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Orbifold Hurwitz numbers and Eynard–Orantin invariants

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We prove that a generalisation of simple Hurwitz numbers due to Johnson, Pandharipande and Tseng satisfies the topological recursion of Eynard and Orantin. This generalises the Bouchard–Mariño conjecture and places Hurwitz–Hodge integrals, which arise in the Gromov–Witten theory of target curves with orbifold structure, in the context of the Eynard–Orantin topological recursion.

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1. Introduction

The topological recursion of Eynard and Orantin produces invariants of a Riemann surface C equipped with two meromorphic functions $x : C \to \mathbb{C}$ and $y : C \to \mathbb{C}$ [17]. For integers $g \geq 0$ and $n > 0$, the Eynard–Orantin invariant ω_n^g is a multidifferential — in other words, a tensor product of meromorphic 1-forms — on the Cartesian product C^n. See Section 4 for a precise definition of the topological recursion and further details.

For various choices of spectral curve (C, x, y), the Eynard–Orantin invariants store intersection numbers on moduli spaces of curves [15, 17]; Weil–Petersson volumes [18]; simple Hurwitz numbers [2, 4, 16]; the enumeration of lattice points in moduli spaces of curves [30]; Gromov–Witten invariants of $\mathbb{P}^1$ [10, 31]; and the open and closed Gromov–Witten invariants of toric Calabi–Yau threefolds [3, 20, 21, 29]. The main result of this paper adds to this list an infinite family of examples, which generalises the relation between simple Hurwitz numbers and Eynard–Orantin invariants known as the Bouchard–Mariño conjecture [4]. The methods herein generalise the proof of this result by Eynard, Mulase, and Safnuk [16].

A great deal of attention in the literature has been paid to simple Hurwitz numbers and their relation to various moduli spaces [4, 12, 16, 26, 32]. The simple Hurwitz number $H_{g;\mu}$ is the weighted count of connected genus g branched covers of $\mathbb{P}^1$ with ramification profile over ∞ given by the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and simple ramification elsewhere. In this paper, we consider the generalisation $H^{[a]}_{g;\mu}$, which is the weighted count of connected genus g branched covers of $\mathbb{P}^1$ with ramification profile over ∞ given by μ, ramification profile over 0 given by a partition of the form $(a, a, \ldots, a)$, and simple ramification elsewhere. We refer to these as *orbifold Hurwitz numbers* and note that we recover the simple Hurwitz numbers in the case $a = 1$. See Section 2 for a precise definition of orbifold Hurwitz numbers.

Assemble the orbifold Hurwitz numbers into the following generating function.

$$H^{[a]}_{g,n}(x_1, \ldots, x_n) = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} H^{[a]}_{g;\mu} \, \frac{|\mathrm{Aut}\ \mu|}{(2g-2+n+\frac{|\mu|}{a})!} \, x_1^{\mu_1} \cdots x_n^{\mu_n}. \tag{1}$$

We use the notation $|\mu|$ to denote the sum $\mu_1 + \mu_2 + \cdots + \mu_n$ and Aut μ to denote the group of permutations that leave the tuple $(\mu_1, \mu_2, \ldots, \mu_n)$ invariant.

Theorem 1. *For any positive integer a, consider the rational spectral curve C given by*

$$x(z) = z\exp(-z^a) \qquad \text{and} \qquad y(z) = z^a.$$

For $2g-2+n>0$, the analytic expansion of the Eynard–Orantin invariant ω_n^g of C around the point $x_1 = x_2 = \cdots = x_n = 0$ is given by

$$\Omega_{g,n}^{[a]}(x_1,\ldots,x_n) = \frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_n} H_{g,n}^{[a]}(x_1,\ldots,x_n)\; dx_1 \otimes \cdots \otimes dx_n. \tag{2}$$

Here and throughout the paper, we consider z_k to be the rational parameter on the kth copy of C in the Cartesian product C^n and we adopt the shorthand x_k for $x(z_k)$. For notational convenience, we usually omit the $\otimes$ symbol in the tensor product of 1-forms.

Our results are motivated by and reliant upon the work of Johnson, Pandharipande and Tseng concerning Hurwitz–Hodge integrals [28]. They describe spaces of admissible covers and interpret them as moduli spaces of stable maps. This allows for virtual localisation to be used, which leads to the following relation between orbifold Hurwitz numbers and intersection numbers on moduli spaces of stable maps to the classifying stack $\mathcal{B}\mathbb{Z}_a$, given by a point with trivial $\mathbb{Z}_a$ action.

$$\begin{aligned} H_{g,\mu}^{[a]} = {} & \frac{(2g-2+n+\frac{|\mu|}{a})!}{|\mathrm{Aut}\ \mu|}\, a^{1-g+\sum\{\mu_i/a\}} \\ & \times \prod_{i=1}^{n} \frac{\mu_i^{\lfloor \mu_i/a\rfloor}}{\lfloor \mu_i/a\rfloor!} \int_{\overline{\mathcal{M}}_{g,[-\mu]}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty}(-a)^i\lambda_i^U}{\prod_{i=1}^{n}(1-\mu_i\bar{\psi}_i)} \end{aligned}$$

Here, $r = \lfloor r\rfloor + \{r\}$ gives the integer and fractional parts of the real number r. The case $a=1$ is the famous ELSV formula, which expresses simple Hurwitz numbers as intersection numbers on moduli spaces of curves [12]. See Section 3 for a precise definition of the moduli spaces and characteristic classes appearing in the formula above.

One consequence of this ELSV-type formula is that the so-called Hurwitz–Hodge integrals appearing on the right side can be calculated from the knowledge of the orbifold Hurwitz numbers. So our main theorem places Hurwitz–Hodge integrals, which arise in the Gromov–Witten theory of target curves with orbifold structure, in the context of Eynard–Orantin topological recursion.

Sections 2, 3 and 4 contain preparatory material. The heart of the proof of Theorem 1 is contained in Section 5. In Section 6, we conclude the paper with some applications of our main theorem. In particular, the general theory of Eynard–Orantin topological recursion involves string and dilaton equations. In the context of orbifold Hurwitz numbers, these lead to relations between Hurwitz–Hodge integrals on $\overline{\mathcal{M}}_{g,n+1}(\mathcal{B}\mathbb{Z}_a)$ and those on $\overline{\mathcal{M}}_{g,n}(\mathcal{B}\mathbb{Z}_a)$.

2. Hurwitz numbers

Hurwitz numbers count branched covers $\Sigma \to \Sigma'$ of Riemann surfaces with specified branch points and ramification profiles. Variants may require further conditions to be satisfied — for example, that the branched covers be connected or that the preimages of a branch point be labelled. Two branched covers $\Sigma_1 \to \Sigma'$ and $\Sigma_2 \to \Sigma'$ are isomorphic if there exists an isomorphism of Riemann surfaces $f : \Sigma_1 \to \Sigma_2$ that covers the identity on Σ'. Similarly, an automorphism of a branched cover $\Sigma \to \Sigma'$ is an automorphism of the Riemann surface Σ that covers the identity on Σ'. For a degree d branched cover, the ramification profile at a branch point is given by a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ consisting of non-increasing positive integers that sum to d. Suppose that we fix $\{p_1, p_2, \ldots, p_r\} \subset \Sigma'$ together with partitions $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)}$ of d. The associated *Hurwitz number* is the weighted count of branched covers $\pi : \Sigma \to \Sigma'$ with ramification profile $\mu^{(i)}$ at p_i, where the weight of a branched cover π is $\frac{1}{|\mathrm{Aut}\ \pi|}$. There are two distinct flavours of Hurwitz theory corresponding to the enumerations of connected covers and possibly disconnected covers. Although our primary goal is to understand connected Hurwitz numbers, it is often necessary to deal with disconnected Hurwitz numbers in the first instance.

The Riemann existence theorem allows a branched cover of $\mathbb{P}^1$ to be described by its monodromy at the branch points. It follows that a disconnected Hurwitz number is equal to $\frac{1}{|\mu|!}$ multiplied by the number of tuples $(\sigma_1, \sigma_2, \ldots, \sigma_r)$ of permutations in the symmetric group $S_{|\mu|}$ such that

- $\sigma_1 \sigma_2 \cdots \sigma_r = (1)$; and
- σ_i has cycle type given by the partition $\mu^{(i)}$.

One obtains connected Hurwitz numbers by further requiring that the permutations $\sigma_1, \sigma_2, \ldots, \sigma_r$ generate a transitive subgroup of $S_{|\mu|}$.

Definition 2. For a positive integer a, let the *orbifold Hurwitz number* $H^{[a]}_{g;\mu}$ be the weighted count of connected genus g branched covers of $\mathbb{P}^1$ such that

- the ramification profile over ∞ is given by the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$;

- the ramification profile over 0 is given by a partition of the form $(a, a, \ldots, a)$; and
- the only other ramification is simple and occurs at m fixed points.

The Hurwitz number is zero unless a divides $|\mu|$ and the Riemann–Hurwitz formula implies that m must be equal to $2g - 2 + n + \frac{|\mu|}{a}$. We will consistently use m to denote the expression $2g - 2 + n + \frac{|\mu|}{a}$ throughout the paper. We also consider a to be fixed and often drop the superscript $[a]$.

2.1. Cut-and-join recursion

The cut-and-join recursion provides a simple method for the calculation of Hurwitz numbers. It was originally conceived for the case of simple Hurwitz numbers [22] and has since been generalised in various ways [33, 36]. The basic premise is to determine the behaviour of a branched cover as two of the branch points come together. For simple ramification, this translates into understanding the behaviour of permutations multiplied by transpositions. We state the cut-and-join recursion using the following normalisation of the simple Hurwitz numbers, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $m = 2g - 2 + n + |\mu|$.

$$H_g(\mu) = H_{g;\mu} \times \frac{|\mathrm{Aut}\ \mu|}{m!}.$$

Proposition 3 (Cut-and-join recursion for simple Hurwitz numbers [23]). *The normalised simple Hurwitz numbers satisfy the following recursion, where $m = 2g - 2 + n + |\mu|$. We use the notation $S = \{1, 2, \ldots, n\}$ and $\mu_I = (\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_N})$ for $I = \{i_1, i_2, \ldots, i_N\}$.*

$$(3)\qquad mH_g(\mu_S) = \sum_{i<j} (\mu_i + \mu_j) H_g(\mu_{S\setminus\{i,j\}}, \mu_i + \mu_j) + \sum_{i=1}^{n} \sum_{\alpha+\beta=\mu_i} \frac{\alpha\beta}{2} \Bigg[H_{g-1}(\mu_{S\setminus\{i\}}, \alpha, \beta) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S\setminus\{i\}}} H_{g_1}(\mu_I, \alpha) H_{g_2}(\mu_J, \beta) \Bigg].$$

We can state the cut-and-join recursion for orbifold Hurwitz numbers using the following analogous normalisation, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and

$m = 2g - 2 + n + \frac{|\mu|}{a}$.

$$H_g^{[a]}(\mu) = H_{g;\mu}^{[a]} \times \frac{|\mathrm{Aut}\ \mu|}{m!}.$$

Proposition 4 (Cut-and-join recursion for orbifold Hurwitz numbers). *The normalised orbifold Hurwitz numbers satisfy the cut-and-join recursion* (3), *where* $m = 2g - 2 + n + \frac{|\mu|}{a}$.

In Appendix A, we provide a proof of this proposition via a graphical representation of branched covers. An immediate use of Proposition 4 is the calculation of the generating functions appearing in equations (1) and (2) in the base cases $(g, n) = (0, 1)$ and $(g, n) = (0, 2)$. Here and throughout the paper, we use the fact that for $x = z \exp(-z^a)$

$$x \frac{d}{dx} = \frac{z}{1 - az^a} \frac{d}{dz}$$

is a differential operator that maps rational functions in z to rational functions in z.

Lemma 5. *The generating function* $H_{0,1}^{[a]}(x)$ *satisfies the differential equation*

$$x \frac{d}{dx} H_{0,1}^{[a]}(x) = z^a.$$

Proof. In the case $(g, n) = (0, 1)$, Proposition 4 states that

$$\left(\frac{\mu}{a} - 1\right) H_0^{[a]}(\mu) = \sum_{\alpha+\beta=\mu} \frac{\alpha\beta}{2} H_0^{[a]}(\alpha) H_0^{[a]}(\beta).$$

This may be equivalently expressed at the level of generating functions in the following way.

$$\begin{aligned} \frac{x}{a} \frac{dH_{0,1}^{[a]}}{dx} - H_{0,1}^{[a]} &= \frac{x^2}{2} \left(\frac{dH_{0,1}^{[a]}}{dx}\right)^2 \\ \Rightarrow \qquad \frac{z}{a(1 - az^a)} \frac{dH_{0,1}^{[a]}}{dz} - H_{0,1}^{[a]} &= \frac{z^2}{2(1 - az^a)^2} \left(\frac{dH_{0,1}^{[a]}}{dz}\right)^2. \end{aligned}$$

Observe that the recursion above uniquely defines all coefficients of $H_{0,1}^{[a]}$ from the initial values $H_0^{[a]}(\mu) = 0$ for $\mu = 0, 1, 2, \ldots, a - 1$ and $H_0^{[a]}(a) = \frac{1}{a}$.

Therefore, we seek the unique solution to this nonlinear differential equation of the form $\frac{x^a}{a} + O(x^{a+1})$. This is satisfied by $H_{0,1}^{[a]} = \frac{z^a}{a} - \frac{z^{2a}}{2}$, from which we immediately obtain the desired result. $\square$

From the preceding lemma one obtains the differential form

$$\Omega_{0,1}^{[a]}(x) = dH_{0,1}^{[a]}(x) = z^{a-1}(1 - az^a)\, dz.$$

Lemma 6. *The bidifferential $\Omega_{0,2}^{[a]}$ satisfies the following equation.*

$$\Omega_{0,2}^{[a]}(x_1, x_2) = \frac{dz_1 \ dz_2}{(z_1 - z_2)^2} - \frac{dx_1 \ dx_2}{(x_1 - x_2)^2}.$$

Proof. In the case $(g, n) = (0, 2)$, Proposition 4 states that at the level of generating functions,

$$\left[\frac{1}{a}(1 - az_1^a)x_1 \frac{\partial}{\partial x_1} + \frac{1}{a}(1 - az_2^a)x_2 \frac{\partial}{\partial x_2}\right] H_{0,2}^{[a]}(x_1, x_2) = \frac{x_2 z_1^a - x_1 z_2^a}{x_1 - x_2}.$$

We note that this is a special case of Proposition 7 below. Note that the differential operator on the left side of the equation may be written as $\frac{1}{a}\left[z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}\right]$, which is simply a constant multiplied by the degree operator. It follows that $H_{0,2}^{[a]}$ is defined by this differential equation up to an additive constant.

Since $H_{0,2}^{[a]}(x_1, x_2)$ must be symmetric in x_1 and x_2, the equation is satisfied by

$$x_1 \frac{\partial}{\partial x_1} H_{0,2}^{[a]}(x_1, x_2) = \frac{x_2}{x_2 - x_1} - \frac{z_2}{(z_2 - z_1)(1 - az_1^a)}. \tag{4}$$

Now apply $x_2 \frac{\partial}{\partial x_2} = \frac{z_2}{1 - az_2^a} \frac{\partial}{\partial z_2}$ to this equation to obtain

$$\begin{aligned} x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} H_{0,2}^{[a]}(x_1, x_2) &= -\frac{x_1 x_2}{(x_1 - x_2)^2} + \frac{z_1 x_2}{(z_1 - z_2)^2 (1 - az_1^a)} \frac{dz_2}{dx_2} \\ &= -\frac{x_1 x_2}{(x_1 - x_2)^2} + \frac{x_1 x_2}{(z_1 - z_2)^2} \frac{dz_1}{dx_1} \frac{dz_2}{dx_2}, \end{aligned}$$

from which we immediately obtain the desired result. $\square$

The following proposition expresses the cut-and-join recursion in terms of the orbifold Hurwitz number generating functions and generalises Theorem 4.4 from [23].

Proposition 7. *For $2g-2+n>1$, the orbifold Hurwitz number generating functions satisfy the following partial differential equation. We use the notation $S=\{1,2,\ldots,n\}$ and $x_I=(x_{i_1},x_{i_2},\ldots,x_{i_N})$ for $I=\{i_1,i_2,\ldots,i_N\}$.*

$$\begin{aligned}(5)\quad & \left[2g-2+n+\frac{1}{a}\sum_{i=1}^{n}(1-az_i^a)x_i\frac{\partial}{\partial x_i}\right]H_{g,n}^{[a]}(x_S)\\ &=\sum_{i<j}\frac{1}{z_i-z_j}\left(\frac{z_j\,x_i}{1-az_i^a}\frac{\partial}{\partial x_i}-\frac{z_i\,x_j}{1-az_j^a}\frac{\partial}{\partial x_j}\right)\\ &\qquad\times\left[H_{g,n-1}^{[a]}(x_{S\setminus\{j\}})+H_{g,n-1}^{[a]}(x_{S\setminus\{i\}})\right]\\ &\quad+\frac{1}{2}\sum_{i=1}^{n}\left[u_1u_2\frac{\partial^2}{\partial u_1\partial u_2}H_{g-1,n+1}^{[a]}(u_1,u_2,x_{S\setminus\{i\}})\right]_{\substack{u_1=x_i\\u_2=x_i}}\\ &\quad+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}}\left[x_i\frac{\partial}{\partial x_i}H_{g_1,|I|+1}^{[a]}(x_i,x_I)\right]\left[x_i\frac{\partial}{\partial x_i}H_{g_2,|J|+1}^{[a]}(x_i,x_J)\right].\end{aligned}$$

The final summation is stable in the sense that we omit terms involving $H_{0,1}^{[a]}$ or $H_{0,2}^{[a]}$.

Proof. Apply the operator

$$\sum_{\mu_1,\ldots,\mu_n=1}^{\infty}[\,\cdot\,]\;x_1^{\mu_1}\cdots x_n^{\mu_n}$$

to both sides of the cut-and-join recursion to obtain the following partial differential equation satisfied by the expansion of $H_{g,n}^{[a]}$ around $x_1=\cdots=x_n=0$.

$$\begin{aligned}(6)\quad & \left[2g-2+n+\frac{1}{a}\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}\right]H_{g,n}^{[a]}(x_S)\\ &=\sum_{i\neq j}\left[x_i\frac{\partial}{\partial x_i}H_{g,n-1}^{[a]}(x_{S\setminus\{j\}})\right]\left[x_i\frac{\partial}{\partial x_i}H_{0,2}^{[a]}(x_i,x_j)\right]\\ &\quad+\sum_{i<j}\frac{x_ix_j}{x_i-x_j}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\left[H_{g,n-1}^{[a]}(x_{S\setminus\{j\}})+H_{g,n-1}^{[a]}(x_{S\setminus\{i\}})\right]\\ &\quad+\sum_{i=1}^{n}\left[x_i\frac{\partial}{\partial x_i}H_{g,n}^{[a]}(x_S)\right]\left[x_i\frac{\partial}{\partial x_i}H_{0,1}^{[a]}(x_i)\right]+\end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\left[u_1u_2\frac{\partial^2}{\partial u_1\partial u_2}H^{[a]}_{g-1,n+1}(u_1,u_2,x_{S\setminus\{i\}})\right]_{\substack{u_1=x_i\\u_2=x_i}}$$
$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{g_1+g_2=g\\I\sqcup J=S\setminus\{i\}}}^{\text{stable}}\left[x_i\frac{\partial}{\partial x_i}H^{[a]}_{g_1,|I|+1}(x_i,x_I)\right]\left[x_i\frac{\partial}{\partial x_i}H^{[a]}_{g_2,|J|+1}(x_i,x_J)\right]$$

We have used here the fact that

$$\frac{x_1x_2}{x_1-x_2}\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)\left[x_1^N+x_2^N\right]=N(x_1^{N-1}x_2+x_1^{N-2}x_2^2+\cdots+x_1x_2^{N-1}).$$

Substitute the expression $x_i\frac{\partial}{\partial x_i}H^{[a]}_{0,2}(x_i,x_j)=\frac{x_j}{x_j-x_i}-\frac{z_j}{(z_j-z_i)(1-az_i^a)}$ from equation (4) into equation (6) to get cancellation of all terms involving $\frac{x_ix_j}{x_i-x_j}$. Furthermore, move the terms involving $x_i\frac{\partial}{\partial x_i}H_0(x_i)=z_i^a$ calculated in Lemma 5 to the left side in order to obtain the desired result. □

Many of the cancellations in the proof of Proposition 7 do not occur in the special cases $(g,n)=(0,3)$ and $(g,n)=(1,1)$. Nevertheless, equation (6) is still satisfied in these special cases and simplifies to the following PDEs.

Proposition 8. *The orbifold Hurwitz generating functions $H^{[a]}_{0,3}$ and $H^{[a]}_{1,1}$ satisfy the following, where the subscripts in the first equation are to be interpreted modulo 3.*

$$\left[1+\frac{1}{a}\sum_{i=1}^{3}z_i\frac{\partial}{\partial z_i}\right]H^{[a]}_{0,3}(x_1,x_2,x_3)=\frac{z_1z_2z_3}{(z_1-z_2)(z_2-z_3)(z_3-z_1)}\sum_{i=1}^{3}\frac{z_{i-1}-z_{i+1}}{z_i(1-az_i^a)^2}\tag{7}$$

$$\left[1+\frac{1}{a}z_1\frac{\partial}{\partial z_1}\right]H^{[a]}_{1,1}(x_1)=\frac{1}{2}\frac{x_1^2}{dx_1\ dx_1}\left(\frac{dz_1\ dz_2}{(z_1-z_2)^2}-\frac{dx_1\ dx_2}{(x_1-x_2)^2}\right)\bigg|_{z_2=z_1}.\tag{8}$$

Equations (7) and (8) uniquely determine $H^{[a]}_{0,3}(x_1,x_2,x_3)$ and $H^{[a]}_{1,1}(x_1)$. In equation (8), the right side is meromorphic at $z_1=\alpha$ with no other poles, due to equation (19).

2.2. Double Hurwitz numbers

An alternative proof of the cut-and-join equations arises by considering the action of transpositions in the symmetric group and so is more natural from the disconnected Hurwitz number viewpoint. We describe it here for completeness.

Orbifold Hurwitz numbers are particular examples of double Hurwitz numbers, which count branched covers of $\mathbb{P}^1$ with specified ramification over two points and simple ramification elsewhere.

Definition 9. The *double Hurwitz number* $H_{g;\mu,\nu}$ is the number of genus g branched covers of $\mathbb{P}^1$ such that

- the ramification profile over ∞ is given by the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$;
- the ramification profile over 0 is given by a partition $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$; and
- the only other ramification is simple and occurs over m fixed points.

The Hurwitz number must be zero unless $|\mu| = |\nu|$ and the Riemann–Hurwitz formula implies that $m = 2g - 2 + \ell(\mu) + \ell(\nu)$.

Shadrin, Spitz and Zvonkine [33] use the infinite wedge space formalism together with calculations involving characters of the symmetric group to prove that double Hurwitz numbers satisfy the cut-and-join equation. The cut-and-join equation takes the form of a partial differential equation satisfied by the following generating functions for double Hurwitz numbers, written here in the special case of orbifold Hurwitz numbers. Note that $H_{g;\mu}^{[a]\bullet}$ is the analogous Hurwitz number to $H_{g;\mu}^{[a]}$, where the enumeration includes possibly disconnected covers.

$$\mathbf{H}^{[a]\bullet}(s; p_1, p_2, \ldots) = \sum_{m=0}^{\infty} \sum_{\mu} H_{g;\mu}^{[a]\bullet} \frac{s^m}{m!} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n}$$
$$\mathbf{H}^{[a]}(s; p_1, p_2, \ldots) = \sum_{m=0}^{\infty} \sum_{\mu} H_{g;\mu}^{[a]} \frac{s^m}{m!} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n}$$

Here, the inner summations are over all partitions μ, including the empty partition. The relation between disconnected and connected Hurwitz numbers can be succinctly stated using the exponential–logarithm trick.

$$\mathbf{H}^{[a]\bullet} = \exp \mathbf{H}^{[a]}$$

The cut-and-join recursion can be naturally expressed as the following partial differential equation, due to Shadrin, Spitz and Zvonkine [33].

Proposition 10. *The generating function for connected orbifold Hurwitz numbers satisfies the following partial differential equation.*

$$\frac{\partial \mathbf{H}^{[a]}}{\partial s} = \frac{1}{2} \sum_{i,j=1}^{\infty} \left[(i+j)p_i p_j \frac{\partial \mathbf{H}^{[a]}}{\partial p_{i+j}} + ijp_{i+j} \frac{\partial^2 \mathbf{H}^{[a]}}{\partial p_i \partial p_j} + ijp_{i+j} \frac{\partial \mathbf{H}^{[a]}}{\partial p_i} \frac{\partial \mathbf{H}^{[a]}}{\partial p_j} \right].$$

3. An ELSV-type formula

Simple Hurwitz numbers are involved in the remarkable ELSV formula due to Ekedahl, Lando, Shapiro and Vainshtein [11, 12, 24]. For $i = 1, \ldots, n$, define the line bundle $\mathcal{L}_i$ on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ whose fibre over $[(C, p_1, \ldots, p_n)]$ is the cotangent space at the marked point p_i. Denote its first Chern class by $\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ and refer to these as *descendent classes*. If π is the universal curve over $\overline{\mathcal{M}}$ and $\mathcal{K}_\pi$ is its relative dualizing line bundle, then the *Hodge bundle* is defined to be $\mathbb{E}_{g,n} = \pi_* \mathcal{K}_\pi$. Moreover, we denote its Chern classes by $\lambda_k = c_k(\mathbb{E}_{g,n}) \in H^{2k}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ and set $\Lambda = 1 - \lambda_1 + \cdots + (-1)^g \lambda_g$.

For $2g - 2 + n > 0$, the *linear Hodge integrals* are the top intersection products of λ-classes and ψ-classes that are of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_k \psi_1^{j_1} \cdots \psi_n^{j_n}.$$

The ELSV formula expresses Hurwitz numbers using linear Hodge integrals thus.

$$H_{g,\mu} = \frac{|\mathrm{Aut}\ \mu|}{m!} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \frac{\Lambda}{\prod_{i=1}^{\ell(\mu)} (1 - \mu_i \psi_i)}$$

Note that simple Hurwitz numbers are well defined for $g = 0$ and $\ell(\mu) > 0$, so one defines by analogy the notation

$$\int_{\overline{\mathcal{M}}_{0,1}} \frac{\lambda_0}{1 - \mu_1 \psi_1} = \frac{1}{\mu_1^2}$$

$$\int_{\overline{\mathcal{M}}_{0,2}} \frac{\lambda_0}{(1 - \mu_1 \psi_1)(1 - \mu_2 \psi_2)} = \frac{1}{\mu_1 + \mu_2}.$$

Johnson, Pandharipande and Tseng introduced a generalisation of the ELSV formula, which uses generalisations of descendent classes, Hodge classes and linear Hodge integrals [28]. For a positive integer a, we consider the cyclic group $\mathbb{Z}_a$ and the *moduli space of admissible covers* $\overline{\mathcal{A}}_{g,\gamma}(\mathbb{Z}_a)$, which is a compact moduli space introduced by Harris and Mumford [25].

For $\gamma = (\gamma_1, \ldots, \gamma_n)$ with each $\gamma_i \in \mathbb{Z}_a$, an *admissible cover* is a pair $[\pi, \tau]$, where

- $\pi : D \to (C, p_1, \ldots, p_n)$ is a degree a finite map of complete curves; and
- $\tau : \mathbb{Z}_a \times D \to D$ is a $\mathbb{Z}_a$-action;

where the following conditions are satisfied.

- The curve D is possibly disconnected and nodal.
- The class of curves $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$ is stable.
- The map π takes non-singular points to non-singular points, and nodes to nodes.
- The pair $[\pi, \tau]$ restricts to a principal $\mathbb{Z}_a$-bundle over the punctured non-singular locus with monodromy γ_i at p_i.
- Distinct branches of nodes in D map to distinct branches of nodes in C, with equal ramification orders.
- The monodromies of the $\mathbb{Z}_a$-bundle at the two branches of a node lie in opposite conjugacy classes.

This moduli space is isomorphic to the moduli space of stable maps $\overline{\mathcal{A}}_{g,\gamma}(\mathbb{Z}_a) \cong \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_a)$, where $\mathcal{B}\mathbb{Z}_a$ is the classifying stack of $\mathbb{Z}_a$ given by a point with trivial $\mathbb{Z}_a$-action [27]. One can obtain this isomorphism by viewing an admissible cover as a principle $\mathbb{Z}_a$-bundle over the stack quotient $[D/\mathbb{Z}_a]$, which induces a stable map to the classifying stack. Note that when $a = 1$, we have $\mathbb{Z}_1 = \{0\}$ and so $\overline{\mathcal{M}}_{g,(0,\ldots,0)}(\mathcal{B}\mathbb{Z}_1) \cong \overline{\mathcal{M}}_{g,n}$.

Define descendent classes (sometimes known as *ancestor classes* in analogous contexts) by the pullback of the forgetful map $\varepsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_a) \to \overline{\mathcal{M}}_{g,n}$.

$$\bar{\psi}_i = \varepsilon^*(\psi_i) \in H^2(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_a); \mathbb{Q})$$

Let U be the irreducible representation $U : \mathbb{Z}_a \to \mathbb{C}^*$ defined on a cyclic generator g by $U(g) = \exp\left(\frac{2\pi i}{a}\right)$. For each map $[f : [D/\mathbb{Z}_a] \to \mathcal{B}\mathbb{Z}_a] \in \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_a)$, the $\mathbb{Z}_a$-action on D and the functoriality of the global sections functor gives that $H^0(D, \omega_D)$ is a $\mathbb{Z}_a$-representation. Associate the

U-summand of this representation to $[f] \in \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_a)$, which yields the *generalised Hodge bundle*

$$\mathbb{E}^U \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_a).$$

Define the *generalised Hodge classes* to be the Chern classes of this vector bundle

$$\lambda_k^U = c_k(\mathbb{E}^U) \in H^{2k}(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_a); \mathbb{Q})$$

and set $\Lambda^U = \sum_{i=0}^{\infty}(-1)^i \lambda_i^U$. For $2g-2+n>0$, the *linear Hodge integrals* over $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ are the top intersection products of these generalised λ-classes and $\bar{\psi}$-classes that are of the form

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)} \lambda_k^U \bar{\psi}_1^{j_1} \cdots \bar{\psi}_n^{j_n}.$$

Johnson, Pandharipande and Tseng expressed the orbifold Hurwitz number $H_{g,\mu}^{[a]}$ in terms of these generalised linear Hodge integrals. Such integrals have been referred to as *Hurwitz–Hodge integrals* [5] and they have been extensively studied in the literature [6–8, 34].

Theorem 11 (Johnson, Pandharipande and Tseng [28]). *For $\mu = (\mu_1, \ldots, \mu_n)$, the orbifold Hurwitz number $H_{g;\mu}^{[a]}$ satisfies*

$$H_{g,\mu}^{[a]} = \frac{m!}{|\mathrm{Aut}\ \mu|}\, a^{1-g+\sum\{\mu_i/a\}} \prod_{i=1}^{n} \frac{\mu_i^{\lfloor \mu_i/a \rfloor}}{\lfloor \mu_i/a \rfloor!} \int_{\overline{\mathcal{M}}_{g,[-\mu]}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty}(-a)^i \lambda_i^U}{\prod_{i=1}^{n}(1-\mu_i\bar{\psi}_i)},$$

where $[-\mu] = (-\mu_1 \bmod a, \ldots, -\mu_n \bmod a)$*. One can interpret the unstable cases by defining*

$$\int_{\overline{\mathcal{M}}_{g,[-\mu]}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty}(-a)^i \lambda_i^U}{1-\mu_1\bar{\psi}_1} = \begin{cases} \frac{1}{a}\cdot\frac{1}{\mu_1^2} & \textit{if } \mu_1 \equiv 0 \pmod{a}, \\ 0 & \textit{otherwise;} \end{cases}$$

$$\int_{\overline{\mathcal{M}}_{g,[-\mu])}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty}(-a)^i \lambda_i^U}{(1-\mu_1\bar{\psi}_1)(1-\mu_2\bar{\psi}_2)} = \begin{cases} \frac{1}{a}\cdot\frac{1}{\mu_1+\mu_2} & \textit{if } \mu_1+\mu_2 \equiv 0 \pmod{a}, \\ 0 & \textit{otherwise.} \end{cases}$$

The proof of this theorem is by virtual localisation on the moduli space of maps $\overline{\mathcal{M}}_g(\mathbb{P}^1[a], \mu)$, where $\mathbb{P}^1[a]$ is the projective space $\mathbb{P}^1$ with an a-fold orbifold point at 0. In fact, Johnson, Pandaripande and Tseng proved a more general statement than Theorem 11. Their formula can be used to give an expression for any double Hurwitz number by choosing a sufficiently large.

3.1. Orbifold Hurwitz generating function

For a partition μ, define an analogue of Witten's notation by

$$\left\langle \lambda_i^U \bar{\tau}_{m_1} \cdots \bar{\tau}_{m_n} \right\rangle_{g,n}^{(\mu)} = \int_{\overline{\mathcal{M}}_{g,[-\mu]}(\mathcal{B}G)} \lambda_i^U \bar{\psi}_1^{m_1} \cdots \bar{\psi}_n^{m_n}.$$

Using this notation in Theorem 11 and rearranging terms yields

$$H_{g,\mu}^{[a]} = \frac{m! a^m}{|\mathrm{Aut}\ \mu|} \sum_{j_1,\ldots,j_{\ell(\mu)} \in \mathbb{Z}_{\geq 0}} \left\langle \Lambda^U \bar{\tau}_{j_1} \cdots \bar{\tau}_{j_{\ell(\mu)}} \right\rangle_{g,\ell(\mu)}^{(\mu)} \prod_{i=1}^{\ell(\mu)} \frac{(\mu_i/a)^{\lfloor \mu_i/a \rfloor + j_i}}{\lfloor \mu_i/a \rfloor !},$$

where we have used the fact that

$$m = 2g - 2 + n + \frac{|\mu|}{a} \quad \text{and}$$
$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,[-\mu]}(\mathcal{B}G) = 3g - 3 + n.$$

So the orbifold Hurwitz generating function of equation (1) becomes

$$H_{g,n}^{[a]}(x_1, \ldots, x_n) = \sum_{\substack{\mu_1,\ldots,\mu_n \in \mathbb{Z}_+ \\ j_1,\ldots,j_n \in \mathbb{Z}_{\geq 0}}} a^m \left\langle \Lambda^U \bar{\tau}_{j_1} \cdots \bar{\tau}_{j_n} \right\rangle_{g,n}^{(\mu)} \prod_{i=1}^{n} \frac{(\mu_i/a)^{\lfloor \mu_i/a \rfloor + j_i}}{\lfloor \mu_i/a \rfloor !} x_i^{\mu_i}.$$

By Theorem 11, we know that $\mu \equiv \nu \pmod a$ implies that

$$\left\langle \Lambda^U \bar{\tau}_{j_1} \cdots \bar{\tau}_{j_n} \right\rangle_{g,n}^{(\mu)} = \left\langle \Lambda^U \bar{\tau}_{j_1} \cdots \bar{\tau}_{j_n} \right\rangle_{g,n}^{(\nu)}.$$

So we can sum over the mod-classes of μ

$$H_{g,n}^{[a]}(x_1, \ldots, x_n) = \sum_{\substack{j_1,\ldots,j_n \in \mathbb{Z}_{\geq 0} \\ \beta \in \mathbb{Z}_a^n}} \left\langle \Lambda^U \bar{\tau}_{j_1} \cdots \bar{\tau}_{j_n} \right\rangle_{g,n}^{(\beta)}$$
$$\times \sum_{\mu \in \mathbb{Z}_+^n, [\mu] = \beta} a^{m - |\mathbf{j}|} \prod_{i=1}^{n} \frac{(\mu_i/a)^{\lfloor \mu_i/a \rfloor}}{\lfloor \mu_i/a \rfloor !} \mu_i^{j_i} x_i^{\mu_i}.$$

Now, for $\mu \in \mathbb{Z}_+^n, [\mu] = \beta$ we set $b_i = \frac{\mu_i - \beta_i}{a} = \lfloor \mu_i/a \rfloor$. The generating function becomes

$$H_{g,n}^{[a]}(x_1, \ldots, x_n) = a^{2g-2+n} \sum_{\substack{j_1,\ldots,j_n \in \mathbb{Z}_{\geq 0} \\ \beta_1,\ldots,\beta_n \in \mathbb{Z}_a}} a^{\frac{|\beta|}{a} - |j|} \left\langle \Lambda^U \bar{\tau}_{j_1} \cdots \bar{\tau}_{j_n} \right\rangle_{g,n}^{(\beta)} \prod_{i=1}^{n} f_{\beta_i, j_i}(x_i), \tag{9}$$

where

$$f_{r,k}(x) := \sum_{b=0}^{\infty} \frac{(ab+r)^{b+k}}{b!} x^{ab+r}.$$

Note that the intersection number vanishes for $j_1 + \cdots + j_n > 3g - 3 + n$, so that the summation in equation (9) is finite. Therefore, the generating function $H^{[a]}_{g,n}(x_1, \ldots, x_n)$ can be written as a linear combination of products of the form $\prod_{i=1}^{n} f_{\beta_i, j_i}(x_i)$.

For $r = 1, \ldots, a$, define

$$\xi^{(r)}_{k+1}(z) = x\frac{d}{dx}\xi^{(r)}_{k}(z),$$
$$\xi^{(r)}_{-1}(z) = \begin{cases} z^r/r & r = 1, \ldots, a-1 \\ z^a & r = a. \end{cases}$$

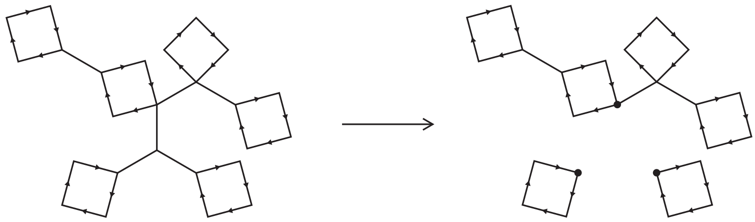
Lemma 12. *The function* $z(x) = \sum_{b=0}^{\infty} \frac{(ab+1)^{b-1}}{b!} x^{ab+1}$ *satisfies* $x = z(x)\exp(-z(x)^a)$. *Furthermore,* $\frac{z(x)^r}{r} = \sum_{b=0}^{\infty} \frac{(ab+r)^{b-1}}{b!} x^{ab+r}$ *and* $f_{r,k}(x) = \xi^{(r)}_k(z(x))$.

Proof. We recall some properties of exponential generating functions. Let $f(x)$ and $g(x)$ be exponential generating functions for collections of labelled objects F and G respectively. Then

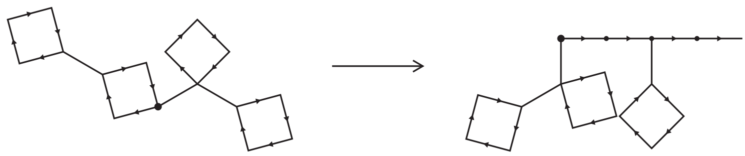
- $f(x)g(x)$ is the exponential generating function for sequences (A_F, A_G) where $A_F \in F$ and $A_G \in G$.
- $f(x)^k$ is the exponential generating function for sequences of k objects from F.
- $\exp f(x)$ is the exponential generating function for sets of elements from F of all cardinalities.

We now use these properties to prove the result.

- $z(x)$ is the exponential generating function for cactus-node trees of type $(1, a, \ldots, a)$ (see Appendix B, Definition 27 and Proposition 29). Removing the node of type 1 we obtain a pair (x, C) where x is a point representing the node and C is a collection of rooted-cactus-node trees of type $(a, \ldots, a)$.

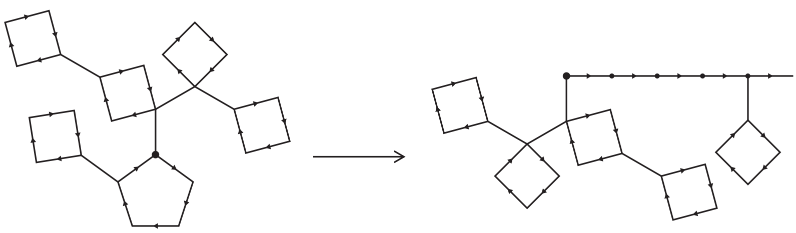


A rooted-cactus-node trees of type $(a,\ldots,a)$ is a seqence $(T_1,\ldots,T_a)$ of cactus-node trees of type $(1,a,\ldots,a)$.



Hence $z(x)$ satisfies $z(x) = x\exp(z(x)^a)$.

- $r\sum\limits_{b=0}^{\infty}\frac{(ab+r)^{b-1}}{b!}x^{ab+r}$ is the exponential generating function for cactus-node trees of type $(r,a,\ldots,a)$ with one of the points in the r-node marked. A cactus-node trees of this type is a sequence $(T_1,\ldots,T_r)$ of cactus-node trees of type $(1,a,\ldots,a)$.



Hence, $z(x)^r = r\sum\limits_{b=0}^{\infty}\frac{(ab+r)^{b-1}}{b!}x^{ab+r}$.

- Finally, we have

$$\begin{aligned}
f_{r,k}(x) = \sum_{b=0}^{\infty}\frac{(ab+r)^{b+k}}{b!}x^{ab+r} &= \left(x\frac{d}{dx}\right)^{k+1}\sum_{b=0}^{\infty}\frac{(ab+r)^{b-1}}{b!}x^{ab+r}\\
&= \left(x\frac{d}{dx}\right)^{k+1}\frac{z^r}{r} = \xi_k^{(r)}(z(x))
\end{aligned}$$

$\square$

Define

$$F_{g,n}^{[a]}(z_1,\ldots,z_n) := H_{g,n}^{[a]}(x_1(z_1),\ldots,x_n(z_n)). \tag{10}$$

Applying Lemma 12 to the orbifold generating function (9), we obtain

$$H_{g,n}^{[a]}(x_1,\ldots,x_n) = a^{2g-2+n} \sum_{\substack{j_1,\ldots,j_n\in\mathbb{Z}_{\geq 0}\\ \beta_1,\ldots,\beta_n\in\mathbb{Z}_a}} a^{\frac{|\beta|}{a}-|j|} \left\langle \Lambda^U \bar{\tau}_{j_1}\cdots\bar{\tau}_{j_n}\right\rangle_{g,n}^{(\beta)} \prod_{i=1}^n \xi_{j_i}^{(\beta_i)}(z_i(x_i)),$$

which gives

$$F_{g,n}^{[a]}(z_1,\ldots,z_n) = a^{2g-2+n} \sum_{\substack{j_1,\ldots,j_n\in\mathbb{Z}_{\geq 0}\\ \beta_1,\ldots,\beta_n\in\mathbb{Z}_a}} a^{\frac{|\beta|}{a}-|j|} \left\langle \Lambda^U \bar{\tau}_{j_1}\cdots\bar{\tau}_{j_n}\right\rangle_{g,n}^{(\beta)} \prod_{i=1}^n \xi_{j_i}^{(\beta_i)}(z_i) \tag{11}$$

A key consequence is that the generating functions $H_{g,n}^{[a]}(x_1,\ldots,x_n)$ are *rational* in $(z_1,\ldots,z_n)$, or more accurately are local expansions around $x_i = 0$ of rational functions.

The function $x = z\exp(-z^a)$ defines local involutions $z \mapsto \sigma_\alpha(z)$ near each root α of $dx(\alpha) = 0$. Via a local coordinate s_α such that $x = s_\alpha^2 + x(\alpha)$, the involution is given by $\sigma_\alpha(s_\alpha) = -s_\alpha$. It gives rise to the following vector space.

Definition 13. Define the vector space $\mathcal{A}_x$ to consist of rational functions p satisfying:

- p has poles only at $\{\alpha : dx(\alpha) = 0\}$;
- $p(z) + p(\sigma_\alpha(z))$ is analytic at $z = \alpha$.

Lemma 14. *For $k \geq 0$ and $r = 1,\ldots,a$, $\xi_k^{(r)}(z) \in \mathcal{A}_x$ and form a basis.*

Proof. The proof uses the simple fact that $x\frac{d}{dx}$ preserves $\mathcal{A}_x$ — that is,

$$x\frac{d}{dx}\mathcal{A}_x \subset \mathcal{A}_x. \tag{12}$$

This can be seen as follows. Clearly $x\frac{d}{dx} = \frac{z}{1-az^a}\frac{d}{dz}$ introduces no new poles outside $\{\alpha : dx(\alpha) = 0\}$. So we study its behaviour locally around a single pole α. The principal part of any function in $\mathcal{A}_x$ is an odd polynomial in

s_α^{-1} for s_α the local coordinate defined above (since $z \mapsto \sigma_\alpha(z)$ corresponds to $s_\alpha \mapsto -s_\alpha$), and

$$x\frac{d}{dx} = \frac{s_\alpha^2+\alpha}{2s_\alpha}\frac{d}{ds_\alpha}$$

maps odd polynomials in s_α^{-1} to odd polynomials in s_α^{-1} since it preserves the parity of the power of any monomial in s_α. Furthermore, $x\frac{d}{dx}\mathbb{C}[[s_\alpha]] \subset \mathbb{C}s_\alpha^{-1} \oplus \mathbb{C}[[s_\alpha]] \subset \mathcal{A}_x$. Hence (12) is proven.

Now $\xi_{-1}^{(r)}(z) = z^r/r \in \mathcal{A}_x$ (or z^a for $r = a$) since it is analytic at $z = \alpha$. Since $\xi_k^{(r)}(z) = x\frac{d}{dx}\xi_{k-1}^{(r)}(z)$ and $x\frac{d}{dx}$ preserves $\mathcal{A}_x$,by induction $\xi_k^{(r)}(z) \in \mathcal{A}_x$ for all r and k.

A simple dimension argument proves that the $\xi_k^{(r)}(z)$ form a basis. $\square$

Remark. If $f(z)$ is analytic at $z = \alpha$ and satisfies $f(z) = f(\sigma_\alpha(z))$ then $x\frac{d}{dx}f(z)$ is analytic at $z = \alpha$. In the terminology of the proof of Lemma 14, the local expansion of $f(z)$ lies in $\mathbb{C}[[s_\alpha^2]]$ and $x\frac{d}{dx}\mathbb{C}[[s_\alpha^2]] \subset \mathbb{C}[[s_\alpha^2]]$.

Since each $\xi_k^{(r)}(z) \in \mathcal{A}_x$ we have proven:

Corollary 15. *For $2g-2+n > 0$ and all a, $F_{g,n}^{[a]}(z_1,\ldots,z_n) \in \mathcal{A}_{x_i}$ for each $i = 1,\ldots,n$.*

4. Eynard–Orantin invariants

Consider a triple (C, x, y) consisting of a genus 0 Riemann surface C and meromorphic functions $x, y : C \to \mathbb{C}$ with the property that the zeros of dx are simple and disjoint from the zeros of dy. For every $(g, n) \in \mathbb{Z}^2$ with $g \geq 0$ and $n > 0$ the Eynard–Orantin invariant of (C, x, y) is a multidifferential $\omega_n^g(p_1,\ldots,p_n)$ — i.e., a tensor product of meromorphic 1-forms on the Cartesian product C^n, where $p_i \in C$. (More generally, if C has positive genus, it should come equipped with a *Torelli marking*, which is a choice of symplectic basis $\{a_i, b_i\}_{i=1,\ldots,g}$ of the first homology group $H_1(\bar{C})$ of the compact closure $\bar{C}$ of C. In particular, a genus 0 surface C requires no Torelli marking.) When $2g-2+n > 0$, $\omega_n^g(p_1,\ldots,p_n)$ is defined recursively in terms of local information around the poles of $\omega_{n'}^{g'}(p_1,\ldots,p_n)$ for $2g'+2-n' < 2g-2+n$.

Since each zero α of dx is simple, for any point $p \in C$ close to α, there is a unique point $\hat{p} \neq p$ close to α such that $x(\hat{p}) = x(p)$. The recursive definition of $\omega_n^g(p_1,\ldots,p_n)$ uses only local information around zeros of dx and makes use of the well-defined map $p \mapsto \hat{p}$ there. The Eynard–Orantin invariants are defined as follows. Given a rational coordinate z on C, we define

$$\omega_1^0 = -\frac{y(z)\,dx(z)}{x(z)}$$
$$\omega_2^0 = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2}.$$

For $2g - 2 + n > 0$, we define

$$\omega_n^g(z_1, z_{S'}) = \sum_\alpha \operatorname*{Res}_{z=\alpha} K(z_1, z) \Bigg[\omega_{n+1}^{g-1}(z, \hat{z}, z_{S'}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\circ} \omega_{|I|+1}^{g_1}(z, z_I)\omega_{|J|+1}^{g_2}(\hat{z}, z_J)\Bigg], \tag{13}$$

where the sum is over the zeros α of dx and we use the notation $S' = \{2, \ldots, n\}$. The $\circ$ over the inner summation denotes the fact that we exclude the terms with $(g_1, |I|) = (0, 0)$ or $(g_2, |J|) = (0, 0)$. Furthermore, we define

$$\begin{aligned} K(z_1, z) &= \frac{-\int_{\hat{z}}^{z} \omega_2^0(z_1, z')x(z)}{2(y(z) - y(\hat{z}))dx(z)} \\ &= \frac{x(z)}{2(y(\hat{z}) - y(z))x'(z)} \left(\frac{1}{z - z_1} - \frac{1}{\hat{z} - z_1}\right) \frac{dz_1}{dz}, \end{aligned}$$

which is well-defined in the vicinity of each zero of dx. Note that the quotient of a differential by the differential $dx(z)$ is a meromorphic function. The recursion or equation (13) depends only on the meromorphic differential $y(z)\,dx(z)/x(z)$ and the map $p \mapsto \hat{p}$ near zeros of dx. For $2g - 2 + n > 0$, each ω_n^g is a symmetric multidifferential with poles only at the zeros of dx, of order $6g - 4 + 2n$ with zero residues.

Define $\Phi(z)$ up to an additive constant by $d\Phi(z) = y(z)\,dx(z)/x(z)$. For $2g - 2 + n > 0$, the invariants satisfy the dilaton equation [17]

$$\sum_\alpha \operatorname*{Res}_{z=\alpha} \Phi(z)\,\omega_{n+1}^g(z, z_1, \ldots, z_n) = (2 - 2g - n)\,\omega_n^g(z_1, \ldots, z_n),$$

where the sum is over the zeros α of dx. This enables the definition of the so-called *symplectic invariants*

$$F_g = \sum_\alpha \operatorname*{Res}_{z=\alpha} \Phi(z)\omega_1^g(z).$$

Remark. There are variations on the definition of the Eynard–Orantin invariants determined by how x and y appear in the kernel K. Here we have used dx/x and y to define the kernel K but any of the four combinations of dx or dx/x and dy or dy/y can be used. All are equivalent via changes of coordinates to $u = \log x$ and $v = \log y$, but in order to have x appear algebraically in generating functions, the choice here suits best.

4.1. Principal parts

The recursion of equation (13) defines the Eynard–Orantin invariants as a sum

$$\omega_n^g(z_1, z_2, \ldots, z_n) = \sum_\alpha \operatorname*{Res}_{z=\alpha} K(z_1, z)\mathcal{F}(z, z_2, \ldots, z_n)$$

over $\{\alpha \mid dx(\alpha) = 0\}$. We will see below that this expresses $\omega_n^g(z_1, \ldots, z_n)$ as the sum of its principal parts in z_1 over the set of its poles $\{z_1 = \alpha\}$. This is an important feature, so we explain it below after first recalling the definition and properties of principal parts.

Given a local parameter z of a curve C, the *principal part* at a point $\alpha \in C$ of a function or differential $h(z)$ analytic in $U \setminus \{\alpha\}$ for some neighbourhood U of α is

$$[h(z)]_\alpha := \operatorname*{Res}_{w=\alpha} \frac{h(w)dw}{z - w}. \tag{14}$$

(Strictly we might write $z = z(p)$ and $w = w(q)$ for points p and q on C, but we abuse terminology and identify U with $z(U)$.) We have the fact that

(i) $[h(z)]_\alpha$ is analytic on $U \setminus \{\alpha\}$; and

(ii) $h(z) - [h(z)]_\alpha$ is analytic on U.

Thus, $[h(z)]_\alpha$ is given by the negative part of the Laurent series of $h(z)$ at α.

To see (i), given $z \in U \setminus \{\alpha\}$, choose a contour γ_1 around α not containing z to calculate the residue, as in Figure 1.

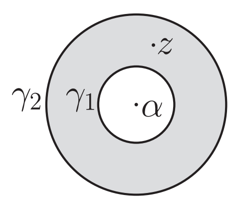


Figure 1.

Now simply differentiate under the integral sign to prove analyticity at z. To see (ii), consider Figure 1 to obtain

$$h(z) = -\operatorname*{Res}_{w=z} \frac{h(w)dw}{z-w} = \frac{1}{2\pi i}\int_{\gamma_1-\gamma_2} \frac{h(w)dw}{z-w} = [h(z)]_\alpha - \frac{1}{2\pi i}\int_{\gamma_2} \frac{h(w)dw}{z-w}$$

The integral around γ_2 is analytic in $z \in U$ again, since we can differentiate under the integral sign.

Suppose that C is a rational curve and that z is a rational parameter, as will be the case in our applications. If $h(z)$ has a pole at α then $[h(z)]_\alpha$ is a polynomial in $1/(z-\alpha)$ (or in z when $\alpha = \infty$.) Up to a constant, any rational function is the sum of its principal parts, and such a decomposition is commonly known as its partial fraction decomposition. It follows that any rational differential is equal to the sum of its principal parts (without any constant ambiguity.)

The principal part of a function or multidifferential $h(z_1, \ldots, z_n)$ of several variables at the point $z_1 = \alpha$ is defined via equation (14), where we interpret z_j for $j > 1$ as constants. We denote it by $[h(z_1, \ldots, z_n)]_{z_1=\alpha}$ or simply $[h(z_1, \ldots, z_n)]_\alpha$, when z_1 is understood.

For $h(z)$ analytic on $U \setminus \{\alpha\}$, we have

$$\left[\frac{h(z_1)}{z_1 - z_2}\right]_{z_1=\alpha} = \operatorname*{Res}_{w=\alpha} \frac{h(w)\,dw}{(w-z_2)(z_1-w)}.$$

Consider the contour containing z_1 and not z_2 as in Figure 2 below.

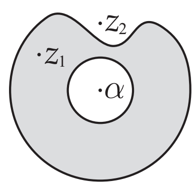


Figure 2.

Integrating over this contour, we have the fact that

(i) $[h(z_1)/(z_1 - z_2)]_\alpha$ is analytic in z_1 and z_2 on $U \setminus \{\alpha\}$; and

(ii) $h(z_1)/(z_1 - z_2) - [h(z_1)/(z_1 - z_2)]_\alpha$ is analytic in z_1 on U.

Note that in (ii), we allow z_1 to take the value $z_1 = \alpha$ whereas we do not allow z_2 to take the value $z_2 = \alpha$. Note also that

$$\left[\frac{h(z_1)}{z_1 - z_2}\right]_{z_1=\alpha} = \left[\frac{h(z_1) - h(z_2)}{z_1 - z_2}\right]_{z_1=\alpha} = \left[\frac{h(z_2)}{z_2 - z_1}\right]_{z_2=\alpha}. \tag{15}$$

In particular, if $h(z)$ is analytic at α, then $[h(z_1)/(z_1-z_2)]_\alpha$.

The principal part of a function with respect to more than one variable depends on the order and so is slightly subtle. For example,

$$\left[\left[\frac{1}{z_1z_2}\right]_{z_1=0}\right]_{z_2=0}=\left[\frac{1}{z_1z_2}\right]_{z_2=0}=\frac{1}{z_1z_2},$$

so we see that principal parts at 0 with respect to z_1 and z_2 commute on $\frac{1}{z_1z_2}$ and more generally for any product $h_1(z_1)h_2(z_2)$. On the other hand,

$$\begin{aligned}\left[\left[\frac{1}{z_1(z_1-z_2)}\right]_{z_1=0}\right]_{z_2=0}&=\left[-\frac{1}{z_1z_2}\right]_{z_2=0}=-\frac{1}{z_1z_2}\\ \left[\left[\frac{1}{z_1(z_1-z_2)}\right]_{z_2=0}\right]_{z_1=0}&=\Big[0\Big]_{z_1=0}=0,\end{aligned}$$

so they do not commute on $\frac{1}{z_1(z_1-z_2)}$. In this paper, whenever we have a function of several variables, we only take the principal part with respect to one of the variables, so the subtlety described here never arises.

4.2. Principal parts of Eynard–Orantin invariants

An important property of the stable Eynard–Orantin invariants is that they are meromorphic multidifferentials with poles at the zeros of dx. In particular, on a rational curve they are rational multidifferentials and hence equal to the sum of their principal parts.

Each summand at a zero α of dx on the right side of the defining recursion (13) for $\omega_n^g(z_1,\ldots,z_n)$ has dependence on z_1 only occurring as $1/(z-z_1)$ and $1/(\hat{z}-z_1)$. Now express equation (13) as $\omega_n^g=\sum_\alpha I_\alpha$ with

$$\begin{aligned}I_\alpha=\operatorname*{Res}_{z=\alpha}&\left(\frac{dz_1}{z-z_1}-\frac{dz_1}{\hat{z}-z_1}\right)\frac{x(z)}{2(y(\hat{z})-y(z))dx(z)}\\ &\times\left[\omega_{n+1}^{g-1}(z,\hat{z},z_{S'})+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S'}}\omega_{|I|+1}^{g_1}(z,z_I)\omega_{|J|+1}^{g_2}(\hat{z},z_J)\right].\end{aligned}$$

One can differentiate I_α with respect to z_1 under the integral sign, showing that it is analytic everywhere except possibly at $z_1=\alpha$. Note that I_α is analytic at $z_1 = \infty$ (assuming $\alpha \neq \infty$), since

$$K(z_1, z) \sim \frac{(\hat{z} - z)x(z)}{2(y(\hat{z}) - y(z))x'(z)} \frac{1}{dz} \frac{dz_1}{z_1^2}.$$

So for some C constant in z_1, $I_\alpha \sim \frac{C dz_1}{z_1^2}$, which is analytic at $z_1 = \infty$. Thus, I_α is rational and equal to its principal part at $z_1 = \alpha$. It follows that I_α is the principal part of $\omega_n^g(z_1, \ldots, z_n)$ at $z_1 = \alpha$.

Furthermore, we can calculate I_α since

$$\begin{aligned} I_\alpha &= \operatorname*{Res}_{z=\alpha} \left(\frac{\eta(z)}{z - z_1} - \frac{\eta(\hat{z})}{\hat{z} - z_1} \right) dz_1 = 2 \operatorname*{Res}_{z=\alpha} \frac{\eta(z)}{z - z_1} dz_1 = -2[\eta(z_1)]_\alpha \\ &= -\Bigg[\frac{x(z_1)}{(y(\hat{z}_1) - y(z_1)) dx(z_1)} \\ &\qquad \times \Bigg(\omega_{n+1}^{g-1}(z_1, \hat{z}_1, z_{S'}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S'}} \omega_{|I|+1}^{g_1}(z_1, z_I) \omega_{|J|+1}^{g_2}(\hat{z}_1, z_J) \Bigg) \Bigg]_\alpha . \end{aligned}$$

Here, $\eta(z)$ is a differential form that satisfies $\eta(z) = -\eta(\hat{z})$, since $x(z) = x(\hat{z})$ and ω_{n+1}^{g-1} is symmetric in its arguments. Observe that $I_\alpha(z_1) = -I_\alpha(\hat{z}_1)$. In summary, we have proven the following result.

Proposition 16. *The recursion of equation* (13) *expresses any Eynard–Orantin invariant as the sum of its principal parts.*

$$\begin{aligned} \omega_n^g(z_1, z_{S'}) = -\sum_\alpha \Bigg[\frac{x(z_1)}{(y(\hat{z}_1) - y(z_1)) dx(z_1)} \Bigg(&\omega_{n+1}^{g-1}(z_1, \hat{z}_1, z_{S'}) \\ &+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S'}}^{\circ} \omega_{|I|+1}^{g_1}(z_1, z_I) \omega_{|J|+1}^{g_2}(\hat{z}_1, z_J) \Bigg) \Bigg]_\alpha \end{aligned} \tag{16}$$

Here, the sum is over the zeros α of dx and we use the notation $S' = \{2, \ldots, n\}$. The $\circ$ over the inner summation denotes the fact that we exclude the terms with $(g_1, |I|) = (0, 0)$ or $(g_2, |J|) = (0, 0)$.

The principal parts of the stable ω_n^g in the right side of equation (16) are straightforward because their pole structure is rather simple. The terms involving ω_2^0 are less straightforward, so later we will require the following result.

Lemma 17.

$$\left[\left(\frac{dz_1}{z_1-z_2}+\frac{d\hat{z}_1}{\hat{z}_1-z_2}\right)\frac{x_1}{dx_1}\right]_\alpha=\left[\frac{x_1}{x_1-x_2}\right]_\alpha.$$

Proof. Note that we express the terms on the left side as quotients of differentials instead of functions for later ease. Locally, we have

$$x(z_1)-x(z_2)=(z_1-z_2)(\hat{z}_1-z_2)h(z_1,z_2), \tag{17}$$

where $h(z_1,z_2)$ is analytic and non-zero at $z_1=\alpha$ and $h(z_1,z_2)=h(\hat{z}_1,z_2)$. By the remark after Lemma 14, since $\log h(z_1,z_2)$ is analytic at $z_1=\alpha$ and invariant under $z_1\mapsto\hat{z}_1$, then $x_1\frac{d}{dx_1}\log h(z_1,z_2)$ is analytic at $z_1=\alpha$. Hence, the right side of

$$x_1\frac{d}{dx_1}\log h(z_1,z_2)=\frac{x_1}{x_1-x_2}-\left(\frac{dz_1}{z_1-z_2}+\frac{d\hat{z}_1}{\hat{z}_1-z_2}\right)\frac{x_1}{dx_1} \tag{18}$$

is analytic at $z_1=\alpha$ and the lemma follows. □

Take the exterior derivative of equation (18) with respect to the second variable to obtain

$$\left[\left(\frac{dz_1dz_2}{(z_1-z_2)^2}+\frac{d\hat{z}_1dz_2}{(\hat{z}_1-z_2)^2}\right)\frac{x_1}{dx_1}\right]_\alpha=\left[\frac{x_1dx_2}{(x_1-x_2)^2}\right]_\alpha. \tag{19}$$

Hence, the expression

$$\frac{dx_1dx_2}{(x_1-x_2)^2}-\frac{dz_1dz_2}{(z_1-z_2)^2}-\frac{d\hat{z}_1dz_2}{(\hat{z}_1-z_2)^2}$$

is analytic at $z_1=\alpha$ and vanishes there for all z_2 and, in particular, for $z_2=\alpha$.

Observe that x appears in the definition only via the expression $\frac{dx}{x}$, which is rational for the spectral curve

$$x(z)=z\exp(-z^a)\qquad\text{and}\qquad y(z)=z^a,$$

which we are interested in. Hence, the kernel K is also rational and is given by

$$K(z_1,z)=\frac{z}{2(\hat{z}^a-z^a)(1-az^a)}\left(\frac{1}{z-z_1}-\frac{1}{\hat{z}-z_1}\right)\frac{dz_1}{dz}.$$

By the construction of Eynard–Orantin invariants as a sum of their principal parts, one easily sees that ω_n^g has invariance properties under the local involutions $\sigma_\alpha(z)$ defined near each zero α of dx. In the language of Section 3.1, we see that for $x = z\exp(-z^a)$ and $2g-2+n>0$, $\omega_n^g(z_1,\ldots,z_n) \in \mathcal{A}_{x_i}$ for each $i = 1,\ldots,n$.

Remark. Eynard chooses a basis $d\xi_{\alpha,m}$ of $\mathcal{A}_x$, linearly related to the basis $d\xi_k^{(r)}(z)$ defined in Section 3.1 [13, 14]. He identifies the coefficients in terms of intersection numbers over a moduli space $\overline{\mathcal{M}}_{g,n}^a$ of a-coloured Riemann surfaces via the formula

$$\omega_n^g(z_1,\ldots,z_n) = \sum_{i_1,\ldots,i_n}\sum_{d_1,\ldots,d_n} A_n^{(g)}(i_1,d_1;\ldots;i_n,d_n)\prod d\xi_{\alpha_{i_k},d_k}(z_k),$$

where $\alpha_1,\ldots,\alpha_a$ are the zeros of dx. Essentially, Eynard showed that the Eynard–Orantin invariants give cohomological field theories, a statement that was made more precise in the work of Dunin-Barkowski, Orantin, Shadrin and Spitz [10]. Eynard described his result as a generalised ELSV formula. It is intriguing that the ELSV-type formula in this paper transforms linearly to Eynard's formula and hence, we see a relationship between intersection numbers over $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_a)$ and intersection numbers over $\overline{\mathcal{M}}_{g,n}^a$.

5. Proof of main theorem

In this section, we prove that the Eynard–Orantin invariants of the spectral curve given by

$$x(z) = z\exp(-z^a) \qquad \text{and} \qquad y(z) = z^a$$

coincide with the total derivatives of the orbifold Hurwitz number generating functions. The strategy of proof is quite natural. Since the Eynard–Orantin recursion expresses the invariants as a sum over the principal parts in the first variable, we will analyse the principal parts of equation (5). Furthermore, the principal parts of the Eynard–Orantin invariants are antisymmetric with respect to the local involutions at each zero of dx, so we take the antisymmetric part of the principal part of equation (5). (In actual fact, we take the symmetric part of the principal part of equation (5), since it contains an extra antisymmetric factor.) This is exactly the strategy of proof used in the proof of the Bouchard–Mariño conjecture [16].

Proof of Theorem 1. Recall from Section 3 that

$$F_{g,n}^{[a]}(z_1,\ldots,z_n) := H_{g,n}^{[a]}(x(z_1),\ldots,x(z_n))$$

is a rational function of the z_i. Equivalently, $H_{g,n}^{[a]}(x_1, \ldots, x_n)$ gives a local expansion of the rational function $F_{g,n}^{[a]}(z_1, \ldots, z_n)$ in the local coordinate $x(z_i)$ around $z_i = 0$. Furthermore, $x = z\exp(-z^a)$ defines local involutions $\sigma_\alpha(z)$ near each zero α of dx. For $2g - 2 + n > 0$, it follows from Corollary 15 that

- $F_{g,n}^{[a]}(z_1, \ldots, z_n)$ has poles only at $\{z_i = \alpha \mid dx(\alpha) = 0\}$; and
- $F_{g,n}^{[a]}(z_1, \ldots, z_i, \ldots, z_n) + F_{g,n}^{[a]}(z_1, \ldots, \sigma_\alpha(z_i), \ldots, z_n)$ is analytic at $z_i = \alpha$.

Equation (5) is satisfied locally by $H_{g,n}^{[a]}(x_1, \ldots, x_n)$ but is satisfied globally by $F_{g,n}^{[a]}(z_1, \ldots, z_n)$. Recall the notation $S = \{1, \ldots, n\}$ and $S' = \{2, \ldots, n\}$. For $2g - 2 + n > 1$, we have

$$\begin{aligned}
(20)\qquad &\left(2g-2+n+\frac{1}{a}\sum_{i=1}^{n}(1-az_i^a)x_i\frac{\partial}{\partial x_i}\right)F_{g,n}^{[a]}(z_S))\\
&= \sum_{i<j}\frac{\left(\frac{z_j}{1-az_i^a}x_i\frac{\partial}{\partial x_i} - \frac{z_i}{1-az_j^a}x_j\frac{\partial}{\partial x_j}\right)}{z_i - z_j}\left(F_{g,n-1}^{[a]}(z_{S\setminus\{j\}}) + F_{g,n-1}^{[a]}(z_{S\setminus\{i\}})\right)\\
&\quad + \frac{1}{2}\sum_{i=1}^{n} x(t_1)x(t_2)\frac{\partial^2}{\partial x(t_1)\partial x(t_2)}F_{g-1,n+1}^{[a]}(t_1, t_2, z_{S\setminus\{i\}})\Big|_{t_1=t_2=z_i}\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} x_i\frac{\partial}{\partial x_i}F_{g_1,|I|+1}^{[a]}(z_i, z_I)\,x_i\frac{\partial}{\partial x_i}F_{g_2,|J|+1}^{[a]}(z_i, z_J).
\end{aligned}$$

Take the principal part of equation (20) at $z_1 = \alpha$ and take the invariant part under the involution $\hat{z}_1 := \sigma_\alpha(z_1)$. We have

$$\begin{aligned}
&\left[F_{g,n}^{[a]}(z_1, z_I) + F_{g,n}^{[a]}(\hat{z}_1, z_I)\right]_\alpha\\
&= 0 = \left[x_i\frac{\partial}{\partial x_i}F_{g,n}^{[a]}(z_1, z_I) + x_i\frac{\partial}{\partial x_i}F_{g,n}^{[a]}(\hat{z}_1, z_I)\right]_\alpha,
\end{aligned}$$

for $i = 1, \ldots, n$ and any $I \subset S'$, since $F_{g,n}^{[a]}(z_1, z_I) + F_{g,n}^{[a]}(\hat{z}_1, z_I)$ and $x_i\frac{\partial}{\partial x_i}F_{g,n}^{[a]}(z_1, z_I) + x_i\frac{\partial}{\partial x_i}F_{g,n}^{[a]}(\hat{z}_1, z_I)$ are analytic at $z_1 = \alpha$. This annihilates the factor $(2g - 2 + n)$ in the first line as well as all summands not involving z_1 in all four lines of equation (20).

The principal part of the terms in equation (20) involving $1/(z_1 - z_j)$ are calculated as follows. For any $j \neq 1$, put $\mathcal{F}_j(z_1) = x_1\frac{\partial}{\partial x_1}F_{g,n-1}^{[a]}(z_1, z_{S'\setminus\{j\}})$.

Then $\mathcal{F}_j(z_j) = x_j \frac{\partial}{\partial x_j} F^{[a]}_{g,n-1}(z_{S\setminus\{1\}})$ and

$$\begin{aligned}
&\left[\frac{\frac{z_j}{1-az_1^a}\mathcal{F}_j(z_1) - \frac{z_1}{1-az_j^a}\mathcal{F}_j(z_j)}{z_1 - z_j}\right]_\alpha \\
&= \left[\frac{\frac{z_1}{1-az_1^a}\mathcal{F}_j(z_1) - \frac{z_j}{1-az_j^a}\mathcal{F}_j(z_j)}{z_1 - z_j} - \frac{\mathcal{F}_j(z_1)}{1-az_1^a} - \frac{\mathcal{F}_j(z_j)}{1-az_j^a}\right]_\alpha \\
&= \left[\frac{\frac{z_1}{1-az_1^a}\mathcal{F}_j(z_1)}{z_1 - z_j}\right]_\alpha - \left[\frac{\mathcal{F}_j(z_1)}{1-az_1^a}\right]_\alpha \\
&= \left[\frac{\frac{z_1}{1-az_1^a}\mathcal{F}_j(z_1)}{z_1 - z_j}\right]_\alpha + c_j.
\end{aligned}$$

Here, we have used equation (15) and the fact that $\frac{\mathcal{F}_j(z_j)}{1-az_j^a}$ is independent of z_1 and hence, annihilated by taking principal parts. Note that $c_j := -\left[\frac{\mathcal{F}_j(z_1)}{1-az_1^a}\right]_\alpha$ is independent of z_j. Therefore, the invariant part of the principal part of equation (20) can now be written as

$$\begin{aligned}
(21)\qquad &\left[(\hat{z}_1^a - z_1^a)x_1\frac{\partial}{\partial x_1}F^{[a]}_{g,n}(z_1, z_{S'})\right]_\alpha \\
&= \left[\sum_{j=2}^n \frac{\frac{z_1}{1-az_1^a}x_1\frac{\partial}{\partial x_1}F^{[a]}_{g,n-1}(z_1, z_{S'\setminus\{j\}})}{z_1 - z_j} + \frac{\frac{\hat{z}_1}{1-a\hat{z}_1^a}x_1\frac{\partial}{\partial x_1}F^{[a]}_{g,n-1}(\hat{z}_1, z_{S'\setminus\{j\}})}{\hat{z}_1 - z_j}\right]_\alpha \\
&+ \left[\frac{1}{2}x(t_1)x(t_2)\left(\frac{\partial^2}{\partial x(t_1)\partial x(t_2)}F^{[a]}_{g-1,n+1}(t_1, t_2, z_{S'})\Big|_{t_1=t_2=z_1}\right.\right. \\
&\qquad\left.\left. + \frac{\partial^2}{\partial x(t_1)\partial x(t_2)}F^{[a]}_{g-1,n+1}(t_1, t_2, z_{S'})\Big|_{t_1=t_2=\hat{z}_1}\right)\right]_\alpha \\
&+ \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S'}}^{\text{stable}} \left[x_1\frac{\partial}{\partial x_1}F^{[a]}_{g_1,|I|+1}(z_1, z_I)x_1\frac{\partial}{\partial x_1}F^{[a]}_{g_2,|J|+1}(z_1, z_J)\right. \\
&\qquad\left. + x_1\frac{\partial}{\partial x_1}F^{[a]}_{g_1,|I|+1}(\hat{z}_1, z_I)x_1\frac{\partial}{\partial x_1}F^{[a]}_{g_2,|J|+1}(\hat{z}_1, z_J)\right]_\alpha + \sum_{j=2}^n c_j'.
\end{aligned}$$

Here, $c_j' := c_j(z_1) + c_j(\hat{z}_1)$ is independent of z_j. There is now a one-to-one correspondence between terms in equation (21) and terms in the Eynard–Orantin recursion (13), as long as we ignore the c_j' terms, which will later be annihilated. We can simplify equation (21) further to obtain the following.

$$\begin{aligned}(22)\quad &\left[(\hat{z}_1^a - z_1^a)x_1\frac{\partial}{\partial x_1}F^{[a]}_{g,n}(z_1,z_{S'})\right]_\alpha \\ &=\sum_{j=2}^{n}\left[\frac{\frac{z_1}{1-az_1^a}x_1\frac{\partial}{\partial x_1}F^{[a]}_{g,n-1}(\hat{z}_1,z_{S'\setminus\{j\}})}{z_1-z_j}+\frac{\frac{\hat{z}_1}{1-a\hat{z}_1^a}x_1\frac{\partial}{\partial x_1}F^{[a]}_{g,n-1}(z_1,z_{S'\setminus\{j\}})}{\hat{z}_1-z_j}\right]_\alpha \\ &+\left[x(t_1)x(t_2)\frac{\partial^2}{\partial x(t_1)\partial x(t_2)}F^{[a]}_{g-1,n+1}(t_1,t_2,z_{S'})\Big|_{t_1=z_1,t_2=\hat{z}_1}\right]_\alpha \\ &+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S'}}^{\text{stable}}\left[x_1\frac{\partial}{\partial x_1}F^{[a]}_{g_1,|I|+1}(z_1,z_I)x_1\frac{\partial}{\partial x_1}F^{[a]}_{g_2,|J|+1}(\hat{z}_1,z_J)\right]_\alpha+\sum_{j=2}^{n}c'_j\end{aligned}$$

Define $\mathcal{F}(z_1)=x_1\frac{\partial}{\partial x_1}F^{[a]}_{g,n-1}(z_1,z_{S\setminus\{j\}})$. The replacement in the first line of (21) to get the first line of (22) uses:

$$(23)\quad \left[\frac{\frac{z_1}{1-az_1^a}\mathcal{F}(z_1)}{z_1-z_j}+\frac{\frac{\hat{z}_1}{1-a\hat{z}_1^a}\mathcal{F}(\hat{z}_1)}{\hat{z}_1-z_j}+\frac{\frac{z_1}{1-az_1^a}\mathcal{F}(\hat{z}_1)}{z_1-z_j}+\frac{\frac{\hat{z}_1}{1-a\hat{z}_1^a}\mathcal{F}(z_1)}{\hat{z}_1-z_j}\right]_\alpha=0$$

which is true since (23) factorises into

$$\begin{aligned}&\left[\Big(\mathcal{F}(z_1)+\mathcal{F}(\hat{z}_1)\Big)\left(\frac{\frac{z_1}{1-az_1^a}}{\hat{z}_1-z_j}+\frac{\frac{\hat{z}_1}{1-a\hat{z}_1^a}}{z_1-z_j}\right)\right]_\alpha \\ &+\left[\Big(\mathcal{F}(z_1)+\mathcal{F}(\hat{z}_1)\Big)\left(\frac{dz_1}{\hat{z}_1-z_j}+\frac{d\hat{z}_1}{z_1-z_j}\right)\frac{x_1}{dx_1}\right]_\alpha \\ =&\left[\Big(\mathcal{F}(z_1)+\mathcal{F}(\hat{z}_1)\Big)\frac{x_1}{x_1-x_j}\right]_\alpha \\ =&\left[\frac{x_1(\mathcal{F}(z_1)+\mathcal{F}(\hat{z}_1))-x_j(\mathcal{F}(z_j)+\mathcal{F}(\hat{z}_j))}{x_1-x_j}\right]_\alpha=0\end{aligned}$$

where we have used $\frac{dx_1}{x_1}=\frac{dz_1(1-az_1^a)}{z_1}=\frac{d\hat{z}_1(1-a\hat{z}_1^a)}{\hat{z}_1}$ in the first line and the second line uses Lemma 17 together with the fact that $(\mathcal{F}(z_1)+\mathcal{F}(\hat{z}_1)$ is analytic at $z_1=\alpha$. The final expression does not have a pole at $z_1=z_j$ and vanishes since it is an analytic function.

Define the symmetric function of two variables

$$\mathcal{F}_3(t_1,t_2)=x(t_1)x(t_2)\frac{\partial^2}{\partial x(t_1)\partial x(t_2)}F^{[a]}_{g-1,n+1}(t_1,t_2,z_S).$$

so $\mathcal{F}_3(t_1,t_2)+\mathcal{F}_3(\hat{t}_1,t_2)$ is analytic at $t_1=\alpha$ and $\mathcal{F}_3(t_1,t_2)+\mathcal{F}_3(t_1,\hat{t}_2)$ is analytic at $t_2=\alpha$. Thus $\mathcal{F}_3(t_1,t_2)+\mathcal{F}_3(\hat{t}_1,t_2)+\mathcal{F}_3(t_1,\hat{t}_2)+\mathcal{F}_3(\hat{t}_1,\hat{t}_2)$ is analytic at $t_1=\alpha$ and $t_2=\alpha$ so

$$\Big[\mathcal{F}_3(z_1,z_1)+\mathcal{F}_3(\hat{z}_1,\hat{z}_1)+\mathcal{F}_3(z_1,\hat{z}_1)+\mathcal{F}_3(\hat{z}_1,z_1)\Big]_\alpha=0$$

which gives the second line of (22). For any choice of $I\subset S$, and $J=S-I$, put

$$\mathcal{F}_1(z_1)=x_1\frac{\partial}{\partial x_1}F^{[a]}_{g_1,|I|+1}(z_1,z_I),\quad \mathcal{F}_2(z_1)=x_1\frac{\partial}{\partial x_1}F^{[a]}_{g_2,|J|+1}(\hat{z}_1,z_J)$$

Then $\mathcal{F}_1(z_1)+\mathcal{F}_1(\hat{z}_1)$ and $\mathcal{F}_2(z_1)+\mathcal{F}_2(\hat{z}_1)$ are analytic at $z_1=\alpha$ and so is their product. Hence

$$\Big[\mathcal{F}_1(z_1)\mathcal{F}_2(z_1)+\mathcal{F}_1(\hat{z}_1)\mathcal{F}_2(\hat{z}_1)+\mathcal{F}_1(z_1)\mathcal{F}_2(\hat{z}_1)+\mathcal{F}_1(\hat{z}_1)\mathcal{F}_2(z_1)\Big]_\alpha=0$$

which gives the third line of (22).

Note that

$$d_{z_2}\cdots d_{z_n}x_1\frac{\partial}{\partial x_1}F^{[a]}_{g,n}(z_1,\dots,z_n)=\Omega^{[a]}_{g,n}(z_1,\dots,z_n)\frac{x_1}{dx_1}.$$

So act on (22) by $d_{z_2}\cdots d_{z_n}$. (The d_{z_1} derivatives are already present.)

$$\begin{aligned}
&\left[(\hat{z}_1^a-z_1^a)\,\frac{x_1}{dx_1}\Omega^{[a]}_{g,n}(z_1,z_{S'})\right]_\alpha \\
&=\sum_{j=2}^n\left[\frac{x_1}{dx_1}\left(\frac{\frac{z_1}{1-az_1^a}\Omega^{[a]}_{g,n-1}(\hat{z}_1,z_{S'\setminus\{j\}})dz_j}{(z_1-z_j)^2}\right.\right. \\
&\qquad\left.\left.+\frac{\frac{\hat{z}_1}{1-a\hat{z}_1^a}\Omega^{[a]}_{g,n-1}(z_1,z_{S'\setminus\{j\}})dz_j}{(\hat{z}_1-z_j)^2}\right)\right]_\alpha \\
&\quad+\left[\frac{x_1^2}{dx_1^2}\Omega^{[a]}_{g-1,n+1}(z_1,\hat{z}_1,z_{S'})\right]_\alpha \\
&\quad+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S'}}^{\text{stable}}\left[\frac{x_1^2}{dx_1^2}\Omega^{[a]}_{g_1,|I|+1}(z_1,z_I)\Omega^{[a]}_{g_2,|J|+1}(\hat{z}_1,z_J)\right]_\alpha.
\end{aligned}\tag{24}$$

Note that $d_{z_2}\cdots d_{z_n}\sum_{j=2}^n c'_j=0$ since $d_{z_j}c'_j=0$.

Take

$$\left[\frac{dx_1}{x_1}\frac{1}{(\hat{z}_1^a - z_1^a)}(24)\right]_\alpha$$

i.e., multiply the principal parts in (24) by $\frac{dx_1}{x_1}\frac{1}{(\hat{z}_1^a - z_1^a)}$ which is analytic at $z_1 = \alpha$ so can pass inside principal parts by taking principal parts again. Substitute the identity

$$\frac{z_1 dz_j}{(1 - az_1^a)(z_1 - z_j)^2} = \frac{x_1}{dx_1}\frac{dz_1 dz_j}{(z_1 - z_j)^2} = \frac{x_1}{dx_1}\omega_{0,2}(z_1, z_j)$$

to get for $2g - 2 + n > 1$,

$$\begin{aligned}
\text{(25)}\qquad &\left[\Omega^{[a]}_{g,n}(z_1, z_{S'})\right]_\alpha \\
&= \sum_{j=2}^{n}\left[\frac{1}{\hat{z}_1^a - z_1^a}\frac{x_1}{dx_1}\Big(\Omega^{[a]}_{g,n-1}(\hat{z}_1, z_{S'\setminus\{j\}})\omega_{0,2}(z_1, z_j)\right. \\
&\qquad\left. + \Omega^{[a]}_{g,n-1}(z_1, z_{S'\setminus\{j\}})\omega_{0,2}(\hat{z}_1, z_j)\Big)\right]_\alpha \\
&\quad + \left[\frac{1}{\hat{z}_1^a - z_1^a}\frac{x_1}{dx_1}\Omega^{[a]}_{g-1,n+1}(z_1, \hat{z}_1, z_{S'})\right]_\alpha \\
&\quad + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=S'}}^{\text{stable}}\left[\frac{1}{\hat{z}_1^a - z_1^a}\frac{x_1}{dx_1}\Omega^{[a]}_{g_1,|I|+1}(z_1, z_I)\Omega^{[a]}_{g_2,|J|+1}(\hat{z}_1, z_J)\right]_\alpha.
\end{aligned}$$

Now (25) agrees with the Eynard–Orantin recursion expressed in terms of its principal parts in (16) for the kernel

$$K(z_1, z) = \frac{z}{2(\hat{z}^a - z^a)(1 - az^a)}\left(\frac{1}{z - z_1} - \frac{1}{\hat{z} - z_1}\right)\frac{dz_1}{dz}.$$

Hence $\Omega^{[a]}_{g,n}(z)$ satisfies the Eynard–Orantin recursion (13) as required.

To complete the proof we need to show the base cases $(g, n) = (0, 3)$ and $(1, 1)$ agree since until now we have used Proposition 7 which requires $2g - 2 + n > 1$.

For $\Omega^{[a]}_{0,3}(z_1, z_2, z_3)$ we use Proposition 8. We consider only the principal parts of $\Omega^{[a]}_{0,3}(z_1, z_2, z_3)$ since it is rational and hence determined by its principal parts. It is clear from (7) that $F^{[a]}_{0,3}$ can only have simple poles hence has principal part

$$\left[F^{[a]}_{0,3}\right]_\alpha = \frac{\lambda}{(z_1-\alpha)(z_2-\alpha)(z_3-\alpha)}$$

for some λ which is easily calculated using (7) to be $\lambda = -\alpha^3/a$. Then the differential $\Omega^{[a]}_{0,3}(z_1, z_2, z_3) = d_{z_1} d_{z_2} d_{z_3} F^{[a]}_{0,3}$ agrees with the Eynard–Orantin invariant which can be given via the direct formula Theorem 4.1 in [17].

$$\begin{aligned}\left[\omega^0_3(z_1, z_2, z_3)\right]_\alpha &= \operatorname*{Res}_{z=\alpha} \frac{\omega^0_2(z, z_1)\omega^0_2(z, z_2)\omega^0_2(z, z_3)x(z)}{dx(z)dy(z)}\\ &= d_{z_1} d_{z_2} d_{z_3} \frac{1}{(z_1-\alpha)(z_2-\alpha)(z_3-\alpha)(\ln x)''(\alpha)y'(\alpha)}\end{aligned}$$

since $-\alpha^3/a = 1/(\ln x)''(\alpha)y'(\alpha)$.

For $\Omega^{[a]}_{1,1}(z_1)$ we take the invariant part of the principal part of (8).

$$\left[(z_1^a - \hat{z}_1^a)\, x_1 \frac{d}{dx_1} F^{[a]}_{1,1}(z_1)\right]_\alpha = \left[\frac{x_1^2}{dx_1^2} \frac{dz_1\, d\hat{z}_1}{(z_1-\hat{z}_1)^2}\right]_\alpha \tag{26}$$

where we have used Lemma 17 to replace the invariant part of the right side of (8). Hence

$$\left[\Omega^{[a]}_{1,1}(z_1)\right]_\alpha = \left[\frac{1}{z_1^a - \hat{z}_1^a} \frac{x_1}{dx_1} \omega_{0,2}(z_1, \hat{z}_1)\right]_\alpha$$

which agrees with the Eynard–Orantin recursion expressed in terms of its principal parts so $\Omega^{[a]}_{1,1}(z_1)$ satisfies the Eynard–Orantin recursion (13) as required. Alternatively, we can use [28] where all 1-point functions on the right side of the ELSV-type formula in Theorem 11 have been calculated. This yields

$$F^{[a]}_{1,1}(z_1) = \frac{a}{24}\xi^{(a)}_1(z) - \frac{1}{24}\xi^{(a)}_0(z)$$

so $\Omega^{[a]}_{1,1}(z_1) = dF^{[a]}_{1,1}(z_1)$ agrees with $\omega^1_1(z_1)$ by a direct calculation of (13).

Hence the base cases agree and $\Omega^{[a]}_{g,n}(z_1, \ldots, z_n) = \omega^g_n(z_1, \ldots, z_n)$ as required. □

6. String and dilaton equations

The general Eynard–Orantin theory of topological recursion includes string and dilaton equations, which relate ω^g_{n+1} and ω^g_n.

Theorem 18 (String and dilaton equations). *The Eynard–Orantin invariants satisfy the following, where the summations are over the zeros of dx on the spectral curve and Φ satisfies $d\Phi = y\frac{dx}{x}$.*

$$\sum_\alpha \operatorname*{Res}_{z=\alpha} y(z)\,\omega^g_{n+1}(z, z_S) = -\sum_{k=1}^{n} dz_k \frac{\partial}{\partial z_k}\left[\omega^g_n(z_S)\frac{x_k}{dx_k}\right] \tag{27}$$

$$\sum_\alpha \operatorname*{Res}_{z=\alpha} \Phi(z)\,\omega^g_{n+1}(z_S, z) = (2-2g-n)\,\omega^g_n(z_S). \tag{28}$$

These are modified versions of equation (A.26) and Theorem 4.7 from the original paper of Eynard and Orantin [17]. The adjustment is due to our use of the exponentiated form of x, which effectively requires us to use $\frac{dx}{x}$ in place of dx.

A consequence of Theorem 11 is that orbifold Hurwitz numbers can be expressed as

$$H^{[a]}_g(\mu_1, \ldots, \mu_n) = a^{1-g+\sum\{\mu_i/a\}} Q^{[a]}_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} C(\mu_i),$$

where $C(\mu) = \frac{\mu^{\lfloor \mu/a \rfloor}}{\lfloor \mu/a \rfloor!}$ and $Q^{[a]}_{g,n}$ is a quasi-polynomial modulo a. The string and dilaton equations for orbifold Hurwitz numbers provide a relation between the quasi-polynomials $Q^{[a]}_{g,n+1}$ and $Q^{[a]}_{g,n}$. The above equation defines the values $Q^{[a]}_{g,n}(\mu_1, \ldots, \mu_n)$ for positive integers $\mu_1, \ldots, \mu_n$. Since $Q^{[a]}_{g,n}$ is a quasi-polynomial, we can naturally extend its domain to all integers $\mu_1, \mu_2, \ldots, \mu_n$. In particular, it makes sense to evaluate these quasi-polynomials at zero.

Theorem 19 (String equation for orbifold Hurwitz numbers).

$$Q^{[a]}_{g,n+1}(\mu_1, \ldots, \mu_n, 0) = (\mu_1 + \cdots + \mu_n)\, Q^{[a]}_{g,n}(\mu_1, \ldots, \mu_n).$$

Theorem 20 (Dilaton equation for orbifold Hurwitz numbers).

$$\frac{\partial Q^{[a]}_{g,n+1}}{\partial \mu_{n+1}}(\mu_1, \ldots, \mu_n, 0) = (2-2g-n)\, Q^{[a]}_{g,n}(\mu_1, \ldots, \mu_n).$$

The quasi-polynomial behaviour of $Q^{[a]}_{g,n}$ allows us to express the Eynard–Orantin invariants in the following way. The constants $A^{r_1,\ldots,r_n}_{k_1,\ldots,k_n}$ are the coefficients of the polynomials governing $Q^{[a]}_{g,n}$ for particular classes modulo a, which are essentially Hurwitz–Hodge integrals by Theorem 11.

$$
\begin{aligned}
\omega_n^g &= \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} a^{1-g+\sum\{\mu_i/a\}} Q_{g,n}^{[a]}(\mu_1,\ldots,\mu_n) \prod_{i=1}^{n} C(\mu_i)\mu_i x_i^{\mu_i} \frac{dx_i}{x_i} \\
&= \sum_{r_1,\ldots,r_n=1}^{a} a^{1-g+\sum\{r_i/a\}} \sum_{\mu_1\equiv r_1,\ldots,\mu_n\equiv r_n} \sum_{k_1,\ldots,k_n=0}^{\text{finite}} A_{k_1,\ldots,k_n}^{r_1,\ldots,r_n} \prod_{i=1}^{n} C(\mu_i)\mu_i^{k_i+1} x_i^{\mu_i} \frac{dx_i}{x_i} \\
&= \sum_{r_1,\ldots,r_n=1}^{a} a^{1-g+\sum\{r_i/a\}} \sum_{k_1,\ldots,k_n=0}^{\text{finite}} A_{k_1,\ldots,k_n}^{r_1,\ldots,r_n} \prod_{i=1}^{n} \xi_{k_i+1}^{(r_i)}(x_i) \frac{dx_i}{x_i}.
\end{aligned}
$$

Proof of Theorem 19. We begin with the following residue calculation.

$$
\sum_{\alpha} \operatorname*{Res}_{z=\alpha} y(z)\xi_k^{(r)}(z)\frac{dx}{x} = \begin{cases} 1 & \text{for } k=1 \text{ and } r=a \\ 0 & \text{otherwise.} \end{cases}
$$

One can prove by induction that

$$
\xi_k^{(r)}(z) = \frac{z^r p_k(z^a)}{(1-az^a)^{2k+1}}
$$

for positive integers k, where p_k is a polynomial of degree k for $1 \le r \le a-1$ and of degree $k-1$ for $r=a$. It follows that the residue

$$
\sum_{\alpha} \operatorname*{Res}_{z=\alpha} y(z)\xi_k^{(r)}(z)\frac{dx}{x} = -\operatorname*{Res}_{z=\infty} z^a \frac{1-az^a}{z} \xi_k^{(r)}(z)\ dz
$$

can be non-zero only when $k=1$ and $r=a$. In this case, the residue can be computed explicitly.

$$
\begin{aligned}
-\operatorname*{Res}_{z=\infty} z^a \frac{1-az^a}{z} \xi_1^{(a)}(z)\ dz &= -\operatorname*{Res}_{z=\infty} \frac{a^2 z^{2a-1}}{(1-az^a)^2}\ dz \\
&= \operatorname*{Res}_{z=0} \frac{a^2}{(z^a-a)^2}\frac{1}{z}\ dz = 1.
\end{aligned}
$$

Now consider the left side of equation (27).

$$\begin{aligned}
&\sum_{\alpha} \operatorname*{Res}_{z=\alpha} y(z)\omega_{n+1}^{g}(z, z_S) \\
&= \sum_{r,r_1,\dots,r_n=1}^{a} a^{1-g+\{r/a\}+\sum\{r_i/a\}} \\
&\quad \times \sum_{k,k_1,\dots,k_n=0}^{\text{finite}} A_{k_1,\dots,k_n,k}^{r_1,\dots,r_n,r} \prod_{i=1}^{n} \xi_{k_i+1}^{(r_i)}(x_i)\frac{dx_i}{x_i} \sum_{\alpha} \operatorname*{Res}_{z=\alpha} y(z)\xi_{k+1}^{(r)}\frac{dx}{x} \\
&= \sum_{r_1,\dots,r_n=1}^{a} a^{1-g+\sum\{r_i/a\}} \sum_{k_1,\dots,k_n=0}^{\text{finite}} A_{k_1,\dots,k_n,0}^{r_1,\dots,r_n,a} \prod_{i=1}^{n} \xi_{k_i+1}^{(r_i)}(x_i)\frac{dx_i}{x_i} \\
&= \sum_{\mu_1,\dots,\mu_n=1}^{\infty} a^{1-g+\sum\{r_i/a\}} Q_{g,n+1}^{[a]}(\mu_1,\dots,\mu_n,0) \prod_{i=1}^{n} C(\mu_i)\mu_i x_i^{\mu_i}\frac{dx_i}{x_i}.
\end{aligned}$$

For the right side of equation (27), we use the fact that $dz_k \frac{\partial}{\partial z_k} x_k^{\mu_k} = \mu_k x_k^{\mu_k} \frac{dx_k}{x_k}$.

$$\begin{aligned}
&\sum_{k=1}^{n} dz_k \frac{\partial}{\partial z_k}\left[\omega_n^g(z_S)\frac{x_k}{dx_k}\right] \\
&= \sum_{k=1}^{n} dz_k \frac{\partial}{\partial z_k}\left[\frac{x_k}{dx_k} \sum_{\mu_1,\dots,\mu_n=1}^{\infty} H_g^{[a]}(\mu_1,\dots,\mu_n) \prod_{i=1}^{n} \mu_i x_i^{\mu_i}\frac{dx_i}{x_i}\right] \\
&= \sum_{k=1}^{n} \sum_{\mu_1,\dots,\mu_n=1}^{\infty} \mu_k H_g^{[a]}(\mu_1,\dots,\mu_n) \prod_{i=1}^{n} \mu_i x_i^{\mu_i}\frac{dx_i}{x_i} \\
&= \sum_{\mu_1,\dots,\mu_n=1}^{\infty} (\mu_1+\cdots+\mu_n) a^{1-g+\sum\{\mu_i/a\}} Q_{g,n}^{[a]}(\mu_1,\dots,\mu_n) \prod_{i=1}^{n} C(\mu_i)\mu_i x_i^{\mu_i}\frac{dx_i}{x_i}.
\end{aligned}$$

Now compare coefficients of $\prod x_i^{\mu_i} \frac{dx_i}{x_i}$ for both of these expressions to yield the desired result. □

Proof of Theorem 20. We begin with the following residue calculation.

$$\sum_{\alpha} \operatorname*{Res}_{z=\alpha} \Phi(z)\xi_k^{(r)}(z)\frac{dx}{x} = \begin{cases} -1 & \text{for } k=2 \text{ and } r=a \\ 0 & \text{otherwise.} \end{cases}$$

The equation $d\Phi = y\frac{dx}{x}$ implies that we may write $\Phi = \frac{1}{a}z^a - \frac{1}{2}z^{2a}$. For brevity, we omit the details of the remainder of the proof, which uses the same strategy as the proof of Theorem 19. □

Let $\widehat{Q}^{[a]}_{g,n}$ denote the polynomial which governs the quasi-polynomial $Q^{[a]}_{g,n}$ in the case that all entries are divisible by a. Although the string and dilaton equations are not recursive by nature, they do allow us to uniquely determine these polynomials for low genus.

Corollary 21. *In genus 0, we have the closed formula* $\widehat{Q}_{0,n}(\mu_1, \ldots, \mu_n) = \frac{1}{a}(\mu_1 + \cdots + \mu_n)^{n-3}$*. In genus 1, the polynomial* $\widehat{Q}_{1,n+1}$ *can be effectively determined from* $\widehat{Q}_{1,n}$ *by the string and dilaton equations.*

Proof. The formula certainly holds for the base cases $n = 1$ and $n = 2$, which correspond to the unstable cases of Theorem 11. Now suppose that the formula is true for some $n \geq 2$. Then the string equation and the inductive hypothesis imply that

$$\begin{aligned}\widehat{Q}^{[a]}_{0,n+1}(\mu_1, \ldots, \mu_n, 0) &= (\mu_1 + \cdots + \mu_n)\, \widehat{Q}^{[a]}_{g,n}(\mu_1, \ldots, \mu_n) \\ &= \frac{1}{a}(\mu_1 + \cdots + \mu_n)^{n-2}.\end{aligned}$$

It follows that

$$\widehat{Q}^{[a]}_{0,n+1}(\mu_1, \ldots, \mu_n, \mu_{n+1}) = \frac{1}{a}(\mu_1 + \cdots + \mu_n)^{n-2} + \mu_{n+1}F(\mu_1, \ldots, \mu_n, \mu_{n+1}).$$

Now use the fact that $\widehat{Q}_{0,n+1}$ is symmetric of degree at most $n-2$, a consequence of Theorem 11. Suppose that it is possible to write down another symmetric polynomial of degree at most $n-2$, which has the form

$$\frac{1}{a}(\mu_1 + \cdots + \mu_n)^{n-2} + \mu_{n+1}G(\mu_1, \ldots, \mu_n, \mu_{n+1}).$$

Then the difference $\mu_{n+1}[F(\mu_1, \ldots, \mu_n, \mu_{n+1}) - G(\mu_1, \ldots, \mu_n, \mu_{n+1})]$ must also be symmetric of degree at most $n-2$. Symmetry implies that, since it is divisible by μ_{n+1}, it must also be divisible by $\mu_1 \cdots \mu_n$. The degree condition now forces the difference to be equal to zero. In other words, the symmetry and degree condition on $\widehat{Q}^{[a]}_{0,n+1}$ uniquely determine F and it follows by induction that $\widehat{Q}^{[a]}_{0,n}(\mu_1, \ldots, \mu_n) = \frac{1}{a}(\mu_1 + \cdots + \mu_n)^{n-3}$.

Now use the same argument and the fact that $\widehat{Q}^{[a]}_{1,n+1}$ is symmetric of degree at most $n+1$. This allows us to determine $\widehat{Q}^{[a]}f_{1,n+1}(\mu_1, \ldots, \mu_n, \mu_{n+1})$

up to the addition of $c\mu_1 \cdots \mu_n \mu_{n+1}$ for some constant c. Now invoke the dilaton equation to determine the value of c. $\square$

Appendix A. Graphical interpretation of Hurwitz numbers

Let us introduce some notation for the set of branched covers enumerated by the orbifold Hurwitz numbers.

Definition 22. For a positive integer a, let $\mathrm{Cov}_g^{[a]}(\mu_1, \mu_2, \ldots, \mu_n)$ be the set of connected genus g branched covers $f : C \to \mathbb{P}^1$ such that

- the preimages of ∞ are labelled $p_1, p_2, \ldots, p_n$ and the divisor $f^{-1}(\infty)$ is equal to $\mu_1 p_1 + \mu_2 p_2 + \cdots + \mu_n p_n$;
- the ramification profile over 0 is given by a partition of the form $(a, a, \ldots, a)$; and
- the only other ramification is simple and occurs over the mth roots of unity.

Note that the weighted count of the branched covers in $\mathrm{Cov}_g^{[a]}(\mu_1, \mu_2, \ldots, \mu_n)$ is equal to $H_{g;\mu}^{[a]} \times |\mathrm{Aut}\ \mu|$. The extra factor appears since we require the branched covers to have labelled preimages of ∞. Thus, it is natural to define the following normalisation of the orbifold Hurwitz numbers.

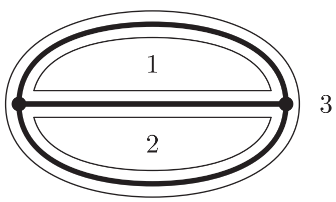
$$K_g^{[a]}(\mu) = H_{g;\mu}^{[a]} \times |\mathrm{Aut}\ \mu|.$$

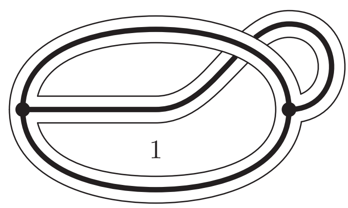
In this appendix, we prove Proposition 4 using an interpretation of Hurwitz numbers as the weighted count of fatgraphs, which appears in the work of Okounkov and Pandharipande [32]. One can informally think of a fatgraph as the 1-skeleton of a finite cell decomposition of a connected oriented surface, where the faces are labelled from 1 up to n. It is useful to consider each edge as the union of two half-edges. The orientation of the underlying surface allows us to define the permutation σ_0 on the set of half-edges that cyclically permutes the half-edges adjacent to a common vertex. The permutation σ_1 denotes the fixed point free involution that swaps two half-edges comprising the same edge. The product $\sigma_2 = \sigma_0\sigma_1$ is the permutation that cyclically permutes the half-edges adjacent to a common face. The following precise definition generalises this notion of a fatgraph by allowing σ_1 to have fixed points, which correspond to half-edges that do not get paired to create an edge. We refer to such half-edges in the fatgraph as *leaves*.

Definition 23. A *fatgraph* is a triple (X, σ_0, σ_1) where X is a finite set, $\sigma_0 : X \to X$ is a permutation, and $\sigma_1 : X \to X$ is an involution. We require that the group generated by σ_0 and σ_1 acts transitively on X and that the elements of $X/\langle\sigma_2\rangle$ are labelled from 1 up to n.

The set $X/\langle\sigma_0\rangle$ is canonically equivalent to the set of vertices of the fatgraph. The set $X/\langle\sigma_1\rangle$ is canonically equivalent to the set of leaves and edges of the fatgraph. Furthermore, the set $X/\langle\sigma_2\rangle$ is canonically equivalent to the set of faces of the fatgraph. The *perimeter* of a face is defined to be the length of the cycle of σ_2 corresponding to the face. We consider two fatgraphs (X, σ_0, σ_1) and (Y, τ_0, τ_1) to be equivalent if there exists a bijection $\phi : X \to Y$ satisfying $\phi \circ \sigma_0 = \tau_0 \circ \phi$ and $\phi \circ \sigma_1 = \tau_1 \circ \phi$, which preserves the face labels. Thus, each fatgraph Γ is endowed with a natural automorphism group $\operatorname{Aut} \Gamma$.

The structure of a fatgraph allows one to thicken the underlying graph to a connected oriented surface with boundary, where the boundary components naturally correspond to the faces. In particular, a fatgraph acquires a type (g, n), where g denotes the genus and n the number of faces. The following diagram shows two distinct fatgraphs — the first of type $(0, 3)$ and the second of type $(1, 1)$ — whose underlying graphs are isomorphic. We use the usual convention whereby the cyclic ordering of the half-edges adjacent to a vertex is induced by the orientation of the page.





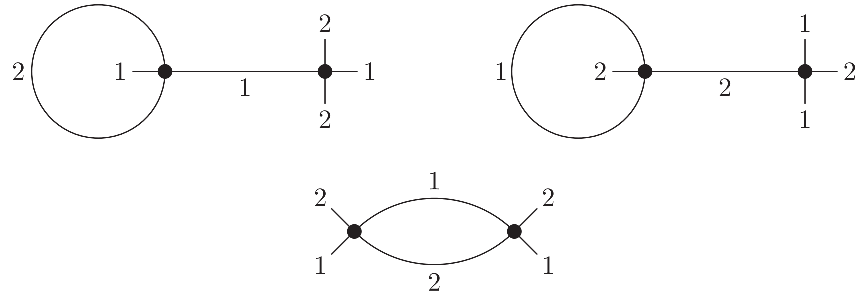
Definition 24. For a positive integer a, let $\mathrm{Fat}_g^{[a]}(\mu_1, \mu_2, \ldots, \mu_n)$ be the set of edge-labelled fatgraphs of type (g, n) such that

- there are $\frac{|\mu|}{a}$ vertices and at each of them there are am adjacent half-edges that are cyclically labelled

$$1, 2, 3, \ldots, m, 1, 2, 3, \ldots, m, \ldots, 1, 2, 3, \ldots, m;$$

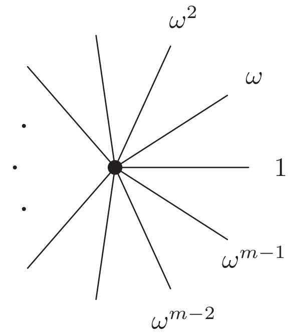
- there are exactly m edges that are labelled $1, 2, 3, \ldots, m$; and
- the perimeters of the faces are given by the tuple $(\mu_1 m, \mu_2 m, \ldots, \mu_n m)$.

Here, we say that an edge is labelled k if its constituent half-edges are both labelled k. For example, the set $\mathrm{Fat}_0^{[2]}(3,1)$ consists of the following three fatgraphs, where the face labels have been omitted for clarity.



Proposition 25. *There is a one-to-one correspondence between* $\mathrm{Cov}_g^{[a]}(\mu_1, \ldots, \mu_n)$ *and* $\mathrm{Fat}_g^{[a]}(\mu_1, \ldots, \mu_n)$ *that preserves automorphism groups. Consequently, the normalised orbifold Hurwitz number* $K_g^{[a]}(\mu)$ *is the weighted count of the fatgraphs in* $\mathrm{Fat}_g^{[a]}(\mu_1, \ldots, \mu_n)$.

Proof. Let Γ_m denote the fatgraph with one vertex obtained by connecting 0 to the mth roots of unity in $\mathbb{P}^1$ by half-edges, as shown in the diagram below. The one-to-one correspondence between $\mathrm{Cov}_g^{[a]}(\mu_1, \mu_2, \ldots, \mu_n)$ and $\mathrm{Fat}_g^{[a]}(\mu_1, \mu_2, \ldots, \mu_n)$ is given by $f \mapsto f^{-1}(\Gamma_m)$.



The vertices correspond to the preimages of 0 and the faces to the preimages of ∞. Preimages of the segment connecting 0 to ω^k correspond to half-edges labelled k, so that the edges correspond to the points where branching occurs. The conditions for a branched cover to be in $\mathrm{Cov}_g^{[a]}(\mu_1, \mu_2, \ldots, \mu_n)$ translate into the conditions for a fatgraph to be in $\mathrm{Fat}_g^{[a]}(\mu_1, \mu_2, \ldots, \mu_n)$.

An isomorphism of fatgraphs is equivalent to an orientation-preserving homeomorphism of the underlying surface that maps vertices, edges, and faces to vertices, edges, and faces, while preserving all labels. It follows that the notion of equivalence and automorphism of branched covers descends to the notion of equivalence and automorphism of fatgraphs. $\square$

Continuing the previous example, we note that the three fatgraphs in $\mathrm{Fat}_0^{[2]}(3,1)$ have trivial automorphism groups. Therefore, we have calculated the orbifold Hurwitz number $H^{[2]}_{0;(3,1)} = K_0^{[2]}(3,1) = 3$.

The remainder of this section will be devoted to the proof of Proposition 4, the cut-and-join recursion for orbifold Hurwitz numbers. It will be useful to define the following normalisation of the orbifold Hurwitz numbers.

$$\overline{K}_g^{[a]}(\mu) = K_g^{[a]}(\mu) \times \prod_{i=1}^{n} \mu_i.$$

The cut-and-join recursion can then be stated in terms of this normalisation in the following way.

$$\begin{aligned}\overline{K}_g^{[a]}(\mu_S) = {} & \sum_{i<j} \mu_i\mu_j \overline{K}_g^{[a]}(\mu_{S\setminus\{i,j\}}, \mu_i+\mu_j) \\ & + \sum_{i=1}^{n} \sum_{\alpha+\beta=\mu_i} \frac{\mu_i}{2} \Bigg[\overline{K}_{g-1}^{[a]}(\mu_{S\setminus\{i\}}, \alpha, \beta) \\ & \qquad + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{i\}}} \frac{(m-1)!}{m_1!m_2!} \overline{K}_{g_1}^{[a]}(\mu_I, \alpha) \overline{K}_{g_2}^{[a]}(\mu_J, \beta) \Bigg],\end{aligned}$$

Here, we use the notation $m_1 = 2g_1 - 1 + |I| + \frac{|\mu_I|+\alpha}{a}$ and $m_2 = 2g_2 - 1 + |J| + \frac{|\mu_J|+\beta}{a}$. The conditions $g_1 + g_2 = g$, $I \sqcup J = S \setminus \{i\}$, and $\alpha + \beta = \mu_i$ imply that $m_1 + m_2 = m - 1$.

Definition 26. For a positive integer a, let $\overline{\mathrm{Fat}}_g^{[a]}(\mu_1, \mu_2, \ldots, \mu_n)$ be the set of edge-labelled fatgraphs of type (g, n) such that

- there are $\frac{|\mu|}{a}$ vertices and at each of them there are am adjacent half-edges that are cyclically labelled

$$1, 2, 3, \ldots, m, 1, 2, 3, \ldots, m, \ldots, 1, 2, 3, \ldots, m;$$

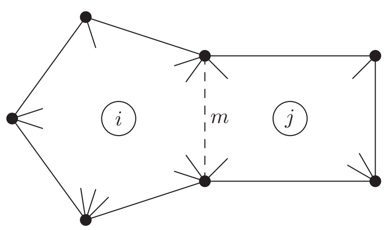
- there are exactly m edges that are labelled $1, 2, 3, \ldots, m$;
- the perimeters of the faces are given by the tuple $(\mu_1 m, \mu_2 m, \ldots, \mu_n m)$; and

- there is a marked angle between two consecutive edges labelled m and 1 in each face.

The advantage of the marked angles is that they remove the possibility of non-trivial automorphisms. Note that there are μ_k choices for the marked angle in the face labelled k. It follows that we can interpret $\overline{K}_g^{[a]}(\mu)$ as the number of labelled fatgraphs in $\overline{\mathrm{Fat}}_g^{[a]}(\mu)$ and that $\overline{K}_g^{[a]}(\mu)$ is always an integer.

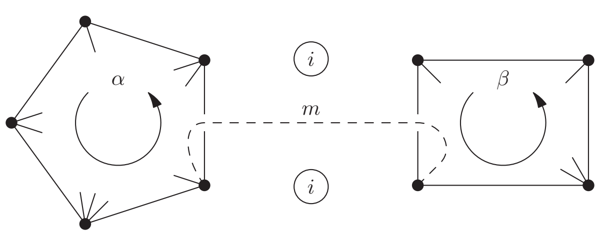
Proof of Proposition 4. Recall that $\overline{K}_g^{[a]}(\mu_S)$ is the number of fatgraphs in $\overline{\mathrm{Fat}}_g^{[a]}(\mu_S)$. Choose a fatgraph in $\overline{\mathrm{Fat}}_g^{[a]}(\mu_S)$ and remove the half-edges and the edge labelled m from it. Then one of the following three cases must arise.

- *The edge labelled m is adjacent to two distinct faces labelled i and j.* The removal of the half-edges and the edge labelled m leaves a fatgraph in $\overline{\mathrm{Fat}}_g^{[a]}(\mu_{S\setminus\{i,j\}}, \mu_i + \mu_j)$.



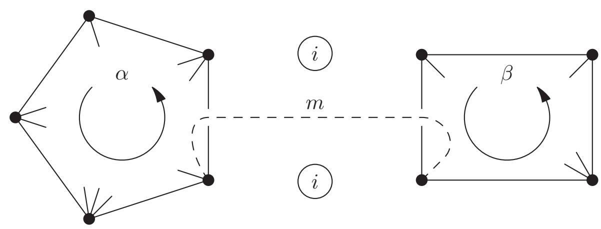
  Conversely, there are $\mu_i\mu_j$ ways to reconstruct a fatgraph in $\overline{\mathrm{Fat}}_g^{[a]}(\mu_S)$ from a fatgraph in $\overline{\mathrm{Fat}}_g^{[a]}(\mu_{S\setminus\{i,j\}}, \mu_i + \mu_j)$ by adding half-edges and one edge labelled m. The edge labelled m has one endpoint in the marked angle and the other is uniquely defined by the fact that the faces created must have perimeters $\mu_i m$ and $\mu_j m$. There are then $\mu_i\mu_j$ ways to choose marked angles in these two faces.

- *The edge labelled m is adjacent to the face labelled i on both sides and its removal leaves a fatgraph.*
  The removal of the half-edges and the edge labelled m leaves a fatgraph in $\mathrm{Fat}_{g-1}^{[a]}(\mu_{S\setminus\{i\}}, \alpha, \beta)$, where $\alpha + \beta = \mu_i$.



Conversely, there are μ_i ways to reconstruct a fatgraph in $\overline{\mathrm{Fat}}_g^{[a]}(\mu_S)$ from a fatgraph in $\overline{\mathrm{Fat}}_{g-1}^{[a]}(\mu_{S\setminus\{i\}}, \alpha, \beta)$ where $\alpha + \beta = \mu_i$ by adding half-edges and one edge labelled m. The edge labelled m has its endpoints in the marked angles of the faces with perimeters α and β. There are then μ_i ways to choose a marked angle in the new face. There is an additional factor of $\frac{1}{2}$ to adjust for the overcounting due to the symmetry in α and β.

- *The edge labelled m is adjacent to the face labelled i on both sides and its removal leaves the disjoint union of two fatgraphs.*
  The removal of the half-edges and the edge labelled m leaves the disjoint union of two fatgraphs Γ_1 and Γ_2. Remove from Γ_1 all leaves whose label does not appear on an edge of Γ_1 and replace all labels with the numbers from 1 up to m_1, preserving the order. Similarly, remove from Γ_2 all leaves whose label does not appear on an edge of Γ_2 and replace all labels with the numbers from 1 up to m_2, preserving the order. We are left with the disjoint union of two fatgraphs in $\overline{\mathrm{Fat}}_{g_1}^{[a]}(\mu_I, \alpha)$ and $\overline{\mathrm{Fat}}_{g_2}^{[a]}(\mu_J, \beta)$. We necessarily have the conditions $g_1 + g_2 = g$, $I \sqcup J = S \setminus \{i\}$, and $\alpha + \beta = \mu_i$.



  Conversely, there are $\mu_i \times \frac{(m-1)!}{m_1!m_2!}$ ways to reconstruct a fatgraph in $\overline{\mathrm{Fat}}_g^{[a]}(\mu_S)$ from a pair of fatgraphs in $\overline{\mathrm{Fat}}_{g_1}^{[a]}(\mu_I, \alpha)$ and $\overline{\mathrm{Fat}}_{g_2}^{[a]}(\mu_J, \beta)$ by adding half-edges and one edge labelled m. We have assumed here that $g_1 + g_2 = g$, $I \sqcup J = S \setminus \{i\}$, and $\alpha + \beta = \mu_i$. The edge labelled m has its endpoints in the marked angles of the faces with perimeters α and β. There are then μ_i ways to choose a marked angle in the new face. The factor $\frac{(m-1)!}{m_1!m_2!}$ accounts for the distribution of the labels $\{1, 2, \ldots, m-1\}$ between the two fatgraphs. There is an additional factor of $\frac{1}{2}$ to adjust for the overcounting due to the symmetry in (g_1, I, α) and (g_2, J, β).

To obtain all fatgraphs in $\overline{\mathrm{Fat}}_g^{[a]}(\mu)$, it is necessary to perform the reconstruction process in the first case for all possible values of i and j; in the second case for all possible values of i and $\alpha + \beta = \mu_i$; and in the third case for all possible values of i, $\alpha + \beta = \mu_i$, $g_1 + g_2 = g$, and $I \sqcup J = S \setminus \{i\}$. We obtain the cut-and-join recursion for orbifold Hurwitz numbers by summing up over all these contributions. $\square$

Appendix B. Combinatorics of exponential generating functions

Definition 27. Let d be a positive integer and ν be a partition of d. A *cactus-node tree of type* ν is a connected graph D such that:

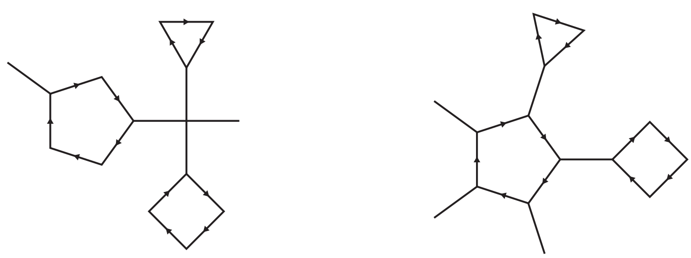
- There exists a collection N called the *nodes* (or *cactus-nodes*):

$$N = \left\{ g_i \;\middle|\; \begin{array}{l} i \in \{1, \ldots, l\}, \text{ if } i \neq j \text{ then } g_i \cap g_j = \emptyset, \text{ and} \\ g_i \text{ is a } \textit{directed } \nu_i\text{-cycle in } D \text{ if } \nu_i > 1, \\ \text{or a vertex of } D \text{ if } \nu_i = 1, \end{array} \right\}$$

- There exists a collection of edges B with $|B| = \ell(\nu) - 1$ and $B \cap E(N) = \emptyset$ called the *branches*.
- If c is a cycle in D then $c \in N$.
- $|\mathrm{Edges}(D)| = |\mathrm{Edges}(N)| + |\mathrm{Edges}(B)|$

Call a node that is connected to exactly one branch a *leaf*.

Example 28. These are examples of cactus-node trees of type $\{1, 1, 1, 3, 4, 5\}$.



Proposition 29. *Let d be a positive integer and ν be a partition of d. The number of cactus-node trees of type ν on a set of d marked points is*

$$\frac{d!}{|\mathrm{Aut}\ \nu|} d^{\ell(\nu)-2}.$$

Proof. We generalise the Prüfer encoding used to prove Cayley's formula for the number of labelled trees. Let M be the set of collections of the form:

$$\left\{ C_i \;\middle|\; \begin{array}{l} i \in \{1, \dots, l\}, \text{ if } i \neq j \text{ then } C_i \cap C_j = \emptyset, \text{ and} \\ C_i \text{ is a \textit{rooted} } \nu_i\text{-cycle on } \{1, \dots, d\} \text{ if } \nu_i > 1, \\ \text{or a marked point on } \{1, \dots, d\} \text{ if } \nu_i = 1, \end{array} \right\}$$

We claim that there is a bijection

$$\left\{ \begin{array}{c} \text{Cactus-node trees} \\ \text{of type } \nu \end{array} \right\} \longleftrightarrow M \times \{1, \dots, d\}^{l-2}$$

To see this we use ideas from the Prüfer encoding:

- Locate the leaf with with the largest label.
- Mark the leaf where the branch is connected.
- Write down the label the where branch is connected to the non-leaf component.
- Remove the branch connecting this to the graph.
- Repeat this until two leaves are left.
- Remove the branch connecting the final two leaves.
- We are left with a collection in M and a sequence in $\{1, \dots, \}$ of length $l-2$.

This encoding can be reversed. Let C be a collection in M, and K be a sequence in $\{1, \dots, \}$ of length $l-2$.

- Locate $b \in C$ with the largest label not in K.
- Locate $c \in C$ that contains the label K_1.
- Connect b and c with a branch at the marked points.
- Replace C with $C \setminus b$ and K with $(K_2, \dots, K_{l-2})$.
- Continue until K is empty.
- Connect the marked points of the remaining two elements of C.

Each $C \in M$ can be uniquely specified by

- Partitioning S into sets of size determined by ν. The number of ways to do this is:

$$\binom{d}{\nu_1}\binom{d-\nu_1}{\nu_2}\cdots\binom{\nu_l}{\nu_l} = \frac{d!}{\nu_1!(d-\nu_1)!}\frac{(d-\nu_1)!}{\nu_2!(d-\nu_1-\nu_2)!}\cdots 1 = \frac{d!}{\nu_1!\cdots\nu_l!}.$$

- Specifying the cycle structure and marked point of each set. For each set of size ν_i there are $\nu_i!$ possible marked cycle structures.
- We must divide by $|\text{Aut } \nu|$ because these sets are unlabelled.

Hence $|M| = \frac{d!}{\text{Aut}(\nu)}$ and the desired result follows immediately. □

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References

[1] A. Bayer and C. Cadman, *Quantum cohomology of* $[\mathbb{C}^N/\mu_r]$, Compos. Math. **146** (2010), no. 5, 1291–1322.

[2] G. Borot, B. Eynard, M. Mulase, and B. Safnuk, *A matrix model for simple Hurwitz numbers, and topological recursion*, J. Geom. Phys. **61** (2011), no. 2, 522–540.

[3] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, *Remodeling the B-model*, Comm. Math. Phys. **287** (2009), no. 1, 117–178.

[4] V. Bouchard and M. Mariño, *Hurwitz numbers, matrix models and enumerative geometry*, in From Hodge theory to integrability and TQFT tt*-geometry, Vol. 78 ofProc. Sympos. Pure Math., 263–283, Amer. Math. Soc., Providence, RI (2008).

[5] J. Bryan, T. Graber, and R. Pandharipande, *The orbifold quantum cohomology of* $\mathbb{C}^2/Z_3$ *and Hurwitz-Hodge integrals*, J. Algebraic Geom. **17** (2008), no. 1, 1–28.

[6] C. Cadman and R. Cavalieri, *Gerby localization,* Z_3*-Hodge integrals and the GW theory of* $[\mathbb{C}^3/Z_3]$, Amer. J. Math. **131** (2009), no. 4, 1009–1046.

[7] R. Cavalieri, *Generating functions for Hurwitz-Hodge integrals*, Adv. Math. **218** (2008), no. 5, 1419–1429.

[8] T. Coates, A. Corti, H. Iritani, and H.-H. Tseng, *Computing genus-zero twisted Gromov-Witten invariants*, Duke Math. J. **147** (2009), no. 3, 377–438.

[9] R. Dijkgraaf, H. Fuji, and M. Manabe, *The volume conjecture, perturbative knot invariants, and recursion relations for topological strings*, Nuclear Phys. B **849** (2011), no. 1, 166–211.

[10] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, *Identification of the Givental formula with the spectral curve topological recursion procedure*, Comm. Math. Phys. **328** (2014), no. 2, 669–700.

[11] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *On Hurwitz numbers and Hodge integrals*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 12, 1175–1180.

[12] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001), no. 2, 297–327.

[13] B. Eynard, *Intersection numbers of spectral curves*, (2011).

[14] B. Eynard, *Invariants of spectral curves and intersection theory of moduli spaces of complex curves*, Commun. Number Theory Phys. **8** (3), (2014), 541–588.

[15] B. Eynard, *Recursion between Mumford volumes of moduli spaces*, Ann. Henri Poincaré **12** (2011), no. 8, 1431–1447.

[16] B. Eynard, M. Mulase, and B. Safnuk, *The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 2, 629–670.

[17] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1** (2007), no. 2, 347–452.

[18] B. Eynard and N. Orantin, *Weil–Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models*, (2007).

[19] B. Eynard and N. Orantin, *Topological recursion in enumerative geometry and random matrices*, J. Phys. A **42** (2009), no. 29, 293001, 117.

[20] B. Eynard and N. Orantin, *Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture*, Comm. Math. Phys. **337** (2015), no. 2, 483–567.

[21] B. Fang, C.-C. M. Liu, and Z. Zong, *All Genus Open-Closed Mirror Symmetry for Affine Toric Calabi-Yau 3-Orbifolds*, (2013).

[22] I. P. Goulden and D. M. Jackson, *Transitive factorisations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. **125** (1997), no. 1, 51–60.

[23] I. P. Goulden, D. M. Jackson, and A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. Comb. **4** (2000), no. 1, 27–46.

[24] T. Graber and R. Vakil, *Hodge integrals and Hurwitz numbers via virtual localization*, Compositio Math. **135** (2003), no. 1, 25–36.

[25] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88. With an appendix by William Fulton.

[26] A. Hurwitz, *Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. **39** (1891), no. 1, 1–60.

[27] T. J. Jarvis and T. Kimura, *Orbifold quantum cohomology of the classifying space of a finite group*, in Orbifolds in mathematics and physics (Madison, WI, 2001), Vol. 310 of Contemp. Math., 123–134, Amer. Math. Soc., Providence, RI (2002).

[28] P. Johnson, R. Pandharipande, and H.-H. Tseng, *Abelian Hurwitz-Hodge integrals*, Michigan Math. J. **60** (2011), no. 1, 171–198.

[29] M. Mariño, *Open string amplitudes and large order behavior in topological string theory*, J. High Energy Phys. (2008), no. 3, 060, 34.

[30] P. Norbury, *String and dilaton equations for counting lattice points in the moduli space of curves*, Trans. Amer. Math. Soc. **365** (2013), no. 4, 1687–1709.

[31] P. Norbury and N. Scott, *Gromov-Witten invariants of* $\mathbb{P}^1$ *and Eynard-Orantin invariants*, Geom. Topol. **18** (2014), no. 4, 1865–1910.

[32] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and matrix models*, in Algebraic geometry—Seattle 2005. Part 1, Vol. 80 of *Proc. Sympos. Pure Math.*, 325–414, Amer. Math. Soc., Providence, RI (2009).

[33] S. Shadrin, L. Spitz, and D. Zvonkine, *On double Hurwitz numbers with completed cycles*, J. Lond. Math. Soc. (2) **86** (2012), no. 2, 407–432.

[34] H.-H. Tseng, *Orbifold quantum Riemann-Roch, Lefschetz and Serre*, Geom. Topol. **14** (2010), no. 1, 1–81.

[35] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, in Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA (1991).

[36] H. Zhang and J. Zhou, *Wreath Hurwitz numbers, colored cut-and-join equations, and 2-Toda hierarchy*, Sci. China Math. **55** (2012), no. 8, 1627–1646.

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