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Yamabe invariants and the $\mathrm{Pin}^-(2)$-monopole equations

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We compute the Yamabe invariants for a new infinite class of closed 4-dimensional manifolds by using a "twisted" version of the Seiberg-Witten equations, the $\mathrm{Pin}^-(2)$-monopole equations. The same technique also provides a new obstruction to the existence of Einstein metrics or long-time solutions of the normalised Ricci flow with uniformly bounded scalar curvature.

1. Introduction

The Yamabe invariant is a diffeomorphism invariant of smooth manifolds, which arises from a variational problem for the total scalar curvature of Riemannian metrics. The $\mathrm{Pin}^-(2)$-monopole equations are a "twisted" version of the Seiberg-Witten equations. In this paper we will compute the Yamabe invariants for a new infinite class of closed 4-dimensional manifolds by using the $\mathrm{Pin}^-(2)$-monopole equations.

We begin by recalling the Yamabe invariant. Let X be a closed, oriented, connected manifold of $\dim X = m \geq 3$, and $\mathcal{M}(X)$ the space of all smooth Riemannian metrics on X. For each metric $g \in \mathcal{M}(X)$, we denote by s_g the scalar curvature and by $d\mu_g$ the volume form. Then the normalised Einstein-Hilbert functional $E_X \colon \mathcal{M}(X) \to \mathbb{R}$ is defined by

$$E_X \colon g \mapsto \frac{\int_X s_g \, d\mu_g}{\left(\int_X d\mu_g\right)^{\frac{m-2}{m}}}.$$

The classical Yamabe problem is to find a metric $\check{g}$ in a given conformal class C such that the normalised Einstein-Hilbert functional attains its minimum on C: $E_X(\check{g}) = \inf_{g\in C} E_X(g)$. This minimising metric $\check{g}$ is called a Yamabe metric, and a conformal invariant $\mathcal{Y}(X, C) := E_X(\check{g})$ the Yamabe constant.

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We define a diffeomorphism invariant $\mathcal{Y}(X)$ by the supremum of $\mathcal{Y}(X,C)$ of all the conformal classes C on X:

$$\mathcal{Y}(X) := \sup_C \mathcal{Y}(X,C) = \sup_C \inf_g \frac{\int_X s_g \, d\mu_g}{\left(\int_X d\mu_g\right)^{\frac{m-2}{m}}}.$$

We call it the Yamabe invariant of X; it is also referred to as the σ-constant. See [16] and [28].

It is a natural problem to compute the Yamabe invariant. In dimension 4, Seiberg-Witten theory and LeBrun's curvature estimates have played a prominent role in this problem. LeBrun used the ordinary Seiberg-Witten equations to compute the Yamabe invariants of most algebraic surfaces [17, 19]. In particular, he showed that a compact Kähler surface is of general type if and only if its Yamabe invariant is negative. He also showed $\mathcal{Y}(\mathbb{CP}^2) = 12\sqrt{2}\pi$ via the perturbed Seiberg-Witten equations [18]. Bauer and Furuta's stable cohomotopy Seiberg-Witten invariant [2] or Sasahira's spin bordism Seiberg-Witten invariant [27] enable us to compute the Yamabe invariants of connected sums of some compact Kähler surfaces [13–15, 27]. In this paper, we will employ a recently introduced "twisted" version of the Seiberg-Witten invariant, the $\mathrm{Pin}^-(2)$-monopole invariant [24], to compute the Yamabe invariants for a new infinite class of 4-dimensional manifolds. The advantage of using this new invariant lies in the fact that it can be non-trivial even when the ordinary Seiberg-Witten invariants, the spin bordism Seiberg-Witten invariants, and the stable cohomotopy Seiberg-Witten invariants all vanish. Example 6 lies at the heart of this paper.

We now state the main theorems of this paper. In what follows, $\chi(X)$ and $\tau(X)$ denote the Euler number and the signature of a manifold X respectively, and $mX := X\#\cdots\#X$ denotes the m-fold connected sum.

Theorem 1. *Let M be a compact, connected, minimal Kähler surface with $b_+(M) \geq 2$ and $c_1^2(M) = 2\chi(M) + 3\tau(M) \geq 0$. Let N be a closed, oriented, connected 4-manifold with $b_+(N) = 0$ and $\mathcal{Y}(N) \geq 0$. Let Z be a connected sum of arbitrary positive number of 4-manifolds, each of which belongs to one of the following types:*

1) *$S^2 \times \Sigma$, where Σ is a compact Riemann surface with positive genus, or*
2) *$S^1 \times Y$, where Y is a closed oriented 3-manifold.*

The Yamabe invariant of the connected sum $M\#N\#Z$ is equal to $-4\pi\sqrt{2c_1^2(M)}$.

Theorem 2. *Let M be an Enriques surface. Let N and Z satisfy the assumptions in Theorem 1. The Yamabe invariant of $M\#N\#Z$ is equal to* 0.

The key ingredients of the proofs are Proposition 9 and Proposition 12, the non-vanishing of the $\mathrm{Pin}^-(2)$-monopole invariants of $M\#N\#Z$. We emphasise that the ordinary Seiberg-Witten invariants, the spin bordism Seiberg-Witten invariants, and the stable cohomotopy Seiberg-Witten invariants all vanish if Z contains at least one $S^2 \times \Sigma$ as a connected-summand.

Much more subtle is the following theorem. In general, the moduli spaces of the $\mathrm{Pin}^-(2)$-monopole equations are, in contrast to ordinary Seiberg-Witten theory, not orientable, and only $\mathbb{Z}_2$-valued invariants are defined; these invariants are powerful enough to prove the theorems above.

Theorem 3. *Let M be an Enriques surface. Let N be a closed, oriented, connected 4-manifold with $b_+(N) = 0$ and $\mathcal{Y}(N) \geq 0$. For any $m \geq 2$, the Yamabe invariant of $mM\#N$ is equal to* 0*; moreover, it does not admit Riemannian metrics of non-negative scalar curvature.*

The ordinary Seiberg-Witten invariants of mM are trivial; furthermore, its $\mathbb{Z}_2$-valued $\mathrm{Pin}^-(2)$-monopole invariants are also trivial [24, Theorem 1.13]. We need refined $\mathbb{Z}$-valued $\mathrm{Pin}^-(2)$-monopole invariants to prove the last theorem.

2. The $\mathrm{Pin}^-(2)$-monopole equations and LeBrun's curvature estimates

2.1. The $\mathrm{Pin}^-(2)$-monopole equations

We briefly review $\mathrm{Pin}^-(2)$-monopole theory; for a thorough treatment, we refer the reader to [24, 25].

Let X be a closed, oriented, connected 4-manifold. Fix a Riemannian metric g on X. Let $\widetilde{X} \to X$ be an unbranched double cover, and $\ell := \widetilde{X} \times_{\{\pm 1\}} \mathbb{Z}$ its associated local system. Let $b_j^\ell(X) := \operatorname{rank} H^j(X; \ell)$ and $b_+^\ell(X) := \operatorname{rank} H^+(X; \ell)$. Recall that $\mathrm{Pin}^-(2) := \mathrm{U}(1) \cup jU(1) \subset \mathrm{Sp}(1)$ and $\mathrm{Spin}^{c-}(4) := \mathrm{Spin}(4) \times_{\{\pm 1\}} \mathrm{Pin}^-(2)$. A Spin^{c-}-structure on $\widetilde{X} \to X$ is defined to be a triple $\mathfrak{s} = (P, \sigma, \tau)$, where

- P is a $\mathrm{Spin}^{c-}(4)$-bundle on X,
- σ is an isomorphism between $\widetilde{X}$ and $P/\mathrm{Spin}^c(4)$, and
- τ is an isomorphism between the frame bundle of X and $P/\mathrm{Pin}^-(2)$.

We call the associated O(2)-bundle $E := P/\mathrm{Spin}(4)$ the characteristic bundle of a Spin^{c-}-structure $\mathfrak{s} = (P, \sigma, \tau)$, and denote its ℓ-coefficient Euler class by $\widetilde{c}_1(\mathfrak{s}) \in H^2(X; \ell)$. If $\widetilde{X} \to X$ is trivial, any Spin^{c-}-structure on $\widetilde{X} \to X$ canonically induces a Spin^c-structure on X [24, 2.4].

Spin^{c-}-structures are in many ways like Spin^c-structures: The Spin^{c-}-structure $\mathfrak{s}$ on $\widetilde{X} \to X$ determines a triple (S^+, S^-, ρ), where $S^\pm$ are the spinor bundles on X and $\rho\colon \Omega^1(X; \ell \otimes \sqrt{-1}\mathbb{R}) \to \mathrm{Hom}(S^+, S^-)$ is the Clifford multiplication. An O(2)-connection A on E gives a Dirac operator $D_A\colon \Gamma(S^+) \to \Gamma(S^-)$. Note that $F_A^+ \in \Omega^+(X; \ell \otimes \sqrt{-1}\mathbb{R})$. The canonical real quadratic map is denoted by $q\colon S^+ \to \Omega^+(X; \ell \otimes \sqrt{-1}\mathbb{R})$.

We denote by $\mathcal{A}$ the space of O(2)-connections on E. Let $\mathcal{C} := \mathcal{A} \times \Gamma(S^+)$ and $\mathcal{C}^* := \mathcal{A} \times (\Gamma(S^+) \setminus \{0\})$. We define the $\mathrm{Pin}^-(2)$-monopole equations to be

$$\begin{cases} D_A\Phi = 0 \\ \dfrac{1}{2}F_A^+ = q(\Phi) \end{cases}$$

for $(A, \Phi) \in \mathcal{C}$. The gauge group $\mathcal{G} := \Gamma(\widetilde{X} \times_{\{\pm 1\}} \mathrm{U}(1))$ acts on the set of solutions of these equations; the moduli space is defined to be the set of solutions modulo $\mathcal{G}$. The formal dimension of the moduli space is given by

$$d(\mathfrak{s}) := \frac{1}{4}\big(\widetilde{c}_1(\mathfrak{s})^2 - \tau(X)\big) - \big(b_+^\ell(X) - b_1^\ell(X) + b_0^\ell(X)\big).$$

Note that $b_0^\ell(X) = 0$ if $\widetilde{X}$ is non-trivial.

Let $\mathcal{B}^* := \mathcal{C}^*/\mathcal{G}$ be the irreducible configuration space. As in ordinary Seiberg-Witten theory, we can define the $\mathrm{Pin}^-(2)$-monopole invariant

$$\mathrm{SW}^{\mathrm{Pin}^-(2)}(X, \mathfrak{s})\colon H^{d(\mathfrak{s})}(\mathcal{B}^*; \mathbb{Z}_2) \to \mathbb{Z}_2$$

via intersection theory on the moduli space. In contrast to ordinary Seiberg-Witten theory, a moduli space of solutions of the $\mathrm{Pin}^-(2)$-monopole equations might not be orientable, and thus the invariant is, in general, $\mathbb{Z}_2$-valued. We remark, however, that, in the case of Theorem 3, the moduli spaces are orientable, and we will use the refined $\mathbb{Z}$-valued invariant [25, Theorem 1.13].

Example 4. Let $\widetilde{T^2} \to T^2$ be a non-trivial double cover, and $\ell := \widetilde{T^2} \times_{\pm 1} \mathbb{Z}$ its associated local system. Set $\Sigma := T^2 \# \cdots \# T^2$. The connected sum $\ell \# \cdots \# \ell$ gives a local system on Σ. We define a local system ℓ_Σ on $S^2 \times \Sigma$

by the pull-back of $\ell\#\cdots\#\ell$ by the projection $S^2\times\Sigma\to\Sigma$. Then, we have

$$b_0^{\ell_\Sigma}(S^2\times\Sigma)=b_2^{\ell_\Sigma}(S^2\times\Sigma)=b_4^{\ell_\Sigma}(S^2\times\Sigma)=0$$
$$b_1^{\ell_\Sigma}(S^2\times\Sigma)=b_3^{\ell_\Sigma}(S^2\times\Sigma)=\chi(\Sigma).$$

In particular, $b_+^{\ell_\Sigma}(S^2\times\Sigma)=0$, while $b_+(S^2\times\Sigma)>0$.

Example 5. Let Y be a closed oriented 3-manifold. Let $\widetilde{S^1}\to S^1$ be a connected double cover. We define a non-trivial double cover $\widetilde{S^1\times Y}\to S^1\times Y$ by the pull-back of $\widetilde{S^1}$ by the projection $S^1\times Y\to S^1$, and denote by ℓ_{S^1} its associated local system. Then, we have $b_j^{\ell_{S^1}}(S^1\times Y)=0$ for all $j=0,\dots,4$.

Example 6. Let Z be a connected sum

$$Z:=\big(S^2\times\Sigma_1\#\cdots\#S^2\times\Sigma_n\big)\#\big(S^1\times Y_1\#\cdots\#S^1\times Y_m\big),$$

where each Σ_j is a Riemann surface of positive genus and each Y_i is a closed oriented 3-manifold. We define a non-trivial double cover $\widetilde{Z}\to Z$ by the connected sum of $\widetilde{S^2\times\Sigma_j}$ and $\widetilde{S^1\times Y_i}$ in Examples 4 and 5, and denote by ℓ_Z its associated local system. We emphasise that $b_+^{\ell_Z}(Z)=0$, even if $b_+(Z)>0$. It follows that $\widetilde{c}_1(\mathfrak{s})$ is a torsion class for every Spin^{c-}-structure $\mathfrak{s}$ on $\widetilde{Z}\to Z$. See [25, Theorem 1.7].

2.2. LeBrun's curvature estimates

Definition 7. Let X be a closed, oriented, connected 4-manifold. Assume that X has a non-trivial double cover $\widetilde{X}\to X$ with $b_+^\ell(X)\geq 2$, where $\ell:=\widetilde{X}\times_{\pm1}\mathbb{Z}$. A cohomology class $\mathfrak{a}\in H^2(X;\ell)/\mathrm{Tor}$ is called a $\mathrm{Pin}^-(2)$-basic class if there exists a Spin^{c-}-structure $\mathfrak{s}$ on $\widetilde{X}\to X$ with $\widetilde{c}_1(\mathfrak{s})=\mathfrak{a}$ modulo torsions for which the $\mathrm{Pin}^-(2)$-monopole invariant is non-trivial.

As in ordinary Seiberg-Witten theory, if X has a $\mathrm{Pin}^-(2)$-basic class, the corresponding $\mathrm{Pin}^-(2)$-monopole equations have at least one solution for every Riemannian metric; hence, X does not admit Riemannian metrics of positive scalar curvature. We have, moreover, LeBrun's curvature estimates, which we will explain. In what follows, given a Riemannian metric g on X, we identify $H^2(X;\ell\otimes\mathbb{R})$ with the space of ℓ-coefficient g-harmonic 2-forms, and denote by $\mathfrak{a}^{+_g}$ the g-self-dual part of $\mathfrak{a}\in H^2(X;\ell)/\mathrm{Tor}\subset H^2(X;\ell\otimes\mathbb{R})$.

Proposition 8. *Let X be a closed, oriented, connected 4-manifold. Assume that X has a non-trivial double cover $\pi\colon \widetilde{X} \to X$ with $b_+^\ell(X) \geq 2$, where $\ell := \widetilde{X} \times_{\{\pm 1\}} \mathbb{Z}$. If there exists a $\mathrm{Pin}^-(2)$-basic class $\mathfrak{a} \in H^2(X;\ell)/\mathrm{Tor}$, then the following hold for every Riemannian metric g on X:*

- *The scalar curvature s_g of g satisfies*

$$\int_X s_g^2\, d\mu_g \geq 32\pi^2(\mathfrak{a}^{+_g})^2. \tag{1}$$

  If $\mathfrak{a}^{+_g} \neq 0$, equality holds if and only if there exists an integrable complex structure on the double cover $\widetilde{X}$ compatible with the pulled-back metric $\widetilde{g} := \pi^* g$ such that the covering transformation $\iota\colon \widetilde{X} \to \widetilde{X}$ is anti-holomorphic and the compatible Kähler form $\widetilde{\omega}$ satisfies $\iota^*\widetilde{\omega} = -\widetilde{\omega}$.

- *The scalar curvature s_g and the self-dual Weyl curvature W_g^+ of g satisfy*

$$\int_X \left(s_g - \sqrt{6}|W_g^+|\right)^2 d\mu_g \geq 72\pi^2(\mathfrak{a}^{+_g})^2. \tag{2}$$

  If $\mathfrak{a}^{+_g} \neq 0$, equality holds if and only if the pulled-back metric $\tilde{g} := \pi^* g$ on $\widetilde{X}$ is an almost-Kähler metric with almost-Kähler form $\tilde{\omega}$ such that $\iota^*\tilde{\omega} = -\tilde{\omega}$.

Proof. LeBrun's arguments [20, 21] or the perturbations introduced in [10] are easily adapted to prove (1) and (2) by using the Weitzenböck formulae of the Dirac operator for Spin^{c-}-spinors and the Hodge Laplacian for ℓ-coefficient self-dual forms.

Assume that equality holds. We lift a solution (A,Φ) of the $\mathrm{Pin}^-(2)$-monopole equations on X to the double cover $\widetilde{X}$. The lifted Spin^{c-}-structure on $\widetilde{X}$ canonically reduces to a Spin^c-structure, and the lifted solution $(\widetilde{A},\widetilde{\Phi})$ can be identified with a solution of the ordinary Seiberg-Witten equations on $\widetilde{X}$ that satisfies

$$\int_{\widetilde{X}} s_{\widetilde{g}}^2\, d\mu_{\widetilde{g}} = 32\pi^2((\pi^*\mathfrak{a})^{+_{\widetilde{g}}})^2, \text{ or } \int_{\widetilde{X}} \left(s_{\widetilde{g}} - \sqrt{6}|W_{\widetilde{g}}^+|\right)^2 d\mu_{\widetilde{g}} = 72\pi^2((\pi^*\mathfrak{a})^{+_{\widetilde{g}}})^2.$$

If the former (resp. latter) equality holds, the $\widetilde{g}$-self-dual form $\widetilde{\omega} = \sqrt{2}iq(\widetilde{\Phi})/|q(\widetilde{\Phi})|$ is a Kähler (resp. almost-Kähler) form compatible with $\widetilde{g}$. See [22, Proposition 3.2 and Proposition 3.8]. Since $q(\widetilde{\Phi}) = \pi^* q(\Phi)$ and $iq(\Phi) \in$

$\Omega^2(X;\ell \otimes \mathbb{R})$, we have $\iota^*\widetilde{\omega} = -\widetilde{\omega}$. In the former case, moreover, ι is anti-holomorphic because $\iota^*\widetilde{g} = \widetilde{g}$. □

3. Gluing formulae and Pin$^-$(2)-basic classes

Based on gluing formulae for the Pin$^-$(2)-monopole invariant [25], we will establish the existence of Pin$^-$(2)-basic classes on some classes of closed 4-manifolds.

3.1. Irreducible U(1) and reducible Pin$^-$(2).

We first establish a non-vanishing result based on a gluing formula for irreducible U(1)-monopoles and reducible Pin$^-$(2)-monopoles [25, Theorem 3.8]. It will play a pivotal role in the proof of Theorem 1.

Proposition 9. *Let M be a closed, oriented, connected 4-manifold that satisfies the following:*

- *$b_+(M) \geq 2$, and*
- *there exists a Spin^c-structure $\mathfrak{s}_M$ such that $c_1(\mathfrak{s}_M)^2 = 2\chi(M) + 3\tau(M)$ and its ordinary Seiberg-Witten invariant is odd.*

Let N be a closed, oriented, connected 4-manifold with $b_+(N) = 0$. Let Z be a connected sum of arbitrary positive number of 4-manifolds, each of which belongs to one of the following types:

1) *$S^2 \times \Sigma$, where Σ is a compact Riemann surface with positive genus,*
2) *$S^1 \times Y$, where Y is a closed oriented 3-manifold.*

Set $X := M\#N\#Z$. Then, there exists a non-trivial double cover $\widetilde{X} \to X$ and a Pin$^-$(2)-basic class $\mathfrak{a} \in H^2(X;\ell_X)$, where $\ell_X := \widetilde{X} \times_{\{\pm 1\}} \mathbb{Z}$, such that

$$(\mathfrak{a}^{+_g})^2 \geq 2\chi(M) + 3\tau(M)$$

for any Riemannian metric g on X.

Proof. Set $X_1 := M\#N$ and $X_2 := Z$. We will apply [25, Theorem 3.8] to $X = X_1\#X_2$ as follows.

We can choose a set of non-trivial smooth loops $\gamma_1, \ldots, \gamma_b$ in N so that surgery along them produces a 4-manifold N' with $b_1(N') = 0$ and $b_+(N') = 0$. Conversely, we can find a set of homologically trivial embedded 2-spheres in N' so that surgery along them recovers N. We will identify $H^2(N;\mathbb{Z})$ with $H^2(N';\mathbb{Z})$.

Set $X_1' := M\#N'$. Let $e_1,\dots,e_k$ be a set of generators for $H^2(N';\mathbb{Z})/\mathrm{Tor}$ relative to which the intersection form is diagonal [5]. By Froyshov's generalised blow-up formula [9, Corollary 14.1.1], X_1' has a Spin^c-structure $\mathfrak{s}_1'$ such that

$$c_1(\mathfrak{s}_1') = c_1(\mathfrak{s}_M) + (\pm e_1 + \cdots + \pm e_k),$$

its ordinary Seiberg-Witten moduli space is 0-dimensional, and its ordinary Seiberg-Witten invariant is equal to that of $(M,\mathfrak{s}_M)$. Here, the signs of $\pm e_i$ are arbitrary and independent of one another.

By Ozsváth and Szabó's surgery formula [26, Proposition 2.2], X_1 has a Spin^c-structure $\mathfrak{s}_1$ such that $c_1(\mathfrak{s}_1) = c_1(\mathfrak{s}_1')$ and

$$\mathrm{SW}^{\mathrm{U}(1)}(X_1,\mathfrak{s}_1)(\mu(\gamma_1)\cdots\mu(\gamma_b)) = \mathrm{SW}^{\mathrm{U}(1)}(X_1',\mathfrak{s}_1')(1)$$

for some homology orientation on X_1', where $\mathrm{SW}^{\mathrm{U}(1)}$ denotes the ordinary Seiberg-Witten invariant and $\mu\colon H_1(X_1;\mathbb{Z}) \to H^1(\mathcal{B}^*;\mathbb{Z})$ is a "μ-map" to the irreducible configuration space $\mathcal{B}^* = \mathcal{B}^*(\mathfrak{s}_1)$.

We take a non-trivial double cover $\widetilde{X_2} \to X_2$ as described in Example 6, and choose any Spin^{c-}-structure $\mathfrak{s}_2$ on $\widetilde{X_2} \to X_2$. Note that $\widetilde{c}_1(\mathfrak{s}_2)^2 = 0$.

Set $\widetilde{X} := X_1\#X_1\#\widetilde{X_2}$. It now follows from [25, Theorem 3.8] that

$$\mathfrak{a} := c_1(\mathfrak{s}_M) + (\pm e_1 + \cdots + \pm e_k) + \widetilde{c}_1(\mathfrak{s}_2)$$

is a Pin$^-(2)$-basic class. Given a Riemannian metric g on X, we can choose the signs of $\pm e_i$ so that

$$(\mathfrak{a}^{+_g})^2 \geq (c_1(\mathfrak{s}_M) + \widetilde{c}_1(\mathfrak{s}_2))^2 = 2\chi(M) + 3\tau(M)$$

holds [13, Corollary 11]. This completes the proof. □

3.2. Surgery formulae for the Pin$^-(2)$-monopole invariant

We digress to generalise Ozsváth and Szabó's surgery formula to the Pin$^-(2)$-monopole invariant.

We first describe a surgery formula for the $\mathbb{Z}_2$-valued Pin$^-(2)$-monopole invariant, which will be used to prove Proposition 12. Let X be a closed, oriented, connected 4-manifold and $\pi\colon \widetilde{X} \to X$ a non-trivial double cover. Fix a Spin^{c-}-structure $\mathfrak{s}$ on $\widetilde{X} \to X$. Let $S \subset X$ be an embedded 2-sphere with zero self-intersection number. Note that the restriction of $\mathfrak{s}$ to a tubular neighbourhood of S is untwisted; therefore, it canonically induces a usual Spin^c-structure on the neighbourhood. We denote by X' the manifold obtained by surgery on S, and let $C \subset X'$ be the core of the added $S^1 \times D^3$. The inverse image $\pi^{-1}(S) \subset \widetilde{X}$ consists of disjoint embedded 2-spheres S_1 and S_2. Equivariant surgery on S_1 and S_2 produces a double covering $\widetilde{X'} \to X'$. Let $\{C_1, C_2\} := \pi^{-1}(C) \subset \widetilde{X'}$.

$$\begin{array}{ccc} \widetilde{X} \setminus \{S_1 \cup S_2\} & \xrightarrow{\cong} & \widetilde{X'} \setminus \{C_1 \cup C_2\} \\ \downarrow & & \downarrow \\ X \setminus S & \xrightarrow{\cong} & X' \setminus C. \end{array}$$

There is a unique Spin^{c-}-structure $\mathfrak{s}'$ on $\widetilde{X'} \to X'$ with the property that

$$\mathfrak{s}'|_{\widetilde{X'}\setminus\{C_1\cup C_2\}\to X\setminus S} = \mathfrak{s}|_{\widetilde{X}\setminus\{S_1\cup S_2\}\to X'\setminus C}.$$

Note that the restriction of $\mathfrak{s}'$ to a tubular neighbourhood of C is untwisted; therefore, it canonically induces a usual Spin^c-structure on the neighbourhood. We define a "μ-map" associated with $\mathfrak{s}'$ by

$$\mu_{\mathcal{E}} \colon H_1(X'; \mathbb{Z}_2) \to H^1(\mathcal{B}^*; \mathbb{Z}_2), \quad \alpha \mapsto w_2(\mathcal{E})/\alpha,$$

where $\mathcal{E}$ is the universal characteristic O(2)-bundle on $X' \times \mathcal{B}^*$.

Proposition 10.

$$\mathrm{SW}^{\mathrm{Pin}^-(2)}(X', \mathfrak{s}')(\xi \cdot \mu_{\mathcal{E}}(C)) = \mathrm{SW}^{\mathrm{Pin}^-(2)}(X, \mathfrak{s})(\xi)$$

for any $\xi \in H^*(\mathcal{B}^*; \mathbb{Z}_2)$.

Proof. Fix a cylindrical-end metric on $X \setminus S$ modelled on the standard product metric on $[0, \infty) \times S^1 \times S^2$. This metric on $X \setminus S$ can be extended over both $S^1 \times D^3$ and $D^2 \times S^2$ to give metrics with non-negative scalar curvature. As noted above, the Spin^{c-}-structure $\mathfrak{s}$ induces a usual Spin^c-structure on a neighbourhood of S, and so does $\mathfrak{s}'$ on a neighbourhood of C. Thus, the moduli spaces of solution of the Pin$^-$(2)-monopole equations over $S^1 \times S^2$, $S^1 \times D^3$, and $D^2 \times S^2$ can be identified with the moduli spaces of reducible solutions of the ordinary Seiberg-Witten equations. We also observe that each solution of the Pin$^-$(2)-monopole equations on X and X' restricts to a solution of the ordinary Seiberg-Witten equations near S and C respectively. The rest of the proof runs parallel to that of [26, Proposition 2.2]. □

We next describe a surgery formula for the $\mathbb{Z}$-valued $\mathrm{Pin}^-(2)$-monopole invariant, which will be used to prove Theorem 3. Assume that the moduli space on $(X, \mathfrak{s})$ is orientable. As noted above, the restriction of $\mathfrak{s}$ and that of $\mathfrak{s}'$ canonically induce Spin^c-structures on tubular neighbourhoods of S and C respectively. For a Spin^c-structure, the determinant line bundle of its Dirac operators is always trivial. Then, by the excision property for the indices of families (see [7, 7.1.3] and [25, Lemma 6.10]), we can show that the moduli space on $(X', \mathfrak{s}')$ is also orientable. Consequently, if the $\mathbb{Z}$-valued invariant $\mathrm{SW}_{\mathbb{Z}}^{\mathrm{Pin}^-(2)}(X, \mathfrak{s})$ is defined, so does $\mathrm{SW}_{\mathbb{Z}}^{\mathrm{Pin}^-(2)}(X', \mathfrak{s}')$. We define another "$\mu$-map" associated with $\mathfrak{s}'$ by

$$\hat{\mu}_{\mathcal{E}} \colon H_1(X'; \ell') \to H^1(\mathcal{B}^*; \mathbb{Z}), \quad \alpha \mapsto \widetilde{c}_1(\mathcal{E})/\alpha,$$

where $\ell' := \widetilde{X'} \times_{\pm 1} \mathbb{Z}$. The proof of the following surgery formula also runs parallel to that of [26, Proposition 2.2].

Proposition 11. *Assume that the moduli space on $(X, \mathfrak{s})$ is orientable. We have, for any $\xi \in H^*(\mathcal{B}^*; \mathbb{Z})$,*

$$\mathrm{SW}_{\mathbb{Z}}^{\mathrm{Pin}^-(2)}(X', \mathfrak{s}')(\xi \cdot \hat{\mu}_{\mathcal{E}}(C)) = \mathrm{SW}_{\mathbb{Z}}^{\mathrm{Pin}^-(2)}(X, \mathfrak{s})(\xi)$$

for some orientations on the moduli spaces.

3.3. Irreducible $\mathrm{Pin}^-(2)$ and reducible $\mathrm{Pin}^-(2)$

We can establish another non-vanishing result based on a generalised blow-up formula for the $\mathrm{Pin}^-(2)$-monopole invariant [24, Theorem 3.9] and a gluing formula for irreducible $\mathrm{Pin}^-(2)$-monopoles and reducible $\mathrm{Pin}^-(2)$-monopoles [25, Theorem 3.11]. It will play a key role in the proof of Theorem 2.

Proposition 12. *Let M be a closed, oriented, connected 4-manifold that satisfies the following:*

- *there exists a non-trivial double cover $\widetilde{M} \to M$ with $b_+^{\ell_M}(M) \geq 2$, where $\ell_M = \widetilde{M} \times_{\{\pm 1\}} \mathbb{Z}$, and*

- *there exists a* Spin^{c-}*-structure* $\mathfrak{s}_M$ *on* $\widetilde{M} \to M$ *such that* $\widetilde{c}_1(\mathfrak{s}_M)^2 = 2\chi(M) + 3\tau(M)$ *and its* $\mathbb{Z}_2$*-valued* $\mathrm{Pin}^-(2)$*-monopole invariant is non-trivial.*

Let N *be a closed, oriented, connected* 4*-manifold with* $b_+(N) = 0$*. Let* Z *be a connected sum of arbitrary positive number of* 4*-manifolds, each of which belongs to one of the following types:*

1) $S^2 \times \Sigma$*, where* Σ *is a compact Riemann surface with positive genus,*
2) $S^1 \times Y$*, where* Y *is a closed oriented* 3*-manifold.*

Set $X := M\#N\#Z$*. Then, there exist a non-trivial double cover* $\widetilde{X} \to X$ *and a* $\mathrm{Pin}^-(2)$*-basic class* $\mathfrak{a} \in H^2(X; \ell_X)$*, where* $\ell_X := \widetilde{X} \times_{\{\pm 1\}} \mathbb{Z}$*, such that*

$$(\mathfrak{a}^{+_g})^2 \geq 2\chi(M) + 3\tau(M)$$

for any Riemannian metric on X*.*

Proof. Set $X_1 := M\#N$ and $X_2 := Z$. We will first apply [25, Theorem 3.9] to $X_1 = M\#N$, and next [25, Theorem 3.11] to $X = X_1\#X_2$ as follows.

We can choose a set of non-trivial smooth loops $\gamma_1, \dots, \gamma_b$ in N so that surgery along them produces a 4-manifold N' with $b_1(N') = 0$ and $b_+(N') = 0$. Conversely, we can find a set of homologically trivial embedded 2-spheres in N' so that surgery along them recovers N. We will identify $H^2(N; \mathbb{Z})$ with $H^2(N'; \mathbb{Z})$.

Set $X_1' := M\#N'$. Let $e_1, \dots, e_k$ be a set of generators for $H^2(N'; \mathbb{Z})/\mathrm{Tor}$ relative to which the intersection form is diagonal. By a generalised blow-up formula for the $\mathrm{Pin}^-(2)$-monopole invariant [25, Theorem 3.9], we have a double cover $\widetilde{X_1'} \to X_1'$ and a unique Spin^{c-}-structure $\mathfrak{s}_1'$ on it such that

$$\widetilde{c}_1(\mathfrak{s}_1') = \widetilde{c}_1(\mathfrak{s}_M) + (\pm e_1 + \dots + \pm e_k),$$

its $\mathrm{Pin}^-(2)$-monopole moduli space is 0-dimensional, and its $\mathrm{Pin}^-(2)$-monopole invariant is equal to that of $(M, \mathfrak{s}_M)$. Here, the signs of $\pm e_i$ are arbitrary and independent of one another.

By Proposition 10, we have a double cover $\widetilde{X_1} \to X_1$ and a unique Spin^{c-}-structure $\mathfrak{s}_1$ on it such that $\widetilde{c}_1(\mathfrak{s}_1) = \widetilde{c}_1(\mathfrak{s}_1')$ and

$$\mathrm{SW}^{\mathrm{Pin}^-(2)}(X_1, \mathfrak{s}_1)(\mu_{\mathcal{E}}(\gamma_1) \cdots \mu_{\mathcal{E}}(\gamma_b)) = \mathrm{SW}^{\mathrm{Pin}^-(2)}(X_1', \mathfrak{s}_1')(1).$$

We take a non-trivial double cover $\widetilde{X_2} \to X_2$ as described in Example 6, and choose any Spin^{c-}-structure $\mathfrak{s}_2$ on $\widetilde{X_2} \to X_2$. Note that $\widetilde{c}_1(\mathfrak{s}_2)^2 = 0$.

It now follows from [25, Theorem 3.11] that

$$\mathfrak{a} := c_1(\mathfrak{s}_M) + (\pm e_1 + \cdots + \pm e_k) + \widetilde{c}_1(\mathfrak{s}_2)$$

is a $\mathrm{Pin}^-(2)$-basic class. Given a Riemannian metric g on X, we can choose the signs of $\pm e_i$ so that

$$(\mathfrak{a}^{+_g})^2 \geq (c_1(\mathfrak{s}_M) + \widetilde{c}_1(\mathfrak{s}_2))^2 = 2\chi(M) + 3\tau(M)$$

holds [13, Corollary 11]. This completes the proof. □

4. Computations of the Yamabe invariant

Let us recall that we have

$$\mathcal{I}_s(X) := \inf_g \int_X |s_g|^2 \, d\mu_g = \begin{cases} (\mathcal{Y}(X))^2 & \text{if } \mathcal{Y}(X) \leq 0 \\ 0 & \text{if } \mathcal{Y}(X) \geq 0 \end{cases}$$

for any closed oriented 4-manifold X [3, 19].

Proposition 13. *Let M, N, and Z satisfy the assumptions in Proposition 9 or Proposition 12. Set $X = M \# N \# Z$. Then, we have*

$$\mathcal{I}_s(X) \geq 32\pi^2 \big(2\chi(M) + 3\tau(M)\big).$$

Proof. Proposition 9 or Proposition 12 and LeBrun's curvature estimate (1) imply that

$$\int_X s_g^2 \, d\mu_g \geq 32\pi^2 (\mathfrak{a}^{+_g})^2 \geq 32\pi^2 \big(2\chi(M) + 3\tau(M)\big)$$

for any Riemannian metric g on X. □

Proof of Theorem 1 and Theorem 2. Let M, N, and Z satisfy the assumptions in Theorem 1 or Theorem 2. We have $\mathcal{I}_s(M) = 32\pi^2 c_1^2(M)$ by [17, 19], and $\mathcal{I}_s(N) = 0$ by assumption. Note that $\mathcal{Y}(S^1 \times Y) \geq 0$ for any closed oriented 3-manifold Y; thus, $\mathcal{I}_s(Z) = 0$. We remark that an Enriques surface satisfies the assumption for M in Proposition 12 by [25, Theorem 1.3]. Set $X := M \# N \# Z$.

By Proposition 13, we have

$$\mathcal{I}_s(X) \geq 32\pi^2\big(2\chi(M) + 3\tau(M)\big) = 32\pi^2 c_1^2(M).$$

On the other hand, by [13, Proposition 13], we have

$$\mathcal{I}_s(X) \leq \mathcal{I}_s(M) + \mathcal{I}_s(N) + \mathcal{I}_s(Z) = 32\pi^2 c_1^2(M).$$

Since X has a Pin$^-(2)$-basic class, $\mathcal{Y}(X) \leq 0$. Thus,

$$\mathcal{Y}(X) = -\sqrt{\mathcal{I}_s(X)} = -4\pi\sqrt{2c_1^2(M)}.$$

This completes the proof. □

Proof of Theorem 3. Let M be an Enriques surface. By [25, Theorem 1.13], the $\mathbb{Z}$-valued Pin$^-(2)$-monopole invariant of mM is non-trivial for any $m \geq 2$.

If $b_1(N) = 0$, by [24, Theorem 3.9], the $\mathbb{Z}$-valued Pin$^-(2)$-monopole invariant of $mM\#N$ is non-trivial. We remark that [24, Theorem 3.9] holds for the $\mathbb{Z}$-valued Pin$^-(2)$-monopole invariant. If $b_1(N) > 0$, by using Proposition 11 as in the proof of Proposition 9 or that of Proposition 12, we are reduced to the case when $b_1(N) = 0$. Thus, the $\mathbb{Z}$-valued Pin$^-(2)$-monopole invariant of $mM\#N$ is non-trivial. In particular, we have

$$\mathcal{Y}(mM\#N) \leq 0.$$

On the other hand, we have

$$0 \leq \mathcal{I}_s(mM\#N) \leq m\mathcal{I}_s(M) + \mathcal{I}_s(N) = 0.$$

Thus, $\mathcal{Y}(mM\#N) = 0$.

Since $2\chi(mM\#N) + 3\tau(mM\#N) < 0$, by the Hitchin-Thorpe inequality, it does not admit Ricci-flat metrics. Consequently, it does not admit Riemannian metrics of non-negative scalar curvature. □

5. Obstructions to Einstein metrics

We begin by examining LeBrun's inequalities (Cf. [20, Proposition 3.2]).

Lemma 14. *Let M, N, and Z satisfy the assumptions in Proposition 9 or Proposition 12. If equality holds in either (1) or (2) for some Riemannian metric g on $X := M\#N\#Z$, then $\mathfrak{a}^{+_g} = 0$.*

Proof. Suppose that equality holds and $\mathfrak{a}^{+_g} \neq 0$. Proposition 8 implies that the double cover $\widetilde{X} = \widetilde{M \# Z}$ admits an almost-Kähler structure; therefore, its ordinary Seiberg-Witten invariant is non-trivial [29]. On the other hand, $\widetilde{X} = M \# M \# N \# N \# \widetilde{Z}$ or $\widetilde{X} = \widetilde{M} \# N \# N \# \widetilde{Z} \# (S^1 \times S^3)$ according as M satisfies the assumptions of Proposition 9 or those of Proposition 12; in either case, $\widetilde{X}$ has at least two connected-summands with positive b_+; thus, its ordinary Seiberg-Witten invariant is trivial. This is a contradiction. □

Proposition 15. *Let M, N, and Z satisfy the assumptions in Proposition 9 or Proposition 12. Then, we have a strict inequality*

$$\frac{1}{4\pi^2}\int_X \left(\frac{s_g^2}{24} + 2|W_g^+|^2\right) d\mu_g > \frac{2}{3}\big(2\chi(M) + 3\tau(M)\big)$$

for any Riemannian metric g on $X := M \# N \# Z$.

Proof. Combined with Proposition 9 or Proposition 12, the Cauchy-Schwarz inequality and LeBrun's curvature estimate (2) yield

$$\begin{aligned}\frac{1}{4\pi^2}\int_X \left(\frac{s_g^2}{24} + 2|W_g^+|^2\right) d\mu_g &\geq \frac{1}{4\pi^2}\frac{1}{27}\int_X \left(s_g - \sqrt{6}|W_g^+|\right)^2 d\mu_g \\ &\geq \frac{1}{4\pi^2}\frac{1}{27}\cdot 72\pi^2(\mathfrak{a}^{+_g})^2 \\ &\geq \frac{2}{3}\big(2\chi(M) + 3\tau(M)\big)\end{aligned}$$

for any Riemannian metric g on X (Cf. [20, Proposition 3.1]).

We remark that X is not diffeomorphic to a finite quotient of a K3 surface or T^4; in particular, X does not admit a Ricci-flat anti-self-dual metric [11]. Suppose that equality holds for some Riemannian metric g on X. By Lemma 14, we have $\mathfrak{a}^{+_g} = 0$; therefore, $s_g = W_g^+ = 0$. Note that X does not admit a Riemannian metric of positive scalar curvature by Proposition 9 or Proposition 12. Consequently, g is Ricci-flat and anti-self-dual. This is a contradiction. □

Proposition 15 leads to a new obstruction to the existence of Einstein metrics (Cf. [13, Section 6]).

Theorem 16. *Let M, N, and Z satisfy the assumptions in Proposition 9 or Proposition 12. If $X := M \# N \# Z$ admits an Einstein metric, then*

$$\frac{1}{3}\big(2\chi(M) + 3\tau(M)\big) > 4 - \big(2\chi(N \# Z) + 3\tau(N \# Z)\big).$$

Proof. We first note that

$$\begin{aligned}2\chi(X)+3\tau(X) &= 2\big(\chi(M)+\chi(N\#Z)-2\big)+3\big(\tau(M)+\tau(N\#Z)\big)\\ &= 2\chi(M)+3\tau(M)+2\chi(N\#Z)+3\tau(N\#Z)-4.\end{aligned}$$

By the Chern-Gauss-Bonnet formula and the Hirzebruch signature theorem, if X admits an Einstein metric, we have

$$2\chi(X)+3\tau(X)=\frac{1}{4\pi^2}\int_X\left(\frac{s_g^2}{24}+2|W_g^+|^2\right)d\mu_g.$$

By Proposition 15, we have a strict inequality

$$\frac{1}{4\pi^2}\int_X\left(\frac{s_g^2}{24}+2|W_g^+|^2\right)d\mu_g>\frac{2}{3}\big(2\chi(M)+3\tau(M)\big).$$

Thus, we have

$$2\chi(M)+3\tau(M)+2\chi(N\#Z)+3\tau(N\#Z)-4>\frac{2}{3}\big(2\chi(M)+3\tau(M)\big).$$

The proof is completed by rearranging terms. □

Theorem 17. *Let M, N, and Z satisfy the assumptions in Proposition 9 or Proposition 12. If $X:=M\#N\#Z$ admits an anti-self-dual Einstein metric, then*

$$\frac{1}{4}\big(2\chi(M)+3\tau(M)\big)>4-\big(2\chi(N\#Z)+3\tau(N\#Z)\big).$$

Proof. We first note that

$$\begin{aligned}2\chi(X)+3\tau(X) &= 2\big(\chi(M)+\chi(N\#Z)-2\big)+3\big(\tau(M)+\tau(N\#Z)\big)\\ &= 2\chi(M)+3\tau(M)+2\chi(N\#Z)+3\tau(N\#Z)-4.\end{aligned}$$

By the Chern-Gauss-Bonnet formula and the Hirzebruch signature theorem, if X admits an anti-self-dual Einstein metric, we have

$$2\chi(X)+3\tau(X)=\frac{1}{4\pi^2}\int_X\frac{s_g^2}{24}\,d\mu_g=\frac{1}{96\pi^2}\int_X s_g^2\,d\mu_g.$$

We have a strict inequality

$$\int_X s_g^2\,d\mu_g>72\pi^2\big(2\chi(M)+3\tau(M)\big)=96\pi^2\cdot\frac{3}{4}\big(2\chi(M)+3\tau(M)\big),$$

which follows by the same method as in Proposition 15 using LeBrun's curvature estimate (2) and Lemma 14. Thus, we have

$$2\chi(M) + 3\tau(M) + 2\chi(N\#Z) + 3\tau(N\#Z) - 4 > \frac{3}{4}\big(2\chi(M) + 3\tau(M)\big).$$

The proof is completed by rearranging terms. □

Example 18. Mumford constructed a compact complex surface K of general type that is homeomorphic to the complex projective plane [23]. Let M be a closed symplectic manifold with $b_+(M) \geq 2$. Let Z be a connected sum of arbitrary positive number of 4-manifolds, each of which belongs to one of the following types:

1) $S^2 \times \Sigma$, where Σ is a compact Riemann surface with positive genus,
2) $S^1 \times Y$, where Y is a closed oriented 3-manifold.

Then, $M\#m\overline{\mathbb{C}\mathrm{P}^2}\#n\overline{K}\#Z$ does not admit an Einstein metric if

$$4 - 5(n+m) \geq \big(2\chi(Z) + 3\tau(Z)\big) + \frac{1}{3}\big(2\chi(M) + 3\tau(M)\big),$$

and it does not admit an anti-self-dual Einstein metric if

$$4 - 5(n+m) \geq \big(2\chi(Z) + 3\tau(Z)\big) + \frac{1}{4}\big(2\chi(M) + 3\tau(M)\big).$$

We end this section by examining an equality related to Proposition 15, the proof of which is worth mentioning here although it will not play any role in our work.

Proposition 19. *Let $\pi\colon \widetilde{M} \to M$ satisfy the the assumptions in Proposition 12. If there exists a Riemannian metric g on M that satisfies*

$$\frac{1}{4\pi^2}\int_M \left(\frac{s_g^2}{24} + 2|W_g^+|^2\right) d\mu_g = \frac{2}{3}\big(2\chi(M) + 3\tau(M)\big),$$

then $(\widetilde{M}, \pi^* g)$ is a K3 surface or T^4 with hyper-Kähler metric and the covering transformation of $\widetilde{M}$ is anti-holomorphic; moreover, M is an Enriques surface if $\widetilde{M}$ is a K3 surface.

Proof. It follows from a similar argument as in [20, Proposition 3.2] that $(\widetilde{M}, \pi^* g)$ is a K3 surface or T^4 with hyperKähler metric, and that the covering transformation is anti-holomorphic. By "Donaldson's trick" (see [6] and

[4, Section 15.1]), we can show that there exists another complex structure on $\widetilde{M}$ compatible with $\pi^* g$ for which the covering transformation is holomorphic; in particular, M is an Enriques surface if $\widetilde{M}$ is a K3 surface. □

6. Obstructions to long-time Ricci flows

Recall that a long-time solution of the normalised Ricci flow is a family of Riemannian metrics that satisfies

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + \frac{2}{m}\left(\frac{\int_X s_{g(t)}\, d\mu_{g(t)}}{\int_X d\mu_{g(t)}}\right) g(t)$$

for $t \in [0, \infty)$. Proposition 15 also leads to a new obstruction to the existence of long-time solutions of the normalised Ricci flow with uniformly bounded scalar curvature (Cf. [12, Section 5]).

Lemma 20. *Let M, N, and Z satisfy the assumptions in Proposition 9 or Proposition 12. If $X := M\#N\#Z$ admits a long-time solution of the normalised Ricci flow with uniformly bounded scalar curvature, then we have $\mathcal{Y}(X) < 0$.*

Proof. By Proposition 9 or Proposition 12, X has a $\mathrm{Pin}^-(2)$-basic class; hence, $\mathcal{Y}(X) \le 0$. Then, by [1, Theorem A] and [30, Theorem 1.1], we have a Hitchin-Thorpe type inequality

$$2\chi(X) - 3|\tau(X)| \ge \frac{1}{96\pi^2}\mathcal{Y}(X)^2.$$

Thus, $2\chi(X) + 3\tau(X) \ge 0$. Note that $2\chi(N\#Z) + 3\tau(N\#Z) < 0$. Thus, we get

$$2\chi(M) + 3\tau(M) > 0.$$

By Proposition 13, we have

$$\mathcal{Y}(X) = -\mathcal{I}_s(X) \le -32\pi^2\big(2\chi(M) + 3\tau(M)\big) < 0.$$

This completes the proof. □

Theorem 21. *Let M, N, and Z satisfy the assumptions in Proposition 9 or Proposition 12. If $X := M\#N\#Z$ admits a long-time solution of the normalised Ricci flow with uniformly bounded scalar curvature, then*

$$4 - \big(2\chi(N\#Z) + 3\tau(N\#Z)\big) \le \frac{1}{3}\big(2\chi(M) + 3\tau(M)\big).$$

Proof. By Lemma 20, we have $\mathcal{Y}(X) < 0$. Then, by [12, Proposition 5], we have

$$\sup_{t\in[0,\infty)} \min_{x\in X} s_{g(t)}(x) < 0.$$

Thus, by [8, Lemma 3.1], we have

$$\int_0^\infty \int_X \left|\overset{\circ}{r}_{g(t)}\right|^2 d\mu_{g(t)}\, dt < \infty,$$

where we denote by $\overset{\circ}{r}$ the traceless Ricci tensor. Hence, we have

$$\lim_{m\to\infty} \int_m^{m+1} \int_X \left|\overset{\circ}{r}_{g(t)}\right|^2 d\mu_{g(t)}\, dt = 0. \tag{3}$$

By the Chern-Gauss-Bonnet formula and the Hirzebruch signature theorem, we have

$$2\chi(X) + 3\tau(X) = \frac{1}{4\pi^2}\int_X \left(\frac{s_{g(t)}^2}{24} + 2|W_{g(t)}^+|^2 - \frac{\left|\overset{\circ}{r}_{g(t)}\right|^2}{2}\right) d\mu_{g(t)}$$

for any $t \in [0,\infty)$. Hence, we have

$$2\chi(X) + 3\tau(X) = \frac{1}{4\pi^2}\int_m^{m+1}\int_X \left(\frac{s_{g(t)}^2}{24} + 2|W_{g(t)}^+|^2 - \frac{\left|\overset{\circ}{r}_{g(t)}\right|^2}{2}\right) d\mu_{g(t)}\, dt \tag{4}$$

for any $m \in [0,\infty)$. By (3) and (4), we have

$$2\chi(X) + 3\tau(X) = \lim_{m\to\infty}\frac{1}{4\pi^2}\int_m^{m+1}\int_X \left(\frac{s_{g(t)}^2}{24} + 2|W_{g(t)}^+|^2\right) d\mu_{g(t)}\, dt.$$

On the other hand, by Lemma 15, we have

$$\frac{1}{4\pi^2}\int_X \left(\frac{s_{g(t)}^2}{24} + 2|W_{g(t)}^+|^2\right) d\mu_{g(t)} > \frac{2}{3}\big(2\chi(M) + 3\tau(M)\big)$$

for any $t \in [0,\infty)$. Thus,

$$\lim_{m\to\infty}\frac{1}{4\pi^2}\int_m^{m+1}\int_X \left(\frac{s_{g(t)}^2}{24} + 2|W_{g(t)}^+|^2\right) d\mu_{g(t)}\, dt \geq \frac{2}{3}\big(2\chi(M) + 3\tau(M)\big).$$

Consequently, we get

$$2\chi(X) + 3\tau(X) \geq \frac{2}{3}\big(2\chi(M) + 3\tau(M)\big).$$

This completes the proof. □

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