.

To prove the first claim we prove that if the qc-scalar curvature is not locally constant then the qc-conformal curvature W^{qc} of [IV1] vanishes on the open set where $\nabla S \neq 0$. For this we recall the formula for the qc-conformal curvature W^{qc} given in [IV1, Proposition 4.2] which with the assumptions

$T^0 = U = 0$ simplifies to

$$(36)\quad W^{qc}(Z,X,Y,V) = \frac{1}{4}\Big[R(Z,X,Y,V) + \sum_{s=1}^{3} R(I_sZ, I_sX, Y, V)\Big] + \frac{S}{2}\big[g(Z,Y)g(X,V) - g(Z,V)g(X,Y)\big] + \frac{S}{2}\sum_{s=1}^{3}\big(\omega_s(Z,Y)\omega_s(X,V) - \omega_s(Z,V)\omega_s(X,Y)\big).$$

A substitution of (11) in (36) shows $W^{qc} = 0$ on $\nabla S \neq 0$.

Now, [IV1, Theorem 1.2] shows that the open set $\nabla S \neq 0$ is locally qc-conformaly flat, i.e., every point p, $\nabla S(p) \neq 0$ has an open neighborhood O and a qc-conformal transformation $F : O \to \boldsymbol{G}(\mathbb{H})$ to the quaternionic Heisenberg group $\boldsymbol{G}(\mathbb{H})$ equipped with the standard flat qc structure $\tilde{\Theta}$. Thus, $\Theta \overset{def}{=} F^*\eta = \frac{1}{2\mu}\tilde{\Theta}$ for some positive smooth function μ defined on the open set $F(O)$. By its definition Θ is a qc-Einstein structure, hence the proof of [IMV, Theorem 1.1] shows that, with a small change of the parameters in [IMV, Theorem 1.1], μ is given by

$$(37)\qquad \mu(q,\omega) = c_0\Big[\big(\sigma + |q + q_0|^2\big)^2 + |\omega + \omega_o + 2\mathrm{Im} q_o\, \bar{q}|^2\Big],$$

for some fixed $(q_o, \omega_o) \in \boldsymbol{G}(\mathbb{H})$ and constants $c_0 > 0$ and $\sigma \in \mathbb{R}$. A small calculation using (37) and the Yamabe equation [IMV, (5.8)] shows $Scal_\Theta = 128n(n+2)c_0\sigma = const$. Since η is qc-conformal to Θ via the map F, it follows that $Scal_\eta = const$ on O, which is a contradiction. $\square$

An immediate consequence of Theorem 1.1 and [IMV, Theorem 4.9] is the next

Corollary 3.2. *The vertical space V of a seven dimensional qc-Einstein manifold is integrable.*

We note that the integrability of the vertical distribution of a $4n+3$ dimensional qc-Einstein manifold in the case $n > 1$, and when $S = const$ and $n = 1$ was proven earlier in [IMV, Theorem 4.9]. Thus, in any dimension, the vertical distribution V of a qc-Einstein manifold is integrable and we have

$$\rho_s(X,Y) = -S\omega_s(X,Y), \quad Ric(\xi_s, X) = \rho_s(X,\xi_t) = 0, \quad [\xi_s,\xi_t] \in V. \tag{38}$$

Another Corollary of Theorem 1.1 and the analysis of the corresponding results in the case $n > 1$[IV2] is

Corollary 3.3. *If M is a seven dimensional qc-Einstein manifold then $d\Omega = 0$, where $\Omega = \sum_{s=1}^{3} \omega_s \wedge \omega_s$ is the fundamental 4-form defining the quaternionic structure on the horizontal distribution.*

For details, we refer to the proof of the case $n>1$ in [IV3, Theorem 4.4.2. c)] which is valid in the case $n = 1$, as well, due to Theorem 1.1 and Corollary 3.2. We note that the converse to Corollary 3.3 holds true when $n > 1$, see [IV2], while in the case $n = 1$ a counterexample for the implication was found in [CFS].

4. A characterization based on vertical flat connection

In this section we show that for any qc manifold M there is a natural linear connection $\tilde{\nabla}$, defined on the vertical distribution V, the latter considered as a vector bundle over M. This connection has the remarkable property of being flat exactly when M is qc-Einstein, see Theorem 4.3, and will turn out to be a useful technical tool for the geometry of qc-Einstein manifolds in the sequel.

We start by introducing a cross-product on the vertical space V. Recall that h (2) is the natural extension of the horizontal metric g to a Riemannian metric on M, which induces an inner product, denoted by $\langle .,. \rangle$ here, and an orientation on the vertical distribution V. This allows us to introduce also the cross-product operation $\times : \Lambda^2(V) \to V$ in the standard way: $\xi_i \times \xi_j = \xi_k$, $\xi_i \times \xi_i = 0$. The cross product operation is parallel with respect to any connection on V preserving the inner product $\langle .,. \rangle$, in particular, with respect to the restriction of the Biquard connection ∇ to V. For any $\xi, \xi', \xi'' \in V$, we have the standard relations

$$\begin{aligned}
(\xi \times \xi') \times \xi'' &= \langle \xi, \xi'' \rangle \xi' - \langle \xi', \xi'' \rangle \xi, \\
\xi \times (\xi' \times \xi'') &= (\xi \times \xi') \times \xi'' + \xi' \times (\xi \times \xi''), \\
\nabla_A(\xi \times \xi') &= (\nabla_A \xi) \times \xi' + \xi \times (\nabla_A \xi').
\end{aligned} \tag{39}$$

In the next lemma we collect some formulas, which will be used in the proof of Theorem 4.3.

Lemma 4.1. *The curvature R and torsion T of the Biquard connection ∇ of a qc-Einstein manifold satisfy the following identities*

$$T(\xi,\xi') = -S\xi\times\xi', \quad T(\xi,X)=0, \tag{40}$$
$$R(A,B)\xi = -2S\sum_{s=1}^{3}\omega_s(A,B)\xi_s\times\xi.$$

Proof. The first two identities follow directly from (4) and the integrability of the vertical distribution V, see Corollary 3.2 and the paragraph after it. The last identity follows from (13), (5) and (38). In particular, the three Ricci 2-forms $\rho_s(A,B)$ vanish unless A and B are both horizontal, in which case we have (38). The proof is complete. □

Definition 4.2. We define a connection $\widetilde{\nabla}$ on the vertical vector bundle V of a qc manifold M as follows

$$\widetilde{\nabla}_X\xi = \nabla_X\xi, \qquad \widetilde{\nabla}_\xi\xi' = \nabla_\xi\xi' + S(\xi\times\xi'). \tag{41}$$

The main result of this section is

Theorem 4.3. *A qc manifold M is qc-Einstein iff the connection $\widetilde{\nabla}$ is flat, $R^{\widetilde{\nabla}}=0$.*

Proof. We start by relating the curvature $R^{\widetilde{\nabla}}$ of the connection $\widetilde{\nabla}$, cf. (41), to the curvature of the Biquard connection ∇. To this end, let $L=(\widetilde{\nabla}-\nabla)\in\Gamma(M,T^*M\otimes V^*\otimes V)$ be the difference between the two connections on V. Then (41) implies $L_A\xi = L(A,\xi) = S[A]_V\times\xi$, where $[A]_V$ is the orthogonal projection of A on V. The curvature tensor $R^{\widetilde{\nabla}}$ of the new connection $\widetilde{\nabla}$ is given in terms of R and L by the well known general formula

$$R^{\widetilde{\nabla}}(A,B)\xi = R(A,B)\xi + \big(\nabla_A L\big)(B,\xi) - \big(\nabla_B L\big)(A,\xi) \tag{42}$$
$$+\big[L_A,L_B\big]\xi + L\big(T(A,B),\xi\big).$$

We proceed by considering each of the terms on the right hand side of (42) separately. We have, cf. (9),

$$R(A,B)\xi = \left(\sum_{s=1}^{3}2\rho_s(A,B)\xi_s\right)\times\xi. \tag{43}$$

Using (39) and the obvious identity $\nabla_A\big([B]_V\big) = \big[\nabla_A B\big]_V$ we obtain

$$\begin{aligned}(\nabla_A L)(B,\xi) &= \nabla_A\big(L(B,\xi)\big) - L\big(\nabla_A B,\xi\big) - L\big(B,\nabla_A\xi\big)\\ &= dS(A)[B]_V\times\xi.\end{aligned}\tag{44}$$

From (39) it follows

$$\big[L_A, L_B\big]\xi = \big(L_A\times L_B\big)\times\xi = S^2\big([A]_V\times[B]_V\big)\times\xi.\tag{45}$$

The torsion identities (4) imply

$$\begin{aligned}L\big(T(A,B),\xi\big) &= S\big[T(A,B)\big]_V\times\xi\\ &= S\left(-S[A]_V\times[B]_V + 2\sum_{s=1}^{3}\omega_s(A,B)\xi_s\right)\times\xi.\end{aligned}\tag{46}$$

Finally, a substitution of (43), (44), (45) and (46) in the right hand side of formula (42) gives the equivalent relation

$$\begin{aligned}R^{\widetilde{\nabla}}(A,B)\xi = {}&\left(\sum_{s=1}^{3}2\rho_s(A,B)\xi_s + dS(A)[B]_V - dS(B)[A]_V\right)\times\xi\\ &+2S\left(\sum_{s=1}^{3}\omega_s(A,B)\xi_s\right)\times\xi.\end{aligned}\tag{47}$$

We are now ready to complete the proof of the theorem. Suppose first that M is a qc-Einstein manifold. By Theorem 1.1 when $n=1$ and [IMV] when $n>1$ it follows that the qc-scalar curvature is constant. Lemma 4.1 implies that

$$\sum_{s=1}^{3}\rho_s(A,B)\xi_s = -S\sum_{s=1}^{3}\omega_s(A,B)\xi_s.$$

Since $dS=0$, (47) gives $R^{\widetilde{\nabla}}=0$, and thus $\widetilde{\nabla}$ is a flat connection on V.

Conversely, if $\widetilde{\nabla}$ is flat, then by applying (47) with $(A,B)=(X,Y)$ we obtain $\rho_s(X,Y) = -S\omega_s(X,Y)$. Applying the second formula of (6) we derive $T^0=0$ and $U=0$ by comparing the $Sp(n)Sp(1)$ components of the obtained equalities. Thus, (M,η) is a qc-Einstein manifold taking into account the first formula in (6). $\square$

5. The structure equations of a qc-Einstein manifold

Let M be a qc manifold with normalized qc-scalar curvature S. From [IV2, Proposition 3.1] we have the structure equations

$$
\begin{aligned}
d\eta_i &= 2\omega_i - \eta_j \wedge \alpha_k + \eta_k \wedge \alpha_j - S\eta_j \wedge \eta_k, \\
d\omega_i &= \omega_j \wedge (\alpha_k + S\eta_k) - \omega_k \wedge (\alpha_j + S\eta_j) - \rho_k \wedge \eta_j \\
&\quad + \rho_j \wedge \eta_k + \frac{1}{2} dS \wedge \eta_j \wedge \eta_k,
\end{aligned} \tag{48}
$$

where (η_1, η_2, η_3) is a local $\mathbb{R}^3$-valued 1-form defining the given qc-structure and α_s are the corresponding connection 1-forms. If, locally, there is an $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ defining the given qc-structure, such that, we have the structure equations $d\eta_i = 2\omega_i + S\eta_j \wedge \eta_k$ with $S = const$ or the connection 1-forms vanish on the horizontal space, $\alpha_i|_H = 0$, then M is a qc-Einstein manifold of normalized qc-scalar curvature S, see [IV2, Proposition 3.1] and [IMV, Lemma 4.18].

Conversely, on a qc-Einstein manifold of nowhere vanishing qc-scalar curvature the structure equations (49) hold true by [IV2] and [IV3, Section 4.4.2], taking into account Corollary 6.1. The purpose of this section is to give the corresponding results in the case $Scal = 0$. The proof of Theorem 5.1 which is based on the connection defined in Section 4 rather than the cone over a 3-Sasakian manifold employed in [IV2] and [IV3, Theorem 4.4.4] works also in the case $Scal \neq 0$, thus in the statement of the Theorem we will not make an explicit note of the condition $Scal = 0$.

Theorem 5.1. *Let M be a qc manifold. The following conditions are equivalent:*

a) M is a qc-Einstein manifold;

b) locally, the given qc-structure is defined by 1-form (η_1, η_2, η_3) such that for some constant S we have

$$
d\eta_i = 2\omega_i + S\eta_j \wedge \eta_k; \tag{49}
$$

c) locally, the given qc-structure is defined by 1-form (η_1, η_2, η_3) such that the corresponding connection 1-forms vanish on H, $\alpha_s = -S\eta_s$.

Proof. As explained above, the implication c) $\Rightarrow$ a) is known, while b) $\Rightarrow$ c) is an immediate consequence of (48). Thus, only the implication a) implies b) needs to be proven, see also the paragraph preceding the Theorem.

Assume a) holds. We will show that the structure equation in b) are satisfied. By Theorem 1.1 when $n = 1$ and [IMV] when $n > 1$ it follows M is of constant qc-scalar curvature. Let V be the vertical distribution. Clearly, the connection $\widetilde{\nabla}$ defined in Theorem 4.3 is a flat metric connection on V with respect to the inner product $\langle .,. \rangle$. Therefore the bundle V admits a local orthonormal oriented frame K_1, K_2, K_3 which is $\widetilde{\nabla}$-parallel, i.e., we have

$$\nabla_A K_i = -S[A]_V \times K_i. \tag{50}$$

There exists a triple of local 1-forms (η_1, η_2, η_3) on M vanishing on H, which satisfy $\eta_s(K_t) = \delta_{st}$. We rewrite (50) as

$$\nabla_A K_i = S\big(\eta_j(A)K_k - \eta_k(A)K_j\big). \tag{51}$$

Since K_1, K_2, K_3 is an orthonormal and oriented frame of V, we can complete the dual triple (η_1, η_2, η_3) to one defining the given qc-structure. By differentiating the equalities $\eta_s(K_i) = \delta_{si}$ we obtain using (51) that

$$\begin{aligned} 0 &= \big(\nabla_A \eta_s\big)(K_i) + \eta_s\big(\nabla_A K_i\big) \\ &= \big(\nabla_A \eta_s\big)(K_i) + \eta_s\left(S\big(\eta_j(A)K_k - \eta_k(A)K_j\big)\right) \\ &= \big(\nabla_A \eta_s\big)(K_i) + S\big(\eta_j(A)\delta_{sk} - \eta_k(A)\delta_{sj}\big). \end{aligned}$$

Hence, $\big(\nabla_A \eta_i\big)(B) = S\eta_j \wedge \eta_k(A,B)$, which together with Lemma 4.1 allows the computation of the exterior derivative of η_i,

$$\begin{aligned} d\eta_i(A,B) &= \big(\nabla_A \eta_i\big)(B) - \big(\nabla_B \eta_i\big)(A) + \eta_i\big(T(A,B)\big) \\ &= S\eta_j \wedge \eta_k(A,B) - S\eta_j \wedge \eta_k(B,A) \\ &\quad + \eta_i\left(g - S[A]_V \times [B]_V + 2\sum_s \omega_s(A,B)\xi_s\right) \\ &= \big(2\omega_i + S\eta_j \wedge \eta_k\big)(A,B), \end{aligned} \tag{52}$$

which proves (49). Now $\alpha_s|_H = 0$ shows that K_s satisfy (1) and therefore K_s are the Reeb vector fields, which completes the proof of the Theorem. □

We finish the section with another condition characterizing qc-Einstein manifolds, which is useful in some calculations.

Proposition 5.2. *Let M be a qc manifold. M is qc-Einstein iff for some η compatible with the given qc-structure*

$$d\omega_s(X,Y,Z)=0. \tag{53}$$

Proof. If (49) are satisfied, then we have $0=d(d\eta_i)=d\big(2\omega_i+S\eta_j\wedge\eta_k\big)$, which implies (53).

Conversely, suppose the given qc-structure is locally defined by 1-form (η_1,η_2,η_3) which satisfies (53). By (48) we have $\big(\omega_j\wedge\alpha_k-\omega_k\wedge\alpha_j\big)|_H=0$, which after a contraction with the endomorphism I_i gives

$$\begin{aligned}
0&=(\omega_j\wedge\alpha_k-\omega_k\wedge\alpha_j)(X,e_a,I_ie_a)\\
&=\omega_j(X,e_a)\alpha_k(I_ie_a)+\omega_j(e_a,I_ie_a)\alpha_k(X)+\omega_j(I_ie_a,X)\alpha_k(e_a)\\
&\quad-\omega_k(X,e_a)\alpha_j(I_ie_a)-\omega_k(e_a,I_ie_a)\alpha_j(X)-\omega_k(I_ie_a,X)\alpha_j(e_a)\\
&=2\omega_j(X,e_a)\alpha_k(I_ie_a)-2\omega_k(X,e_a)\alpha_j(I_ie_a)\\
&=2\alpha_k(I_kX)+2\alpha_j(I_jX).
\end{aligned}$$

Since the above calculation is valid for any even permutation (i,j,k), it follows that $\alpha_s(X)=0$ which completes the proof of the Proposition. □

6. The related Riemannian geometry

A (4n + 3)-dimensional (pseudo) Riemannian manifold (M,g) is 3-Sasakian if the cone metric is a (pseudo) hyper-Kähler metric [BG, BGN]. We note explicitly that in this paper 3-Sasakian manifolds are to be understood in the wider sense of positive (the usual terminology) or negative 3-Sasakian structures, cf. [IV2, Section 2] and [IV3, Section 4.4.1] where the "negative" 3-Sasakian term was adopted in the case when the Riemannian cone is hyper-Kähler of signature $(4n,4)$. Every 3-Sasakian manifold is a qc-Einstein manifold of constant qc-scalar curvature, [Biq1], [IMV] and [IV2]. As well known, a positive 3-Sasakian manifold is Einstein with a positive Riemannian scalar curvature [Kas] and, if complete, it is compact with finite fundamental group due to Myers theorem. The negative 3-Sasakian structures are Einstein with respect to the corresponding pseudo-Riemannian metric of signature $(4n,3)$ [Kas, Tan]. In this case, by a simple change of signature, we obtain a positive definite nS metric on M, [Tan, Jel, Kon].

By [IMV, Theorem 1.3] when $Scal>0$, and [IV2] and [IV3, Theorem 4.4.4] when $Scal<0$ a qc-Einstein of dimension at least eleven is locally qc-homothetic to a 3-Sasakian structure. The corresponding result in the seven dimensional case was proven with the extra assumption that the qc-scalar is constant. Thanks to Theorem 1.1 the additional hypothesis is redundant, hence we have the following

Corollary 6.1. *A seven dimensional qc-Einstein manifold of nowhere vanishing qc-scalar curvature is locally qc-homothetic to a 3-Sasakian structure.*

There are many known examples of positive 3-Sasakian manifold, see [BG] and references therein for a nice overview of 3-Sasakian spaces. On the other hand, certian SO(3)-bundles over quaternionic Kähler manifolds with negative scalar curvature constructed in [Kon, Tan, Jel] are examples of negative 3-Sasakian manifolds. Explicit examples of negative 3-Sasakian manifolds on Lie groups are constructed in [AFIV]. It was also shown in [AFIV] that to any qc structure in dimension seven one can associate a globally defined G_2-structure. It is interesting to compare the CR structure on the twistor space over the seven dimensional qc manifold constructed by Biquard [Biq1, D] with the CR twistor space associated with the corresponding G_2 structure described by Verbitsky [Ver].

Complete and regular 3-Sasakian manifolds, resp. nS-structures, fiber over a quaternionic Kähler manifold with positive, resp. negative, scalar curvature [Is, BGN, Tan, Jel] with fiber $SO(3)$. Conversely, a quaternionic Kähler manifold with positive (resp. negative) scalar curvature has a canonical $SO(3)$ principal bundle, the total space of which admits a natural 3-Sasakian (resp. nS-) structure [Is, Kon, Tan, BGN, Jel].

In this section we describe the properties of qc-Einstein structures of zero qc-scalar curvature, which complement the well known results in the 3-Sasakian case. A common feature of the $Scal = 0$ and $Scal \neq 0$ cases is the existence of Killing vector fields.

Lemma 6.2. *Let M be a qc-Einstein manifold with zero qc-scalar curvature. If (η_1, η_2, η_3) is an $\mathbb{R}^3$-valued local 1-form defining the qc structure as in* (49)*, then the corresponding Reeb vector fields ξ_1, ξ_2, ξ_3 are Killing vector fields for the Riemannian metric h, cf.* (2).

Proof. By Theorem 5.1 c) we have $\alpha_i = 0$, hence $\nabla_A \xi_i = 0$ while Lemma 4.1 yields $T(\xi_s, \xi_t) = 0$. Therefore, $[\xi_s, \xi_t] = \nabla_{\xi_s}\xi_t - \nabla_{\xi_t}\xi_s - T(\xi_s, \xi_t) = 0$, which implies for any $i, s, t \in \{1, 2, 3\}$ we have $(\mathcal{L}_{\xi_i} h)(\xi_s, \xi_t) = -h([\xi_i, \xi_s], \xi_t) - h(\xi_s, [\xi_i, \xi_t]) = 0$. Furthermore, using $d\eta_j(\xi_i, X) = \alpha_k(X) = 0$ we compute

$$(\mathcal{L}_{\xi_s} h)(\xi_t, X) = -h(\xi_t, [\xi_s, X]) = d\eta_t(\xi_s, X) = 0.$$

Finally, (3) gives $(\mathcal{L}_{\xi_i}h)(X,Y) = (\mathcal{L}_{\xi_i}g)(X,Y) = 2T^0_{\xi_i}(X,Y) = 0$, which completes the proof. $\square$

6.1. The quotient space of a qc-Einstein manifold with $S = 0$

The total space of an $\mathbb{R}^3$-bundle over a hyper-Kähler manifold with closed and locally exact Kähler forms $2\omega_s = d\eta_s$ with connection 1-forms η_s is a qc-structure determined by the three 1-forms η_s, which is qc-Einstein of vanishing qc-scalar curvature, see [IV2]. In fact, we characterize qc-Einstein manifold with vanishing qc-scalar curvature as $\mathbb{R}^3$-bundle over hyper-Kähler manifold.

Let M be a qc-Einstein manifold. As observed in Corollary 3.2 and the paragraph after it the vertical distribution V is completely integrable hence defines a foliation on M. We recall, taking into account [Pal], that the quotient space $P = M/V$ is a manifold when the foliation is regular and the quotient topology is Hausdorff.

If P is a manifold and all the leaves of V are compact, then by Ehresmann's fibration theorem [Ehr, Pal] it follows that $\Pi : M \to P$ is a locally trivial fibration and all the leaves are isomorphic. By [Pal], examples of such foliations are given by regular foliations on compact manifolds. In the case of a qc-Einstein manifold of non-vanishing qc-scalar curvature, the leaves of the foliation generated by V are Riemannian 3-manifold of positive constant curvature. Hence, if the associated (pseudo) Riemannian metrics on M is complete, then the leaves of the foliation are compact. On the other hand, in the case of vanishing qc-scalar curvature, the leaves of the foliation are flat Riemannian manifolds that may not be compact as is, for example, the case of the quaternionic Heisenberg group. We summarize the properties of the Reeb foliation on a qc-Einstein manifold of vanishing qc-scalar curvature case in the following

Proposition 6.3. *Let M be a qc-Einstein manifold with zero qc-scalar curvature.*

a) If the vertical distribution V is regular and the space of leaves $P = M/V$ with the quotient topology is Hausdorff, then P is a locally hyper-Kähler manifold.

b) If the leaves of the foliation generated by V are compact then there exists an open dense subset $M_o \subset M$ such that $P_o = M_o/V$ is a locally hyper-Kähler manifold.

Proof. We begin with the proof of a). By Theorem 5.1 we can assume, locally, the structure equations given in Theorem 5.1. This, together with [IMV, Lemma 3.2 & Theorem 3.12] imply that the horizontal metric g, see also (3), and the closed local fundamental 2-forms ω_s, see (49) with $S = 0$, are projectable. The claim of part a) follows from Hitchin's lemma [Hit].

We turn to the proof of part b). Lemma 6.2 implies that, in particular, the Riemannian metric h on M is bundle-like, i.e., for any two horizontal vector fields X and Y in the normalizer of $\mathcal{V}$ under the Lie bracket, the equation $\xi h(X,Y) = 0$ holds for any vector field ξ in $\mathcal{V}$. Since all the leaves of the vertical foliation are assumed to be compact, we can apply [Mo, Proposition 3.7], which shows that $P = M/V$ is a 4n-dimensional orbifold. In particular P is a Hausdorff space. The regular points of any orbifold are an open dense set. Thus, if we let P_o to be the set of all regular points of P, then P_o is an open dense subset of P which is also a manifold. It follows that if $M_o = \Pi^{-1}(P_o)$ then all the leaves of the restriction of the vertical foliation to M_o are regular and hence the claim of b) follows. □

6.2. The Riemannian curvature

Let M be a qc-Einstein manifold. Note that, by applying an appropriate qc homothetic transformation, we can aways reduce a general qc-Einstein structure to one whose normalized qc-scalar curvature S equals $0, 2$ or -2. Consider the one-parameter family of (pseudo) Riemannian metrics h^λ, $\lambda \neq 0$ on M by letting $h^\lambda(A,B) = h(A,B) + (\lambda - 1)h|_V$. Let ∇^λ be the Levi-Civita connection of h^λ. Note that h^λ is a positive-definite metric when $\lambda > 0$ and has signature $(4n, 3)$ when $\lambda < 0$.

Let us recall that, if $S = 2$ and $\lambda = 1$ the Riemannian metric $h = h^\lambda$ is a 3-Sasakian metric on M. In particular, it is an Einstein metric of positive Riemannian scalar curvature (4n + 2)(4n + 3) [Kas]. There is also a second Einstein metric, the "squashed" metric, in the family h^λ when $\lambda = 1/(2n+3)$, see [BG]. The case $S = -2$ is completely analogous. Here we have two distinct pseudo-Riemannian Einstein metrics corresponding to $\lambda = -1$ and $\lambda = -1/(2n+3)$. The first one defines a negative 3-Sasaskian structure. On the other hand, the metric h^λ with $\lambda = 1$ (assuming $S = -2$) gives an nS structure on M. In [Tan], it was shown that the Riemannian Ricci tensor of the latter has precisely two constant eigenvalues, $-4n - 14$ (of multiplicity $4n$) and $4n + 2$ (of multiplicity 3), and that the Riemannian scalar curvature is the negative constant $-16n^2 - 44n + 6$. In particular, in this case, (M, h^λ) is an example of an A-manifold in the terminology of [Gr].

The following proposition addresses the case $S = 0$. However, the argument is valid for all values of S and $\lambda \neq 0$. In particular, we obtain new proofs of the above mentioned results concerning the cases of positive and negative 3-Sasakian structures.

Proposition 6.4. *Let M be a qc-Einstein manifold with normalized qc-scalar curvature S. For a vector field A, let $[A]_V$ denote the orthogonal projection of A to the vertical space V. The (pseudo) Riemannian Ricci and scalar curvatures of h^λ are given by*

$$\begin{aligned} Ric^\lambda(A,B) &= \left(4n\lambda + \frac{S^2}{2\lambda}\right) h^\lambda\big([A]_V,[B]_V\big) \\ &\quad + \big(2S(n+2) - 6\lambda\big) h^\lambda\big([A]_H,[B]_H\big), \\ Scal^\lambda &= \frac{1}{\lambda}\left(-12n\lambda^2 + 8n(n+2)S\lambda + \frac{3}{2}S^2\right). \end{aligned} \tag{54}$$

In particular, if $S = 0$, the Ricci curvature of each metric in the family h^λ has exactly two different constant eigenvalues of multiplicities $4n$ and 3 respectively.

Proof. Since the Biquard connection preserves the metric h^λ it is connected with the Levi-Civita connection ∇^λ of the metric h^λ by the general formula $h^\lambda(M(A,B),C) = h^\lambda(\nabla^{h^\lambda}_A B, C) - h^\lambda(\nabla_A B, C)$ where

$$2h^\lambda(M(A,B),C) = -h^\lambda(T(A,B),C) + h^\lambda(T(B,C),A) - h^\lambda(T(C,A),B).$$

Applying the formula for the torsion in Lemma 4.1 and the first equality of (4) we obtain by a straitforward calculations that

$$\begin{aligned} M(A,B) &\equiv \nabla^\lambda_A B - \nabla_A B \\ &= \frac{S}{2}[A]_V \times [B]_V + \sum_{s=1}^{3}\big\{-\omega_s(A,B)\xi_s + \lambda\eta_s(A)I_sB + \lambda\eta_s(B)I_sA\big\}. \end{aligned} \tag{55}$$

It follows from (55) that the tensor M is ∇-parallel, $\nabla M = 0$. Applying the well known formula for the difference $R^\lambda - R$ between the corresponding curvature tensors (c.f.(42)), we obtain after standard computations that

$$
\begin{aligned}
(56)\quad R^{\lambda}(A,B)C &= R(A,B)C + h^{\lambda}\big([B]_V,[C]_V\big)\left(\frac{S^2}{4\lambda}[A]_V + \lambda[A]_H\right)\\
&- h^{\lambda}\big([A]_V,[C]_V\big)\left(\frac{S^2}{4\lambda}[B]_V + \lambda[B]_H\right)\\
&+ \sum_{(i,j,k)-\text{cyclic}} \Bigg\{ \left(\frac{S}{2}-\lambda\right)\eta_k(A)\omega_j(B,C)\\
&- \left(\frac{S}{2}-\lambda\right)\eta_k(B)\omega_j(A,C) - \left(\frac{S}{2}-\lambda\right)\eta_j(A)\omega_k(B,C)\\
&+ \left(\frac{S}{2}-\lambda\right)\eta_j(B)\omega_k(A,C) + (S+2\lambda)\eta_k(C)\omega_j(A,B)\\
&- (S+2\lambda)\eta_j(C)\omega_k(A,B) - \lambda\eta_i(B)h^{\lambda}\big([A]_H,[C]_H\big)\\
&+ \lambda\eta_i(A)h^{\lambda}\big([B]_H,[C]_H\big)\Bigg\}\xi_i\\
&+ \sum_{(i,j,k)-\text{cyclic}} \Bigg\{ \left(\frac{\lambda S}{2}-\lambda^2\right)\eta_j\wedge\eta_k(B,C)I_iA\\
&- \left(\frac{\lambda S}{2}-\lambda^2\right)\eta_j\wedge\eta_k(A,C)I_iB\\
&- (\lambda S-2\lambda^2)\eta_j\wedge\eta_k(A,B)I_iC - \lambda\omega_i(B,C)I_iA\\
&+ \lambda\omega_i(A,C)I_iB + 2\lambda\omega_i(A,B)I_iC\Bigg\}.
\end{aligned}
$$

Taking the trace with respect to A and D in equation (56) and applying the qc-Einstein condition (7) we obtain the first formula in (54). The formula for the scalar curvature follows. □

Acknowledgements

The research was partially supported by the Contract Idei, DID 02-39/21.12. 2009. D.V. was partially supported by Simons Foundation grant #279381. S.I and I.M. are partially supported by the Contract DFNI I02/4/12.12.2014 and the Contract 168/2014 with the University of Sofia 'St. Kl. Ohridski'.

I.M. is supported by a SoMoPro II Fellowship which is co-funded by the European Commission[1] from "People" specific programme (Marie Curie Actions) within the EU Seventh Framework Programme on the basis of the grant agreement REA No. 291782. It is further co-financed by the South-Moravian Region.

References

[AFIV] L. de Andres, M. Fernández, S. Ivanov, S. Joseba, L. Ugarte, and D. Vassilev, *Quaternionic Kaehler and Spin(7) metrics arising from quaternionic contact Einstein structures*, Ann. Mat. Pura Appl. (4) **193** (2014), no. 1, 261–290.

[Biq1] O. Biquard, *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000).

[BG] C. Boyer and K. Galicki, 3-*Sasakian manifolds*, Surveys in differential geometry: essays on Einstein manifolds, 123–184, Surv. Differ. Geom. **VI**, Int. Press, Boston, MA, 1999.

[BGN] C. Boyer, K. Galicki, and B. Mann, *The geometry and topology of* 3-*Sasakian manifolds*, J. Reine Angew. Math. **455** (1994), 183–220.

[CFS] D. Conti, M. Fernández, and J. Santisteban, *On seven dimensional quaternionic contact solvable Lie groups*, Forum Math. **26** (2014), no. 2, 547–576.

[D] D. Duchemin, *Quaternionic contact structures in dimension* 7, Ann. Inst. Fourier, Grenoble **56** (2006), no. 4, 851–885.

[Ehr] C. Ehresmann, *Les connexions infinitsimales dans un espace fibré différentiable*, Colloque de Topologie, Bruxelles (1950), 29–55.

[F2] G. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Math. **13** (1975), 161–207.

[FS] G. Folland and E. Stein, *Estimates for the $\bar\partial_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[Gr] A. Gray, *Einstein manifolds which are not Einstein*, Geom. Dedicata **7** (1978), 259–280.

[1]This article reflects only the author's views and the EU is not liable for any use that may be made of the information contained therein.

[Hit] N. J. Hitchin, *The self duality equations on a Riemannian surface*, Proc. London Math. Soc. **55** (1987), 59–126.

[Is] S. Ishihara, *Quaternion Kahler manifolds and fibred Riemannian spaces with Sasakian* 3*-structure*, Kodai Math. Sem. Rep. **25** (1973), 321–329.

[IK] S. Ishihara and M. Konishi, *Fibred Riemannian spaces with Sasakian* 3*-structure*, Differential geometry, in honor of K. Yano, Kinokuniya, Tokyo (1972), 179–194.

[IMV2] S. Ivanov, I. Minchev, and D. Vassilev, *Extremals for the Sobolev inequality on the seven dimensional quaternionic Heisenberg group and the quaternionic contact Yamabe problem*, J. Eur. Math. Soc. **12** (2010), 1041–1067.

[IMV] S. Ivanov, I. Minchev, and D. Vassilev, *Quaternionic contact Einstein structures and the quaternionic contact Yamabe problem*, Mem. Amer. Math. Soc., Vol. 231, no. 1086, (2014).

[IMV3] S. Ivanov, I. Minchev, and D. Vassilev, *The optimal constant in the L^2 Folland-Stein inequality on the quaternionic Heisenberg group*, Ann. Sc. Norm. Super Pisa Cl. Sci. (5), Vol. XI (2012), 635–652.

[IPV] S. Ivanov, A. Petkov, and D. Vassilev, *The sharp lower bound of the first eigenvalue of the sub-Laplacian on a quaternionic contact manifold*, J. Geom. Anal. **24** (2014), no. 2, 756–778.

[IV1] S. Ivanov and D. Vassilev, *Conformal quaternionic contact curvature and the local sphere theorem*, J. Math. Pure Appl **93** (2010), 277–307.

[IV2] S. Ivanov and D. Vassilev, *Quaternionic contact manifolds with a closed fundamental* 4*-form*, Bull. London Math. Soc. **42** (2010), 1021–1030.

[IV3] S. Ivanov and D. Vassilev, *Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. xviii+219 pp.

[Jel] W. Jelonek, *Positive and negative* 3*-K-contact structures*, Proc. Amer. Math. Soc. **129** (2000), 247–256.

[Kas] T. Kashiwada, *A note on Riemannian space with Sasakian* 3*-structure*, Nat. Sci. Reps. Ochanomizu Univ. **22** (1971), 1–2.

[Kon] M. Konishi, *On manifolds with Sasakian 3-structure over quaternion Kähler manifolds*, Kodai Math. Sem. Rep. **26** (1975), 194–200.

[Kul] R. S. Kulkarni, *Curvature structures and conformal transformations*, Bull. Amer. Math. Soc. **75** (1969), 91–94.

[LeB91] C. LeBrun, *On complete quaternionic-Kähler manifolds*, Duke Math. J. **63** (1991), no. 3, 723–743.

[M] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973. v+195 pp.

[P] P. Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. (2) **129** (1989), no. 1, 1–60.

[Pal] R. Palais, *A global formulation of the Lie theory of transformation groups* , Mem. Amer. Math. Soc. **22** (1957).

[Mo] P. Molino, *Riemannian Foliations*, Birkhäuser, Boston, (1988).

[Tan] S. Tanno, *Remarks on a triple of K-contact structures*, Tohoku Math. J. **48** (1996), 519–531

[Ver] M. Verbitsky, *A CR twistor space of a G2-manifold*, Diff. Geom. Appl. **29** (2011), no. 1, 101–107.

Faculty of Mathematics and Informatics, University of Sofia
blvd. James Bourchier 5, 1164, Sofia, Bulgaria
& Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
E-mail address: `ivanovsp@fmi.uni-sofia.bg`

Faculty of Mathematics and Informatics, University of Sofia
blvd. James Bourchier 5, 1164 Sofia, Bulgaria
& Department of Mathematics and Statistics, Masaryk University
Kotlarska 2, 61137 Brno, Czech Republic
E-mail address: `minchev@fmi.uni-sofia.bg`

Department of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico, 87131-0001, USA
E-mail address: `vassilev@unm.edu`

Received July 19, 2014