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S^1-equivariant local index and transverse index for non-compact symplectic manifolds

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We define an S^1-equivariant index for non-compact symplectic manifolds with Hamiltonian S^1-action. We use the perturbation by Dirac-type operator along the S^1-orbits. We give a formulation and a proof of quantization conjecture for this S^1-equivariant index. We also give comments on the relation between our S^1-equivariant index and the index of transverse elliptic operators.

1. Introduction

In [4], Furuta, Yoshida and the author gave a formulation of index theory of Dirac-type operator on open manifolds using torus fibration and the perturbation by Dirac-type operator along fibers. In [5] a refinement of it for a family of torus bundles with some compatibility conditions was given. In [6] the authors used equivariant version of them to give a geometric proof of quantization conjecture for Hamiltonian torus action on closed symplectic manifolds. In this paper we give a formulation of S^1-equivariant index theory for non-compact symplectic manifold with Hamiltonian S^1-action based on the framework of [4]. The resulting index is a homomorphism from $R(S^1)$ to $\mathbb{Z}$, and if the manifold is closed, then the S^1-equivariant index coincides with the Riemann-Roch character as a functional on $R(S^1)$.

We use a perturbation by the Dirac-type operator along S^1-orbits. On the other hand Braverman [2] gave an index theory on open manifolds based on a perturbation by the vector field induced from certain equivariant map, e.g., moment map. His index theory realizes the index of transverse elliptic

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operators developed by Atiyah [1] and Paradan-Vergne [12]. Both our construction and Braverman's construction use perturbation by operators along the orbits, and hence, they have conceptual similarity. We show that they are equal for the proper moment map case. We also show that they have different nature. In fact we will give an example in this paper which shows the difference.

Yoshida [14] gave an expository article on the S^1-equivariant version of the index theory developed in [4–6]. The equivariant local index considered in [14] is a straightforward generalization for the equivariant setting. Then the index in [14] is a finite dimensional object. On the other hand the transverse index in [2], [8, 9], [11] and [13] has infinite dimensional nature. In particular the index in [14] does not coincide with the transverse index. The equivariant index in the present paper is an another kind of generalization of [4] for the equivariant setting, which has an infinite dimensional nature.

In our construction it is straightforward to give a formulation and a proof of quantization conjecture for non-compact symplectic manifolds with Hamiltonian S^1-action. Vergne [13] proposed a quantization conjecture for non-compact symplectic manifolds, which was proved by Ma-Zhang [8, 9] and Paradan gave a new proof in [11]. Vergne's conjecture is based on the index theory of transverse elliptic operators. Ma and Zhang showed Vergne's conjecture under weaker assumption, the properness of the moment map, using Braverman's index theory. We do not assume neither compactness of the fixed point set nor properness of the moment map as in [8, 9], [11] and [13]. We only assume that the inverse image of each integer point is compact.

This paper is organized as follows. In Section 2, we give a brief review of the construction in [4] to define an S^1-equivariant index $\mathrm{ind}_{S^1}(X, V)$. In Section 3, we apply the construction in Section 2 to the symplectic geometry case. We define an S^1-equivariant local Riemann-Roch number $RR_{S^1,\mathrm{loc}}(M, L)$ for Hamiltonian S^1-action on non-compact symplectic manifold. In Section 4 we give a quantization conjecture for non-compact symplectic manifold with a Hamiltonian S^1-action. In Section 5, we give comments on relation between our equivariant index and that developed by Braverman [2] and Ma-Zhang [9]. In Appendix A we give some details of the explicit computation of the kernel of a family of perturbed Dirac operators on the cylinder. The family contains the perturbations in this paper and that in [2], and it shows the difference between two equivariant indices.

1.1. Notations

- For each $n \in \mathbb{Z}$ let $\mathbb{C}_{(n)}$ be the complex line with the standard action of the circle group S^1 of weight n.
- Let ρ be a representation space of S^1. For each $n \in \mathbb{Z}$ we denote by $\rho^{(n)}$ the multiplicity of the weight n representation in ρ i.e., we put

$$\rho^{(n)} := \dim\left(\mathrm{Hom}_{S^1}(\mathbb{C}_{(n)}, \rho)\right).$$

  We will also use the same notation for elements in the representation ring $R(S^1)$.

2. Definition of the S^1-equivariant index

In this section we give a brief review of the construction in [4] to define an index $\mathrm{ind}_{S^1}(X, V)$.

2.1. Setting

Let X be a non-compact Riemannian manifold. Let $W = W^+ \oplus W^-$ be a $\mathbb{Z}/2$-graded $Cl(TX)$-module bundle with the Clifford multiplication c. Suppose that the circle group S^1 acts on X in an isometric way and the action lifts to W so that it commutes with c. We assume that there exists an open subset V of X which satisfies the following assumption.

Assumption 2.1. 1) The complement $X \smallsetminus V$ is compact.

2) S^1 acts on V without fixed points.

3) There exists an S^1-equivariant formally self-adjoint operator $D_{S^1} : \Gamma(W|_V) \to \Gamma(W|_V)$ which satisfies the following conditions.
   a) D_{S^1} contains only the derivatives along the S^1-orbits, and its restriction to each orbit is a Dirac-type operator along the orbit.
   b) For each tangent vector u which is normal to the S^1-orbit, D_{S^1} anti-commutes with the Clifford multiplication of $\widetilde{u}$:

$$c(\widetilde{u}) \circ D_{S^1} + D_{S^1} \circ c(\widetilde{u}) = 0,$$

   where $\widetilde{u}$ is the vector field along the S^1-orbit which is obtained by u and the S^1-action.

4) For all $x \in V$ the kernel of the restriction of D_{S^1} to the orbit $S^1 \cdot x$ is trivial, i.e., $\ker(D_{S^1}|_{S^1 \cdot x}) = 0$.

If the data (X, V, W, D_{S^1}) satisfies these conditions, then we call (X, V, W, D_{S^1}) *acyclic*.

2.2. Definition of $\mathrm{ind}_{S^1}(X, V)$

Following the procedure as in [4], we can define an S^1-equivariant index $\mathrm{ind}_{S^1}(X, V) \in R(S^1)$ for the acyclic data (X, V, W, D_{S^1}). Let us recall the definition. By using an S^1-invariant proper function we can deform the end V of X into a complete Riemannian manifold $\hat{X}$, for instance, with cylindrical end $\hat{V}$ in an S^1-equivariant way. Namely let $f : X \to \mathbb{R}$ be an S^1-invariant smooth function and c its regular value such that $f^{-1}((-\infty, c])$ is compact and contains $X \smallsetminus V$. Then $\hat{X}$ is obtained by attaching the cylinder $f^{-1}(c) \times \mathbb{R}_+$ to $f^{-1}((-\infty, c])$. We also deform W and D_{S^1} into $\hat{W}$ and $\hat{D}_{S^1}$, which have translational invariance on the end $\hat{V}$. Let D be an S^1-equivariant Dirac-type operator on $\Gamma(\hat{W})$ which is translationally invariant on $\hat{V}$. Let ρ_V be an S^1-invariant cut-off function such that $\rho_V = 0$ on $X \smallsetminus V$ and $\rho_V = 1$ on the end of $\hat{X}$. For $t \geq 0$ we consider an analog of Witten's deformation $D_t := D + t\rho_V \hat{D}_{S^1}$. We can show that it gives a Fredholm operator on $L^2(\hat{W})$ for any $t \gg 1$. One of the key is the following estimate. See also [4, Lemma 5.2 and Lemma 5.4].

Lemma 2.2. *There exists a positive constant T such that for any $t > T$ we have*

$$\|D_t s\|_{L^2} \geq \|s\|_{L^2}$$

for any compactly supported section s of $\hat{W}$ whose support is contained in $\hat{V}$.

Proof. Let s be a compactly supported section s of $\hat{W}$ whose support is contained in $\hat{V}$. By integration by parts we have

$$\begin{aligned}
\|D_t s\|_{L^2}^2 &= \int_{\hat{V}} (D_t s, D_t s) = \int_{\hat{V}} ({D_t}^2 s, s) \\
&= \int_{\hat{V}} (D^2 s, s) + t \int_{\hat{V}} ((D\hat{D}_{S^1} + \hat{D}_{S^1} D)s, s) + t^2 \int_{\hat{V}} ((\hat{D}_{S^1})^2 s, s) \\
&\geq t \int_{\hat{V}} ((D\hat{D}_{S^1} + \hat{D}_{S^1} D)s, s) + t^2 \int_{\hat{V}} ((\hat{D}_{S^1})^2 s, s).
\end{aligned}$$

The anti-commutativity (Assumption 2.1(3)-(b)) implies that the anti-commutator $D\hat{D}_{S^1} + \hat{D}_{S^1} D$ is a differential operator along the S^1-orbits.

(See [4, Lemma 5.10].) By Assumption 2.1(4) and a priori estimate, there exists a positive constant C_1 and C_2 such that

$$C_1 \int_{\text{orbit}} ((\hat{D}_{S^1})^2 s, s) \geq \left| \int_{\text{orbit}} ((D\hat{D}_{S^1} + \hat{D}_{S^1} D)s, s) \right|$$

and

$$C_2 \int_{\text{orbit}} ((\hat{D}_{S^1})^2 s, s) \geq \int_{\text{orbit}} (s, s)$$

for all S^1-orbits in $\hat{V}$. Note that since $\hat{V}$ has cylindrical end we may assume that C_1 and C_2 do not depend on the choice of orbits. Then we have

$$\|D_t s\|_{L^2}^2 \geq (t^2 - C_1 t) \int_{\hat{V}} ((\hat{D}_{S^1})^2 s, s) \geq C_2^{-1}(t^2 - C_1 t)\|s\|_{L^2}^2,$$

and hence, by taking a positive number T as $T^2 - C_1 T \geq C_2$ and $T \geq C_1/2$, we obtain the required inequality. □

Remark 2.3. Note that since the principal symbol of D_t is given by a combination of the Clifford action, it has finite propagation speed. It is well-known that the finite propagation speed implies the essentially self-adjointness. See [3] for example. In particular the inequality in Lemma 2.2 holds for any L^2-sections of $\hat{W}$ whose supports are contained in $\hat{V}$, and if $\hat{X} = \hat{V}$, then we have the vanishing of the space of L^2-solutions, i.e., $\ker_{L^2} D_t = \{0\}$.

Since D, ρ_V and $\hat{D}_{S^1}$ are S^1-equivariant S^1 acts on the space of L^2-solutions of $D_t s = 0$, we can define the Fredholm index

$$\operatorname{ind}_{S^1}(\hat{X}, \hat{V}) := \operatorname{ind}(D_t) = \ker(D_t|_{L^2(W^+)}) - \ker(D_t|_{L^2(W^-)})$$

as an element of the equivariant K-group $K_{S^1}(\mathrm{pt}) = R(S^1)$. The index is invariant under the continuous deformation of the given data, it does not depend on the choice of $t \gg 1$. Moreover we have the following.

Proposition 2.4. *The Fredholm index of D_t does not depend on the choice of the completion $\hat{X}$ of X.*

Proof. Suppose that there are two completions $\hat{X}_1$ and $\hat{X}_2$. Namely for $i = 1, 2$ let $f_i : X \to \mathbb{R}$ be an S^1-invariant smooth function and c_i its regular value such that $f_i^{-1}((-\infty, c_i])$ is compact and contains $X \setminus V$. Then $\hat{X}_i$ is obtained by attaching the cylinder $f_i^{-1}(c_i) \times \mathbb{R}_+$ to $f_i^{-1}((-\infty, c_i])$ whose end is $\hat{V}_i := (V \cap f_i^{-1}((-\infty, c_i])) \cup f_i^{-1}(c_i) \times \mathbb{R}_+$. It is enough to show for the case $f_1^{-1}((-\infty, c_1]) \subset f_2^{-1}((-\infty, c_2])$. In this case, by the gluing formula of indices ([5, Lemma 4.8]), the index of $\hat{X}_2$ is the sum of the indices of $\hat{X}_1$ and the completion $\hat{X}_{12}$ of $f_2^{-1}((-\infty, c_2]) \smallsetminus f_1^{-1}((-\infty, c_1))$ with an end $\hat{V}_{12}$:

$$\mathrm{ind}(\hat{X}_2, \hat{V}_2) = \mathrm{ind}(\hat{X}_1, \hat{V}_1) + \mathrm{ind}(\hat{X}_{12}, \hat{V}_{12}).$$

Since $f_2^{-1}((-\infty, c_2]) \smallsetminus f_1^{-1}((-\infty, c_1))$ is contained in V, we may take $\hat{V}_{12} = \hat{X}_{12}$. Then the space of L^2-solutions on $\hat{X}_{12}$ vanishes by Remark 2.3. It implies that $\mathrm{ind}(\hat{X}_{12}, \hat{V}_{12}) = 0$ and $\mathrm{ind}(\hat{X}_2, \hat{V}_2) = \mathrm{ind}(\hat{X}_1, \hat{V}_1)$. □

Definition 2.5. Let T be the positive constant as in Lemma 2.2. We define the equivariant index $\mathrm{ind}_{S^1}(X, V) \in K_{S^1}(\mathrm{pt}) = R(S^1)$ to be the S^1-equivariant Fredholm index of D_t on a completion $(\hat{X}, \hat{V})$ for any $t > T$,

$$\mathrm{ind}_{S^1}(X, V) := \mathrm{ind}_{S^1}(\hat{X}, \hat{V}) = \mathrm{ind}_{S^1}(D_t).$$

The equivariant index $\mathrm{ind}_{S^1}(X, V)$ satisfies the excision formula, gluing formula and product formula.

3. S^1-equivariant local index for symplectic manifolds with Hamiltonian S^1-action

Let (M, ω) be a (possibly non-compact) symplectic manifold with a pre-quantizing line bundle (L, ∇), i.e., L is a Hermitian line bundle over M and ∇ is its Hermitian connection whose curvature form is equal to $-\sqrt{-1}\omega$. Suppose that the circle group S^1 acts on (M, ω) and the action lifts to (L, ∇). Note that for each $x \in M$ the restriction $(L, \nabla)|_{S^1 \cdot x}$ is a flat line bundle. Let $\mu : M \to \mathbb{R}$ be the associated moment map. We assume the following compactness.

Assumption. For each $n \in \mathbb{Z}$, the inverse image $\mu^{-1}(n)$ is a compact subset.

Take and fix an S^1-invariant ω-compatible almost complex structure J on M so that we have the associated metric g^J and $\mathbb{Z}/2$-graded $Cl(TM)$-module bundle $W_L := \wedge^\bullet T^*M^{0,1} \otimes L$. We put V to be the complement of the fixed point set, $V := M \smallsetminus M^{S^1}$. Let $T_{S^1}V \to V$ be the tangent bundle along the S^1-orbits, which is, by definition, a real line bundle over V. Let E be the orthogonal complement of $T_{S^1}V \oplus J(T_{S^1}V) \cong T_{S^1}V \otimes \mathbb{C}$ in $TM|_V$. Note that $J(T_{S^1}V) \oplus E$ is the normal bundle of $T_{S^1}V$ and isomorphic to the pull-back of the tangent bundle of the quotient space V/S^1. We have isomorphisms as

Hermitian vector bundles

$$\wedge^\bullet T^*M^{0,1}|_V \cong \wedge^\bullet TM|_V \cong (\wedge^\bullet T_{S^1}V \otimes \mathbb{C}) \otimes (\wedge^\bullet E), \tag{3.1}$$

and hence, we have

$$W_L|_V \cong W_{S^1,L} \otimes (\wedge^\bullet E),$$

where $W_{S^1,L} := \wedge^\bullet T_{S^1}V \otimes L|_V \cong (\wedge^\bullet T_{S^1}V)^* \otimes L|_V$ is the family of Clifford module bundle over the S^1-orbits. Let $\bar{D}_{S^1} : \Gamma(W_{S^1,L}) \to \Gamma(W_{S^1,L})$ be the family of twisted de Rham operators along S^1-orbits with coefficients in $(L,\nabla)|_V$. Namely $\bar{D}_{S^1}$ is the following degree-one differential operator of order-one.

- $\bar{D}_{S^1}$ does not contain any differentials transverse to the S^1-orbits.
- For each $x \in V$ the restriction of $\bar{D}_{S^1}$ to the orbit $S^1 \cdot x$ is the de Rham operator acting on $W_{S^1,L}|_{S^1\cdot x}$, the Clifford module bundle over $S^1 \cdot x$ with coefficients in the flat line bundle $(L,\nabla)|_{S^1\cdot x}$.

Since for each $x \in V$ the restriction $E|_{S^1\cdot x}$ has canonical flat structure induced by the S^1-action, $\bar{D}_{S^1}$ naturally induces a differential operator along the orbits $D_{S^1} : \Gamma(W_L|_V) \to \Gamma(W_L|_V)$. Note that for each $x \in V$ we have

$$\ker(D_{S^1}|_{S^1\cdot x}) = \ker(\bar{D}_{S^1}|_{S^1\cdot x}) \otimes \wedge^\bullet E|_{S^1\cdot x} \cong H^*(S^1\cdot x, L|_{S^1\cdot x}) \otimes \wedge^\bullet E|_{S^1\cdot x}$$

by Hodge theory, where $H^*(S^1 \cdot x, L|_{S^1\cdot x})$ is the de Rham cohomology with coefficients in the flat line bundle $(L,\nabla)|_{S^1\cdot x}$ over the orbit. We first recall some basic properties.

Lemma 3.1. 1) *For each $x \in V$, the space of global parallel sections $H^0(S^1 \cdot x, L|_{S^1\cdot x})$ vanishes if and only if* $\ker(D_{S^1}|_{S^1\cdot x})$ *vanishes.*

2) *For each $x \in V$ and $n \in \mathbb{Z}$, the multiplicity $H^0(S^1 \cdot x, L|_{S^1\cdot x})^{(n)}$ vanishes if and only if the multiplicity* $\ker(D_{S^1}|_{S^1\cdot x})^{(n)}$ *vanishes.*

3) *If $H^0(S^1 \cdot x, L|_{S^1\cdot x}) \neq 0$, then we have $\mu(x) \in \mathbb{Z}$. In particular we have $\mu(M^{S^1}) \subset \mathbb{Z}$.*

4) *If $H^0(S^1 \cdot x, L|_{S^1\cdot x}) \neq 0$, then we have $H^0(S^1 \cdot x, L|_{S^1\cdot x}) = \mathbb{C}_{(\mu(x))}$. In particular if $x \in M^{S^1}$, then we have $L_x = \mathbb{C}_{(\mu(x))}$.*

We take an S^1-invariant relatively compact open neighborhood $X_{\mu,n}$ of the compact set $\mu^{-1}(n)$ as $\mu^{-1}(n) \subset X_{\mu,n} \subset \mu^{-1}([n-1/2, n+1/2])$. We put $V_{\mu,n} := X_{\mu,n} \smallsetminus \mu^{-1}(n)$.

Proposition 3.2. *$(X_{\mu,n}, V_{\mu,n}, W_L|_{V_{\mu,n}}, D_{S^1}|_{V_{\mu,n}})$ is acyclic.*

Proof. Since $\mu(V_{\mu,n}) \cap \mathbb{Z} = \emptyset$ we have $(V_{\mu,n})^{S^1} = \emptyset$ and $\ker(D_{S^1}|_{S^1\cdot x}) = 0$ for all $x \in V_{\mu,n}$ by (1) and (3) in Lemma 3.1. For each normal tangent vector $u = u_1 + u_2 \in J(T_{S^1}V)_x \oplus E_x$ to the orbit $S^1 \cdot x$, let $S^1 \cdot u = S^1 \cdot u_1 + S^1 \cdot u_2$ be the induced vector field along the orbit. Note that the vector field $J(S^1 \cdot u_1)$ is parallel with respect to the Levi-Civita connection of the induced metric on $S^1 \cdot x$ and the Clifford multiplication of $S^1 \cdot u_1$ on $W|_{S^1\cdot x}$ is identified with that of $J(S^1 \cdot u_1)$ on $T(S^1 \cdot x)$ under the identification (3.1). Since $S^1 \cdot u_2$ is normal to $S^1 \cdot x$ and the restriction $D_{S^1}|_{S^1\cdot x}$ is the de Rham operator with coefficient in $(L, \nabla)|_{S^1\cdot x} \otimes E|_{S^1\cdot x}$, it anti-commutes with the Clifford multiplication of $S^1 \cdot u$. □

We can define the index $\mathrm{ind}_{S^1}(X_{\mu,n}, V_{\mu,n}) \in R(S^1)$ as in Definition 2.5 by applying the construction in Section 2.

Proposition 3.3. *The index $\mathrm{ind}_{S^1}(X_{\mu,n}, V_{\mu,n})$ does not depend on the choice of $X_{\mu,n}$.*

Proof. Suppose that we take two relatively compact neighborhoods $X_{\mu,n}$ and $X'_{\mu,n}$ of $\mu^{-1}(n)$ with the required properties. It is enough to show that if $X_{\mu,n} \subset X'_{\mu,n}$, then we have $\mathrm{ind}_{S^1}(X_{\mu,n}, V_{\mu,n}) = \mathrm{ind}_{S^1}(X'_{\mu,n}, V'_{\mu,n})$. The equality follows from the excision formula of $\mathrm{ind}_{S^1}(\cdot,\cdot)$. □

Definition 3.4. For each $n \in \mathbb{Z}$, we put $RR^{(n)}_{S^1,\mathrm{loc}}(M, L) := \mathrm{ind}_{S^1}(X_{\mu,n}, V_{\mu,n})^{(n)} \in \mathbb{Z}$ and define $RR_{S^1,\mathrm{loc}}(M, L) \in \mathrm{Hom}(R(S^1), \mathbb{Z})$ by putting

$$RR_{S^1,\mathrm{loc}}(M, L) : \mathbb{C}_{(n)} \mapsto RR^{(n)}_{S^1,\mathrm{loc}}(M, L).$$

We call $RR_{S^1,\mathrm{loc}}(M, L)$ the S^1*-equivariant local Riemann-Roch number.*

Remark 3.5. In contrast with the equivariant index considered in [14], the S^1-equivariant local Riemann-Roch number $RR_{S^1,\mathrm{loc}}$ can be defined even if $\mu(M)$ contains infinitely many integral points. For such case the number $RR^{(n)}_{S^1,\mathrm{loc}}(M, L)$ may be non-zero for infinitely many n. We do not know whether S^1-equivariant local Riemann-Roch number $RR_{S^1,\mathrm{loc}}(M, L)$ has distributional nature or not.

Remark 3.6. Using the equivariant version of the *acyclic compatible systems* in [5] it would be possible to define the equivariant local index $RR^{(\xi)}_{G,\mathrm{loc}}(M, L)$ and $RR_{G,loc}(M, L)$ for any compact torus G and ξ in the weight lattice of G.

Suppose that M is closed, i.e., compact manifold without boundary. For the S^1-equivariant data (M,ω,L,∇), the S^1-equivariant Riemann-Roch number $RR_{S^1}(M,L)$ is defined as the index of the S^1-equivariant spin^c Dirac operator twisted by L.

Theorem 3.7. *If M is a closed symplectic manifold, then we have*

$$RR_{S^1,\mathrm{loc}}(M,L)=RR_{S^1}(M,L),$$

where the right hand side is regarded as a functional on $R(S^1)$,

$$RR_{S^1}(M,L):\mathbb{C}_{(n)}\mapsto RR_{S^1}(M,L)^{(n)}.$$

Proof. We show $RR^{(n)}_{S^1,\mathrm{loc}}(M,L)=RR_{S^1}(M,L)^{(n)}$ for each $n\in\mathbb{Z}$. By (2) and (4) in Lemma 3.1 we have $\ker(D_{S^1}|_{S^1\cdot x})^{(n)}=H^0(S^1\cdot x,(L,\nabla)|_{S^1\cdot x})^{(n)}=0$, for each $x\notin X_{\mu,n}$, and hence, by *shifting trick* and the localization theorem for S^1-acyclic compatible system ([6, Theorem 2.41]), we have

$$\begin{aligned}RR_{S^1}(M,L)^{(n)}&=RR_{S^1}(M,L\otimes\mathbb{C}_{(-n)})^{(0)}\\&=\mathrm{ind}_{S^1}(X_{\mu-n,0},V_{\mu-n,0},W_L\otimes\mathbb{C}_{(-n)}|_{X_{\mu-n,0}})^{(0)}\\&\quad+\sum_{k\neq0}\mathrm{ind}_{S^1}(X'_{\mu-n,k},V'_{\mu-n,k},W_L\otimes\mathbb{C}_{(-n)}|_{X'_{\mu-n,k}})^{(0)},\end{aligned}$$

where $X'_{\mu-n,k}$ is an S^1-invariant relatively compact open neighborhood of $(\mu-n)^{-1}(k)\cap M^{S^1}=\mu^{-1}(n+k)\cap M^{S^1}$ and we put $V'_k:=X'_k\smallsetminus\mu^{-1}(k)\cap M^{S^1}$. On the other hand we have $L_x=\mathbb{C}_{(k)}$ for each $x\in\mu^{-1}(k)\cap M^{S^1}$ by (4) in Lemma 3.1. We can apply the vanishing theorem ([6, Theorem 4.1]) and we have $\mathrm{ind}_{S^1}(X'_{\mu-n,k},V'_{\mu-n,k},W_L\otimes\mathbb{C}_{(-n)}|_{X'_{\mu-n,k}})^{(0)}=0$ for all $k\neq0$. Note that we may assume that $X_{\mu-n,0}=X_{\mu,n}$ and $V_{\mu-n,0}=V_{\mu,n}$. So we have

$$\begin{aligned}RR_{S^1}(M,L)^{(n)}&=\mathrm{ind}_{S^1}(X_{\mu,n},V_{\mu,n},W_L\otimes\mathbb{C}_{(-n)}|_{X_{\mu,n}})^{(0)}\\&=RR^{(0)}_{S^1,\mathrm{loc}}(M,L\otimes\mathbb{C}_{(-n)})=RR^{(n)}_{S^1,\mathrm{loc}}(M,L).\end{aligned}$$ □

4. Quantization conjecture for $RR_{S^1,\mathrm{loc}}$

Let $(M,\omega),(L,\nabla)$ and μ be the data as in Section 3. Namely (M,ω) is a symplectic manifold and (L,∇) is a pre-quantizing line bundle with Hamiltonian S^1-action whose moment map is μ. We assume that $\mu^{-1}(n)$ is a compact subset for each $n\in\mathbb{Z}$. Suppose that an integer n is a regular value of μ. Then we have a closed symplectic orbifold $M_{(n)} := \mu^{-1}(n)/S^1$ with the pre-quantizing line bundle $L_{(n)} := (L \otimes \mathbb{C}_{(-n)}, \nabla)|_{\mu^{-1}(n)}/S^1$. One can define the Riemann-Roch number $RR(M_{(n)}, L_{(n)})$ of the pre-quantized symplectic orbifold $M_{(n)}$ as the index of the spin^c Dirac operator twisted by $L_{(n)}$.

Theorem 4.1. *If an integer $n \in \mathbb{Z}$ is a regular value of μ, then we have*

$$RR^{(n)}_{S^1,\mathrm{loc}}(M, L) = RR(M_{(n)}, L_{(n)}).$$

Proof. By the excision formula, the index $RR^{(n)}_{S^1,loc}(M, L) = \mathrm{ind}_{S^1}(X_{\mu,n}, V_{\mu,n}, W_L|_{X_{\mu,n}})^{(n)}$ is localized at any neighborhood of $\mu^{-1}(n)$. On the other hand by the normal form theorem (e.g., [6, Proposition 5.11]) we may assume that the neighborhood has the form $\mu^{-1}(n) \times_{S^1} T^*S^1$. Since the S^1-invariant part of the index of $T^*S^1 = S^1 \times \mathbb{R}$ with the standard structure is equal to 1, we have

$$\begin{aligned} RR^{(n)}_{S^1}(M, L) &= \mathrm{ind}_{S^1}(X_{\mu,n}, V_{\mu,n}, W_L|_{X_{\mu,n}})^{(n)} \\ &= \mathrm{ind}(\mu^{-1}(n)/S^1, W_{L_{(n)}}) = RR(M_{(n)}, L_{(n)}) \end{aligned}$$

by the product formula. □

Remark 4.2. Kirwan [7] and Meinrenken-Sjamaar [10] gave definitions of $RR(M_{(n)}, L_{(n)})$ for a critical value n of μ. We do not understand relation between them and $RR^{(n)}_{S^1,\mathrm{loc}}(M, L)$.

5. Relation with the transverse index

Vergne [13] gave a formulation of quantization conjecture for non-compact symplectic manifolds with Hamiltonian action of a general compact Lie group G, in which the compactness of the zero set of the induced vector field (Kirwan vector filed) is assumed. Her conjecture concerns with the *transverse index* which was defined by Atiyah [1] and studied by Paradan-Vergne [12]. Her conjecture was proved by Ma-Zhang [8, 9] and Paradan gave a new proof in [11]. In [9] they defined an equivariant index $Q(L) : R(G) \to \mathbb{Z}$ under the weaker assumption, the properness of the moment map, and showed that the quantization conjecture for $Q(L)$. Namely for each irreducible representation ρ of G, the number $Q(L)(\rho)$ is equal to the Riemann-Roch number of the symplectic quotient. They used the index theorem due to Braverman [2]. He showed that a perturbation of Dirac operator gives an analytic realization of the transverse index $\chi_G(M) = \chi_G(M, \mu)$. The perturbation term is the Clifford action of the induced vector field. If the induced vector field has compact zero set, then the equivariant index $Q(L)$ is equal to the transverse index $\chi_G(M)$. Since both equivariant indices $Q(L)$ and $RR_{S^1,\mathrm{loc}}(M,L)$ satisfy the quantization conjecture, we have the following.

Proposition 5.1. *Let (M,ω,L,∇) be a pre-quantized symplectic manifold equipped with a Hamiltonian S^1-action. If the moment map μ is proper and an integer n is a regular value of μ, then we have*

$$Q(L)(\mathbb{C}_{(n)}) = \chi_{S^1}(M,\mu)(\mathbb{C}_{(n)}) = RR^{(n)}_{S^1,\mathrm{loc}}(M,L).$$

In particular if μ is proper and it does not have any critical points, then we have

$$Q(L) = \chi_{S^1}(M,\mu) = RR_{S^1,\mathrm{loc}}(M,L)$$

as functionals on $R(S^1)$.

We do not know any direct proof of the second equality which does not use the quantization conjecture. On the other hand the following example implies that our equivariant index $RR_{S^1,\mathrm{loc}}(M,L)$ has different behaviour from the transverse index.

Example 5.2. Let m be a non-zero integer and M the product of the circle S^1 and a small interval centered at m. Consider the standard metric and the symplectic structure on M. Let L be the trivial complex line bundle over M which is equipped with a structure of pre-quantizing line bundle over M. Consider the natural S^1-action on M, and we take its lift to L so that S^1 acts trivially on the fiber direction. One has the associated moment map μ which is equal to the projection to the interval factor. Since m is non-zero μ does not have neither critical points nor zeros, and hence, the associated vector field μ^M on M does not vanish. Then [2, Lemma 3.12] implies that the associated transverse index $\chi_{S^1}(M,\mu)$ vanishes. (In fact one can check that the kernel of the perturbation of the Dirac operator by μ^M vanishes by the direct computation.) On the other hand one can check that the kernel of the perturbation by D_{S^1} is one dimensional and it is isomorphic to $\mathbb{C}_{(n)}$, hence, we have $RR^{(n)}_{S^1,\mathrm{loc}}(M,L) = \delta_{mn}$. In particular we have $RR_{S^1,loc}(M,L) \neq \chi_{S^1}(M,\mu)$. See Appendix A for details of the computation.

Appendix A. Perturbations on the cylinder and some computations

In this appendix we give some details of the computations of the kernel of the perturbed Dirac operator on the cylinder. We consider a family of perturbations which includes perturbations used in [2], [8] and [4].

A.1. Setting

1) m : integer
2) $M = \mathbb{R} \times S^1$ with coordinate functions (r, θ)
3) $g = dr^2 + d\theta^2$: Riemannian metric
4) $\omega = dr \wedge d\theta$: symplectic structure
5) $J : \partial_r \mapsto \partial_\theta, \quad \partial_\theta \mapsto -\partial_r$: almost complex structure
6) We use ∂_θ as a frame of $TM_{\mathbb{C}} = (TM, J)$.
7) $W^+ = M \times \mathbb{C},\ W^- = TM_{\mathbb{C}},\ W = W^+ \oplus W^-$
8) $c : T^*M \to \mathrm{End}(W)$: Clifford action defined by

$$c(dr) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \quad c(d\theta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

9) $\rho : \mathbb{R} \to (m - 1/2, m + 1/2)$: smooth non-decreasing function with

$$\rho(r) = \begin{cases} r & (m - 1/4 < r < m + 1/4)) \\ m - 1/2 & (r < m - 1/2) \\ m + 1/2 & (r > m + 1/2) \end{cases}$$

10) $\nabla^W = d - 2\pi\rho(r) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} d\theta$: Clifford connection of W
11) $D : \Gamma(W) \to \Gamma(W)$: Dirac operator,

$$D = \begin{pmatrix} 0 & -\partial_\theta - \sqrt{-1}\partial_r + 2\pi\sqrt{-1}\rho \\ \partial_\theta - \sqrt{-1}\partial_r - 2\pi\sqrt{-1}\rho & 0 \end{pmatrix}$$

12) Let S^1 acts on M in the standard way, and we take a lift of the S^1-action on W so that the action on the fiber direction is trivial. All the data are preserved by the S^1-action.

13) $D_{S^1} : \Gamma(W) \to \Gamma(W)$: Dirac operator along the S^1-orbits :

$$D_{S^1} = \begin{pmatrix} 0 & -\partial_\theta + 2\pi\sqrt{-1}\rho \\ \partial_\theta - 2\pi\sqrt{-1}\rho & 0 \end{pmatrix}$$

14) $\mu := -2\pi\rho : M \to \mathbb{R}$

15) $\mu^M = -2\pi\rho\partial_\theta \in \Gamma(TM)$: induced vector field

16) $f : M \to \mathbb{R}_+$: smooth positive function on M such that $f(r) = |r|$ for $|r - m| > 1/2$

Remark A.1. The above data is a completion of the standard Hamiltonian S^1-action on the symplectic manifold $(m - 1/4, m + 1/4) \times S^1$ with pre-quantizing line bundle $(L, \nabla) = (\underline{\mathbb{C}}, d + \sqrt{-1}\mu)$, which has a cylindrical end and translational invariance on the end. The map μ gives the moment map of the S^1-action on this symplectic manifold, and f^ε is an admissible function for (W, μ, ∇^W) for any $\varepsilon > 0$ in the sense of [2].

A.2. Perturbation of D

For $s, t, \varepsilon_1, \varepsilon_2 \geq 0$ we consider the following perturbation of D :

$$D_{s,t,\varepsilon_1,\varepsilon_2} := D + \sqrt{-1}sf^{\varepsilon_1}c(\mu^M) + tf^{\varepsilon_2}D_{S^1} = \begin{pmatrix} 0 & D^-_{s,t,\varepsilon_1,\varepsilon_2} \\ D^+_{s,t,\varepsilon_1,\varepsilon_2} & 0 \end{pmatrix},$$

where

$$D^+_{s,t,\varepsilon_1,\varepsilon_2} = (1 + tf^{\varepsilon_2})(\partial_\theta - 2\pi\sqrt{-1}\rho) - \sqrt{-1}\partial_r - 2\pi\sqrt{-1}sf^{\varepsilon_1}\rho$$

and

$$D^-_{s,t,\varepsilon_1,\varepsilon_2} = -(1 + tf^{\varepsilon_2})(\partial_\theta - 2\pi\sqrt{-1}\rho) - \sqrt{-1}\partial_r + 2\pi\sqrt{-1}sf^{\varepsilon_1}\rho.$$

Note that $D_{1,0,\varepsilon_1,\varepsilon_2}$ $(\varepsilon_1 > 0)$ is the perturbation considered in [2] and [8], and $D_{0,t,\varepsilon_1,0}$ is the one considered in [4].

A.3. $\ker_{L^2}(D^+_{s,t,\varepsilon_1,\varepsilon_2})$

For $\phi \in \Gamma(W^+)$ by taking the Fourier expansion we write

$$\phi(r, \theta) = \sum_{n \in \mathbb{Z}} a_n(r)e^{2\pi\sqrt{-1}n\theta}.$$

Then we have

$$D^+_{s,t,\varepsilon_1,\varepsilon_2}\phi=\sum_{n\in\mathbb{Z}}\sqrt{-1}\left(2\pi((1+tf^{\varepsilon_2})(n-\rho)-sf^{\varepsilon_1}\rho)a_n(r)-a_n'(r)\right)e^{2\pi\sqrt{-1}n\theta},$$

and hence,

$$\begin{aligned} & D^+_{s,t,\varepsilon_1,\varepsilon_2}\phi=0 \\ \iff\ & 2\pi((1+tf^{\varepsilon_2})(n-\rho)-sf^{\varepsilon_1}\rho)a_n(r)-a_n'(r)=0 \\ \iff\ & a_n(r)=\alpha_n\exp\left(2\pi\int^r((1+tf^{\varepsilon_2})(n-\rho)-sf^{\varepsilon_1}\rho)dr\right)\ (\alpha_n\in\mathbb{C}). \end{aligned}$$

Now we determine the condition for $\phi\in\ker(D_{s,t,\varepsilon_1,\varepsilon_2})$ to be an L^2-section. Since $\rho=m\pm 1/2$ and $f=|r|$ for $\pm r$ large enough we have

$$\begin{aligned} a_n(r) &= \alpha_n\exp\left(2\pi\int^r((1+t|r|^{\varepsilon_2})(n-m\mp 1/2)-(m\pm 1/2)s|r|^{\varepsilon_1})dr\right) \\ &= \alpha_n\exp\left(2\pi(n-m\mp 1/2)\left(r+\frac{tr|r|^{\varepsilon_2}}{\varepsilon_2+1}\right)-2\pi(m\pm 1/2)\frac{sr|r|^{\varepsilon_1}}{\varepsilon_1+1}\right). \end{aligned}$$

Suppose that ϕ is an L^2-solution, i.e., $\int_{-\infty}^{\infty}|a_n(r)|^2dr<\infty$.

(I) $\varepsilon_1>\varepsilon_2$. In this case when we take $r\gg 0$ we have $m+1/2>0$, and when we take $-r\gg 0$ we have $m-\frac{1}{2}<0$. So we have $m=0$, and hence, we have $\ker_{L^2}(D_{s,t,\varepsilon_1,\varepsilon_2})\neq 0$ if and only if $m=0$. If $m=0$, then $\ker_{L^2}(D_{s,t,\varepsilon_1,\varepsilon_2})$ is an infinite dimensional vector space generated by $\{a_n(r)e^{2\pi\sqrt{-1}n\theta}\mid n\in\mathbb{Z}\}$.

(II) $\varepsilon_1<\varepsilon_2$. In this case as in the same way for (I) we have $m-1/2<n<m+1/2$, and hence, $\ker_{L^2}(D_{s,t,\varepsilon_1,\varepsilon_2})=\mathbb{C}\langle a_m(r)e^{2\pi\sqrt{-1}m\theta}\rangle$.

(III) $\varepsilon_1=\varepsilon_2$. In this case when we take $r\gg 0$ and $-r\gg 0$ we have

$$\left(n-m-\frac{1}{2}\right)t-\left(m+\frac{1}{2}\right)s<0 \text{ and } \left(n-m+\frac{1}{2}\right)t-\left(m-\frac{1}{2}\right)s>0.$$

So we have if $t=0$ and $s>0$, then $m=0$, and if $t>0$, then

$$\left(1+\frac{s}{t}\right)\left(m-\frac{1}{2}\right)<n<\left(1+\frac{s}{t}\right)\left(m+\frac{1}{2}\right).$$

In this case $\dim\ker_{L^2}(D^+_{s,t,\varepsilon_1,\varepsilon_1})$ depends on s/t.

A.4. $\ker_{L^2}(D^-_{s,t,\varepsilon_1,\varepsilon_2})$

For $\phi\partial_\theta \in \Gamma(W^-)$ by taking the Fourier expansion we write

$$\phi(r,\theta) = \sum_{n\in\mathbb{Z}} a_n(r) e^{2\pi\sqrt{-1}n\theta}.$$

Then we have

$$D^-_{s,t,\varepsilon_1,\varepsilon_2}\phi = -\sum_{n\in\mathbb{Z}} \sqrt{-1}\big(2\pi((1+tf^{\varepsilon_2})(n-\rho) - sf^{\varepsilon_1}\rho)a_n(r) + a_n'(r)\big)e^{2\pi\sqrt{-1}n\theta},$$

and hence,

$$\begin{aligned} & D^-_{s,t,\varepsilon_1,\varepsilon_2}\phi = 0 \\ \Longleftrightarrow\ & 2\pi((1+tf^{\varepsilon_2})(n-\rho) - sf^{\varepsilon_1}\rho)a_n(r) + a_n'(r) = 0 \\ \Longleftrightarrow\ & a_n(r) = \alpha_n \exp\left(-2\pi\int^r((1+tf^{\varepsilon_2})(n-\rho) - sf^{\varepsilon_1}\rho)dr\right)\ (\alpha_n\in\mathbb{C}). \end{aligned}$$

As in the same way for $\ker_{L^2}(D^+_{s,t,\varepsilon_1,\varepsilon_2})$ one can check that there are no L^2-solutions of $D^-_{s,t,\varepsilon_1,\varepsilon_2}\phi = 0$.

A.5. Computations of indices

We specialize the parameters and have computations of two indices, the transverse index $\chi_{S^1}(M,\mu)$ in [2] and the equivariant local Riemann-Roch number $RR_{S^1,loc}(M,L)$.

A.5.1. $\chi_{S^1}(M,\mu)$. When we take $s=1$, $t=0$ and $\varepsilon_1 > \varepsilon_2$ we have the following.

Proposition A.2.

$$\mathrm{Ker}_{L^2}(D^+_{1,0,\varepsilon_1,\varepsilon_2}) = \mathbb{C}\langle\{\delta_{m0}a_n(r)e^{2\pi\sqrt{-1}n\theta} \mid n\in\mathbb{Z}\}\rangle, \quad \mathrm{Ker}_{L^2}(D^-_{1,0,\varepsilon_1,\varepsilon_2}) = 0.$$

In particular we have

$$\chi_{S^1}(M,\mu) = \mathbb{C}\langle\{\delta_{m0}a_n(r)e^{2\pi\sqrt{-1}n\theta} \mid n\in\mathbb{Z}\}\rangle = \bigoplus_{n\in\mathbb{Z}}\delta_{m0}\mathbb{C}_{(n)}.$$

A.5.2. $RR_{S^1,\mathrm{loc}}(M,L)$. When we take $s=\varepsilon_2=0$ we have the following.

Proposition A.3.

$$\mathrm{Ker}_{L^2}(D^+_{0,t,\varepsilon_1,0}) = \mathbb{C}\langle a_m(r)e^{2\pi\sqrt{-1}m\theta}\rangle, \quad \mathrm{Ker}_{L^2}(D^-_{0,t,\varepsilon_1,0}) = 0.$$

In particular we have

$$RR_{S^1,\mathrm{loc}}(M,L) = \mathbb{C}\langle a_m(r)e^{2\pi\sqrt{-1}m\theta}\rangle = \mathbb{C}_{(m)}.$$

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