Math. Res. Lett.
Volume 23, Number 5, 1263–1279, 2016

The Euler characteristic of a surface from its Fourier analysis in one direction

Nguyen Viet Dang

In this paper, we prove that we can recover the genus of a closed compact surface S in $\mathbb{R}^3$ from the restriction to a generic line of the Fourier transform of the canonical measure carried by S. We also show that the restriction on some line in Minkowski space of the solution of a linear wave equation whose Cauchy data comes from the canonical measure carried by S, allows to recover the Euler characteristic of S.

1. Introduction

Let us start with a surface S in $\mathbb{R}^3$. To this surface S, we can always associate a natural measure just by integrating test functions on S. Indeed, the surface carried measure μ is the distribution defined as

$$\mu(\varphi) = \int_S \varphi d\sigma \tag{1.1}$$

where $d\sigma$ is the canonical area element on S induced by the Euclidean metric of $\mathbb{R}^3$ ([21, p. 334], [20, p. 321]) and φ a test function. If we assume that S is compact, then we find that the measure μ can be viewed as a *compactly supported distribution* on $\mathbb{R}^3$, μ is thus a tempered distribution and therefore it has a well defined Fourier transform denoted by $\widehat{\mu}$.

In harmonic analysis, we are interested in the analytical properties of the Fourier transform $\widehat{\mu}$ of measures μ which are supported on submanifolds of some given vector space. For instance, in a very recent paper [1], using the concept of wave front set and resolution of singularities, Aizenbud and Drinfeld give a proof that the Fourier transform of an algebraic measure is smooth in some open dense set. More classically, one would like to study the asymptotic behaviour of $\widehat{\mu}$ for large momenta. In the simple case where μ is the measure carried by a plane H in a vector space V, the Fourier transform $\widehat{\mu}$ is a measure carried by the dual plane $H^\perp$ in Fourier space V^* [13, Thm. 7.1.25 p. 173]. Moreover, if the measure μ is compactly supported, then we know by Paley–Wiener that its Fourier transform $\widehat{\mu}$ should be a *bounded* function on $\mathbb{R}^3$. Therefore in both cases, for any test function φ, classical theorems in distribution theory only yield that the Fourier transform $\widehat{\mu\varphi}$ is **bounded**.

However, if we assume that μ is carried by a *compact* hypersurface S of dimension n with *non vanishing Gauss curvature*, then a celebrated result of Stein (see [20, Theorem 1 p. 322] and also [13, Thm. 7.7.14]) gives finer decay properties on $\widehat{\mu}$. Indeed, he proves that $|\widehat{\mu}| \leqslant C(1+|\xi|)^{-\frac{n}{2}}$ for some constant C which depends on the volume of S and the Gauss curvature. Stein's result shows the interplay between geometry and analysis, since a simple assumption on the Gauss curvature of S gives sharper decay properties on $\widehat{\mu}$ than a simple application of the Paley–Wiener theorem.

In this paper, we explore the relationship between the Fourier transform $\widehat{\mu}$ of the surface carried measure μ and the topology of the surface itself. In the same spirit as in the papers [14, 15], we use microlocal analysis and Morse theory in the study of $\widehat{\mu}$. Throughout the paper, the surface S will always be assumed to be smooth and oriented. We state in an informal way the main result of this note:

Theorem 1.1. *Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. For a generic line $\ell \subset \mathbb{R}^3$, we can recover the Euler characteristic of the surface S from the restriction $\widehat{\mu}|_\ell$.*

The space of unoriented lines in $\mathbb{R}^3$ is canonically identified with the projective space $\mathbb{RP}^2$ and generic means that the theorem holds true for an open dense set of lines in $\mathbb{RP}^2$. We refer the reader to Theorem 2.2 which explains in *what sense we recover* $\chi(S)$ and gives a precised version of our main result.

Our main result might look surprising because if we knew the full Fourier transform $\widehat{\mu}$, then it would be easy to reconstruct μ hence S from $\widehat{\mu}$ by Fourier inversion. But to recover the topology of S, it is enough to consider the partial information of the restriction of $\widehat{\mu}$ to a line ℓ. Actually, all the information we need is contained in the asymptotic expansion of $\widehat{\mu}(\xi)$ for large $|\xi|$. In a way, our result is reminiscent of Weyl's tube formula where the asymptotic expansion in ε of the volume of the tube S_ε of "thickness" ε around a surface S gives geometrical data on S: the volume and the Euler characteristic of S.

As a byproduct of our main theorem, we derive similar results for more general integral transforms. First, we connect our result with the Radon transform in the spirit of [5, section 5.3]. We denote by $\mathcal{R}\mu$ the Radon transform of μ.

Theorem 1.2. *Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. For a generic line $\ell \subset \mathbb{R}^3$, we can recover the Euler characteristic of the surface S from the restriction of $\mathcal{R}\mu$ on ℓ.*

In the sequel, a Morse function ψ is called *excellent* if its critical values are distinct.

Theorem 1.3. *Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. If $\psi \in C^\infty(\mathbb{R}^3)$ is such that its restriction on S is Morse excellent then we can recover the Euler characteristic of S from the map $\lambda \longmapsto \mu\left(e^{i\lambda\psi}\right)$.*

And finally, let us state informally an application of this theorem to the following inverse problem:

Theorem 1.4. *Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. Consider the solution $u \in \mathcal{D}'(\mathbb{R}^{3+1})$ of the wave equation $\left(\partial_t^2 - \sum_{i=1}^3 \partial_{x^i}^2\right) u = 0$ with Cauchy data $u(0) = 0$, $\partial_t u(0) = \mu$. For x in some open dense subset of $\mathbb{R}^3$, set $\ell(x)$ to be the line $\{x\} \times \mathbb{R} \subset \mathbb{R}^{3+1}$, then:*

- *the restriction $u_{\ell(x)}$ is a compactly supported distribution of t,*
- *we can recover the Euler characteristic of S from the restriction $u_{\ell(x)}$.*

We give precise statements in Proposition 4.1 and Theorem 4.2 which explain in what sense we recover $\chi(S)$.

In appendix, we gather several useful results in Morse theory and transversality theory which are used in the paper.

The general principle underlying our results. The main idea of our note is to think of the stationary phase principle as an analytic version of Lagrangian intersection. The first observation is that a surface carried measure μ is the simplest example of Lagrangian distribution (in the sense of Maslov Hörmander) with wave front set the conormal $N^*(S)$ of the surface S. To probe the wave front set of a distribution, a natural idea is to calculate its Fourier transform or more generally to pair μ with an oscillatory function of the form $e^{i\lambda\psi}$ where $d\psi \neq 0$. We can think of this operation as a plane wave analysis of μ. By the stationary phase principle, the main contributions to the asymptotics of $\mu(e^{i\lambda\psi})$ when $\lambda \to +\infty$ come from the points where the graph of $d\psi$ meets the wave front set of μ which is the conormal $N^*(S)$. Therefore our strategy is to extract topological information from the Lagrangian intersection (Graph of $d\psi$) $\cap N^*(S)$ by the stationary phase principle. This is strongly related to the work of Kashiwara [15, p. 194] and Fu [11] where the Euler characteristic of subanalytic sets (Kashiwara gives an index formula in the context of *constructible sheaves*) is expressed in terms of Lagrangian intersection.

Let us give the central example of the theory.

Example 1.5. Let ψ be a Morse function on a manifold M of dimension n and Ω a test form in $\Omega^n_c(M)$, then the main contributions to the oscillatory integral $\int_M e^{i\lambda\psi}\Omega$ when $\lambda \to +\infty$ come from the critical points of ψ, in other words, from the intersection of the Lagrangian graph $d\psi$ and the zero section $\underline{0} \subset T^*M$.

2. Proof of Theorem 1.1

2.1. Fourier transform, Morse functions and stationary phase

Recall that S is a closed, compact surface embedded in $\mathbb{R}^3$ and μ denotes the associated surface carried measure. We denote by $d\sigma$ the canonical area element on S induced by the Euclidean metric of $\mathbb{R}^3$. Then the surface carried measure μ is the distribution defined by the formula:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^3), \quad \mu(\varphi) = \int_S \varphi d\sigma. \tag{2.1}$$

The Fourier transform $\widehat{\mu}$ reads

$$\widehat{\mu}(\xi) = \int_S e^{-i\xi(x)} d\sigma(x). \tag{2.2}$$

Since S is compact, μ is compactly supported thus $\widehat{\mu}$ is a real analytic function by Paley–Wiener theorem. In order to study the asymptotics of $\widehat{\mu}(\lambda\xi)$ for λ large, we will think of $\xi \in \mathbb{R}^{3*} \subset C^\infty(\mathbb{R}^3)$ as a linear function on $\mathbb{R}^3$ whose restriction on S is a height function which we denote by $\xi \in C^\infty(S)$.

For every $\xi \in \mathbb{R}^{3*}$, we denote by $[\xi]$ its class in $\mathbb{RP}^2$ and by Theorem 5.2 recalled in Appendix, for an everywhere dense set of $[\xi] \in \mathbb{RP}^2$, the function

ξ is a *Morse function* on S. Set $\omega = \frac{\xi}{|\xi|}$, $\lambda = |\xi|$ and note that ω has the same critical points as ξ. The stationary phase principle states that the main contributions to the asymptotics of $\widehat{\mu}(\lambda\omega)$ when $\lambda \to \infty$ come from the *critical points* of the Morse function ω which are isolated by Theorem 10.4.3 in [7, p. 87]. We denote by Crit ω the set of critical points of ω. Then in the neighborhood of every $x \in \text{Crit }\omega$, we choose local coordinates (y^1, y^2) on S such that $y^1(x) = y^2(x) = 0$ and $d\sigma(x) = |dy^1 dy^2|$ where $d\sigma$ is the canonical area element on S. The method of stationary phase ([9, Lemma (19.4) p.95],[18, p. 124]) yields the asymptotic expansion:

$$\widehat{\mu}(\lambda\omega) \underset{\lambda\to\infty}{\sim} \sum_{x\in\text{Crit }\omega} \left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}} e^{i\frac{\pi}{4}(n_+-n_-)(x)} \times \frac{e^{-i\lambda\omega(x)}}{\sqrt{|\det \omega''(x)|}} \left[1 + \sum_{j=1}^{\infty} b_j(x)\lambda^{-j}\right]. \tag{2.3}$$

We want to comment on the geometric interpretation of each of the terms in the expansion:

- $\frac{n}{2} = 1$ in our case since S is a surface.
- $n_+(x)$ (resp $n_-(x)$) is the number of positive (resp negative) eigenvalues of the Hessian $-\omega''$ of the Morse function $-\omega$ at the critical point x, the number $n_-(x)$ is the *Morse index* of $-\omega$ at x. For surfaces, $e^{i\frac{\pi}{4}(n_+-n_-)(x)}$ can only take the three values $\{i, 1, -i\}$. Observe that
$$\begin{aligned} n_-(x) = 1 \quad \text{mod } (2) &\Leftrightarrow e^{i\frac{\pi}{4}(n_+-n_-)(x)} = 1 \\ n_-(x) = 0 \quad \text{mod } (2) &\Leftrightarrow e^{i\frac{\pi}{4}(n_+-n_-)(x)} = \pm i. \end{aligned}$$
- In the local chart (y_1, y_2) around x, the Hessian $-\omega''(x)$ is non degenerate since ω is Morse and it also coincides with the second fundamental form of the surface S at x. Therefore $|K(x)| = |\det \omega''(x)|$ where $K(x)$ is the *Gauss curvature* of the surface S at x ([19, 3.3.1 p. 67]).

2.2. An oscillatory integral whose singular points are the critical values of the height function

We denote by

$$\mathcal{F}_\lambda^{-1} : v \in \mathcal{S}'(\mathbb{R}) \longmapsto \widehat{v}(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda e^{i\tau\lambda} v(\lambda)$$

the inverse Fourier transform w.r.t. variable λ. We define the oscillatory integral $u=\mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega)\right)$ on $\mathbb{R}$ and show that u is singular at the critical values of the Morse function ω. Recall that $K(x)$ denotes the Gauss curvature of S at x.

Proposition 2.1. Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. Let u be the distribution

$$u=\mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega)\right). \tag{2.4}$$

If $\omega\in C^\infty(S)$ is Morse, then:

- the singular support of u is the set of critical values of ω,
- u is a finite sum of oscillatory integrals on $\mathbb{R}$
- each oscillatory integral has polyhomogeneous symbol whose leading term is $\frac{e^{i\frac{\pi}{4}(n_+-n_-)(x)}}{\sqrt{|K(x)|}}$ where $x\in\text{Crit }\omega$.

Proof. By the stationary phase expansion,

$$\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega)=\sum_{x\in\text{Crit }\omega}e^{-i\lambda\omega(x)}b(x,\lambda)$$

where, for every $x\in\text{Crit }\omega$, $b(x,\lambda)$ is a polyhomogeneous symbol in λ:

$$b(x;\lambda)\sim\frac{e^{i\frac{\pi}{4}(n_+-n_-)(x)}}{\sqrt{|\det\omega''(x)|}}\left[1+\sum_{j=1}^\infty b_j(x)\lambda^{-j}\right] \tag{2.5}$$

with principal symbol $\frac{e^{i\frac{\pi}{4}(n_+-n_-)(x)}}{\sqrt{|\det\omega''(x)|}}$ since $|K(x)|=|\det\omega''(x)|\neq 0$. Therefore:

$$\begin{aligned}u(t)&=\mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega)\right)\\&=\sum_{x\in\text{Crit }\omega}\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\lambda(t-\omega(x))}b(x;\lambda)d\lambda\end{aligned} \tag{2.6}$$

is a finite sum of oscillatory integrals ([13, Theorem (7.8.2) p. 237]). By the general theory of oscillatory integrals [17, Theorem IX.47 p. 102] [13, Theorem 8.1.9 p. 260]:

$$WF(u) = \{(t;\tau) | (\omega(x) - t) = 0, \tau \neq 0\}. \tag{2.7}$$

From [13, Proposition 8.1.3 p. 254], we deduce that the singular support of u is $\{t | (\omega(x) - t) = 0, x \in \text{Crit } \omega\}$ which is exactly the set of critical values of ω. $\square$

We state and prove a precised version of the main theorem 1.1 given in the introduction:

Theorem 2.2. *Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. Then for an everywhere dense set $\omega \in \mathbb{S}^2$, the distribution $u = \mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\widehat{\mu}(\lambda\omega)\right)$ can be canonically decomposed as a sum*

$$u = \sum_{x \in Crit\ \omega} a(x)\delta_{\omega(x)} + r \tag{2.8}$$

such that

- $\forall x \in Crit\ \omega, a(x) \neq 0$
- *r is an oscillatory integral with symbol of degree -1.*

The Euler characteristic of S satisfies the identity:

$$\chi(S) = \sum_{x \in Crit\ \omega} -\frac{a(x)^2}{|a(x)|^2}. \tag{2.9}$$

Proof. By Lemma 5.3 proved in appendix, for an everywhere dense set $\omega \in \mathbb{S}^2$, the corresponding function $\omega \in C^\infty(S)$ is an excellent Morse function which means that if (x_1, x_2) are distinct critical points of ω then $\omega(x_1) \neq \omega(x_2)$.

In particular, ω is Morse therefore we can use the results of Proposition 2.1. From the asymptotic expansion (2.5) of the symbol b and the equation of u (2.6), we find that:

$$u = \sum_{x \in \text{Crit } \omega} \frac{e^{i\frac{\pi}{4}(n_+ - n_-)(x)}}{\sqrt{|\det \omega''(x)|}}\delta_{\omega(x)} + r, \tag{2.10}$$

where $r \in \mathcal{D}'(\mathbb{R})$ is an oscillatory integral whose asymptotic symbol has leading term $\sum_{x \in \text{Crit } \omega} a(x)b_{-1}(x)\lambda^{-1}$ for $a(x) = \frac{e^{i\frac{\pi}{4}(n_+ - n_-)(x)}}{\sqrt{|\det \omega''(x)|}}$. We rewrite the above formula in a simpler form:

$$u = \sum_{x\in \text{Crit }\omega} a(x)\delta_{\omega(x)} + r \tag{2.11}$$

where $a(x) = \frac{e^{i\frac{\pi}{4}(n_+-n_-)(x)}}{\sqrt{|\det \omega''(x)|}}$.

Our goal is to express $\chi(S)$ in terms of the coefficients $a(x), x \in \text{Crit }\omega$. We recall the definition of the Morse counting polynomial $\mathcal{M}_\omega(T)$ for a given Morse function ω ([10, Definition C.4 p. 228]):

$$\mathcal{M}_\omega(T) = \sum_{x\in \text{Crit }\omega} T^{n_-(x)}. \tag{2.12}$$

Observe that

$$\begin{aligned} n_-(x) = 0 \quad \text{mod } (2) &\Leftrightarrow a(x) \in i\mathbb{R} \\ n_-(x) = 1 \quad \text{mod } (2) &\Leftrightarrow a(x) \in \mathbb{R} \\ &\Rightarrow -\frac{a(x)^2}{|a(x)|^2} = (-1)^{n_-(x)}. \end{aligned}$$

This implies that:

$$\mathcal{M}_\omega(-1) = \sum_{x\in \text{Crit }\omega} (-1)^{n_-(x)} = \sum_{x\in \text{Crit }\omega} -\frac{a(x)^2}{|a(x)|^2}. \tag{2.13}$$

To conclude, we use the well known result $\chi(S) = \mathcal{M}_\omega(-1)$ which is a consequence of the Morse inequalities ([10, Theorem C.3 p. 228] and [16, Thm. 5.2 p. 29]) . □

Let $\mathcal{R}$ be the Radon transform. From the identity relating the Fourier transform and the Radon transform (see [5, p. 19]), we find a relationship between the distribution u of Theorem 2.2 and $\mathcal{R}\mu$:

$$\begin{aligned} \mathcal{R}\mu(\omega,\tau) = \mathcal{F}_\lambda^{-1}\left(\widehat{\mu}(\lambda\omega)\right)(\tau) \implies u(\tau) &= \frac{1}{2i\pi}\partial_\tau \mathcal{R}\mu(\omega,\tau) \\ \implies \frac{1}{2i\pi}\partial_\tau \mathcal{R}\mu(\omega,\tau) &= \sum_{x\in \text{Crit }\omega} a(x)\delta_{\omega(x)} + r. \end{aligned}$$

This immediately proves a precised version of Theorem 1.2 and gives some informations on the Radon transform of the measure μ:

Theorem 2.3. *Under the assumptions of Theorem 2.2, let $\mathcal{R}\mu$ be the Radon transform of μ. Then for an everywhere dense set of $\omega \in \mathbb{S}^2$,*

$$\frac{1}{2i\pi}\partial_\tau \mathcal{R}\mu(\omega,\tau) = \sum_{x\in Crit\ \omega} a(x)\delta_{\omega(x)} + r \tag{2.14}$$

where

$$\chi(S) = \sum_{x\in Crit\ \omega} -\frac{a(x)^2}{|a(x)|^2}.$$

2.2.1. Remark. If we are given a Lagrangian distribution $u \in \mathcal{D}'(\mathbb{R})$ with singular point t_0 and u can be written as a sum $u = a\delta_{t_0} + r$ where $a \in \mathbb{C}$ and r is a Lagrangian distribution with asymptotic symbol of degree -1, then we can recover a by scaling around the point t_0. Indeed, a straightforward calculation yields

$$\lim_{\lambda\to 0} \lambda u(\lambda(. - t_0)) = a\delta_{t_0} \tag{2.15}$$

where the limit is understood in the sense of distributions. For any test function φ which is equal to 1 in a sufficiently small neighborhood of t_0, we find that $\lim_{\lambda\to 0}\langle \lambda t(\lambda(. - t_0)), \varphi\rangle = a$.

2.3. The example of the sphere

We show how our proof works in the case of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. First, note that any height function restricted on $\mathbb{S}^2$ is Morse excellent and has exactly two critical points. The Fourier transform of μ is given by the exact formula: $\widehat{\mu}(\xi) = 4\pi\frac{\sin(|\xi|)}{|\xi|}$ therefore

$$\begin{aligned} \frac{\lambda}{2\pi}\widehat{\mu}(\lambda\omega) &= 2\sin(\lambda) \\ \implies \mathcal{F}^{-1}\left(\frac{\lambda}{2\pi}\widehat{\mu}(\lambda\omega)\right) &= \mathcal{F}^{-1}\left(2\sin(\lambda)\right) \\ = \mathcal{F}^{-1}\left(\frac{e^{i\lambda} - e^{-i\lambda}}{i}\right) &= i(\delta_{+1} - \delta_{-1}). \end{aligned}$$

Finally, the identity 2.9 allows to recover the well known result $\chi(\mathbb{S}^2) = -i^2 - (-i)^2 = 2$.

3. Plane wave analysis and topology

Let us recall the statement of Theorem 1.3 before we give a proof:

Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. If $\psi \in C^\infty(\mathbb{R}^3)$ is such that its restriction on S is Morse excellent then we can recover the Euler characteristic of S from the map $\lambda \longmapsto \mu\left(e^{i\lambda\psi}\right)$.

The idea to consider oscillatory integrals of the form $\mu\left(e^{i\lambda\psi}\right)$ comes from the coordinate invariant definition of wave front set due to Gabor [4, 12] then corrected by Duistermaat [8, Proposition 1.3.2]. To prove Theorem 1.3, we just repeat the proof of Theorem 2.2 applied to the distribution $u = \mathcal{F}_\lambda^{-1}\left(\left(\frac{\lambda}{2\pi}\right)\mu\left(e^{i\lambda\psi}\right)\right)$ where we replace ω by ψ.

4. The wave equation, propagation of singularities and topology

Let us recall the statement of Theorem 1.4:

Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. Consider the solution $u \in \mathcal{D}'(\mathbb{R}^{3+1})$ of the wave equation $\left(\partial_t^2 - \sum_{i=1}^3 \partial_{x^i}^2\right) u = 0$ with Cauchy data $u(0) = 0, \partial_t u(0) = \mu$. For x in some open dense subset of $\mathbb{R}^3$, set $\ell(x)$ to be the line $\{x\} \times \mathbb{R} \subset \mathbb{R}^{3+1}$, then:

1) *the restriction $u_{\ell(x)}$ is a compactly supported distribution of t,*
2) *we can recover the Euler characteristic of S from the restriction $u_{\ell(x)}$.*

Let us prove a precised version of claim (1):

Proposition 4.1. Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. Consider the solution u of the wave equation $\Box u = 0$ with Cauchy data $u(0) = 0, \partial_t u(0) = \mu$. For all $x \in \mathbb{R}^3 \setminus S$:

- $t \longmapsto u(t, x)$ is a compactly supported distribution of t
- the singular support of $u(., x)$ is a subset of $\{t \in \mathbb{R} | \exists y \in S \text{ s.t. } |x - y| = |t|, y - x \perp T_y S\}$.

Proof. Recall that $\ell_x = \{(t, x) | t \in \mathbb{R}^3\}$ and denote by $N^*(\ell_x) \subset T^*\mathbb{R}^{3+1}$ its conormal bundle. We want to prove that one can restrict the distribution u

on ℓ_x. First, $\Box u = 0$ implies that the wave front set of u lies in the characteristic set of $\Box$ ([13, Theorem 8.3.1]):

$$WF(u) \subset \text{Char}\,\Box = \{\tau^2 - |\xi|^2\} \implies WF(u) \cap N^*(\ell_x) = \emptyset$$

which means that one can pull–back the distribution u by the embedding $i : \ell_x \hookrightarrow \mathbb{R}^{3+1}$ (see [13, Theorem 8.2.4]). The restriction i^*u is thus well defined, it is compactly supported since the Cauchy data $(0, \mu)$ is compactly supported and by finite propagation speed property for the wave equation.

Secondly, we calculate the wave front set of the restriction i^*u. Elements of the cotangent space $T^*\mathbb{R}^{3+1}$ are denoted by $(t, x; \tau, \xi)$ and π is the projection $T^*\mathbb{R}^{3+1} \mapsto \mathbb{R}^{3+1}$. Since u is solution of $\Box u = 0$ with Cauchy data $(0, \mu)$, u is given by the representation formula ([3, equation (2.21) p. 12], [2, Theorem 5.3 p. 67], [6, p. 3])

$$u(t, x) = \frac{1}{4\pi t} \int_{\mathbb{R}^3} \delta(|t| - |x - y|)\mu(y)dy. \tag{4.1}$$

We want to calculate $WF(u)$ using the integral formula (4.1). By finite propagation speed, the condition $x \in \mathbb{R}^3 \setminus S$ ensures that $i^*u = 0$ in some neighborhood of $t = 0$ which means we do not have to consider the contribution of $t = 0$ to $WF(\delta(|t| - |x - y|))$. Denote by P the projection $P : (t, y) \in \mathbb{R}^{3+1} \mapsto t \in \mathbb{R}$, since $WF(\delta(|t| - |x|)) = \{(t, x; \lambda t, -\lambda x) | |t| = |x|, \lambda \in \mathbb{R} \setminus \{0\}\} \cup T^*_0\mathbb{R}^{3+1}$, the calculus of wave front set yields:

$$\begin{aligned}
WF(u) &\subset P_* WF(\delta(|t| - |x - y|)\mu(y)) \\
&\subset \{(t; \lambda t) | \exists (y; \eta) \in N^*(S), |t| = |x - y|, \lambda(x - y) = \eta\} \\
&= \{(\pm|x - y|; \tau) | (x - y) \perp T_y S, \tau \in \mathbb{R} \setminus \{0\}\} \\
\implies \text{ss } u &= \pi(WF(i^*u)) \\
&\subset \{\pm|x - y| | (x - y) \perp T_y S\}.
\end{aligned}$$

$\Box$

Let us prove that one can recover $\chi(S)$ from $u(., x) \in \mathcal{D}'(\mathbb{R})$ concluding the proof of Theorem 1.4.

Theorem 4.2. *Let S be a closed compact surface embedded in $\mathbb{R}^3$ and μ the associated surface carried measure. Consider the solution $u \in \mathcal{D}'(\mathbb{R}^{3+1})$ of the wave equation $\left(\partial_t^2 - \sum_{i=1}^3 \partial_{x^i}^2\right) u = 0$ with Cauchy data $u(0) = 0$,*

$\partial_t u(0) = \mu$. Then for x in some open dense subset of $\mathbb{R}^3$, we have the canonical decomposition:

$$- 2i\left(t\partial_t + 1\right) u(t,x) = \sum_{y\in Crit\ L_x} a(y)\delta_{|y-x|}(t) + r(t) \tag{4.2}$$

where $L_x : y \in S \mapsto |y-x|$ is Morse excellent, r is a finite sum of oscillatory integrals with symbol of degree -1.

$$\chi(S) = \sum_{y\in Crit\ L_x} -\frac{a(y)^2}{|a(y)|^2}. \tag{4.3}$$

Proof. We again use the representation formula for u:

$$u(t,x) = \frac{1}{4\pi t}\int_{\mathbb{R}^3} \delta(|t| - |x-y|)\mu(y)dy.$$

Recall that $x \in \mathbb{R}^3 \setminus S$ implies $u(.,x) = 0$ in some neighborhood of $t = 0$. Therefore, for $t \geqslant 0$:

$$\begin{aligned} u(t,x) &= \frac{1}{8\pi^2 t}\int_{\mathbb{R}} d\lambda e^{it\lambda}\int_{\mathbb{R}^3}\mu(y)e^{-i\lambda|x-y|}dy \\ \Longrightarrow\ -2i\left(t\partial_t + 1\right) u(t,x) &= \mathcal{F}^{-1}\left(\frac{\lambda}{2\pi}\int_{\mathbb{R}^3}\mu(y)e^{-i\lambda|x-y|}dy\right). \end{aligned}$$

By Lemma 5.5, for x in some open dense set in $\mathbb{R}^3$, the function $L_x : y \in S \mapsto |y-x|$ is Morse excellent. Therefore, repeating the proof of Theorem 1.1 with L_x instead of $-\omega$, we find that

$$-2i\left(t\partial_t + 1\right) u(t,x) = \sum_{y\in \mathrm{Crit}\ L_x} a(y)\delta_{|y-x|}(t) + r(t)$$

where r is a finite sum of oscillatory integrals with symbol of degree -1. Finally $\chi(S) = \sum_{y\in \mathrm{Crit}\ L_x} -\frac{a(y)^2}{|a(y)|^2}$. □

5. Appendix

In this section, we gather several useful results in Morse theory.

5.1. Almost all height functions are Morse

Let S be an embedded surface in $\mathbb{R}^3$. For every $x \in S$, we denote by T_xS the tangent plane to S at x.

Definition 5.1. The map $x \in S \mapsto n(x) \in \mathbb{S}^2$ where $n(x)$ is the oriented unit normal vector to T_xS is called the Gauss map. It induces canonically a *projective* Gauss map denoted by $[n] := x \in S \longmapsto [n](x) \in \mathbb{RP}^2$.

The next theorem [7, Thm. 11.2.2 p. 94] characterizes all height functions which are Morse functions in terms of the Gauss map:

Theorem 5.2. *Let S be an embedded surface in $\mathbb{R}^3$. The height function $\xi \in C^\infty(S) := x \in S \mapsto \xi(x)$ is a Morse function precisely when $[\xi] \in \mathbb{RP}^2$ is a regular value of the projective Gauss map $[n]$. It follows that for an* open everywhere dense set *of $[\xi] \in \mathbb{RP}^2$, the height function $\xi \in C^\infty(S)$ is a Morse function.*

5.2. Almost all Morse height functions are excellent Morse functions

We refine Theorem 5.2 and show that for generic $\xi \in \mathbb{RP}^2$, the height function ξ is an excellent Morse function i.e. all critical values of ξ are distinct.

Lemma 5.3. *Let S be a compact embedded surface in $\mathbb{R}^3$, for an* open everywhere dense set *of $[\xi] \in \mathbb{RP}^2$, the height function $\xi \in C^\infty(S)$ is an excellent Morse function.*

Proof. Let V be the set of regular values of the projective Gauss map $[n]$ in $\mathbb{RP}^2$. By Sard's Theorem V is open dense in $\mathbb{RP}^2$ and

$$[\xi] \in V \Leftrightarrow \text{ the height function } \xi \text{ is Morse.}$$

We give next a characterization of Morse height functions which are not Morse excellent. For all $[\xi] \in V$, there is a neighborhood Ω of $[\xi]$ such that the preimage $[n]^{-1}(\Omega)$ is a disjoint union of open sets $(U_1, \dots, U_k) \subset S^k$ and each U_i is sent diffeomorphically to Ω by the Gauss map. Therefore there is a collection of maps

$$(x_1, \dots, x_k) := [\xi] \in \Omega \subset \mathbb{RP}^2 \longmapsto (x_1([\xi]), \dots, x_k([\xi])) \in S^k$$

such that $\forall i, [n](x_i([\xi])) = [\xi]$. We claim that for $[\xi] \in \Omega$:

$$\xi \text{ is not Morse excellent } \Leftrightarrow \xi.\,(x_i([\xi]) - x_j([\xi])) = 0 \text{ for some } 1 \leq i < j \leq k.$$

From the fact that

$$d_\xi \xi.\,(x_i([\xi]) - x_j([\xi])) = \langle (x_i([\xi]) - x_j([\xi]))\,, .\rangle \neq 0$$

we deduce that the set of $[\xi]$ such that the height function ξ is not Morse excellent is a finite union of submanifolds in Ω. It has thus **empty interior** which proves that Morse excellent ξ are open dense. $\square$

5.3. The distance function to almost every point is Morse

The height function is not the only way to produce Morse functions on S. Recall we considered the distance function to x, $L_x := y \in S \longmapsto |x-y|$. The set of points $x \in \mathbb{R}^3$ where $L_x^2 \in C^\infty(S)$ fails to be a Morse function is called the set of *focal points*. By the result in [7, Section 11.3 p. 95] and [16, Corollary 6.2 p. 33], the set of *focal points* has null measure in $\mathbb{R}^3$. If $\psi \geqslant 0$ is a non negative Morse function on S with **only positive critical values** then

$$d_x\psi(c) = 0 \implies d_x^2\sqrt{\psi}(c) = \frac{d_x^2\psi(c)}{2\sqrt{\psi}} \neq 0.$$

This shows that if $x \notin S$ then L_x^2 is a Morse function iff L_x is Morse. Thus:

Proposition 5.4. For an *open everywhere dense set* of points $x \in \mathbb{R}^3$, the distance function

$$L_x := y \in S \longmapsto |x-y|$$

is Morse.

5.4. The distance function to almost every point is Morse excellent

The next Lemma is needed for the proof of Theorem 1.4.

Lemma 5.5. *Let S be a closed compact embedded surface in $\mathbb{R}^3$. For generic $x \in \mathbb{R}^3$, the function $L_x \in C^\infty(S)$ is an excellent Morse function.*

Proof. Note that if $x \notin S$ and L_x^2 is Morse excellent then so is L_x. Therefore, it suffices to prove that L_x^2 is Morse excellent for generic x. For every $(y,x) \in S \times \mathbb{R}^3$, let use denote by $g_x(y)$ the function $L_x(y)^2$.

For given x_0 s.t. g_{x_0} is Morse, we show there is a neighborhood U of x_0 such that the critical points of L_x depend smoothly on $x \in U$. Let us call $y_1(x_0), \dots, y_k(x_0)$ the isolated critical points of g_{x_0}. For all $l \in \{1, \dots, k\}$, $y_l(x_0)$ is a non degenerate critical point of g_x i.e. the Hessian $d_y^2 g_x$ is invertible. Therefore, we can use the implicit function theorem to express the critical points $(y_i)_{1\leq i\leq k}$ of g_x as functions of $x \in U$ in such a way that

$$\forall x \in U, \forall i \in \{1, \dots, k\}, d_y g_x(y_i(x)) = 0. \tag{5.1}$$

Hence

$$L_x^2 \text{ not Morse excellent } \Leftrightarrow g_x(y_i(x)) = g_x(y_j(x)) \text{ for some } 1 \leq i < j \leq k.$$

Observe that the subset $\Sigma_{ij} = \{x \in U | g_x(y_i(x)) = g_x(y_j(x))\}$ is a surface in U since

$$\partial_x g_x(y_i(x)) - \partial_x g_x(y_j(x)) = 2 \langle ., y_i(x) - y_j(x)\rangle \neq 0,$$

therefore the set of x such that L_x^2 fails to be Morse excellent has empty interior in U which proves the claim. □

Acknowledgements

We would like to thank Frédéric Hélein, Camille Laurent Gengoux, Mathieu Stiénon for enlighting discussions in geometry and Ping Xu who is at the origin of our interest for Lagrangian intersections.

References

[1] A. Aizenbud and V. Drinfeld, *The wave front set of the Fourier transform of algebraic measures*, `arXiv:1212.3630`, (2014).

[2] S. Alinhac, *Hyperbolic Partial Differential Equations*, Universitext, Springer, (2009).

[3] L. Bers, F. John, and M. Schechter, *Partial Differential Equations*, American Mathematical Society, third edition, (1971).

[4] C. Brouder, N. V. Dang, and F. Hélein, *Boundedness and continuity of the fundamental operations on distributions having a specified wave front set (with a counter example by Semyon Alesker)*, `arXiv:1409.7662`, (2014).

[5] C. Brouder, N. V. Dang, and F. Hélein, *A smooth introduction to the wavefront set*, `arXiv:1404.1778`.

[6] C. D.Sogge, *Lectures on Non-Linear Wave equations*, International Press, second edition, (2008).

[7] B. Dubrovin, A. Fomenko, and S. Novikov, *Modern Geometry II–Methods and Applications*, Springer, (1985).

[8] J. J. Duistermaat, *Fourier Integral Operators*, Birkhäuser, Boston, (1996).

[9] G. Eskin, *Lectures on Linear Partial Differential Equations*, Vol. 123 of Graduate Studies in Mathematics, Amer. Math. Soc., Providence, (2011).

[10] M. Farber, *Topology of Closed One-forms*, Vol. 108 of Mathematical surveys and monographs, AMS, (2004).

[11] J. Fu, *Curvature measures of subanalytic sets*, Amer. J. Math. **116** (1994), 819–880.

[12] A. Gabor, *Remarks on the wave front of a distribution*, Trans. Am. Math. Soc. **170** (1972), 239–44.

[13] L. Hörmander, *The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis*, Springer Verlag, Berlin, second edition, (1990).

[14] J.L. Brylinski, A. Dubson, and M. Kashiwara, *Formule de l'indice pour modules holonômes et obstruction d'Euler locale*, C. R. Acad. Sci. Paris Sér. I **293** (1981), 573–576.

[15] M. Kashiwara, *Index theorem for constructible sheaves*, Astérisque **130** (1985), 193–209.

[16] J. Milnor, *Morse Theory*, Princeton University Press, Princeton, (1969).

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II Fourier Analysis, Self-adjointness*, Academic Press, New York, (1975).

[18] S. Bates and A. Weinstein, *Lectures on the Geometry of Quantization*, Vol. 8, AMS, Berkeley mathematics lecture notes, edition (1998).

[19] S. P. Novikov and I. A. Taimanov, *Modern Geometric Structures and Fields*, Vol. 71 of Graduate Studies in Mathematics, AMS, (2006).

[20] E. M. Stein, *Beijing Lectures in Harmonic Analysis*, Annals of Mathematics Studies, Princeton University Press, (1986).

[21] E. M. Stein and R. Shakarchi, *Functional Analysis, An Introduction to Further Topics in Analysis*, Princeton University Press, (2011).

Laboratoire Paul Painlevé (U.M.R. CNRS 8524)
UFR de Mathématiques, Université de Lille 1
59 655 Villeneuve d'Ascq Cédex, France
E-mail address: Nguyen-Viet.Dang@math.univ-lille1.fr

Received September 12, 2014